whether the train has been running uniformly or not entirely depends on the standard of time which we adopt.

Now, for all ordinary purposes of life on the earth, the various astronomical recurrences may be looked on as absolutely consistent; and, furthermore assuming their consistency, and thereby assuming the velocities and changes of velocities possessed by bodies, we find that the laws of motion, which have been considered above, are almost exactly verified. But only *almost* exactly when we come to some of the astronomical phenomena. We find, however, that by assuming slightly different velocities for the rotations and motions of the planets and stars, the laws would be exactly verified. This assumption is then made; and we have, in fact thereby, adopted a measure of time, which is indeed defined by reference to the astronomical phenomena, but not so as to be consistent with the uniformity of any one of them. But the broad fact remains that the uniform flow of time on which so much is based, is itself dependent on the observation of periodic events.

Even phenomena, which on the surface seem casual and exceptional, or, on the other hand, maintain themselves with a uniform persistency, may be due to the remote influence of periodicity. Take for example, the principle of resonance. Resonance arises when two sets of connected circumstances have the same periodicities. It is a dynamical law that the small vibrations of all bodies when left to themselves take place in definite times characteristic of the body. Thus a pendulum with a small swing always vibrates in some definite time, characteristic of its shape and distribution of weight and length. A more complicated body may have many ways of vibrating; but each of its modes of vibration will have its own peculiar "period." Those periods of vibration of a body are called its "free" periods. Thus a pendulum has but one period of vibration, while a suspension bridge will have many. We get a musical instrument, like a violin string, when the periods of vibration are all simple submultiples of the longest; *i.e.* if t seconds be the longest period, the others are $\frac{1}{2}t$, $\frac{1}{3}t$, and so on, where any of these smaller periods may be absent. Now, suppose we excite the vibrations of a body by a cause which is itself periodic; then, if the period of the cause is very nearly that of one of the periods of the body, that mode of vibration of the body is very violently excited; even although the magnitude of the exciting cause is small. This phenomenon is called "resonance." The general reason is easy to understand. Any one wanting to upset a rocking stone will push "in tune" with the oscillations of the stone, so as always to secure a favourable moment for a push. If the pushes are out of tune, some increase the oscillations, but others check them. But when they are in tune, after a time all the pushes are favourable. The word "resonance" comes from considerations of sound: but the phenomenon extends far beyond the region of sound. The laws of absorption and emission of light depend on it, the "tuning" of receivers for wireless telegraphy, the comparative importance of the influences of planets on each other's motion, the danger to a suspension bridge as troops march over it in step, and the excessive vibration of some ships under the rhythmical beat of their machinery at certain speeds. This coincidence of periodicities may produce steady phenomena when there is a constant association of the two periodic events, or it may produce violent and sudden outbursts when the association is fortuitous and temporary.

Again, the characteristic and constant periods of vibration mentioned above are the underlying causes of what appear to us as steady excitements of our senses. We work for hours in a steady light, or we listen to a steady unvarying sound. But, if modern science be correct, this steadiness has no counterpart in nature. The steady light is due to the impact on the eye of a countless number of periodic waves in a vibrating ether, and the steady sound to similar waves in a vibrating air. It is not our purpose here to explain the theory of light or the theory of sound. We have said enough to make it evident that one of the first steps necessary to make mathematics a fit instrument for the investigation of Nature is that it should be able to express the essential periodicity of things. If we have grasped this, we can understand the importance of the mathematical conceptions which we have next to consider, namely, periodic functions.

CHAPTER XIII

TRIGONOMETRY

TRIGONOMETRY did not take its rise from the general consideration of the periodicity of nature. In this respect its history is analogous to that of conic sections, which also had their origin in very particular ideas. Indeed, a comparison of the histories of the two sciences yields some very instructive analogies and contrasts. Trigonometry, like conic sections, had its origin among the Greeks. Its inventor was Hipparchus (born about 160 B.C.), a Greek astronomer, who made his observations at Rhodes. His services to astronomy were very great, and it left his hands a truly scientific subject with important results established, and the right method of progress indicated. Perhaps the invention of trigonometry was not the least of these services to the main science of his study. The next man who extended trigonometry was Ptolemy, the great Alexandrian astronomer, whom we have already mentioned. We now see at once the great contrast between conic sections and trigonometry. The origin of trigonometry was practical; it was invented because it was necessary for astronomical research. The origin of conic sections was purely theoretical. The only reason for its initial study was the abstract interest of the ideas involved. Characteristically enough conic sections were invented about 150 years earlier than trigonometry, during the very best period of Greek thought. But the importance of trigonometry, both to the theory and the application of mathematics, is only one of innumerable instances of the fruitful ideas which the general science has gained from its practical applications.

We will try and make clear to ourselves what trigonometry is, and why it should be generated by the scientific study of astronomy. In the first place: What are the measurements which can be made by an astronomer? They are measurements of time and measurements of angles. The astronomer may adjust a telescope (for it is easier to discuss the familiar instrument of modern astronomers) so that it can only turn about a fixed axis pointing east and west; the result is that the telescope can only point to the south, with a greater or less elevation of direction, or, if turned round beyond the zenith, point to the north. This is the transit instrument, the great instrument for the exact measurement of the times at which stars are due south or due north. But indirectly this instrument measures angles. For when the time elapsed between the transits of two stars has been noted, by the assumption of the uniform rotation of the earth, we obtain the angle through which the earth has turned in that period of time. Again, by other instruments, the angle between two stars can be directly measured. For if E is the eye of the astrono-

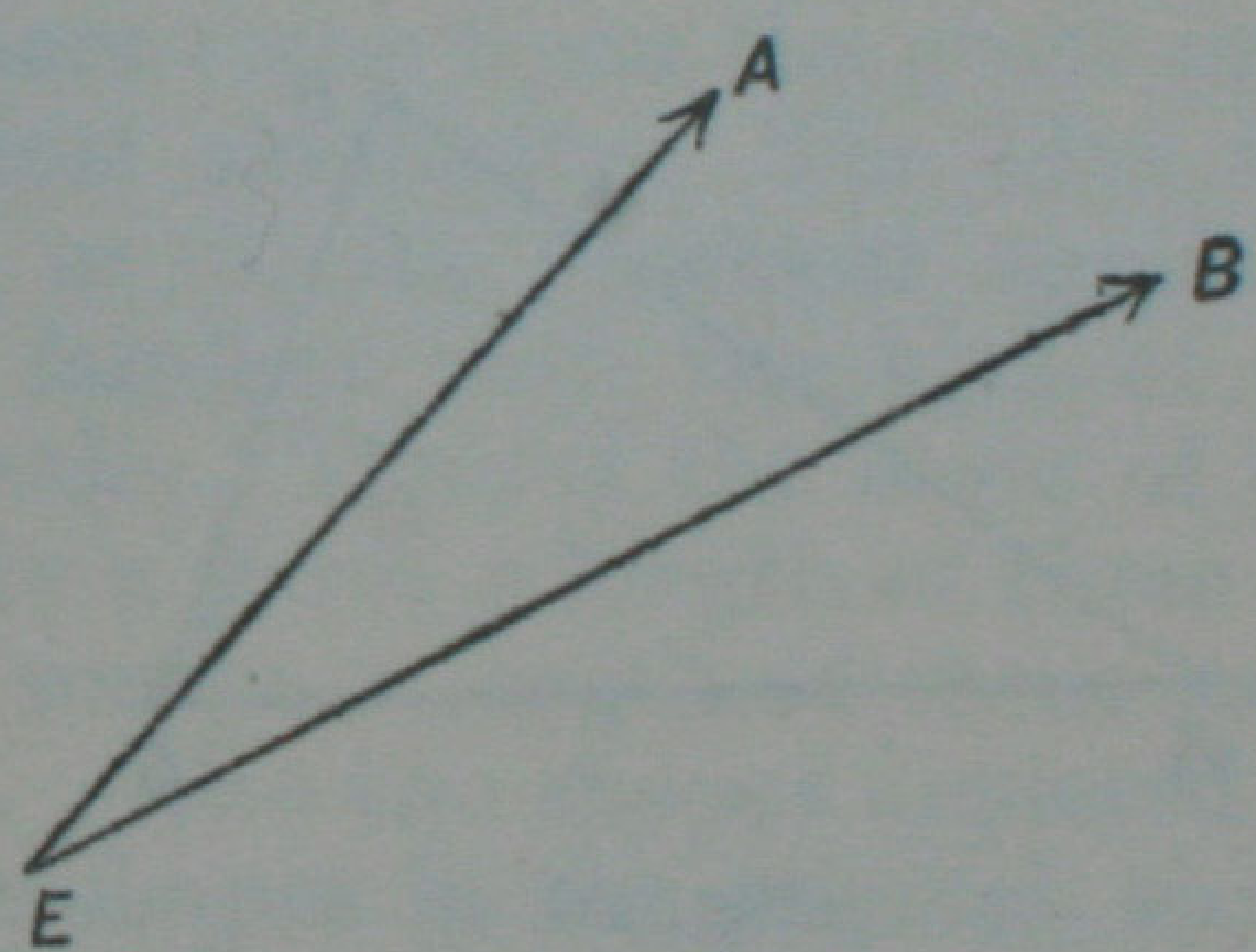

Fig. 22.

mer, and EA and EB are the directions in which the stars are seen, it is easy to devise instruments which shall measure the angle AEB. Hence, when the astronomer is forming a survey of the heavens, he is, in fact, measuring angles so as to fix the relative directions of the stars and planets at any instant. Again, in the analogous problem of land-surveying, angles are the chief subject of measurements. The direct measurements of length are only rarely possible with any accuracy; rivers, houses, forests, mountains, and general irregularities of ground all get in the way. The survey of a whole country will depend only on one or two direct measurements of length, made with the greatest elaboration in selected places like Salisbury Plain. The main work of a survey is the measurement of angles. For example, *A*, *B*, and *C* will be conspicuous points in the dis-

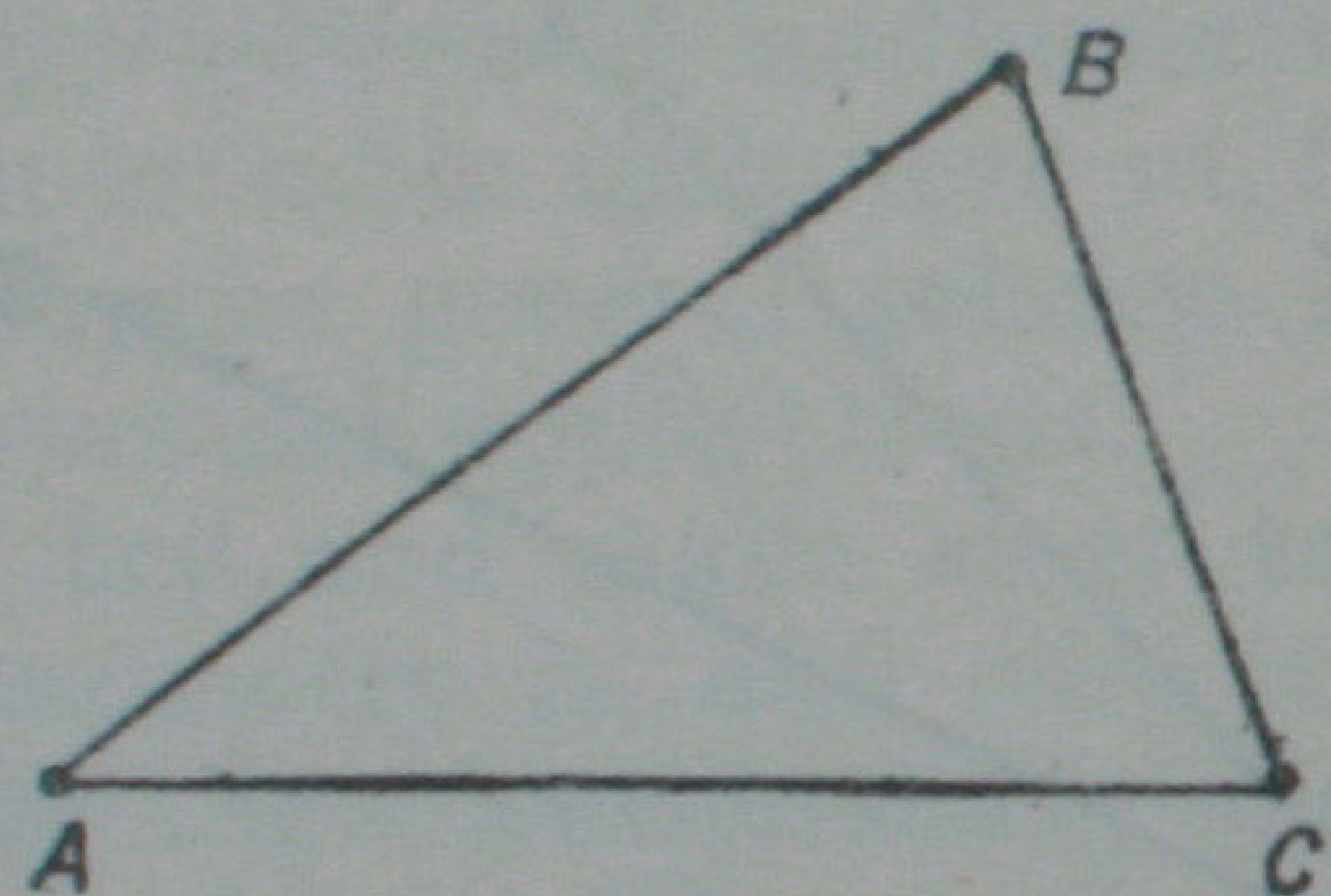

Fig. 23.

trict surveyed, say the tops of church towers. These points are visible each from the others. Then it is a very simple matter at *A* to measure the angle *BAC*, and at *B* to measure the angle *ABC*, and at *C* to measure the angle *BCA*. Theoretically, it is only necessary to measure two of these angles; for, by a well-known proposition in geometry, the sum of the three angles of a triangle amounts to two right-angles, so that when two of the angles are known, the third can be deduced. It is better, however, in practice to measure all three, and then any small errors of observation can be checked. In the process of map-making a country is completely covered with triangles in this way. This process is called triangulation, and is the fundamental process in a survey.

Now, when all the angles of a triangle are known, the shape of the triangle is known—that is, the shape as distinguished from the size. We here come upon the great principle of geometrical similarity. The idea is very familiar to us in its practical applications. We are all familiar with the idea of a plan drawn to scale. Thus if the scale of a plan be an inch to a yard, a length of three inches in the plan means a length of three yards in the original. Also the shapes depicted in the plan are the shapes in the original, so that a right-angle in the original appears as a right-angle in the plan. Similarly in a map, which is only a plan of a country, the proportions of the lengths in the map are the proportions of the distances between the places indicated, and the directions in the map are the directions in the country. For example, if in the map one place is north-north-west of the other, so it is in reality; that is to say, in a map the angles are the same as in reality.

