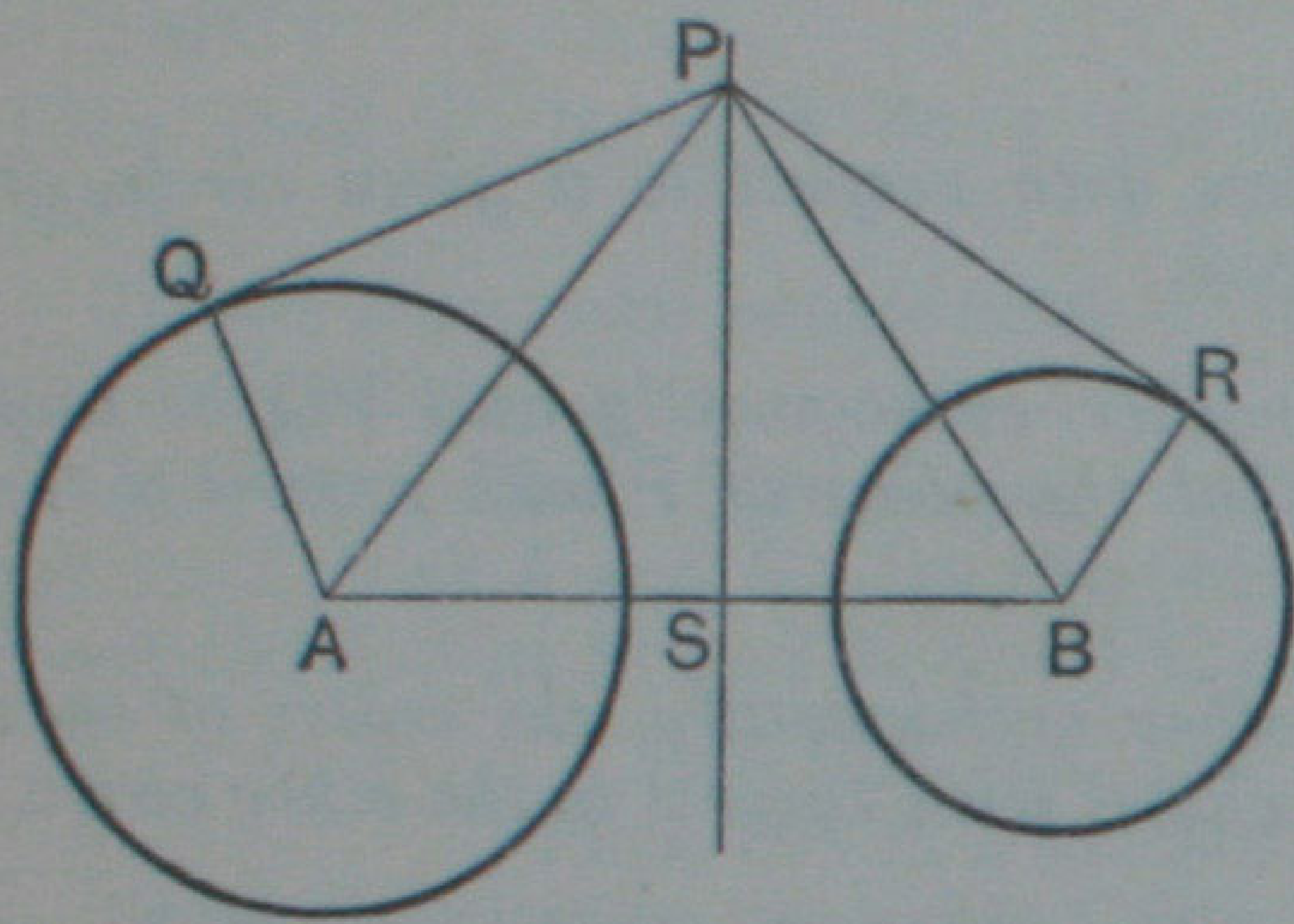
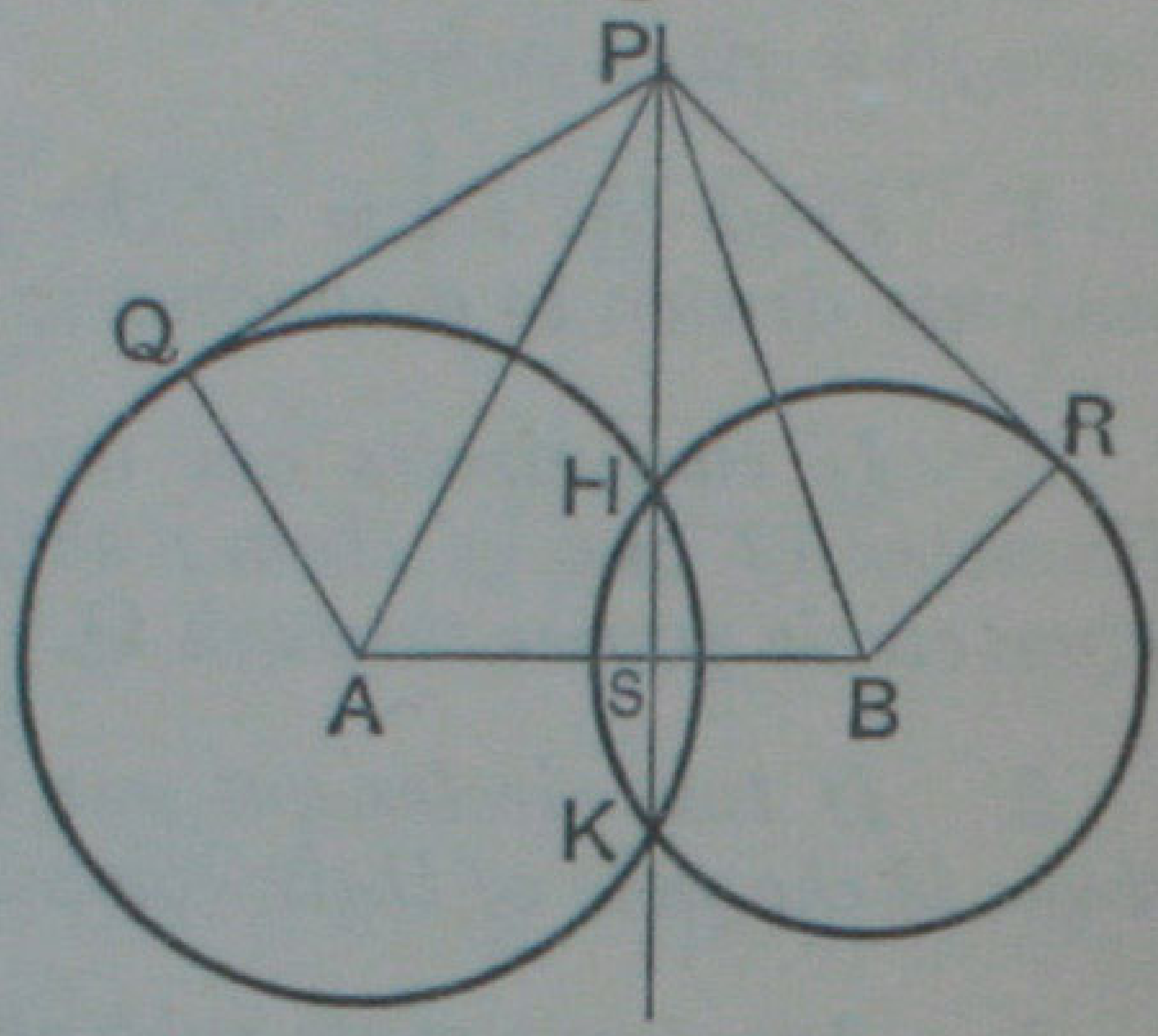
## IV. ON THE RADICAL AXIS.