Geometrical similarity may be defined thus: Two figures are similar (i) if to any point in one figure a point in the other figure corresponds, so that to every line there is a corresponding line, and to every angle a corresponding angle, and (ii) if the lengths of corresponding lines are in a fixed proportion, and the magnitudes of corresponding angles are the same. The fixed proportion of the lengths of corresponding lines in a map (or plan) and in the original is called the scale of the map. The scale should always be indicated on the margin of every map and plan. It has already been pointed out that two triangles whose angles are respectively equal are similar. Thus, if the two triangles

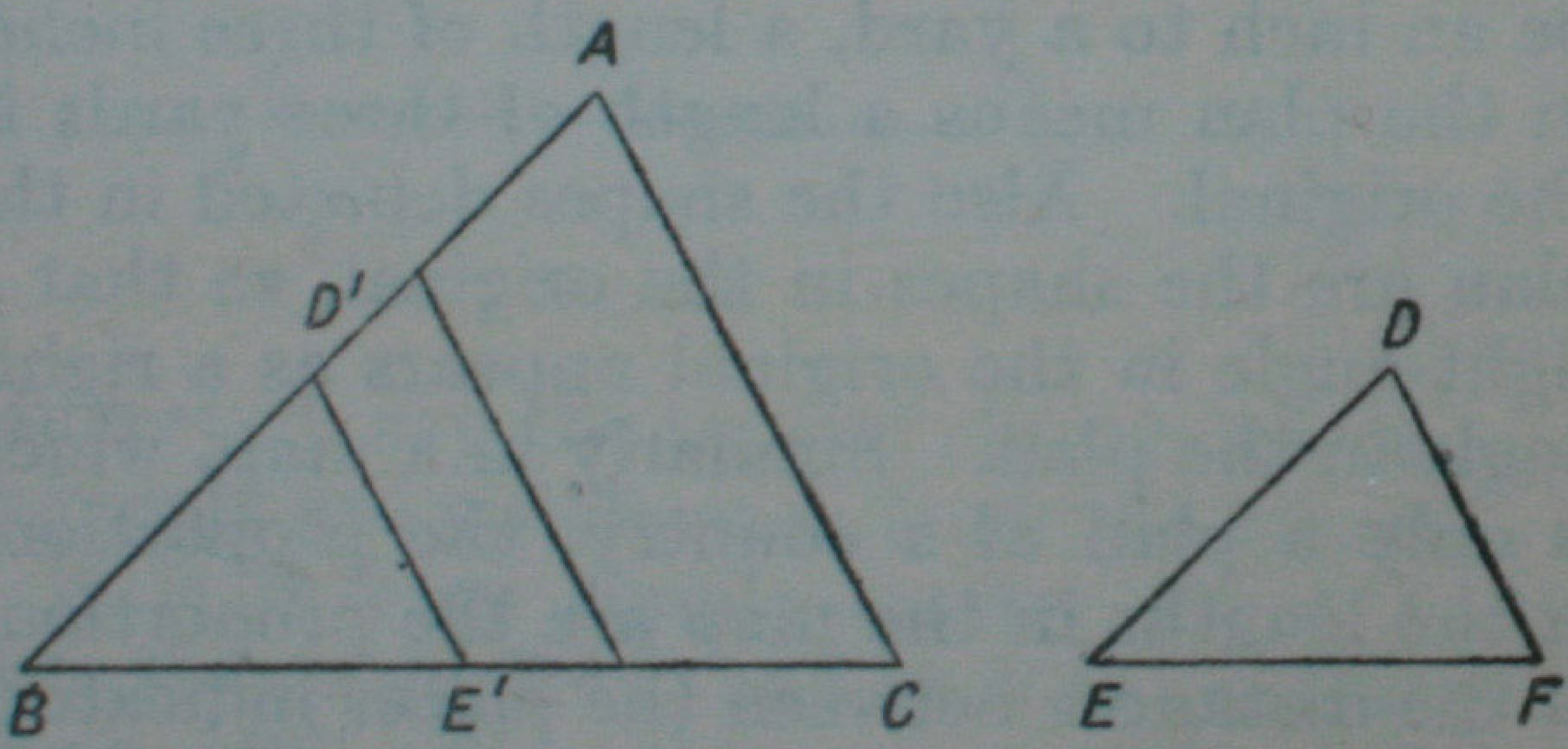

Fig. 24.

ABC and *DEF* have the angles at *A* and *D* equal, and those at *B* and *E*, and those at *C* and *F*, then *DE* is to *AB* in the same proportion as *EF* is to *BC*, and as *FD* is to *CA*. But it is not true of other figures that similarity is guaranteed by the mere equality of angles. Take for example, the familiar cases of a rectangle and a square. Let *ABCD* be a square, and *ABEF* be a rectangle. Then all the corresponding angles are equal. But

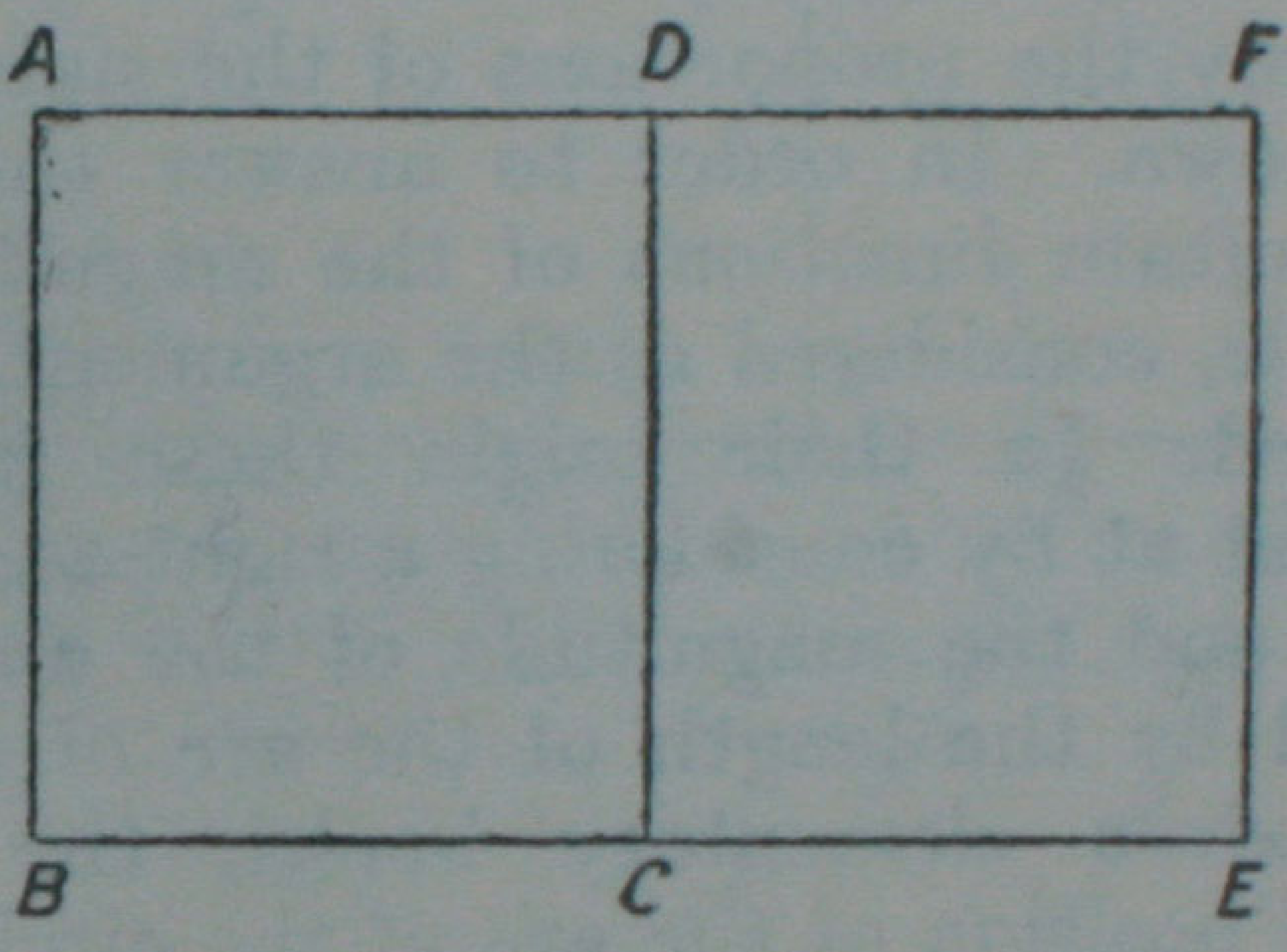

Fig. 25.

whereas the side *AB* of the square is equal to the side *AB* of the rectangle, the side *BC* of the square is about half the size of the side *BE* of the rectangle. Hence it is not true that the square *ABCD* is similar to the rectangle *ABEF*. This peculiar property of the triangle, which is not shared by other rectilinear figures, makes it the fundamental figure in the theory of similarity. Hence in surveys, triangulation is the fundamental process; and hence also arises the word "trigonometry," derived from the two Greek words *trigonon* a triangle and *metria* measurement. The fundamental question from which trigonometry arose is this: Given the magnitudes of the angles of a triangle, what can be stated as to the relative magnitudes of the sides. Note that we say "*relative* magnitudes of the sides," since by the theory of similarity it is only the proportions of the sides which are known. In order to answer this question, certain functions of the magnitudes of an angle, considered as the argument, are introduced. In their origin these functions were got at by considering a right-angled triangle, and the magnitude of the angle was defined by the length of the arc of a circle. In modern elementary books, the fundamental position of the arc of the circle as defining the magnitude of the angle has been pushed somewhat to the background, not to the advantage either of theory or clearness of explanation. It must first be noticed that, in relation to similarity, the circle holds the same fundamental position among curvilinear figures, as does the triangle among rectilinear figures. Any two circles are similar figures; they only differ in scale. The lengths of the circumferences of two circles, such as APA' and $A_1P_1A'_1$ in the fig. 26 are in proportion to the lengths of their radii. Furthermore, if the two circles have the same centre O, as do the two circles in fig. 26, then the arcs AP and A_1P_1 intercepted by the arms of any angle AOP, are also in proportion to their radii. Hence the ratio of the

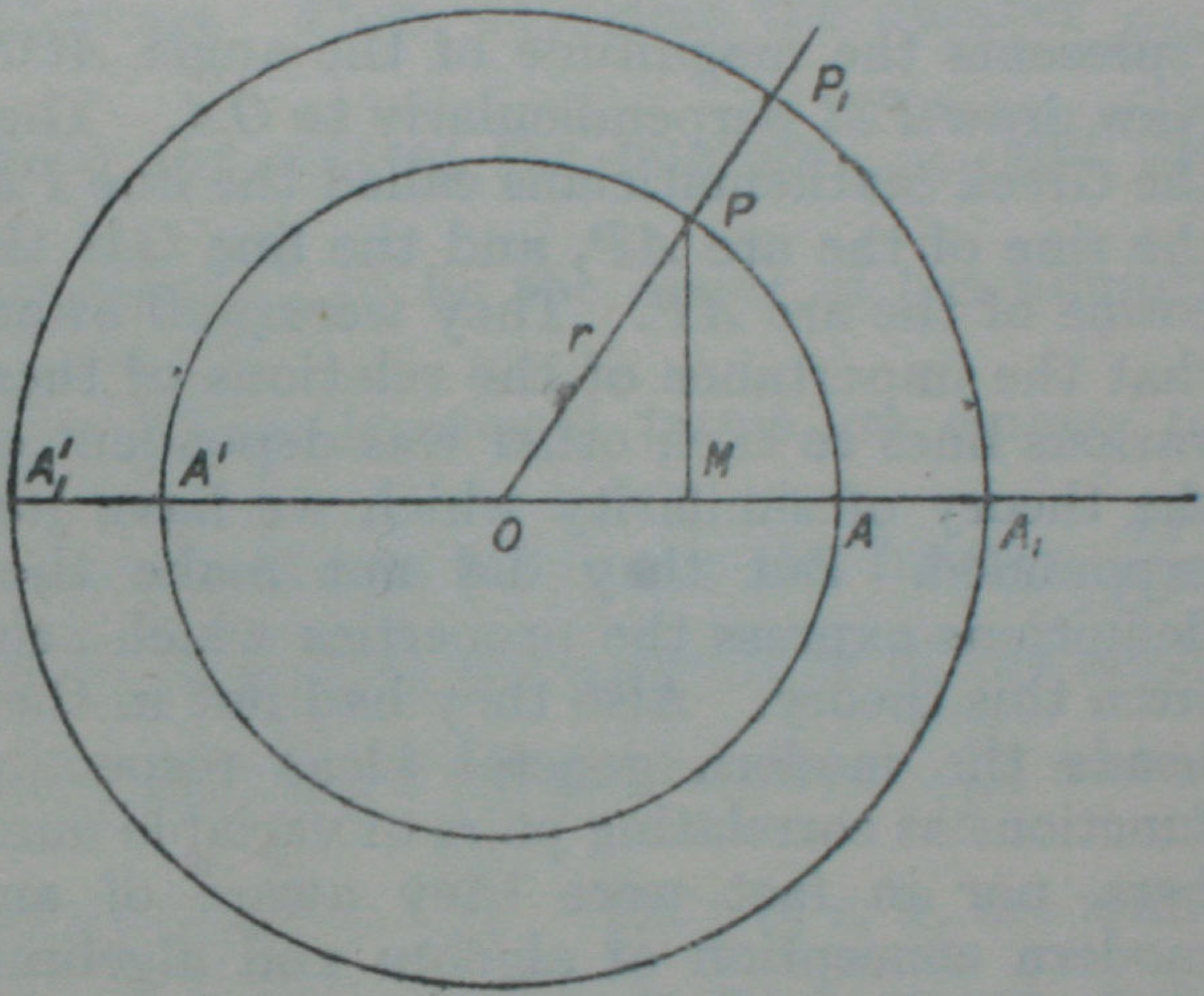

Fig. 26.

length of the arc AP to the length of the radius OP, that is $\frac{\text{arc } AP}{\text{radius } OP}$ is a number which is quite independent of the length OP, and is the same as the fraction $\frac{\text{arc } A_1P_1}{\text{radius } OP_1}$. This fraction of " arc divided by radius " is the proper theoretical way to measure the magnitude of an angle; for it is dependent on no arbitrary unit of length, and on no arbitrary way of dividing up any arbitrarily assumed angle, such as a right-angle. Thus the fraction $\frac{AP}{OA}$ represents the magnitude of the angle AOP. Now draw PM perpendicularly to OA. Then the Greek mathematicians called the line PM the sine of the arc AP, and the line OM the cosine of the arc AP. They were well aware that the importance of the relations of these various lines to each other was dependent on the theory of similarity which we have just expounded. But they did not make their definitions express the properties which arise from this theory. Also they had not in their heads the modern general ideas respecting functions as correlating pairs of variable numbers, nor in fact were they aware of any modern conception of algebra and algebraic analysis. Accordingly, it was natural to them to think merely of the relations between certain lines in a diagram. For us the case is different: we wish to embody our more powerful ideas.