1. *To find the locus of points from which the tangents drawn to two given circles are equal.*

Fig. 1. Fig. 2.




Let A and B be the centres of the given circles, whose radii are $a$ and $b$; and let P be any point such that the tangent PQ drawn to the circle (A) is equal to the tangent PR drawn to the circle (B).

*It is required to find the locus of* P.

Join PA, PB, AQ, BR, AB; and from P draw PS perp. to AB.

Then because $PQ = PR$, $\therefore PQ^2 = PR^2$.

But $PQ^2 = PA^2 - AQ^2$; and $PR^2 = PB^2 - BR^2$: I. 47.

$\therefore PA^2 - AQ^2 = PB^2 - BR^2$;

that is, $PS^2 + AS^2 - a^2 = PS^2 + SB^2 - b^2$; I. 47.

or, $AS^2 - a^2 = SB^2 - b^2$.

Hence AB is divided at S, so that $AS^2 - SB^2 = a^2 - b^2$:

$\therefore$ S is a *fixed point.*

Hence all points from which equal tangents can be drawn to the two circles lie on the straight line which cuts AB at rt. angles, so that the difference of the squares on the segments of AB is equal to the difference of the squares on the radii.

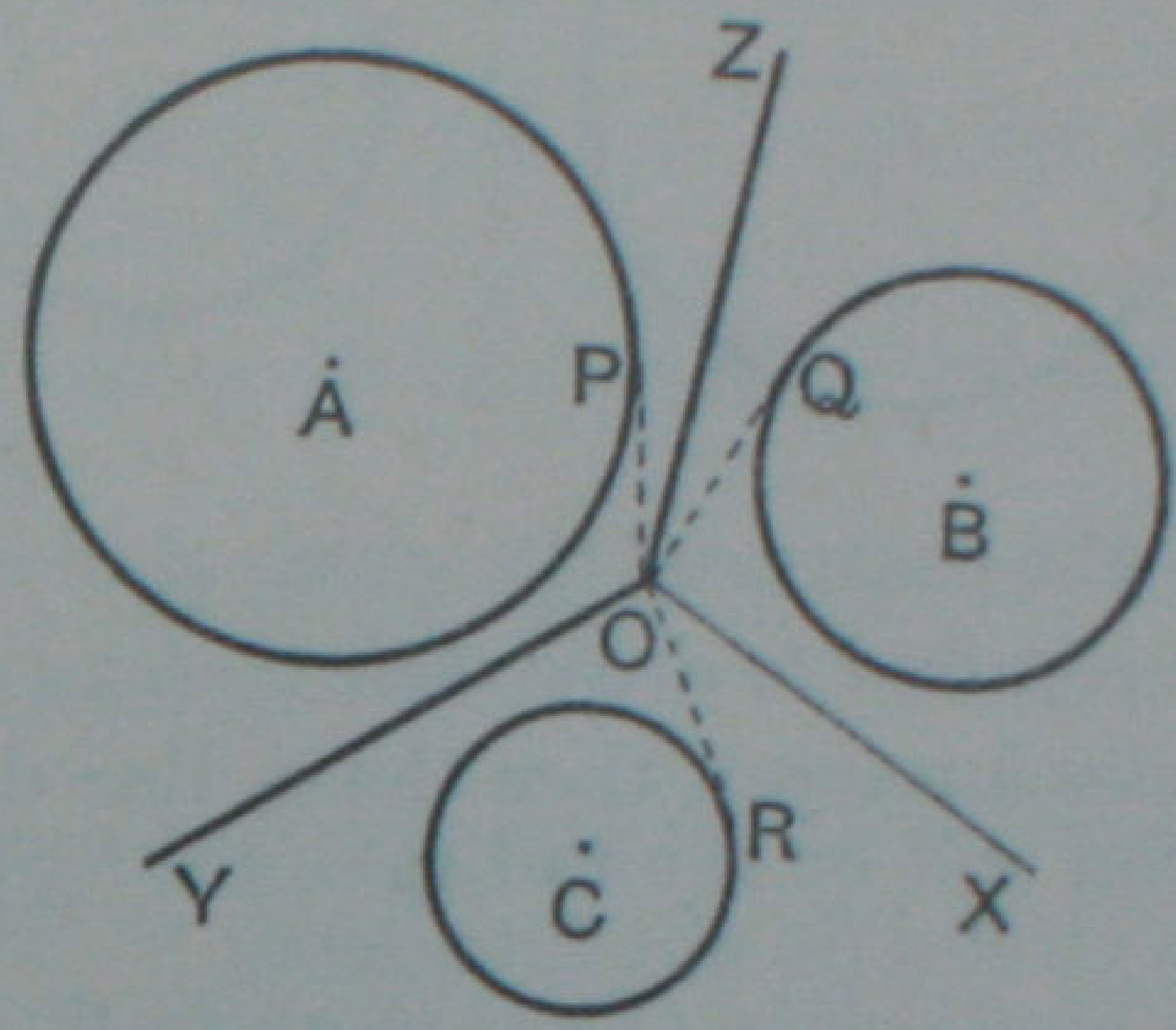
Again, by simply retracing these steps, it may be shewn that in Fig. 1 every point in SP, and in Fig. 2 every point in SP exterior to the circles, is such that tangents drawn from it to the two circles are equal.