Hence, in modern mathematics, instead of considering the arc AP, we consider the fraction $\frac{AP}{OP}$, which is a number the same for all lengths of OP; and, instead of considering the lines PM and OM, we consider the fractions $\frac{PM}{OP}$ and $\frac{OM}{OP}$, which again are numbers not dependent on the length of OP, *i.e.* not dependent on the scale of our diagrams. Then we define the number $\frac{PM}{OP}$ to be the *sine* of the number $\frac{PA}{OP}$, and the number $\frac{OM}{OP}$ to be the *cosine* of the number $\frac{PA}{OP}$. These fractional forms are clumsy to print; so let us put u for the fraction $\frac{AP}{OP}$, which represents the magnitude of the angle AOP, and put v for the fraction $\frac{PM}{OP}$, and w for the fraction $\frac{OM}{OP}$. Then u, v, w, are numbers, and, since we are talking of *any* angle AOP, they are variable numbers. But a correlation exists between their magnitudes, so that when u (*i.e.* the angle AOP) is given the magnitudes of v and w are definitely determined. Hence v and w are functions of the argument u. We have called v the *sine* of u, and w the *cosine* of u. We wish to adapt the general functional notation $y=f(x)$ to these special cases: so in modern mathematics we write *sin* for "f" when we want to indicate the special function of "sine," and "*cos*" for "*f*" when we want to indicate the special function of "cosine." Thus, with the above meanings for u, v, w, we get

$$v = \sin u, \text{ and } w = \cos u,$$

where the brackets surrounding the x in $f(x)$ are omitted for the special functions. The meaning of these functions *sin* and *cos* as correlating the pairs of numbers u and v, and u and w is, that the functional relations are to be found by constructing (*cf.* fig. 26) an angle AOP, whose measure "AP divided by OP" is equal to u, and that then v is the number given by "PM divided by OP" and w is the number given by "OM divided by OP."

It is evident that without some further definitions we shall get into difficulties when the number u is taken too large. For then the arc AP may be greater than one-quarter of the circumference of the circle, and the point M (*cf.* figs. 26 and 27) may fall between O and A' and not between O and A. Also P may be below the line AOA' and not above it as in fig. 26. In order to get over this difficulty we have recourse to the ideas and conventions of coordinate geometry in making our complete definitions of the sine and cosine. Let one arm OA of the angle be the axis OX, and produce the axis backwards to obtain its negative part OX'. Draw the other axis YOY' perpendicular to it. Let any point P at a distance r from O have coordinates x and y. These coordinates are both positive in the first "quadrant" of the plan, *e.g.* the coordinates x and y of

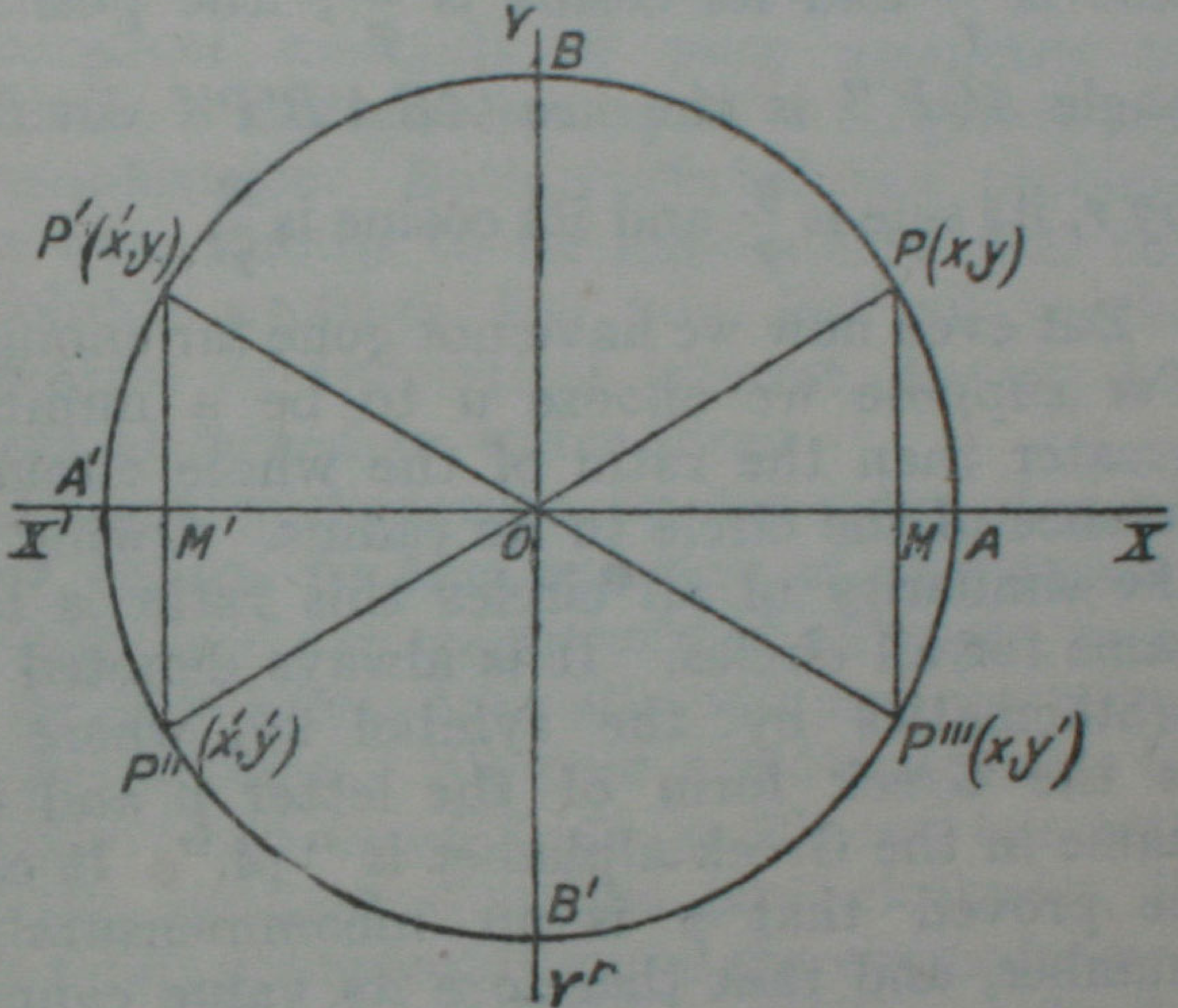

Fig. 27.

P in fig. 27. In the other quadrants, either one or both of the coordinates are negative, for example, x' and y for P', and x' and y' for P'', and x and y' for P''' in fig. 27, where x' and y' are both negative numbers. The positive angle POA is the arc AP divided by r, its sine is $\frac{y}{r}$ and its cosine is $\frac{x}{r}$; the positive angle AOP' is the arc ABP' divided by r, its sine is $\frac{y}{r}$ and cosine $\frac{x'}{r}$; the positive angle AOP'' is the arc $ABA'P''$ divided by r, its sine is $\frac{y'}{r}$ and its cosine is $\frac{x'}{r}$; the positive angle AOP''' is the arc $ABA'B'P'''$ divided by r, its sine is $\frac{y'}{r}$ and its cosine is $\frac{x}{r}$.

But even now we have not gone far enough. For suppose we choose u to be a number greater than the ratio of the whole circumference of the circle to its radius. Owing to the similarity of all circles this ratio is the same for all circles. It is always denoted in mathematics by the symbol 2π, where π is the Greek form of the letter p and its name in the Greek alphabet is "pi." It can be proved that π is an incommensurable number, and that therefore its value cannot be expressed by any fraction, or by any terminating or recurring decimal. Its value to a few decimal places is 3·14159; for many purposes a sufficiently accurate approximate value is $\frac{22}{7}$. Mathematicians can easily calculate π to any degree of accuracy required, just as $\sqrt{2}$ can be so calculated. Its value has been actually given to 707 places of decimals. Such elaboration of calculation is merely a curiosity, and of no practical or theoretical interest. The accurate determination of π is one of the two parts of the famous problem of squaring the circle. The other part of the problem is, by the theoretical methods of pure geometry to describe a straight line equal in length to the circumference. Both parts of the problem are now known to be impossible; and the insoluble problem has now lost all special practical or theoretical interest, having become absorbed in wider ideas.

After this digression on the value of π, we now return to the question of the general definition of the magnitude of an angle, so as to be able to produce an angle corresponding to any value u. Suppose a moving point, Q, to start from A on OX (cf. fig. 27), and to rotate in the positive direction (anti-clockwise, in the figure considered) round the circumference of the circle for any number of times, finally resting at any point, *e.g.* at P or P' or P'' or P'''. Then the total length of the curvilinear circular path traversed, divided by the radius of the circle, r, is the generalized definition of a positive angle of *any* size. Let x, y be the coordinates of the point in which the point Q rests, *i.e.* in one of the four alternative positions mentioned in fig. 27; x and y (as here used) will either x and y, or x' and y, or x' and y', or x

and y'. Then the sign of this generalized angle is $\frac{y}{r}$ and its cosine is $\frac{x}{r}$. With these definitions the functional relations $v = \sin u$ and $w = \cos u$, are at last defined for all positive real values of u. For negative values of u we simply take rotation of Q in the opposite (clockwise) direction; but it is not worth our while to elaborate further on this point, now that the general method of procedure has been explained.

These functions of sine and cosine, as thus defined, enable us to deal with the problems concerning the triangle from which Trigonometry took its rise. But we are now in a position to relate Trigonometry to the wider idea of Periodicity of which the importance was explained in the last chapter. It is easy to see that the functions $\sin u$ and $\cos u$ are periodic functions of u. For consider the position, P (in fig. 27), of a moving point, Q, which has started from A and revolved round the circle. This position, P, marks the angles $\frac{\text{arc } AP}{r}$, and $2\pi + \frac{\text{arc } AP}{r}$, and $4\pi + \frac{\text{arc } AP}{r}$, and $6\pi + \frac{\text{arc } AP}{r}$, and so on indefinitely. Now, all these angles have the same sine and cosine, namely, $\frac{y}{r}$ and $\frac{x}{r}$. Hence it is easy to see that, if u be chosen to have any value, the arguments u and $2\pi+u$, and $4\pi+u$, and $6\pi+u$, and $8\pi+u$ and so on indefinitely, have all the same values for the corresponding sines and cosines. In other words,

$$\sin u=\sin(2\pi+u)=\sin(4\pi+u)=\sin(6\pi+u)=\text{etc.};$$

$$\cos u=\cos(2\pi+u)=\cos(4\pi+u)=\cos(6\pi+u)=\text{etc.}$$

This fact is expressed by saying that *sin u* and cos u are periodic functions with their period equal to 2π.

The graph of the function $y=\sin x$ (notice that we now abandon v and u for the more familiar y and x) is shown in fig. 28. We take on the axis of x any arbitrary length at pleasure to represent the number π, and on the axis of y any arbitrary length at pleasure to represent the number 1. The numerical values of the sine and cosine can never exceed unity. The recurrence of the figure after periods of 2π will be noticed. This graph represents the simplest style of periodic function, out of which all others are constructed. The cosine gives nothing fundamentally different from the sine. For it is easy to prove that $\cos x=\sin(x+\frac{\pi}{2})$; hence it can be seen that the graph of $\cos x$ is simply fig. 28 modified by drawing the axis of OY through the point on OX marked $\frac{\pi}{2}$, instead of drawing it in its actual position on the figure.

It is easy to construct a 'sine' function in

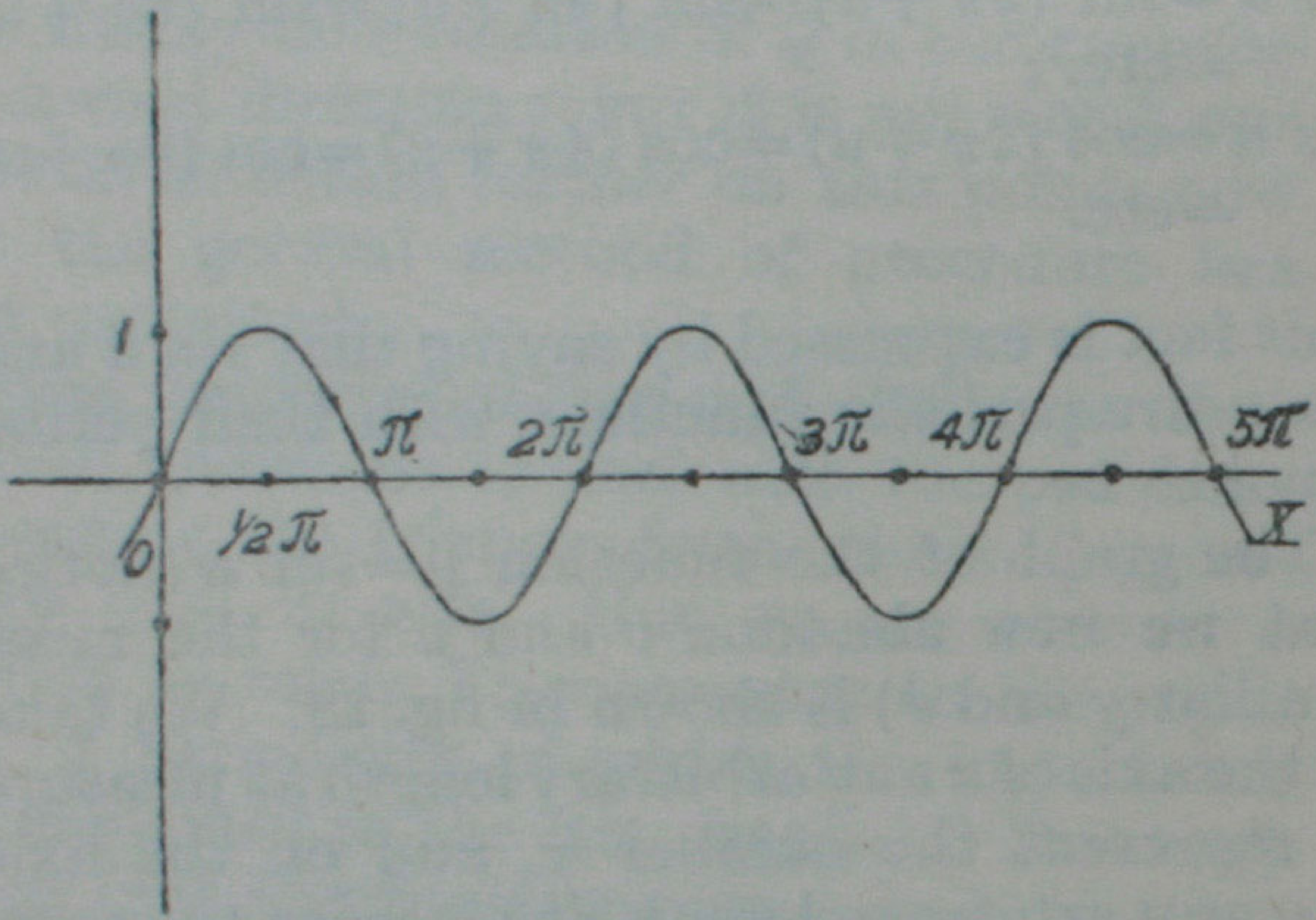

Fig. 28.

which the period has any assigned value a. For we have only to write

$$y=\sin\frac{2\pi x}{a},$$

and then

$$\sin\frac{2\pi(x+a)}{a}=\sin\left\{\frac{2\pi x}{a}+2\pi\right\}=\sin\frac{2\pi x}{a}.$$

Thus the period of this new function is now a. Let us now give a general definition of what we mean by a periodic function. The function $f(x)$ is periodic, with the period a, if (i) for *any* value of x we have $f(x)=f(x+a)$, and (ii) there is no number b smaller than a such that for *any* value of x, $f(x)=f(x+b)$.