Hence we conclude that in Fig. 1 the whole line SP is the required locus, and in Fig. 2 that part of SP which is without the circles.

In either case SP is said to be the **Radical Axis** of the two circles.

Corollary. *If the circles cut one another as in* Fig. 2, *it is clear that the Radical Axis is identical with the straight line which passes through the points of intersection of the circles;* for it follows readily from III. 36 that tangents drawn to two intersecting circles from any point in the common chord produced are equal.

2. *The Radical Axes of three circles taken in pairs are concurrent.*




Let there be three circles whose centres are A, B, C.

Let OZ be the radical axis of the ⊙s (A) and (B);

and OY the Radical Axis of the ⊙s (A) and (C), O being the point of their intersection.

*Then shall the radical axis of the* ⊙s (B) *and* (C) *pass through* O.

It will be found that the point O is either *without* or *within all the circles.*

I. When O is without the circles.

From O draw OP, OQ, OR tangents to the ⊙s (A), (B), (C).

Then because O is a point on the radical axis of (A) and (B); *Hyp.*

∴ OP = OQ.

And because O is a point on the radical axis of (A) and (C), *Hyp.*

∴ OP = OR;

∴ OQ = OR;

∴ O is a point on the radical axis of (B) and (C);

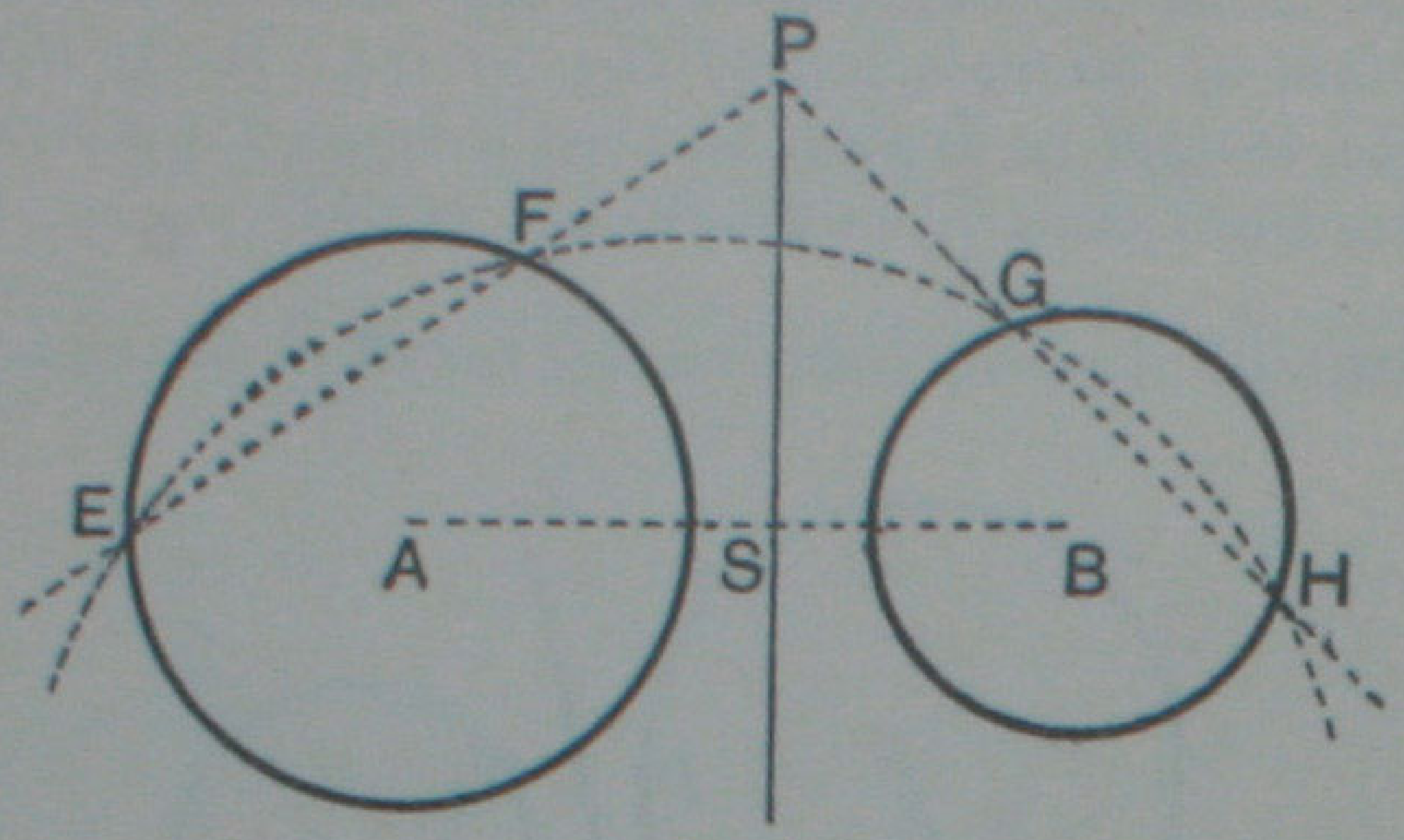
that is, the radical axis of (B) and (C) passes through O.