The second clause is put into the definition because when we have $\sin\frac{2\pi x}{a}$, it is not only periodic in the period a, but also in the periods $2a$ and $3a$, and so on; this arises since

$$\sin\frac{2\pi(x+3a)}{a}=\sin\left(\frac{2\pi x}{a}+6\pi\right)=\sin\frac{2\pi x}{a}.$$

So it is the smallest period which we want to get hold of and call *the* period of the function. The greater part of the abstract theory of periodic functions and the whole of the applications of the theory to Physical Science are dominated by an important theorem called Fourier's Theorem; namely that, if $f(x)$ be a periodic function with the period a and if $f(x)$ also satisfies certain conditions, which practically are always presupposed in functions suggested by natural phenomena, then $f(x)$ can be written as the sum of a set of terms in the form

$$c_0+c_1\sin\left(\frac{2\pi x}{a}+e_1\right)+c_2\sin\left(\frac{4\pi x}{a}+e_2\right)$$
$$+c_3\sin\left(\frac{6\pi x}{a}+e_3\right)+\text{ etc.}$$

In this formula c_0, c_1, c_2, c_3, etc., and also e_1, e_2, e_3, etc., are constants, chosen so as to suit the particular function. Again we have to ask, How many terms have to be chosen? And here a new difficulty arises: for we can prove that, though in some particular cases a definite number will do, yet in general all we can do is to approximate as closely as we like to the value of the function by taking more and more terms. This process of gradual approximation brings us to the consideration of the theory of infinite series, an essential part of mathematical theory which we will consider in the next chapter.

The above method of expressing a periodic function as a sum of sines is called the "harmonic analysis" of the function. For example, at any point on the sea coast the tides rise and fall periodically. Thus at a point near the Straits of Dover there will be two daily tides due to the rotation of the earth. The daily rise and fall of the tides are complicated by the fact that there are two tidal waves, one coming up the English Channel, and the other which has swept round the North of Scotland, and has then come southward down the North Sea. Again some high tides are higher than others: this is due to the fact that the Sun has also a tide-generating influence as well as the Moon. In this way monthly and other periods are introduced.

We leave out of account the exceptional influence of winds which cannot be foreseen. The general problem of the harmonic analysis of the tides is to find sets of terms like those in the expression on page 191 above, such that each set will give with approximate accuracy the contribution of the tide-generating influences of one "period" to the height of the tide at any instant. The argument x will therefore be the *time* reckoned from any convenient commencement.

Again, the motion of vibration of a violin string is submitted to a similar harmonic analysis, and so are the vibrations of the ether and the air, corresponding respectively to waves of light and waves of sound. We are here in the presence of one of the fundamental processes of mathematical physics—namely, nothing less than its general method of dealing with the great natural fact of Periodicity.

CHAPTER XIV

SERIES

No part of Mathematics suffers more from the triviality of its initial presentation to beginners than the great subject of series. Two minor examples of series, namely arithmetic and geometric series, are considered; these examples are important because they are the simplest examples of an important general theory. But the general ideas are never disclosed; and thus the examples, which exemplify nothing, are reduced to silly trivialities.

The general mathematical idea of a series is that of a set of things ranged in order, that is, in sequence. This meaning is accurately represented in the common use of the term. Consider for example, the series of English Prime Ministers during the nineteenth century, arranged in the order of their first tenure of that office within the century. The series commences with William Pitt, and ends with Lord Rosebery, who, appropriately enough, is the biographer of the first member. We might have considered other serial orders for the arrangement of these men; for example, according to their height or their weight. These other suggested orders strike us as trivial in connection with Prime Ministers, and would not naturally occur to the mind; but abstractly they are just as good orders as any other. When one order among terms is very much more important or more obvious than other orders, it is often spoken of as *the* order of those terms. Thus *the* order of the integers would always be taken to mean their order as arranged in order of magnitude. But of course there is an indefinite number of other ways of arranging them. When the number of things considered is finite, the number of ways of arranging them in order is called the number of their permutations. The number of permutations of a set of n things, where n is some finite integer, is

$$n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 4 \times 3 \times 2 \times 1$$

that is to say, it is the product of the first n integers; this product is so important in mathematics that a special symbolism is used for it, and it is always written '$n!$' Thus, $2! = 2 \times 1 = 2$, and $3! = 3 \times 2 \times 1 = 6$, and $4! = 4 \times 3 \times 2 \times 1 = 24$, and $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$. As n increases, the value of $n!$ increases very quickly; thus $100!$ is a hundred times as large as $99!$

It is easy to verify in the case of small values of n that $n!$ is the number of ways of arranging n things in order. Thus consider two things a and b; these are capable of the two orders ab and ba, and $2!=2$.

Again, take three things a, b, and c; these are capable of the six orders, abc, acb, bac, bca, cab, cba, and $3!=6$. Similarly for the twenty-four orders in which four things a, b, c, and d, can be arranged.

When we come to the infinite sets of things—like the sets of all the integers, or all the fractions, or all the real numbers for instance—we come at once upon the complications of the theory of order-types. This subject was touched upon in Chapter VI. in considering the possible orders of the integers, and of the fractions, and of the real numbers. The whole question of order-types forms a comparatively new branch of mathematics of great importance. We shall not consider it any further. All the infinite series which we consider now are of the same order-type as the integers arranged in ascending order of magnitude, namely, with a first term, and such that each term has a couple of next-door neighbours, one on either side, with the exception of the first term which has, of course, only one next-door neighbour. Thus, if m be any integer (not zero), there will be always an mth term. A series with a finite number of terms (say n terms) has the same characteristics as far as next-door neighbours are concerned as an infinite series; it only differs from infinite series in having a last term, namely, the nth.

The important thing to do with a series of numbers—using for the future "series" in the restricted sense which has just been mentioned—is to add its successive terms together.

Thus if $u_1, u_2, u_3, \ldots u_n \ldots$ are respectively the 1st, 2nd, 3rd, 4th, . . . nth, . . . terms of a series of numbers, we form successively the series u_1, u_1+u_2, $u_1+u_2+u_3$, $u_1+u_2+u_3+u_4$, and so on; thus the sum of the 1st n terms may be written.

$$u_1+u_2+u_3+\ldots+u_n.$$

If the series has only a finite number of terms, we come at last in this way to the sum of the whole series of terms. But, if the series has an infinite number of terms, this process of successively forming the sums of the terms never terminates; and in this sense there is no such thing as the sum of an infinite series.

But why is it important successively to add the terms of a series in this way? The answer is that we are here symbolizing the fundamental mental process of approximation. This is a process which has significance far beyond the regions of mathematics. Our limited intellects cannot deal with complicated material all at once, and our method of arrangement is that of approximation. The statesman in framing his speech puts the dominating issues first and lets the details fall naturally into their subordinate places. There is, of course, the converse artistic method of preparing the imagination by the presentation of subordinate or special details, and then gradually rising to a crisis. In either way the process is one of gradual summation of effects; and this is exactly what is done by the successive summation of the terms of a series. Our ordinary method of stating numbers is such a process of gradual summation, at least, in the case of large numbers. Thus 568,213 presents itself to the mind as—

$$500{,}000+60{,}000+8{,}000+200+10+3$$

In the case of decimal fractions this is so more avowedly. Thus 3·14159 is—

$$3+\frac{1}{10}+\frac{4}{100}+\frac{1}{1000}+\frac{5}{10000}+\frac{9}{100000}$$

Also, 3 and $3+\frac{1}{10}$, and $3+\frac{1}{10}+\frac{4}{100}$, and $3+\frac{1}{10}+\frac{4}{100}+\frac{1}{1000}$, and $3+\frac{1}{10}+\frac{4}{100}+\frac{1}{1000}+\frac{6}{10000}$ are successive approximations to the complete result 3·14159. If we read 568,213 backwards from right to left, starting with the 3 units, we read it in the artistic way, gradually preparing the mind for the crisis of 500,000.

The ordinary process of numerical multiplication proceeds by means of the summation of a series. Consider the computation

```
   342
   658
  ----
  2736
 1710
2052
------
225036
```

Hence the three lines to be added form a series of which the first term is the upper line. This series follows the artistic method of presenting the most important term last, not from any feeling for art, but because of the convenience gained by keeping a firm hold on the units' place, thus enabling us to omit some 0's, formally necessary.

But when we approximate by gradually adding the successive terms of an infinite series, what are we approximating to? The difficulty is that the series has no "sum" in the straightforward sense of the word, because the operation of adding together its terms can never be completed. The answer is that we are approximating to the *limit* of the summation of the series, and we must now proceed to explain what the "limit" of a series is.

The summation of a series approximates to a limit when the sum of any number of its terms, provided the number be large enough, is as nearly equal to the limit as you care to approach. But this description of the meaning of approximating to a limit evidently will not stand the vigorous scrutiny of modern mathematics. What is meant by *large enough*, and by *nearly equal*, and by *care to approach*? All these vague phrases must be explained in terms of the simple abstract ideas which alone are admitted into pure mathematics.

Let the successive terms of the series be $u_1, u_2, u_3, u_4, \ldots, u_n$, etc., so that u_n is the nth term of the series. Also let s_n be the sum of the 1st n terms, whatever n may be. So that—

$$s_1 = u_1,\ s_2 = u_1 + u_2,\ s_3 = u_1 + u_2 + u_3, \text{ and}$$
$$s_n = u_1 + u_2 + u_3 + \ldots + u_n.$$

Then the terms $s_1, s_2, s_3, \ldots s_n, \ldots$ form a new series, and the formation of this series is the process of summation of the original series. Then the "approximation" of the *summation* of the original series to a "limit" means the "approximation of the *terms* of this new series to a limit." And we have now to explain what we mean by the approximation to a limit of the terms of a series.

Now, remembering the definition (given in chapter XII.) of a *standard of approximation*, the idea of a limit means this: l is the limit of the terms of the series s_1, s_2, s_3, . . . s_n, . . ., if, corresponding to each real number k, taken as a standard of approximation, a term s_n of the series can be found so that all succeeding terms (*i.e.* s_{n+1}, s_{n+2}, etc.) approximate to l within that standard of approximation. If another smaller standard k^1 be chosen, the term s_n may be too early in the series, and a later term s_m with the above property will then be found.

If this property holds, it is evident that as you go along to series s_1, s_2, s_3, . . ., s_n, . . . from left to right, after a time you come to terms *all of* which are nearer to l than any number which you may like to assign. In other words you approximate to l as closely as you like. The close connection of this definition of the limit of a series with the definition of a continuous function given in chapter XI. will be immediately perceived.

Then coming back to the original series u_1, u_2, u_3, . . ., u_n, . . ., the limit of the terms of the series s_1, s_2, s_3, . . ., s_n, . . ., is called the "sum to infinity" of the original series. But it is evident that this use of the word "sum" is very artificial, and we must not assume the analogous properties to those of the ordinary sum of a finite number of terms without some special investigation.

Let us look at an example of a "sum to infinity." Consider the recurring decimal ·1111. . . . This decimal is merely a way of symbolizing the "sum to infinity" of the series ·1, ·01, ·001, ·0001, etc. The corresponding series found by summation is $s_1 = \cdot 1$, $s_2 = \cdot 11$, $s_3 = \cdot 111$, $s_4 = \cdot 1111$, etc. The limit of the terms of this series is $\frac{1}{9}$; this is easy to see by simple division, for

$$\tfrac{1}{9} = \cdot 1 + \tfrac{1}{90} = \cdot 11 + \tfrac{1}{900} = \cdot 111 + \tfrac{1}{9000} = \text{etc.}$$

Hence, if $\frac{3}{17}$ is given (the *k* of the definition), ·1 and *all* succeeding terms differ from $\frac{1}{9}$ by less than $\frac{3}{17}$; if $\frac{1}{1000}$ is given (another choice for the *k* of the definition), ·111 and all succeeding terms differ from $\frac{1}{9}$ by less than $\frac{1}{1000}$; and so on, whatever choice for *k* be made.

It is evident that nothing that has been said gives the slightest idea as to how the "sum to infinity" of a series is to be found. We have merely stated the conditions which such a number is to satisfy. Indeed, a general method for finding in all cases the sum to infinity of a series is intrinsically out of the question, for the simple reason that such a "sum," as here defined, does not always exist. Series which possess a sum to infinity are called *convergent*, and those which do not possess a sum to infinity are called *divergent*.

An obvious example of a divergent series is 1, 2, 3, . . ., n . . . *i.e.* the series of integers in their order of magnitude. For whatever number l you try to take as its sum to infinity, and whatever standard of approximation k you choose, by taking enough terms of the series you can always make their sum differ from l by more than k. Again, another example of a divergent series is 1, 1, 1, etc., *i.e.* the series of which each term is equal to 1. Then the sum of n terms is n, and this sum grows without limit as n increases. Again, another example of a divergent series is 1, −1, 1, −1, 1, −1, etc., *i.e.* the series in which the terms are alternately 1 and −1. The sum of an odd number of terms is 1, and of an even number of terms is 0. Hence the terms of the series $s_1, s_2, s_3, \ldots s_n, \ldots$ do not approximate to a limit, although they do not increase without limit.

It is tempting to suppose that the condition for $u_1, u_2, \ldots u_n, \ldots$ to have a sum to infinity is that u_n should decrease indefinitely as n increases. Mathematics would be a much easier science than it is, if this were the case. Unfortunately the supposition is not true.

For example the series

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$$

is divergent. It is easy to see that this is the case; for consider the sum of n terms beginning at the $(n+1)^{th}$ term. These n terms are $\frac{1}{n+1}, \frac{1}{n+2}, \frac{1}{n+3}, \ldots \frac{1}{2n}$: there are n of them and $\frac{1}{2n}$ is the least among them. Hence their sum is greater than n times $\frac{1}{2n}$, *i.e.* is greater than $\frac{1}{2}$. Now, without altering the sum to infinity, if it exist, we can add together neighbouring terms, and obtain the series

$$1, \tfrac{1}{2}, \tfrac{1}{3}+\tfrac{1}{4}, \tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}, \text{etc.},$$

that is, by what has been said above, a series whose terms after the 2nd are greater than those of the series,

$$1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \text{etc.},$$

where all the terms after the first are equal. But this series is divergent. Hence the original series is divergent.*

This question of divergency shows how careful we must be in arguing from the pro-

* Cf. Note C, p. 251.

perties of the sum of a finite number of terms to that of the sum of an infinite series. For the most elementary property of a finite number of terms is that of course they possess a sum: but even this fundamental property is not necessarily possessed by an infinite series. This caution merely states that we must not be misled by the suggestion of the technical term "*sum* of an infinite series." It is usual to indicate the sum of the infinite series

$$u_1, u_2, u_3, \ldots u_n. \ldots \text{ by}$$
$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

We now pass on to a generalization of the idea of a series, which mathematics, true to its method, makes by use of the variable. Hitherto, we have only contemplated series in which each definite term was a definite number. But equally well we can generalize, and make each term to be some mathematical expression containing a variable x. Thus we may consider the series $1, x, x^2, x^3, \ldots, x^n, \ldots$, and the series

$$x, \frac{x^2}{2}, \frac{x^3}{3}, \ldots, \frac{x^n}{n}, \ldots$$

In order to symbolize the general idea of any such function, conceive of a function of x, $f_n(x)$ say, which involves in its formation a variable integer n, then, by giving n the values 1, 2, 3, etc., in succession, we get the series

$$f_1(x),\ f_2(x),\ f_3(x),\ \ldots,\ f_n(x),\ \ldots$$

Such a series may be convergent for some values of x and divergent for others. It is, in fact, rather rare to find a series involving a variable x which is convergent for all values of x,—at least in any particular instance it is very unsafe to assume that this is the case. For example, let us examine the simplest of all instances, namely, the "geometrical" series

$$1,\ x,\ x^2,\ x^3,\ \ldots,\ x^n,\ \ldots$$

The sum of n terms is given by

$$s_n = 1 + x + x^2 + x^3 + \ldots + x^n.$$

Now multiply both sides by x and we get

$$xs_n = x + x^2 + x^3 + x^4 + \ldots + x^n + x^{n+1}$$

Now subtract the last line from the upper line and we get

$$s_n(1-x) = s_n - xs_n = 1 - x^{n+1},$$

and hence (if x be not equal to 1)

$$s_n = \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$$

Now if x be numerically less than 1, for sufficiently large values of n, $\frac{x^{n+1}}{1-x}$ is always numerically less than k, however k be chosen. Thus, if x be numerically less than 1, the series 1, x, x^2, . . . x^n, . . . is convergent, and $\frac{1}{1-x}$ is its limit. This statement is symbolized by

$$\frac{1}{1-x}=1+x+x^2+\ldots+x^n+\ldots, (-1 < x < 1).$$

But if x is numerically greater than 1, or numerically equal to 1, the series is divergent. In other words, if x lie between -1 and $+1$, the series is convergent; but if x be equal to -1 or $+1$, or if x lie outside the interval -1 to $+1$, then the series is divergent. Thus the series is convergent at all "points" within the interval -1 to $+1$, exclusive of the end points.

At this stage of our enquiry another question arises. Suppose that the series

$$f_1(x)+f_2(x)+f_3(x)+ \ . \ . \ . \ +f_n(x)+ \ . \ . \ .$$

is convergent for all values of x lying within the interval a to b, *i.e.* the series is convergent for any value of x which is greater than a and less than b. Also, suppose we want to be sure that in approximating to the limit we add together enough terms to come within some standard of approximation k. Can we always state some number of terms, say n, such that, if we take n or more terms to form the sum, then *whatever* value x has within the interval we have satisfied the desired standard of approximation?

Sometimes we can and sometimes we cannot do this for each value of k. When we can, the series is called uniformly convergent throughout the interval, and when we cannot do so, the series is called non-uniformly convergent throughout the interval. It makes a great difference to the properties of a series whether it is or is not uniformly convergent through an interval. Let us illustrate the matter by the simplest example and the simplest numbers.

Consider the geometric series

$$1+x+x^2+x^3+\ .\ .\ .\ +x^n+\ .\ .\ .$$

It is convergent throughout the interval -1 to $+1$, excluding the end values $x=\pm 1$. But it is not uniformly convergent throughout this interval. For if $s_n(x)$ be the sum of n terms, we have proved that the difference between $s_n(x)$ and the limit $\frac{1}{1-x}$ is $\frac{x^{n+1}}{1-x}$. Now suppose n be any given number of terms, say 20, and let k be any assigned standard of approximation, say $\cdot 001$. Then, by taking x near enough to $+1$ or near enough to -1, we can make the numerical value of $\frac{x^{21}}{1-x}$ to be greater than $\cdot 001$. Thus 20 terms will not do over the whole interval, though it is more than enough over some parts of it.

The same reasoning can be applied whatever other number we take instead of 20, and whatever standard of approximation instead of ·001. Hence the geometric series $1+x+x^2+x^3+\ldots+x^n+\ldots$ is non-uniformly convergent over its *whole* interval of convergence -1 to $+1$. But if we take any smaller interval lying at both ends within the interval -1 to $+1$, the geometric series is uniformly convergent within it. For example, take the interval 0 to $+\frac{1}{10}$. Then any value for n which makes $\frac{x^{n+1}}{1-x}$ numerically less than k *at* these limits for x also serves for all values of x between these limits, since it so happens that $\frac{x^{n+1}}{1-x}$ diminishes in numerical value as x diminishes in numerical value. For example, take $k=\cdot001$; then, putting $x=\frac{1}{10}$, we find:

for $n=1$, $\frac{x^{n+1}}{1-x}=\frac{(\frac{1}{10})^2}{1-\frac{1}{10}}=\frac{1}{90}=\cdot0111\ldots$,

for $n=2$, $\frac{x^{n+1}}{1-x}=\frac{(\frac{1}{10})^3}{1-\frac{1}{10}}=\frac{1}{900}=\cdot00111\ldots$,

for $n=3$, $\frac{x^{n+1}}{1-x}=\frac{(\frac{1}{10})^4}{1-\frac{1}{10}}=\frac{1}{9000}=\cdot000111\ldots$,

Thus three terms will do for the whole interval, though, of course, for some parts of the interval it is more than is necessary. Notice that, because $1+x+x^2+\ldots+x^n+\ldots$ is convergent (though not uniformly) throughout the interval -1 to $+1$, for each value of x in the interval some number of terms n can be found which will satisfy a desired standard of approximation; but, as we take x nearer and nearer to either end value $+1$ or -1, larger and larger values of n have to be employed.

It is curious that this important distinction between uniform and non-uniform convergence was not published till 1847 by Stokes—afterwards, Sir George Stokes—and later, independently in 1850 by Seidel, a German mathematician.

The critical points, where non-uniform convergence comes in, are not necessarily at the limits of the interval throughout which convergence holds. This is a speciality belonging to the geometric series.

In the case of the geometric series $1+x+x^2+\ldots+x^n+\ldots$, a simple algebraic expression $\frac{1}{1-x}$ can be given for its limit in its interval of convergence. But this is not always the case. Often we can prove a series to be convergent within a certain interval, though we know nothing more about its limit except that it is the limit of the series.

But this is a very good way of defining a function ; *viz.* as the limit of an infinite convergent series, and is, in fact, the way in which most functions are, or ought to be, defined.

Thus, the most important series in elementary analysis is

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots,$$

where $n!$ has the meaning defined earlier in this chapter. This series can be proved to be absolutely convergent for *all* values of x, and to be uniformly convergent within any interval which we like to take. Hence it has all the comfortable mathematical properties which a series should have. It is called the exponential series. Denote its sum to infinity by $\exp x$. Thus, by definition,

$$\exp x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots$$

$\exp x$ is called the exponential function.

It is fairly easy to prove, with a little knowledge of elementary mathematics, that

$$(\exp x)\times(\exp y)=\exp(x+y) \quad \ldots(A)$$

In other words that

$$(\exp x)\times(\exp y)=$$

$$1+(x+y)+\frac{(x+y)^2}{2!}+\frac{(x+y)^3}{3!}+\ldots+\frac{(x+y)^n}{n!}+\ldots$$

This property (A) is an example of what is called an addition-theorem. When any function [say $f(x)$] has been defined, the first thing we do is to try to express $f(x+y)$ in terms of known functions of x only, and known functions of y only. If we can do so, the result is called an addition-theorem. Addition-theorems play a great part in mathematical analysis. Thus the addition-theorem for the sine is given by

$$\sin(x+y)=\sin x \cos y+\cos x \sin y,$$

and for the cosine by

$$\cos(x+y)=\cos x \cos y-\sin x \sin y.$$

As a matter of fact the best ways of defining $\sin x$ and $\cos x$ are not by the elaborate geometrical methods of the previous chapter, but as the limits respectively of the series

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\text{etc.} \ldots,$$

and $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\text{etc.} \ldots,$

so that we put

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\text{etc.} \ldots,$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\text{etc.} \ldots,$$

These definitions are equivalent to the geometrical definitions, and both series can be proved to be convergent for all values of x, and uniformly convergent throughout any interval. These series for sine and cosine have a general likeness to the exponential series given above. They are, indeed, intimately connected with it by means of the theory of imaginary numbers explained in Chapters VII. and VIII.

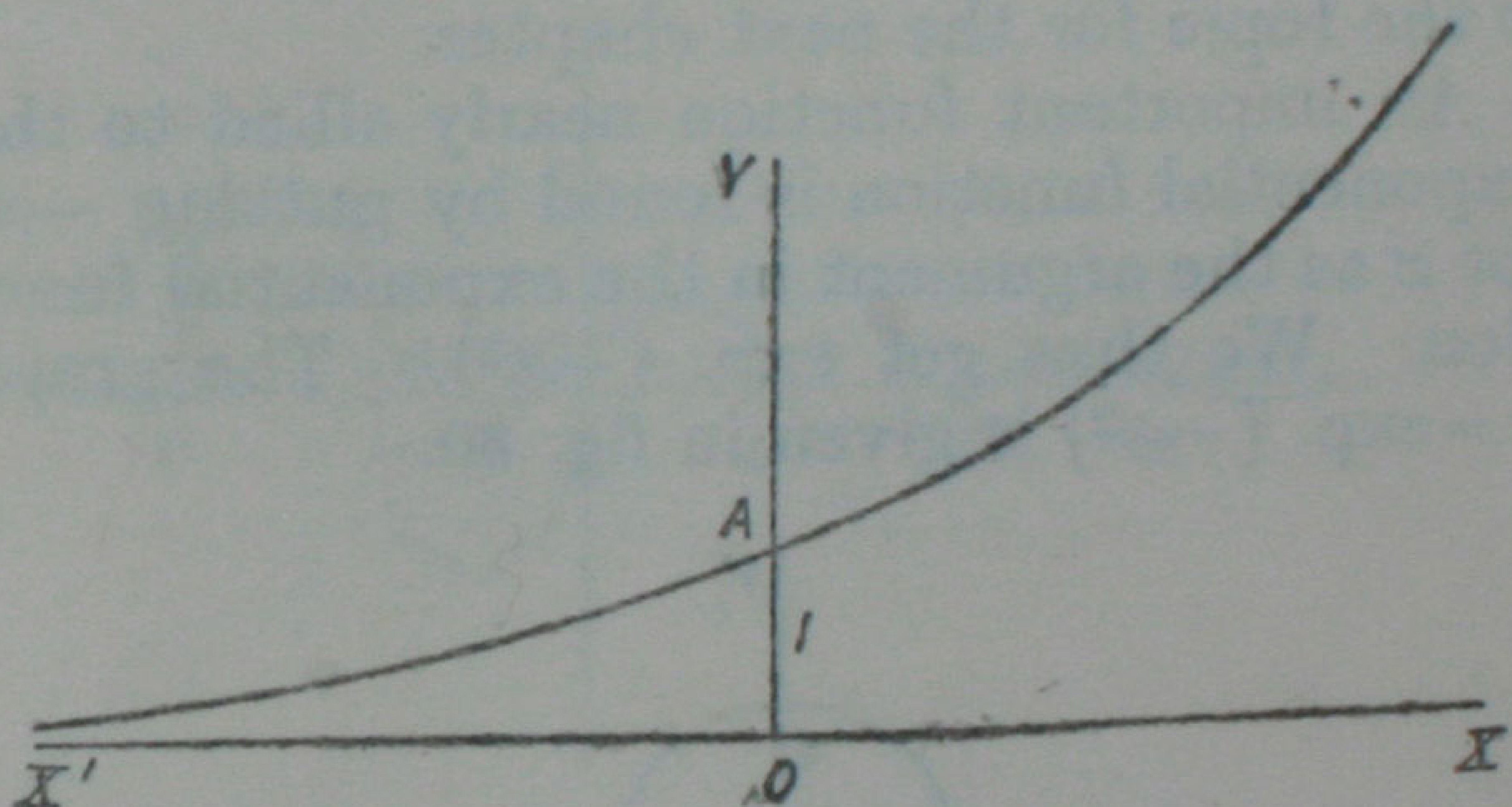

Fig. 29.

The graph of the exponential function is given in fig. 29. It cuts the axis OY at the point $y=1$, as evidently it ought to do, since when $x=0$ every term of the series except the first is zero. The importance of the exponential function is that it represents any changing physical quantity whose rate of increase at any instant is a uniform percentage of its value at that instant. For example, the above graph represents the size at any time of a population with a uniform birth-rate, a uniform death-rate, and no emigration, where the x corresponds to the time reckoned from any convenient day, and the y represents the population to the proper scale. The scale must be such that OA represents the population at the date which is taken as the origin. But we have here come upon the idea of "rates of increase" which is the topic for the next chapter.

An important function nearly allied to the exponential function is found by putting $-x^2$ for x as the argument in the exponential function. We thus get exp. $(-x^2)$. The graph $y=\text{exp.}\ (-x^2)$ is given in fig. 30.

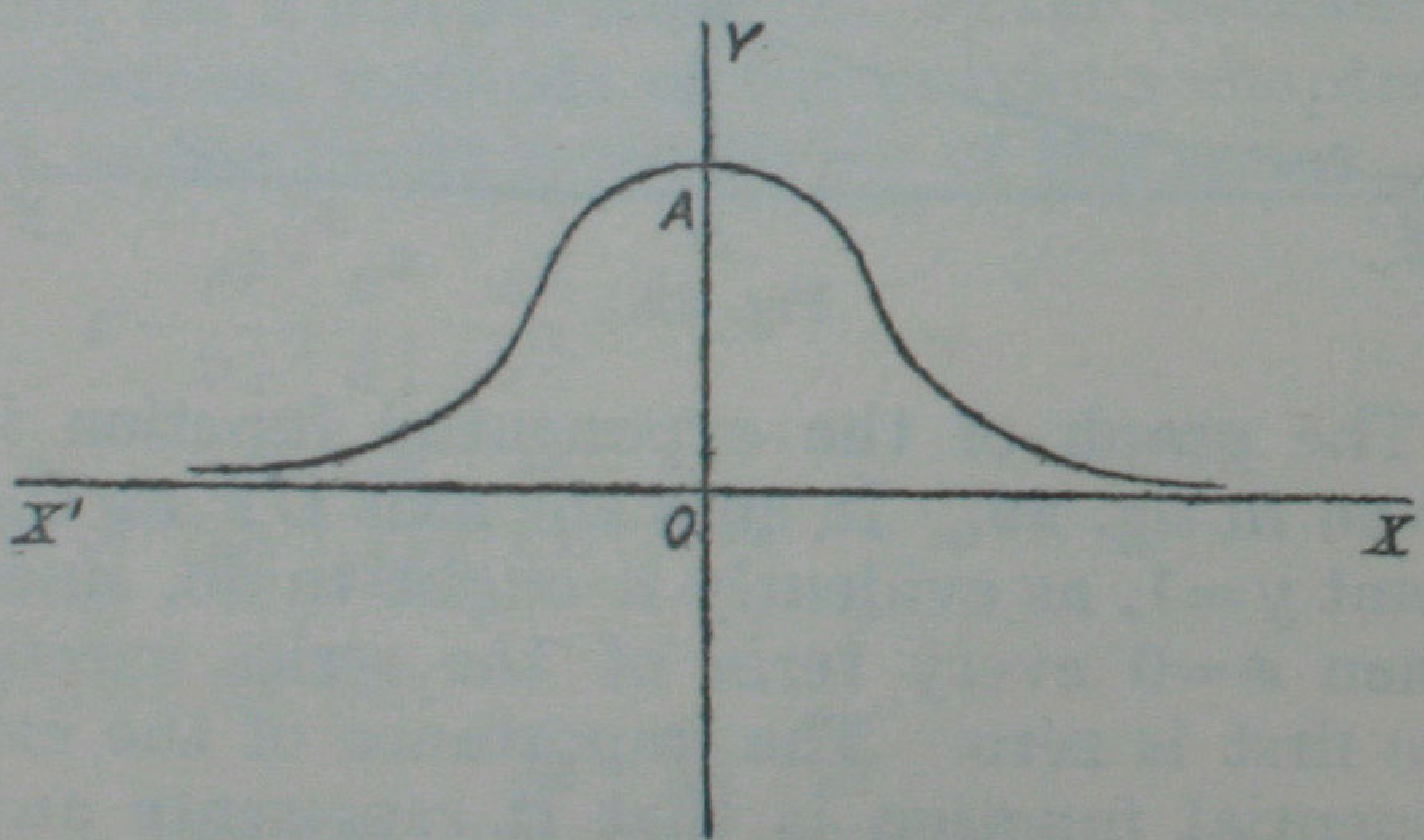

Fig. 30.