II. If the circles intersect in such a way that O is within them all;

the radical axes are then the common chords of the three circles taken two and two; and it is required to prove that these common chords are concurrent. This may be shewn indirectly by III. 35.

Definition. The point of intersection of the radical axes of three circles taken in pairs is called the **radical centre.**

3. *To draw the radical axis of two given circles.*




Let A and B be the centres of the given circles.

*It is required to draw their radical axis.*

If the given circles intersect, then the st. line drawn through their points of intersection will be the radical axis. [Ex. 1, Cor. p. 397.]

But if the given circles do not intersect,

describe any circle so as to cut them in E, F and G, H.

Join EF and HG, and produce them to meet in P.

Join AB; and from P draw PS perp. to AB.

*Then* PS *shall be the radical axis of the* ⊙s (A), (B).

[The proof follows from III. 36 and Ex. 1, p. 396.]

Definition. If each pair of circles in a given system have the same radical axis, the circles are said to be **co-axal.**

## EXAMPLES ON THE RADICAL AXIS.

1. *Shew that the radical axis of two circles bisects any one of their common tangents.*

2. *If tangents are drawn to two circles from any point on their radical axis; shew that a circle described with this point as centre and any one of the tangents as radius, cuts both the given circles orthogonally.*

3. O *is the radical centre of three circles, and from* O *a tangent* OT *is drawn to any one of them: shew that a circle whose centre is* O *and radius* OT *cuts all the given circles orthogonally.*

4. *If three circles touch one another, taken two and two, shew that their common tangents at the points of contact are concurrent.*

5. *If circles are described on the three sides of a triangle as diameter, their radical centre is the orthocentre of the triangle.*

6. *All circles which pass through a fixed point and cut a given circle orthogonally, pass through a second fixed point.*

7. *Find the locus of the centres of all circles which pass through a given point and cut a given circle orthogonally.*

8. *Describe a circle to pass through two given points and cut a given circle orthogonally.* ✓

9. *Find the locus of the centres of all circles which cut two given circles orthogonally.*

10. *Describe a circle to pass through a given point and cut two given circles orthogonally.*

11. *The difference of the squares on the tangents drawn from any point to two circles is equal to twice the rectangle contained by the straight line joining their centres and the perpendicular from the given point on their radical axis.*

12. *In a system of co-axal circles which do not intersect, any point is taken on the radical axis; shew that a circle described from this point as centre, with radius equal to the tangent drawn from it to any one of the circles, will meet the line of centres in two* fixed *points.*

[*These fixed points are called the* **Limiting Points** *of the system.*]

13. *In a system of co-axal circles the two limiting points and the points in which any one circle of the system cuts the line of centres form a harmonic range.*

14. *In a system of co-axal circles a limiting point has the same polar with regard to all the circles of the system.*

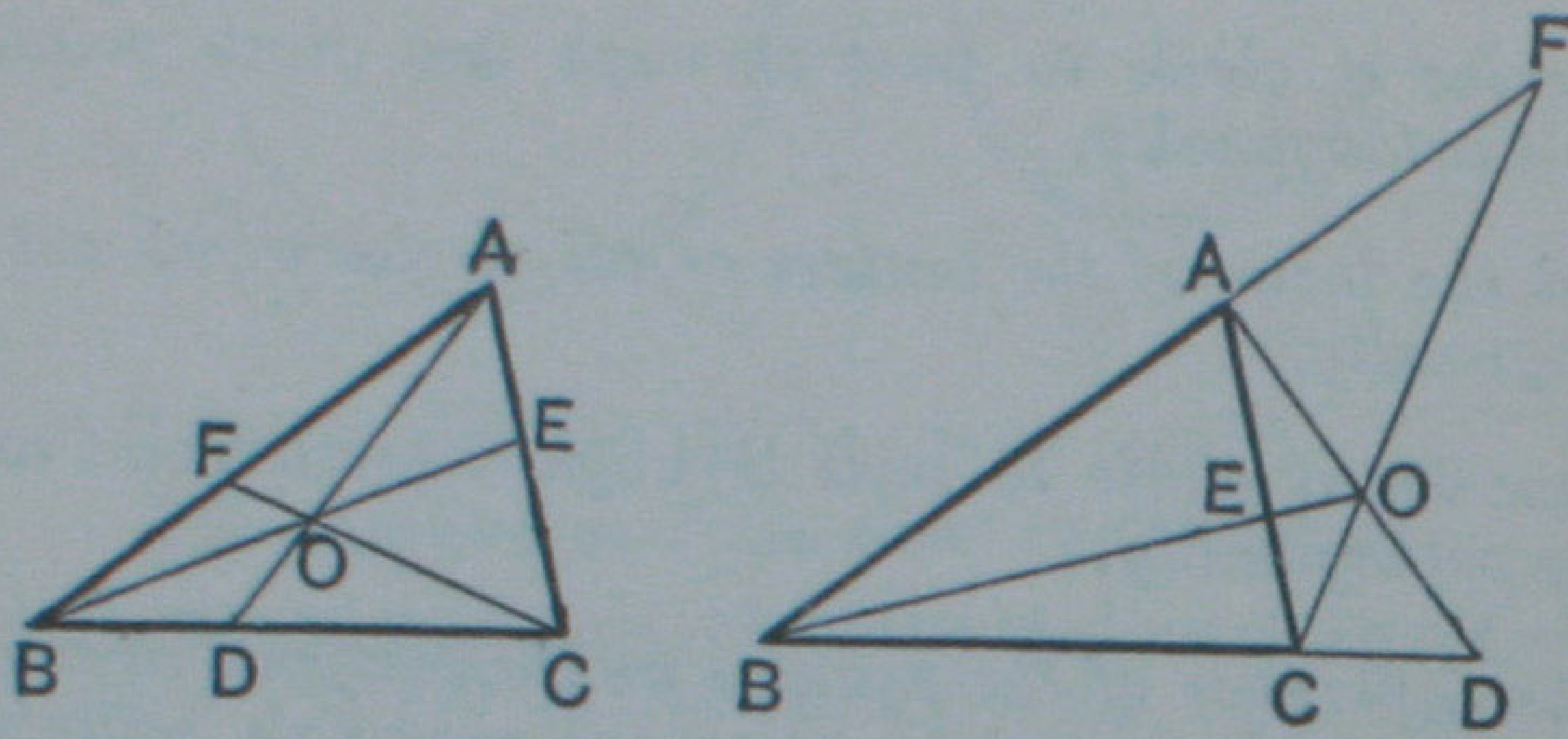
15. *If two circles are orthogonal any diameter of one is cut harmonically by the other.*

## V. ON TRANSVERSALS.

In the two following theorems we are to suppose that the segments of straight lines are expressed numerically in terms of some common unit; and the ratio of one such segment to another will be denoted by the fraction of which the first is the numerator and the second the denominator.