The curve, which is something like a cocked hat, is called the curve of normal error. Its corresponding function is vitally important to the theory of statistics, and tells us in many cases the sort of deviations from the average results which we are to expect.

Another important function is found by combining the exponential function with the sine, in this way:

$$y = \exp(-cx) \times \sin \frac{2\pi x}{p}$$

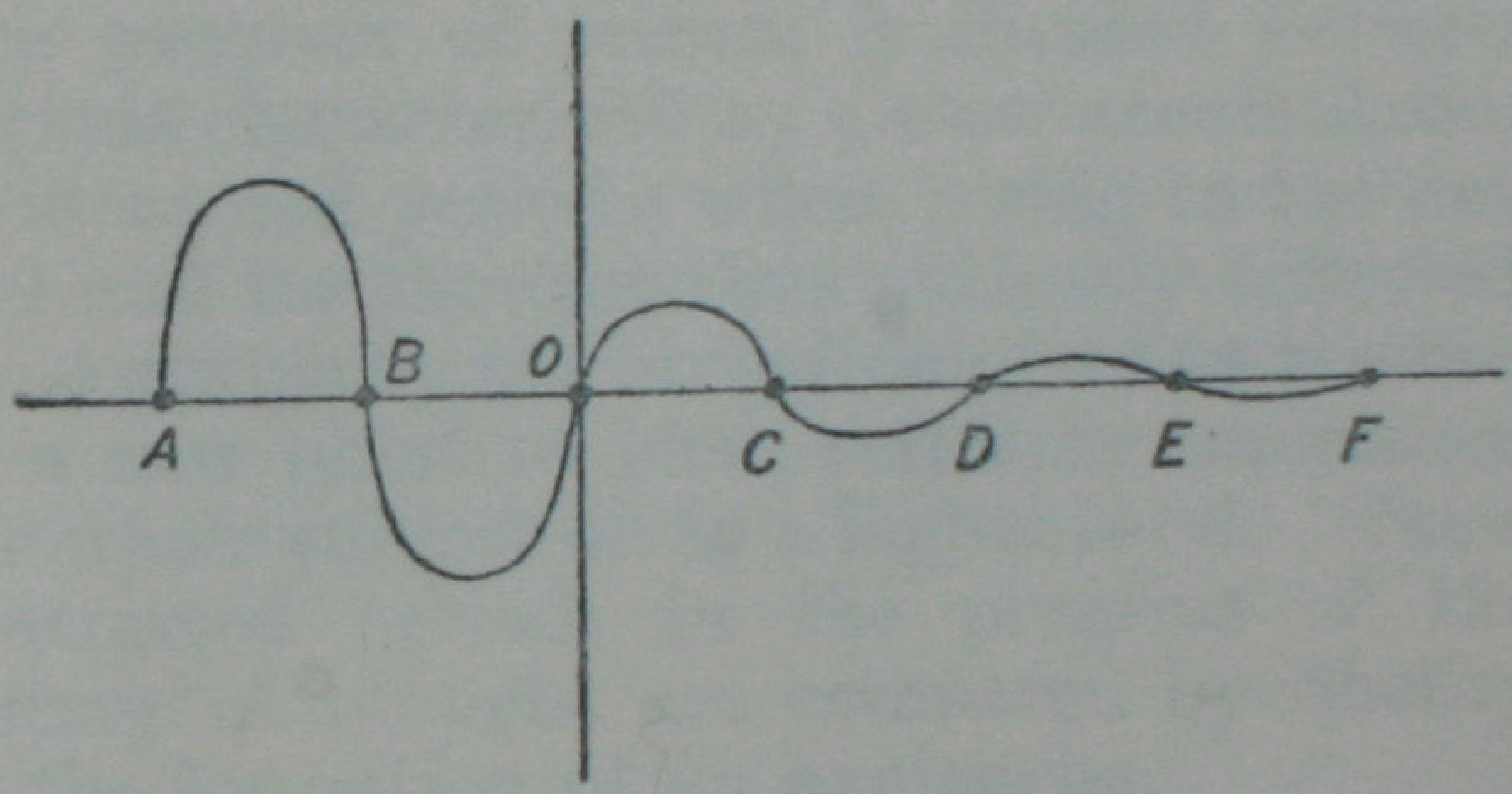

Fig. 31.

Its graph is given in fig. 31. The points *A*, *B*, *O*, *C*, *D*, *E*, *F*, are placed at equal intervals $\frac{1}{2}p$, and an unending series of them should be drawn forwards and backwards. This function represents the dying away of vibrations under the influence of friction or of "damping" forces. Apart from the friction, the vibrations would be periodic, with a period p; but the influence of the friction makes the extent of each vibration smaller than that of the preceding by a constant percentage of that extent. This combination of the idea of "periodicity" (which requires the sine or cosine for its symbolism) and of "constant percentage" (which requires the exponential function for its symbolism) is the reason for the form of this function, namely, its form as a product of a sine-function into an exponential function.

CHAPTER XV

THE DIFFERENTIAL CALCULUS

THE invention of the differential calculus marks a crisis in the history of mathematics. The progress of science is divided between periods characterized by a slow accumulation of ideas and periods, when, owing to the new material for thought thus patiently collected, some genius by the invention of a new method or a new point of view, suddenly transforms the whole subject on to a higher level. These contrasted periods in the progress of the history of thought are compared by Shelley to the formation of an avalanche.

The sun-awakened avalanche ! whose mass,
Thrice sifted by the storm, had gathered there
Flake after flake,—in heaven-defying minds
As thought by thought is piled, till some great truth
Is loosened, and the nations echo round,

. . . . .

The comparison will bear some pressing. The final burst of sunshine which awakens the avalanche is not necessarily beyond comparison in magnitude with the other powers of nature which have presided over its slow formation. The same is true in science. The genius who has the good fortune to produce the final idea which transforms a whole region of thought, does not necessarily excel all his predecessors who have worked at the preliminary formation of ideas. In considering the history of science, it is both silly and ungrateful to confine our admiration with a gaping wonder to those men who have made the final advances towards a new epoch

In the particular instance before us, the subject had a long history before it assumed its final form at the hands of its two inventors. There are some traces of its methods even among the Greek mathematicians, and finally, just before the actual production of the subject, Fermat (born 1601 A.D., and died 1665 A.D.), a distinguished French mathematician, had so improved on previous ideas that the subject was all but created by him. Fermat, also, may lay claim to be the joint inventor of coordinate geometry in company with his contemporary and countryman, Descartes. It was, in fact, Descartes from whom the world of science received the new ideas, but Fermat had certainly arrived at them independently.

We need not, however, stint our admiration either for Newton or for Leibniz. Newton was a mathematician and a student of physical science, Leibniz was a mathematician and a philosopher, and each of them in his own department of thought was one of the greatest men of genius that the world has known. The joint invention was the occasion of an unfortunate and not very creditable dispute. Newton was using the methods of Fluxions, as he called the subject, in 1666, and employed it in the composition of his *Principia*, although in the work as printed any special algebraic notation is avoided. But he did not print a direct statement of his method till 1693. Leibniz published his first statement in 1684. He was accused by Newton's friends of having got it from a MS. by Newton, which he had been shown privately. Leibniz also accused Newton of having plagiarized from him. There is now not very much doubt but that both should have the credit of being independent discoverers. The subject had arrived at a stage in which it was ripe for discovery, and there is nothing surprising in the fact that two such able men should have independently hit upon it.

These joint discoveries are quite common in science. Discoveries are not in general made before they have been led up to by the previous trend of thought, and by that time many minds are in hot pursuit of the important idea. If we merely keep to discoveries in which Englishmen are concerned, the simultaneous enunciation of the law of natural selection by Darwin and Wallace, and the simultaneous discovery of Neptune by Adams and the French astronomer, Leverrier, at once occur to the mind. The disputes, as to whom the credit ought to be given, are often influenced by an unworthy spirit of nationalism. The really inspiring reflection suggested by the history of mathematics is the unity of thought and interest among men of so many epochs, so many nations, and so many races. Indians, Egyptians, Assyrians, Greeks, Arabs, Italians, Frenchmen, Germans, Englishmen, and Russians, have all made essential contributions to the progress of the science. Assuredly the jealous exaltation of the contribution of one particular nation is not to show the larger spirit.

The importance of the differential calculus arises from the very nature of the subject, which is the systematic consideration of the rates of increase of functions. This idea is immediately presented to us by the study of nature; velocity is the rate of increase of the distance travelled, and acceleration is the rate of increase of velocity. Thus the fundamental idea of change, which is at the basis of our whole perception of phenomena, immediately suggests the enquiry as to the rate of change. The familiar terms of "quickly" and "slowly" gain their meaning from a tacit reference to rates of change. Thus the differential calculus is concerned with the very key of the position from which mathematics can be successfully applied to the explanation of the course of nature.

This idea of the rate of change was certainly in Newton's mind, and was embodied in the

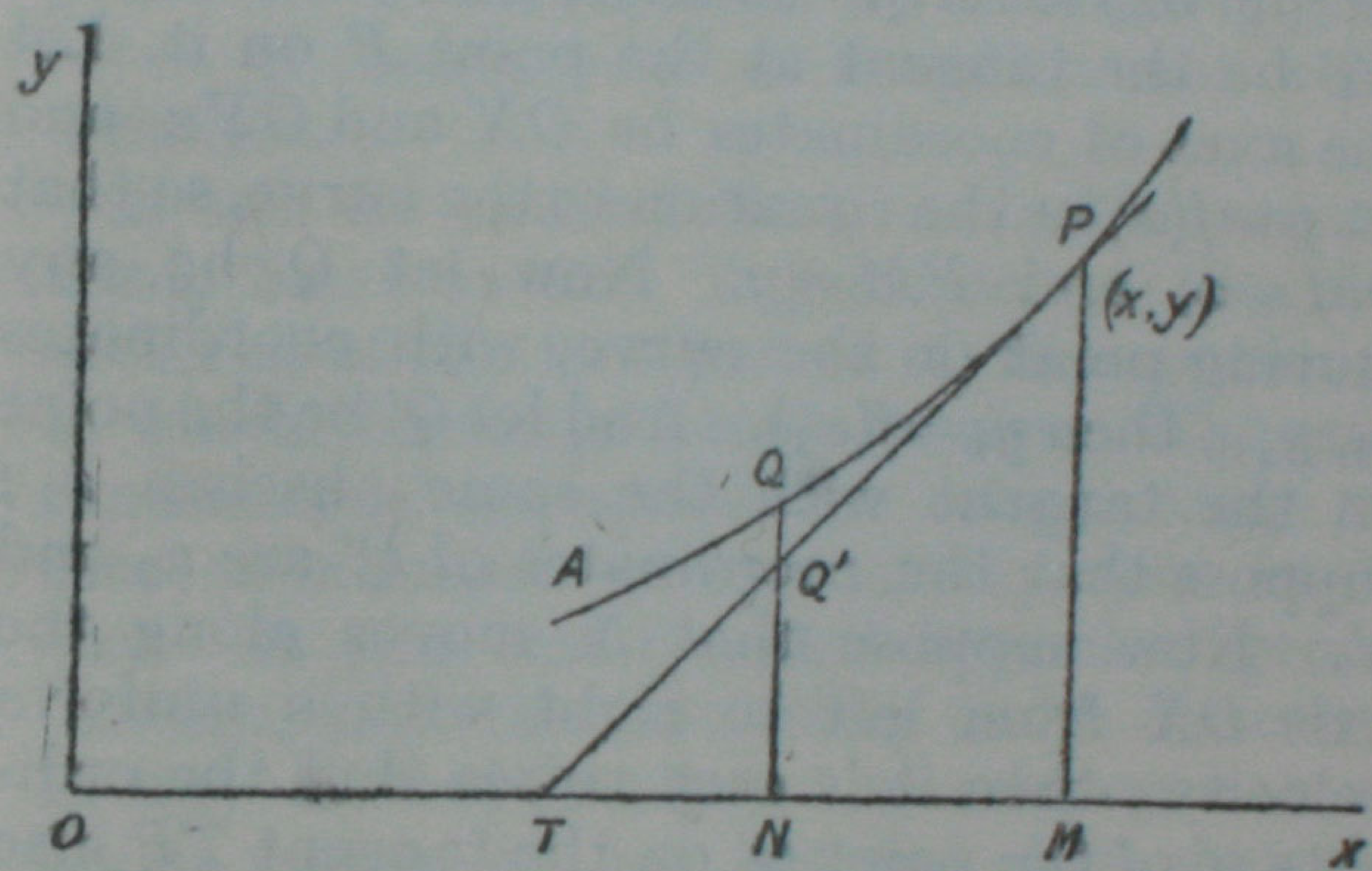

Fig. 32.

language in which he explained the subject. It may be doubted, however, whether this point of view, derived from natural phenomena, was ever much in the minds of the preceding mathematicians who prepared the subject for its birth. They were concerned with the more abstract problems of drawing tangents to curves, of finding the lengths of curves, and of finding the areas enclosed by curves. The last two problems, of the rectification of curves and the quadrature of curves as they are named, belong to the Integral Calculus, which is however involved in the same general subject as the Differential Calculus.

The introduction of coordinate geometry makes the two points of view coalesce. For (cf. fig. 32) let AQP be any curved line and let PT be the tangent at the point P on it. Let the axes of coordinates be OX and OY; and let $y=f(x)$ be the equation to the curve, so that $OM=x$, and $PM=y$. Now let Q be any moving point on the curve, with coordinates x_1, y_1,; then $y_1=f(x_1)$. And let Q' be the point on the tangent with the same abscissa x_1; suppose that the coordinates of Q' are x_1 and y'. Now suppose that N moves along the axis OX from left to right with a uniform velocity; then it is easy to see that the ordinate y' of the point Q' on the tangent TP also increases uniformly as Q' moves along the tangent in a corresponding way. In fact it is easy to see that the ratio of the rate of increase of $Q'N$ to the rate of increase of ON is in the ratio of $Q'N$ to TN, which is the same at all points of the straight line. But the rate of increase of QN, which is the rate of increase of $f(x_1)$, varies from point to point of the curve so long as it is not straight. As Q passes through the point P, the rate of increase of $f(x_1)$ (where x_1 coincides with x for the moment) is the same as the rate of increase of y' on the tangent at P. Hence, if we have a general method of determining the rate of increase of a function $f(x)$ of a variable x, we can determine the slope of the tangent at any point $(x, y,)$ on a curve, and thence can draw it. Thus the problems of drawing tangents to a curve, and of determining the rates of increase of a function are really identical.

It will be noticed that, as in the cases of Conic Sections and Trigonometry, the more artificial of the two points of view is the one in which the subject took its rise. The really fundamental aspect of the science only rose into prominence comparatively late in the day. It is a well-founded historical generalization, that the last thing to be discovered in any science is what the science is really about. Men go on groping for centuries, guided merely by a dim instinct and a puzzled curiosity, till at last "some great truth is loosened."

Let us take some special cases in order to familiarize ourselves with the sort of ideas which we want to make precise. A train is in motion—how shall we determine its velocity at some instant, let us say, at noon? We can take an interval of five minutes which includes noon, and measure how far the train has gone in that period. Suppose we find it to be five miles, we may then conclude that the train was running at the rate of 60 miles per hour. But five miles is a long distance, and we cannot be sure that just at noon the train was moving at this pace. At noon it may have been running 70 miles per hour, and afterwards the break may have been put on. It will be safer to work with a smaller interval, say one minute, which includes noon, and to measure the space traversed during that period. But for some purposes greater accuracy may be required, and one minute may be too long. In practice, the necessary inaccuracy of our measurements makes it useless to take too small a period for measurement. But in theory the smaller the period the better, and we are tempted to say that for ideal accuracy an infinitely small period is required. The older mathematicians, in particular Leibniz, were not only tempted, but yielded to the temptation, and did say it. Even now it is a useful fashion of speech, provided that we know how to interpret it into the language of common sense. It is curious that, in his exposition of the foundations of the calculus, Newton, the natural scientist, is much more philosophical than Leibniz, the philosopher, and on the other hand, Leibniz provided the admirable notation which has been so essential for the progress of the subject.

Now take another example within the region of pure mathematics. Let us proceed to find the rate of increase of the function x^2 for any value x of its argument. We have not yet really defined what we mean by rate of increase. We will try and grasp its meaning in relation to this particular case. When x increases to $x+h$, the function x^2 increases to $(x+h)^2$; so that the total increase has been $(x+h)^2-x^2$, due to an increase h in the argument. Hence throughout the interval x to $(x+h)$ the average increase of the function per unit increase of the argument is $\frac{(x+h)^2-x^2}{h}$.

But

$$(x+h)^2=x^2+2hx+h^2,$$

and therefore

$$\frac{(x+h)^2-x^2}{h}=\frac{2hx+h^2}{h}=2x+h.$$

Thus $2x+h$ is the average increase of the function x^2 per unit increase in the argument, the average being taken over by the interval x to $x+h$. But $2x+h$ depends on h, the size of the interval. We shall evidently get what we want, namely the *rate* of increase at the value x of the argument, by diminishing h more and more. Hence *in the limit* when h

has *decreased indefinitely*, we say that $2x$ is the rate of increase of x^2 at the value x of the argument.

Here again we are apparently driven up against the idea of infinitely small quantities in the use of the words "in the limit when h has decreased indefinitely." Leibniz held that, mysterious as it may sound, there were actually existing such things as infinitely small quantities, and of course infinitely small numbers corresponding to them. Newton's language and ideas were more on the modern lines; but he did not succeed in explaining the matter with such explicitness so as to be evidently doing more than explain Leibniz's ideas in rather indirect language. The real explanation of the subject was first given by Weierstrass and the Berlin School of mathematicians about the middle of the nineteenth century. But between Leibniz and Weierstrass a copious literature, both mathematical and philosophical, had grown up round these mysterious infinitely small quantities which mathematics had discovered and philosophy proceeded to explain. Some philosophers, Bishop Berkeley, for instance, correctly denied the validity of the whole idea, though for reasons other than those indicated here. But the curious fact remained that, despite all criticisms of the foundations of the subject, there could be no doubt but that the mathematical procedure was substantially right. In fact, the subject was right, though the explanations were wrong. It is this possibility of being right, albeit with entirely wrong explanations as to what is being done, that so often makes external criticism—that is so far as it is meant to stop the pursuit of a method—singularly barren and futile in the progress of science. The instinct of trained observers, and their sense of curiosity, due to the fact that they are obviously getting at something, are far safer guides. Anyhow the general effect of the success of the Differential Calculus was to generate a large amount of bad philosophy, centring round the idea of the infinitely small. The relics of this verbiage may still be found in the explanations of many elementary mathematical text-books on the Differential Calculus. It is a safe rule to apply that, when a mathematical or philosophical author writes with a misty profundity, he is talking nonsense.

Newton would have phrased the question by saying that, as h approaches zero, in the limit $2x+h$ becomes $2x$. It is our task so to explain this statement as to show that it does not in reality covertly assume the existence of Leibniz's infinitely small quantities. In reading over the Newtonian method of statement, it is tempting to seek simplicity by saying that $2x+h$ is $2x$, when h is zero. But this will not do; for it thereby abolishes the interval from x to $x+h$, over which the average increase was calculated. The problem is, how to keep an interval of length h over which to calculate the average increase, and at the same time to treat h as if it were zero. Newton did this by the conception of a limit, and we now proceed to give Weierstrass's explanation of its real meaning.

In the first place notice that, in discussing $2x+h$, we have been considering x as fixed in value and h as varying. In other words x has been treated as a "constant" variable, or parameter, as explained in Chapter IX.; and we have really been considering $2x+h$ as a function of the argument h. Hence we can generalize the question on hand, and ask what we mean by saying that the function $f(h)$ tends to the limit l, say, as its argument h tends to the value zero. But again we shall see that the special value *zero* for the argument does not belong to the essence of the subject; and again we generalize still further, and ask, what we mean by saying that the function $f(h)$ tends to the limit l as h tends to the value a.

Now, according to the Weierstrassian explanation the whole idea of h tending to the value a, though it gives a sort of metaphorical picture of what we are driving at, is really off the point entirely. Indeed it is fairly obvious that, as long as we retain anything like "h tending to a," as a fundamental idea, we are really in the clutches of the infinitely small; for we imply the notion of h being infinitely near to a. This is just what we want to get rid of.

Accordingly, we shall yet again restate our phrase to be explained, and ask what we mean by saying that the limit of the function $f(h)$ at a is l.

The limit of $f(h)$ at a is a property of the neighbourhood of a, where "neighbourhood" is used in the sense defined in Chapter XI. during the discussion of the continuity of functions. The value of the function $f(h)$ at a is $f(a)$; but the limit is distinct in idea from the value, and may be different from it, and may exist when the value has not been defined. We shall also use the term "standard of approximation" in the sense in which it is defined in Chapter XI. In fact, in the definition of "continuity" given towards the end of that chapter we have practically defined a limit. The definition of a limit is:—

A function $f(x)$ has the limit l at a value a of its argument x, when in the neighbourhood of a its values approximate to l within *every* standard of approximation.

Compare this definition with that already given for continuity, namely:—

A function $f(x)$ is continuous at a value a of its argument, when in the neighbourhood of a its values approximate to its value at a within *every* standard of approximation.

It is at once evident that a function is continuous at a when (i) it possesses a limit at a, and (ii) that limit is equal to its value at a. Thus the illustrations of continuity which have been given at the end of Chapter XI. are illustrations of the idea of a limit, namely, they were all directed to proving that $f(a)$ was the limit of $f(x)$ at a for the functions considered and the value of a considered. It is really more instructive to consider the limit at a point where a function is not continuous. For example, consider the function of which the graph is given in fig. 20 of Chapter XI. This function $f(x)$ is defined to have the value 1 for all values of the argument except the integers 0, 1, 2, 3, etc., and for these integral values it has the value 0. Now let us think of its limit when $x=3$. We notice that in the definition of the limit the value of the function at a (in this case, $a=3$) is excluded. But, excluding $f(3)$, the values of $f(x)$, when x lies within any interval which (i) contains 3 not as an end-point, and (ii) does not extend so far as 2 and 4, are all equal to 1; and hence these values approximate to 1 within every standard of approximation. Hence 1 is the limit of $f(x)$ at the value 3 of the argument x, but by definition $f(3)=0$.

This is an instance of a function which possesses both a value and a limit at the value 3 of the argument, but the value is not equal to the limit. At the end of Chapter XI. the function x^2 was considered at the value 2 of the argument. Its value at 2 is 2^2, *i.e.* 4, and it was proved that its limit is also 4. Thus here we have a function with a value and a limit which are equal.

Finally we come to the case which is essentially important for our purposes, namely, to a function which possesses a limit, but no defined value at a certain value of its argument. We need not go far to look for such a function, $\frac{2x}{x}$ will serve our purpose. Now in any mathematical book, we might find the equation, $\frac{2x}{x}=2$, written without hesitation or comment. But there is a difficulty in this; for when x is zero, $\frac{2x}{x}=\frac{0}{0}$; and $\frac{0}{0}$ has no defined meaning. Thus the value of the function $\frac{2x}{x}$ at $x=0$ has no defined meaning. But for every other value of x, the value of the function $\frac{2x}{x}$ is 2. Thus the limit of $\frac{2x}{x}$ at $x=0$ is 2, and it has no value at $x=0$. Similarly the limit of $\frac{x^2}{x}$ at $x=a$ is a whatever a may be, so that the limit of $\frac{x^2}{x}$ at $x=0$ is 0. But the value of $\frac{x^2}{x}$ at $x=0$ takes the form $\frac{0}{0}$, which has no defined meaning. Thus the function $\frac{x^2}{x}$ has a limit but no value at 0.

We now come back to the problem from which we started this discussion on the nature of a limit. How are we going to define the rate of increase of the function x^2 at any value x of its argument. Our answer is that this rate of increase is the limit of the function $\frac{(x+h)^2-x^2}{h}$ at the value zero for its argument h. (Note that x is here a "constant.") Let us see how this answer works in the light of our definition of a limit. We have

$$\frac{(x+h)^2-x^2}{h}=\frac{2hx+h^2}{h}=\frac{h(2x+h)}{h}$$

Now in finding the limit of $\frac{h(2x+h)}{h}$ at the value 0 of the argument h, the value (if any) of the function at $h=0$ is excluded. But for all values of h, except $h=0$, we can divide through by h. Thus the limit of $\frac{h(2x+h)}{h}$ at $h=0$ is the same as that of $2x+h$ at $h=0$. Now, whatever standard of approximation k we choose to take, by considering the interval from $-\frac{1}{2}k$ to $+\frac{1}{2}k$ we see that, for values of h which fall within it, $2x+h$ differs from $2x$ by less than $\frac{1}{2}k$, that is by less than k. This is true for *any* standard k. Hence in the neighbourhood of the value 0 for h, $2x+h$ approximates to $2x$ within *every* standard of approximation, and therefore $2x$ is the limit of $2x+h$ at $h=0$. Hence by what has been said above $2x$ is the limit of $\frac{(x+h)^2-x^2}{h}$ at the value 0 for h. It follows, therefore, that $2x$ is what we have called the rate of increase of x^2 at the value x of the argument. Thus this method conducts us to the same rate of increase for x^2 as did the Leibnizian way of making h grow "infinitely small."

The more abstract terms "differential coefficient," or "derived function," are generally used for what we have hitherto called the "rate of increase" of a function. The general definition is as follows: the differential coefficient of the function $f(x)$ is the limit, if it exist, of the function $\frac{f(x+h)-f(x)}{h}$ of the argument h at the value 0 of its argument.

How have we, by this definition and the subsidiary definition of a limit, really managed to avoid the notion of "infinitely small numbers" which so worried our mathematical forefathers? For them the difficulty arose because on the one hand they had to use an interval x to $x+h$ over which to calculate the average increase, and, on the other hand, they finally wanted to put $h=0$. The result was they seemed to be landed into the notion of an existent interval of zero size. Now how do we avoid this difficulty? In this way—we use the notion that corresponding to *any* standard of approximation, *some* interval with such and such properties can be found. The difference is that we have grasped the importance of the notion of "the variable," and they had not done so. Thus, at the end of our exposition of the essential notions of mathematical analysis, we are led back to the ideas with which in Chapter II. we commenced our enquiry—that in mathematics the fundamentally important ideas are those of "*some* things" and "*any* things."

CHAPTER XVI

GEOMETRY

GEOMETRY, like the rest of mathematics, is abstract. In it the properties of the shapes and relative positions of things are studied. But we do not need to consider who is observing the things, or whether he becomes acquainted with them by sight or touch or hearing. In short, we ignore all particular sensations. Furthermore, particular things such as the Houses of Parliament, or the terrestrial globe are ignored. Every proposition refers to any things with such and such geometrical properties. Of course it helps our imagination to look at particular examples of spheres and cones and triangles and squares. But the propositions do not merely apply to the actual figures printed in the book, but to any such figures.

Thus geometry, like algebra, is dominated by the ideas of "any" and "some" things. Also, in the same way it studies the interrelations of sets of things. For example, consider any two triangles *ABC* and *DEF*.

What relations must exist between some of the parts of these triangles, in order that the triangles may be in all respects equal? This is one of the first investigations undertaken in all elementary geometries. It is a study

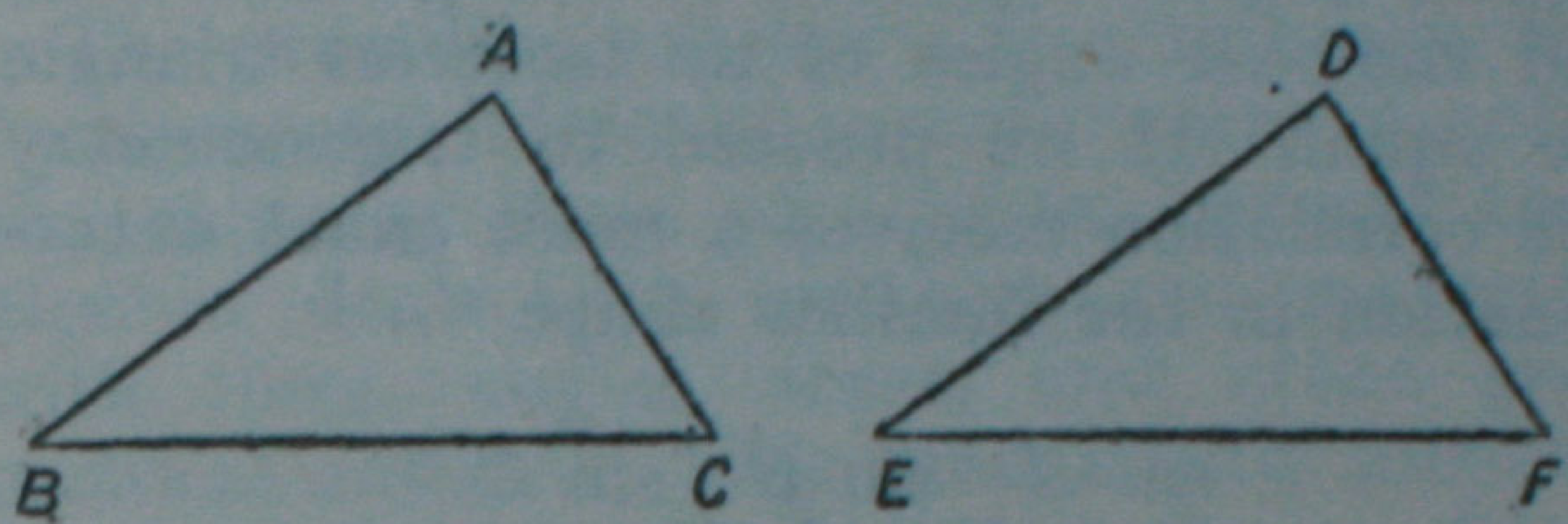

Fig. 33.

of a certain set of possible correlations between the two triangles. The answer is that the triangles are in all respects equal, if:—Either, (*a*) Two sides of the one and the included angle are respectively equal to two sides of the other and the included angle:

Or, (*b*) Two angles of the one and the side joining them are respectively equal to two angles of the other and the side joining them:

Or, (*c*) Three sides of the one are respecttively equal to three sides of the other.