DEFINITION. A straight line drawn to cut a given system of lines is called a **transversal.**

1. *If three concurrent straight lines are drawn from the angular points of a triangle to meet the opposite sides, then the product of three alternate segments taken in order is equal to the product of the other three segments.*




Let AD, BE, CF be drawn from the vertices of the △ ABC to intersect at O, and cut the opposite sides at D, E, F.

*Then shall* BD . CE . AF = DC . EA . FB.

Now the △s AOB, AOC have a common base AO; and it may be shewn that

BD : DC = the alt. of △ AOB : the alt. of △ AOC;

$$\therefore \frac{BD}{DC} = \frac{\triangle AOB}{\triangle AOC};$$

similarly,
$$\frac{CE}{EA} = \frac{\triangle BOC}{\triangle BOA};$$

and
$$\frac{AF}{FB} = \frac{\triangle COA}{\triangle COB}.$$

Multiplying these ratios, we have

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1;$$

or,
$$BD \,.\, CE \,.\, AF = DC \,.\, EA \,.\, FB.$$

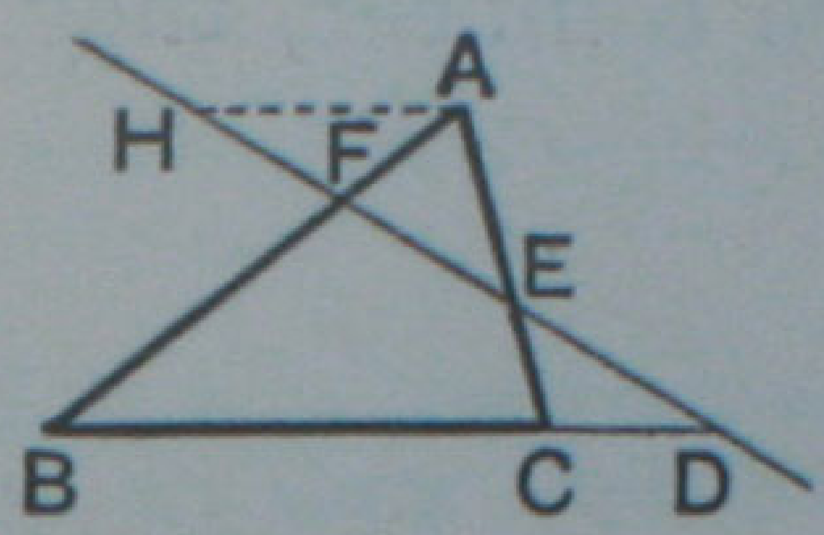
Q.E.D.

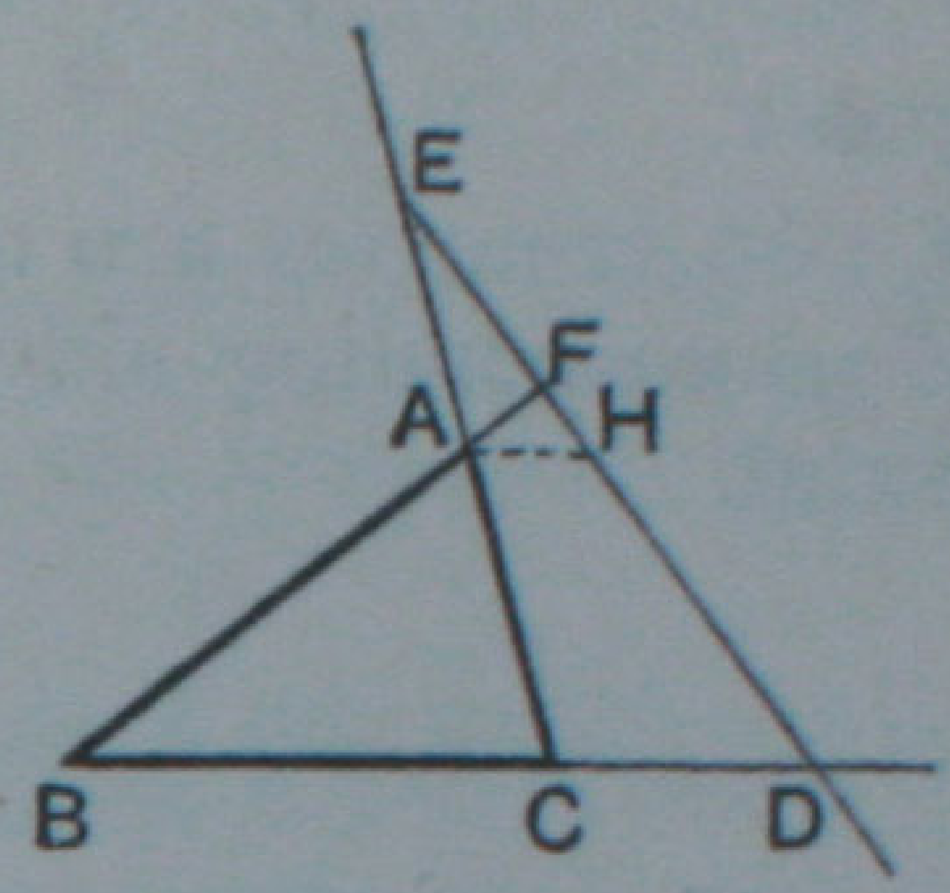
NOTE. The converse of this theorem, which may be proved indirectly, is very important: it may be enunciated thus:

*If three straight lines drawn from the vertices of a triangle cut the opposite sides so that the product of three alternate segments taken in order is equal to the product of the other three, then the three straight lines are concurrent.*

*That is, if* BD . CE . AF = DC . EA . FB,
*then* AD, BE, CF *are concurrent.*

2. *If a transversal is drawn to cut the sides, or the sides produced, of a triangle, the product of three alternate segments taken in order is equal to the product of the other three segments.*




Let ABC be a triangle, and let a transversal meet the sides BC, CA, AB, or these sides produced, at D, E, F.

*Then shall* $BD \,.\, CE \,.\, AF = DC \,.\, EA \,.\, FB.$

Draw AH par[1] to BC, meeting the transversal at H.

Then from the similar $\triangle^s$ DFB, HAF,

$$\frac{BD}{FB} = \frac{HA}{AF}:$$

and from the similar $\triangle^s$ DCE, HAE,

$$\frac{CE}{DC} = \frac{EA}{HA}:$$

$\therefore$, by multiplication, $\dfrac{BD}{FB} \cdot \dfrac{CE}{DC} = \dfrac{EA}{AF};$

that is, $$\frac{BD \,.\, CE \,.\, AF}{DC \,.\, EA \,.\, FB} = 1,$$

or, $$BD \,.\, CE \,.\, AF = DC \,.\, EA \,.\, FB.$$

Q.E.D.

NOTE. In this theorem the transversal must either meet two sides and the third side produced, as in Fig. 1; or all three sides produced, as in Fig. 2.

The converse of this theorem may be proved indirectly:

*If three points are taken in two sides of a triangle and the third side produced, or in all three sides produced, so that the product of three alternate segments taken in order is equal to the product of the other three segments, the three points are collinear.*

## DEFINITIONS.

1. If two triangles are such that three straight lines joining corresponding vertices are concurrent, they are said to be **co-polar**.

2. If two triangles are such that the points of intersection of corresponding sides are collinear, they are said to be **co-axial**.

The propositions given on pages 111–114 relating to the concurrence of straight lines in a triangle, may be proved by the method of transversals, and in addition to these the following important theorems may be established.

## Theorems to be proved by Transversals.

1. *The straight lines which join the vertices of a triangle to the points of contact of the inscribed circle* (*or any of the three escribed circles*) *are concurrent.*

2. *The middle points of the diagonals of a complete quadrilateral are collinear.* [See Def. 4, p. 387.]

3. *Co-polar triangles are also co-axial; and conversely co-axial triangles are also co-polar.*

4. *The six centres of similitude of three circles lie three by three on four straight lines.*

# MISCELLANEOUS EXAMPLES ON BOOK VI.

1. Through D, any point in the base of a triangle ABC, straight lines DE, DF are drawn parallel to the sides AB, AC, and meeting the sides at E, F: shew that the triangle AEF is a mean proportional between the triangles FBD, EDC.

2. If two triangles have one angle of the one equal to one angle of the other, and a second angle of the one supplementary to a second angle of the other, then the sides about the third angles are proportional.

3. AE bisects the vertical angle of the triangle ABC and meets the base in E; shew that if circles are described about the triangles ABE, ACE, the diameters of these circles are to each other in the same ratio as the segments of the base.

4. Through a fixed point O draw a straight line so that the parts intercepted between O and the perpendiculars drawn to the straight line from two other fixed points may have a given ratio.

5. The angle A of a triangle ABC is bisected by AD meeting BC in D, and AX is the median bisecting BC: shew that XD has the same ratio to XB as the difference of the sides has to their sum.

6. AD and AE bisect the vertical angle of a triangle internally and externally, meeting the base in D and E; shew that if O is the middle point of BC, then OB is a mean proportional between OD and OE.

7. P and Q are fixed points; AB and CD are fixed parallel straight lines; any straight line is drawn from P to meet AB at M, and a straight line is drawn from Q parallel to PM meeting CD at N: shew that the ratio of PM to QN is constant, and thence shew that the straight line through M and N passes through a fixed point.

8. If C is the middle point of an arc of a circle whose chord is AB, and D is any point in the conjugate arc; shew that

$$AD + DB : DC :: AB : AC.$$

9. In the triangle ABC the side AC is double of BC. If CD, CE bisect the angle ACB internally and externally meeting AB in D and E, shew that the areas of the triangles CBD, ACD, ABC, CDE are as 1, 2, 3, 4.

10. AB, AC are two chords of a circle; a line parallel to the tangent at A cuts AB, AC in D and E respectively: shew that the rectangle AB, AD is equal to the rectangle AC, AE.

11. If from any point on the hypotenuse of a right-angled triangle perpendiculars are drawn to the two sides, the rectangle contained by the segments of the hypotenuse will be equal to the sum of the rectangles contained by the segments of the sides.

12. D is a point in the side AC of the triangle ABC, and E is a point in AB. If BD, CE divide each other into parts in the ratio 4 : 1, then D, E divide CA, BA in the ratio 3 : 1.

13. If the perpendiculars from two fixed points on a straight line passing between them be in a given ratio, the straight line must pass through a third fixed point.

14. PA, PB are two tangents to a circle; PCD any chord through P: shew that the rectangle contained by one pair of opposite sides of the quadrilateral ACBD is equal to the rectangle contained by the other pair.

15. A, B, C are any three points on a circle, and the tangent at A meets BC produced in D: shew that the diameters of the circles circumscribed about ABD, ACD are as AD to CD.

16. AB, CD are two diameters of the circle ADBC at right angles to each other, and EF is any chord; CE, CF are drawn meeting AB produced in G and H; prove that

the rect. CE, HG = the rect. EF, CH.

17. From the vertex A of any triangle ABC draw a line meeting BC produced in D so that AD may be a mean proportional between the segments of the base.

18. Two circles touch internally at O; AB a chord of the larger circle touches the smaller in C which is cut by the lines OA, OB in the points P, Q: shew that OP : OQ :: AC : CB.