This answer at once suggests a further enquiry. What is the nature of the correlation between the triangles, when the three angles of the one are respectively equal to the three angles of the other? This further investigation leads us on to the whole theory of similarity (cf. Chapter XIII.), which is another type of correlation.

Again, to take another example, consider the internal structure of the triangle ABC. Its sides and angles are inter-related—the greater angle is opposite to the greater side, and the base angles of an isosceles triangle are equal. If we proceed to trigonometry this correlation receives a more exact determination in the familiar shape

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c},$$

$a^2 = b^2 + c^2 - 2bc\cos A$, with two similar formulæ.

Also there is the still simpler correlation between the angles of the triangle, namely, that their sum is equal to two right angles; and between the three sides, namely, that the sum of the lengths of any two is greater than the length of the third

Thus the true method to study geometry is to think of interesting simple figures, such as the triangle, the parallelogram, and the circle, and to investigate the correlations between their various parts. The geometer has in his mind not a detached proposition, but a figure with its various parts mutually inter-dependent. Just as in algebra, he generalizes the triangle into the polygon, and the side into the conic section. Or, pursuing a converse route, he classifies triangles according as they are equilateral, isosceles, or scalene, and polygons according to their number of sides, and conic sections according as they are hyperbolas, ellipses, or parabolas.

The preceding examples illustrate how the fundamental ideas of geometry are exactly the same as those of algebra; except that algebra deals with numbers and geometry with lines, angles, areas, and other geometrical entities. This fundamental identity is one of the reasons why so many geometrical truths can be put into an algebraic dress. Thus if A, B, and C are the numbers of degrees respectively in the angles of the triangle ABC, the correlation between the angles is represented by the equation

$$A+B+C=180°;$$

and if a, b, c are the number of feet respectively in the three sides, the correlation between the sides is represented by $a < b+c$, $b < c+a$, $c < a+b$. Also the trigonometrical formulæ quoted above are other examples of the same fact. Thus the notion of the variable and the correlation of variables is just as essential in geometry as it is in algebra.

But the parallelism between geometry and algebra can be pushed still further, owing to the fact that lengths, areas, volumes, and angles are all measurable; so that, for example, the size of any length can be determined by the number (not necessarily integral) of times which it contains some arbitrarily known unit, and similarly for areas, volumes, and angles. The trigonometrical formulæ, given above, are examples of this fact. But it receives its crowning application in analytical geometry. This great subject is often misnamed as Analytical Conic Sections, thereby fixing attention on merely one of its subdivisions. It is as though the great science of Anthropology were named the Study of Noses, owing to the fact that noses are a prominent part of the human body.

Though the mathematical procedures in geometry and algebra are in essence identical and intertwined in their development, there is necessarily a fundamental distinction between the properties of space and the properties of number—in fact all the essential difference between space and number. The "spaciness" of space and the "numerosity" of number are essentially different things, and must be directly apprehended. None of the applications of algebra to geometry or of geometry to algebra go any step on the road to obliterate this vital distinction.

One very marked difference between space and number is that the former seems to be so much less abstract and fundamental than the latter. The number of the archangels can be counted just because they are things. When we once know that their names are Raphael, Gabriel, and Michael, and that these distinct names represent distinct beings, we know without further question that there are three of them. All the subtleties in the world about the nature of angelic existences cannot alter this fact, granting the premisses.

But we are still quite in the dark as to their relation to space. Do they exist in space at all? Perhaps it is equally nonsense to say that they are here, or there, or anywhere, or everywhere. Their existence may simply have no relation to localities in space. Accordingly, while numbers must apply to all things, space need not do so.

The perception of the locality of things would appear to accompany, or be involved in many, or all, of our sensations. It is independent of any particular sensation in the sense that it accompanies many sensations. But it is a special peculiarity of the things which we apprehend by our sensations. The direct apprehension of what we mean by the positions of things in respect to each other is a thing *sui generis*, just as are the apprehensions of sounds, colours, tastes, and smells. At first sight therefore it would appear that mathematics, in so far as it includes geometry in its scope, is not abstract in the sense in which abstractness is ascribed to it in Chapter I.

This, however, is a mistake; the truth being that the "spaciness" of space does not enter into our geometrical *reasoning* at all. It enters into the geometrical intuitions of mathematicians in ways personal and peculiar to each individual. But what enter into the reasoning are merely certain properties of things in space, or of things forming space, which properties are completely abstract in the sense in which abstract was defined in Chapter I.; these properties do not involve any peculiar space-apprehension or space-intuition or space-sensation. They are on exactly the same basis as the mathematical properties of number. Thus the space-intuition which is so essential an aid to the study of geometry is logically irrelevant: it does not enter into the premisses when they are properly stated, nor into any step of the reasoning. It has the practical importance of an example, which is essential for the stimulation of our thoughts. Examples are equally necessary to stimulate our thoughts on number. When we think of "two" and "three" we see strokes in a row, or balls in a heap, or some other physical aggregation of particular things. The peculiarity of geometry is the fixity and overwhelming importance of the one particular example which occurs to our minds. The abstract logical form of the propositions when fully stated is, "If any collections of things have such and such abstract properties, they also have such and such other abstract properties." But what appears before the mind's eye is a collection of points, lines, surfaces, and volumes in the space: this example inevitably appears, and is the sole example which lends to the proposition its interest. However, for all its overwhelming importance, it is but an example.

Geometry, viewed as a mathematical science, is a division of the more general science of order. It may be called the science of dimensional order; the qualification "dimensional" has been introduced because the limitations, which reduce it to only a part of the general science of order, are such as to produce the regular relations of straight lines to planes, and of planes to the whole of space.

It is easy to understand the practical importance of space in the formation of the scientific conception of an external physical world. On the one hand our space-perceptions are intertwined in our various sensations and connect them together. We normally judge that we touch an object in the same place as we see it; and even in abnormal cases we touch it in the same space as we see it, and this is the real fundamental fact which ties together our various sensations. Accordingly, the space perceptions are in a sense the common part of our sensations. Again it happens that the abstract properties of space form a large part of whatever is of spatial interest. It is not too much to say that to every property of space there corresponds an abstract mathematical statement. To take the most unfavourable instance, a curve may have a special beauty of shape: but to this shape there will correspond some abstract mathematical properties which go with this shape and no others.

Thus to sum up: (1) the properties of space which are investigated in geometry, like those of number, are properties belonging to things as things, and without special reference to any particular mode of apprehension: (2) Space-perception accompanies our sensations, perhaps all of them, certainly many; but it does not seem to be a necessary quality of things that they should all exist in one space or in any space.

CHAPTER XVII

QUANTITY

In the previous chapter we pointed out that lengths are measurable in terms of some unit length, areas in term of a unit area, and volumes in terms of a unit volume.

When we have a set of things such as lengths which are measurable in terms of any one of them, we say that they are quantities of the same kind. Thus lengths are quantities of the same kind, so are areas, and so are volumes. But an area is not a quantity of the same kind as a length, nor is it of the same kind as a volume. Let us think a little more on what is meant by being measurable, taking lengths as an example.

Lengths are measured by the foot-rule. By transporting the foot-rule from place to place we judge of the equality of lengths. Again, three adjacent lengths, each of one foot, form one whole length of three feet. Thus to measure lengths we have to determine the equality of lengths and the addition of lengths. When some test has been applied, such as the transporting of a foot-rule, we say that the lengths are equal; and when some process has been applied, so as to secure lengths being contiguous and not overlapping, we say that the lengths have been added to form one whole length. But we cannot arbitrarily take any test as the test of equality and any process as the process of addition. The results of operations of addition and of judgments of equality must be in accordance with certain preconceived conditions. For example, the addition of two greater lengths must yield a length greater than that yielded by the addition of two smaller lengths. These preconceived conditions when accurately formulated may be called axioms of quantity. The only question as to their truth or falsehood which can arise is whether, when the axioms are satisfied, we necessarily get what ordinary people call quantities. If we do not, then the name "axioms of quantity" is ill-judged —that is all.

These axioms of quantity are entirely abstract, just as are the mathematical properties of space. They are the same for all quantities, and they presuppose no special mode of perception. The ideas associated with the notion of quantity are the means by which a continuum like a line, an area, or a volume can be split up into definite parts. Then these parts are counted; so that numbers can be used to determine the exact properties of a continuous whole.

Our perception of the flow of time and of the succession of events is a chief example of the application of these ideas of quantity. We measure time (as has been said in considering periodicity) by the repetition of similar events—the burning of successive inches of a uniform candle, the rotation of the earth relatively to the fixed stars, the rotation of the hands of a clock are all examples of such repetitions. Events of these types take the place of the foot-rule in relation to lengths. It is not necessary to assume that events of any one of these types are exactly equal in duration at each recurrence. What is necessary is that a rule should be known which will enable us to express the relative durations of, say, two examples of some type. For example, we may if we like suppose that the rate of the earth's rotation is decreasing, so that each day is longer than the preceding by some minute fraction of a second. Such a rule enables us to compare the length of any day with that of any other day. But what is essential is that one series of repetitions, such as successive days, should be taken as the standard series; and, if the various events of that series are not taken as of equal duration, that a rule should be stated which regulates the duration to be assigned to each day in terms of the duration of any other day.

What then are the requisites which such a rule ought to have? In the first place it should lead to the assignment of nearly equal durations to events which common sense judges to possess equal durations. A rule which made days of violently different lengths, and which made the speeds of apparently similar operations vary utterly out of proportion to the apparent minuteness of their differences, would never do. Hence the first requisite is general agreement with common sense. But this is not sufficient absolutely to determine the rule, for common sense is a rough observer and very easily satisfied. The next requisite is that minute adjustments of the rule should be so made as to allow of the simplest possible statements of the laws of nature. For example, astronomers tell us that the earth's rotation is slowing down, so that each day gains in length by some inconceivably minute fraction of a second. Their only reason for their assertion (as stated more fully in the discussion of periodicity) is that without it they would have to abandon the Newtonian laws of motion. In order to keep the laws of motion simple, they alter the measure of time. This is a perfectly legitimate procedure so long as it is thoroughly understood.

What has been said above about the abstract nature of the mathematical properties of space applies with appropriate verbal changes to the mathematical properties of time. A sense of the flux of time accompanies all our sensations and perceptions, and practically all that interests us in regard to time can be paralleled by the abstract mathematical properties which we ascribe to it. Conversely what has been said about the two requisites for the rule by which we determine the length of the day, also applies to the rule for determining the length of a yard measure—namely, the yard measure appears to retain the same length as it moves about. Accordingly, any rule must bring out that, apart from minute changes, it does remain of invariable length. Again, the second requisite is this, a definite rule for minute changes shall be stated which allows of the simplest expression of the laws of nature. For example, in accordance with the second requisite the yard measures are supposed to expand and contract with changes of temperature according to the substances which they are made of.

Apart from the facts that our sensations are accompanied with perceptions of locality and of duration, and that lines, areas, volumes, and durations, are each in their way quantities, the theory of numbers would be of very subordinate use in the exploration of the laws of the Universe. As it is, physical science reposes on the main ideas of number, quantity, space, and time. The mathematical sciences associated with them do not form the whole of mathematics, but they are the substratum of mathematical physics as at present existing.

NOTES

A (p. 60).—In reading these equations it must be noted that a bracket is used in mathematical symbolism to mean that the operations within it are to be performed first. Thus $(1+3)+2$ directs us first to add 3 to 1, and then to add 2 to the result; and $1+(3+2)$ directs us first to add 2 to 3, and then to add the result to 1. Again a numerical example of equation (5) is

$$2 \times (3+4) = (2 \times 3) + (2 \times 4).$$

We perform first the operations in brackets and obtain

$$2 \times 7 = 6 + 8$$

which is obviously true.

B (p. 136).—This fundamental ratio $\frac{SP}{PN}$ is called the eccentricity of the curve. The shape of the curve, as distinct from its scale or size, depends upon the value of its eccentricity. Thus it is wrong to think of ellipses in general or of hyperbolas in general as having in either case one definite shape. Ellipses with different eccentricities have different shapes, and their sizes depend upon the lengths of their major axes. An ellipse with small eccentricity is very nearly a circle, and an ellipse of eccentricity only slightly less than unity is a long flat oval. All parabolas have the same eccentricity and are therefore of the same shape, though they can be drawn to different scales.

C (p. 204).—If a series with all its terms positive is convergent, the modified series found by making some terms positive and some negative according to any definite rule is also convergent. Each one of the set of series thus found, including the original series, is called "absolutely convergent." But it is possible for a series with terms partly positive and partly negative to be convergent, although the corresponding series with all its terms positive is divergent. For example, the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\text{ etc.}$$

is convergent though we have just proved that

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\text{ etc.}$$

is divergent. Such convergent series, which are not absolutely convergent, are much more difficult to deal with than absolutely convergent series.

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NOTE ON THE STUDY OF MATHEMATICS

The difficulty that beginners find in the study of this science is due to the large amount of technical detail which has been allowed to accumulate in the elementary text-books, obscuring the important ideas.

The first subjects of study, apart from a knowledge of arithmetic which is presupposed, must be elementary geometry and elementary algebra. The courses in both subjects should be short, giving only the necessary ideas; the algebra should be studied graphically, so that in practice the ideas of elementary coordinate geometry are also being assimilated. The next pair of subjects should be elementary trigonometry and the coordinate geometry of the straight line and circle. The latter subject is a short one; for it really merges into the algebra. The student is then prepared to enter upon conic sections, a very short course of geometrical conic sections and a longer one of analytical conics. But in all these courses great care should be taken not to overload the mind with more detail than is necessary for the exemplification of the fundamental ideas.

The differential calculus and afterwards the integral calculus now remain to be attacked on the same system. A good teacher will already have illustrated them by the consideration of special cases in the course on algebra and coordinate geometry. Some short book on three-dimensional geometry must be also read.

This elementary course of mathematics is sufficient for some types of professional career. It is also the necessary preliminary for any one wishing to study the subject for its intrinsic interest. He is now prepared to commence on a more extended course. He must not, however, hope to be able to master it as a whole. The science has grown to such vast proportions that probably no living mathematician can claim to have achieved this.

Passing to the serious treatises on the subject to be read *after* this preliminary course, the following may be mentioned: Cremona's *Pure Geometry* (English Translation, Clarendon Press, Oxford), Hobson's *Treatise on Trigonometry*, Chrystal's *Treatise on Algebra* (2 volumes), Salmon's *Conic Sections*, Lamb's *Differential Calculus*, and some book on *Differential Equations*. The student will probably not desire to direct equal attention to all these subjects, but will study one or more of them, according as his interest dictates. He will then be prepared to select more advanced works for himself, and to plunge into the higher parts of the subject. If his interest lies in analysis, he should now master an elementary treatise on the theory of Functions of the Complex Variable; if he prefers to specialize in Geometry, he must now proceed to the standard treatises on the Analytical Geometry of three dimensions. But at this stage of his career in learning he will not require the advice of this note.

I have deliberately refrained from mentioning any elementary works. They are very numerous, and of various merits, but none of such outstanding superiority as to require special mention by name to the exclusion of all the others.

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