19. AB is any chord of a circle; AC, BC are drawn to any point C in the circumference and meet the diameter perpendicular to AB at D, E: if O is the centre, shew that the rect. OD, OE is equal to the square on the radius.

20. YD is a tangent to a circle drawn from a point Y in the diameter AB produced; from D a perpendicular DX is drawn to the diameter; shew that the points X, Y divide AB internally and externally in the same ratio.

21. Determine a point in the circumference of a circle, from which lines drawn to two other given points shall have a given ratio.

22. O is the centre and OA a radius of a given circle, and V is a fixed point in OA; P and Q are two points on the circumference on opposite sides of A and equidistant from it; QV is produced to meet the circle in L; shew that, whatever be the length of the arc PQ, the chord LP will always meet OA produced in a fixed point.

23. EA, EA′ are diameters of two circles touching each other externally at E; a chord AB of the former circle, when produced, touches the latter at C′, while a chord A′B′ of the latter touches the former at C: prove that the rectangle, contained by AB and A′B′, is four times as great as that contained by BC′ and B′C.

24. If a circle be described touching externally two given circles, the straight line passing through the points of contact will intersect the line of centres of the given circles at a fixed point.

25. Two circles touch externally in C; if any point D be taken without them so that the radii AC, BC subtend equal angles at D, and DE, DF be tangents to the circles, shew that DC is a mean proportional between DE and DF.

26. If through the middle point of the base of a triangle any line be drawn intersecting one side of the triangle, the other produced, and the line drawn parallel to the base from the vertex, it will be divided harmonically.

27. If from either base angle of a triangle a line be drawn intersecting the median from the vertex, the opposite side, and the line drawn parallel to the base from the vertex, it will be divided harmonically.

28. Any straight line drawn to cut the arms of an angle and its internal and external bisectors is cut harmonically.

29. P, Q are harmonic conjugates of A and B, and C is an external point; if the angle PCQ is a right angle, shew that CP, CQ are the internal and external bisectors of the angle ACB.

30. From C, one of the base angles of a triangle, draw a straight line meeting AB in G, and a straight line through A parallel to the base in E, so that CE may be to EG in a given ratio.

31. P is a given point outside the angle formed by two given lines AB, AC; shew how to draw a straight line from P such that the parts of it intercepted between P and the lines AB, AC may have a given ratio.

32. Through a given point within a given circle, draw a straight line such that the parts of it intercepted between that point and the circumference may have a given ratio. How many solutions does the problem admit of?

33. If a common tangent be drawn to any number of circles which touch each other internally, and from any point of this tangent as a centre a circle be described, cutting the other circles; and if from this centre lines be drawn through the intersections of the circles, the segments of the lines within each circle shall be equal.

34. APB is a quadrant of a circle, SPT a line touching it at P; C is the centre, and PM is perpendicular to CA; prove that

the △ SCT : the △ ACB :: the △ ACB : the △ CMP.

35. ABC is a triangle inscribed in a circle, AD, AE are lines drawn to the base BC parallel to the tangents at B, C respectively; shew that AD=AE, and BD : CE :: $AB^2$ : $AC^2$.

36. AB is the diameter of a circle, E the middle point of the radius OB; on AE, EB as diameters circles are described; PQL is a common tangent touching the circles at P and Q, and AB produced at L: shew that BL is equal to the radius of the smaller circle.

37. The vertical angle C of a triangle is bisected by a straight line which meets the base at D, and is produced to a point E, such that the rectangle contained by CD and CE is equal to the rectangle contained by AC and CB: shew that if the base and vertical angle be given, the position of E is invariable.

38. ABC is an isosceles triangle having the base angles at B and C each double of the vertical angle: if BE and CD bisect the base angles and meet the opposite sides in E and D, shew that DE divides the triangle into figures whose ratio is equal to that of AB to BC.

39. If AB, the diameter of a semicircle, be bisected in C, and on AC and CB circles be described, and in the space between the three circumferences a circle be inscribed, shew that its diameter will be to that of the equal circles in the ratio of 2 to 3.

40. O is the centre of a circle inscribed in a quadrilateral ABCD; a line EOF is drawn and making equal angles with AD and BC, and meeting them in E and F respectively: shew that the triangles AEO, BOF are similar, and that

AE : ED=CF : FB.

41. From the last exercise deduce the following: The inscribed circle of a triangle ABC touches AB in F; XOY is drawn through the centre making equal angles with AB and AC, and meeting them in X and Y respectively: shew that BX : XF = AY : YC.

42. Inscribe a square in a given semicircle.

43. Inscribe a square in a given segment of a circle.

44. Describe an equilateral triangle equal to a given isosceles triangle.

45. Describe a square having given the difference between a diagonal and a side.

46. Given the vertical angle, the ratio of the sides containing it, and the diameter of the circumscribing circle, construct the triangle.

47. Given the vertical angle, the line bisecting the base, and the angle the bisector makes with the base, construct the triangle.

48. In a given circle inscribe a triangle so that two sides may pass through two given points and the third side be parallel to a given straight line.

49. In a given circle inscribe a triangle so that the sides may pass through the three given points.

50. A, B, X, Y are four points in a straight line, and O is such a point in it that the rectangle OA, OY is equal to the rectangle OB, OX; if a circle is described with centre O and radius equal to a mean proportional between OA and OY, shew that at every point on this circle AB and XY will subtend equal angles.

51. O is a fixed point, and OP is any line drawn to meet a fixed straight line in P; if on OP a point Q is taken so that OQ to OP is a constant ratio, find the locus of Q.

52. O is a fixed point, and OP is any line drawn to meet the circumference of a fixed circle in P; if on OP a point Q is taken so that OQ to OP is a constant ratio, find the locus of Q.

53. If from a given point two straight lines are drawn including a given angle, and having a fixed ratio, find the locus of the extremity of one of them when the extremity of the other lies on a fixed straight line.

54. On a straight line PAB, two points A and B are marked and the line PAB is made to revolve round the fixed extremity P. C is a fixed point in the plane in which PAB revolves; prove that if CA and CB be joined and the parallelogram CADB be completed, the locus of D will be a circle.

55. Find the locus of a point whose distances from two fixed points are in a given ratio.

56. Find the locus of a point from which two given circles subtend the same angle.

57. Find the locus of a point such that its distances from two intersecting straight lines are in a given ratio.

58. In the figure on page 389, shew that QT, P′T′ meet on the radical axis of the two circles.

59. ABC is any triangle, and on its sides equilateral triangles are described externally: if X, Y, Z are the centres of their inscribed circles, shew that the triangle XYZ is equilateral.

60. If S, I are the centres, and R, $r$ the radii of the circumscribed and inscribed circles of a triangle, and if N is the centre of its nine-points circle,

prove that (i) $SI^2 = R^2 - 2Rr$,

(ii) $NI = \frac{1}{2}R - r$.

Establish corresponding properties for the escribed circles, and hence prove that the nine-points circle touches the inscribed and escribed circles of a triangle.

# SOLID GEOMETRY.

## EUCLID. BOOK XI.

### Definitions.

From the Definitions of Book I. it will be remembered that

(i) **A line** is that which has *length*, without breadth or thickness.

(ii) **A surface** is that which has *length* and *breadth*, without thickness.

To these definitions we have now to add:

(iii) **Space** is that which has *length*, *breadth*, and *thickness*.

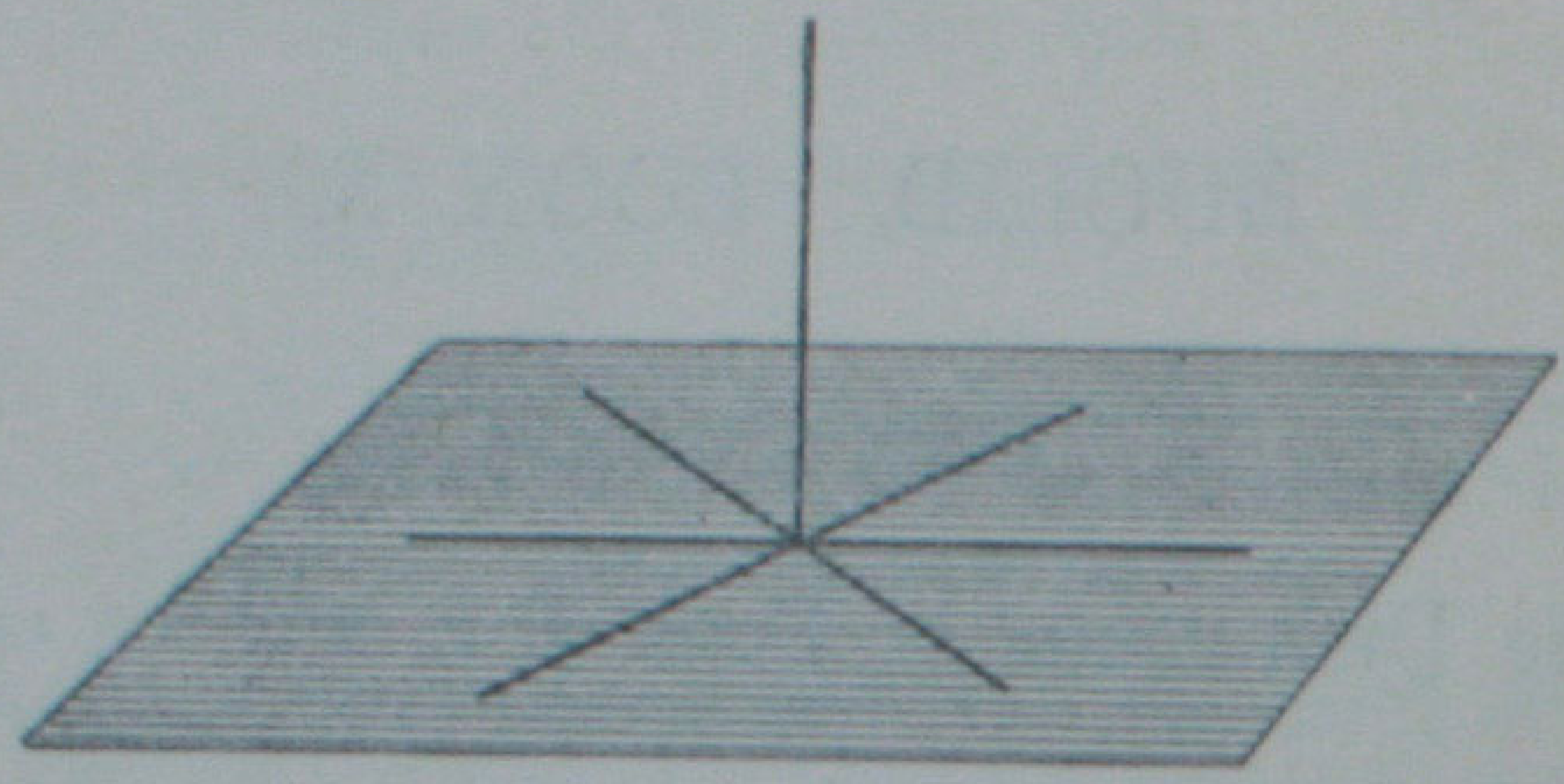
Thus a line is said to be of **one dimension**;
a surface is said to be of **two dimensions**;
and space is said to be of **three dimensions**.

The Propositions of Euclid's Eleventh Book here given establish the first principles of the *geometry of space*, or *solid geometry*. They deal with the properties of straight lines which are not all in the same plane, the relations which straight lines bear to planes which do not contain those lines, and the relations which two or more planes bear to one another. Unless the contrary is stated the straight lines are supposed to be of indefinite length, and the planes of infinite extent.

Solid geometry then proceeds to discuss the properties of solid figures, of surfaces which are not planes, and of lines which cannot be drawn on a plane surface.

## Lines and Planes.

1. A straight line is **perpendicular to a plane** when it is perpendicular to *every* straight line which meets it in that plane.

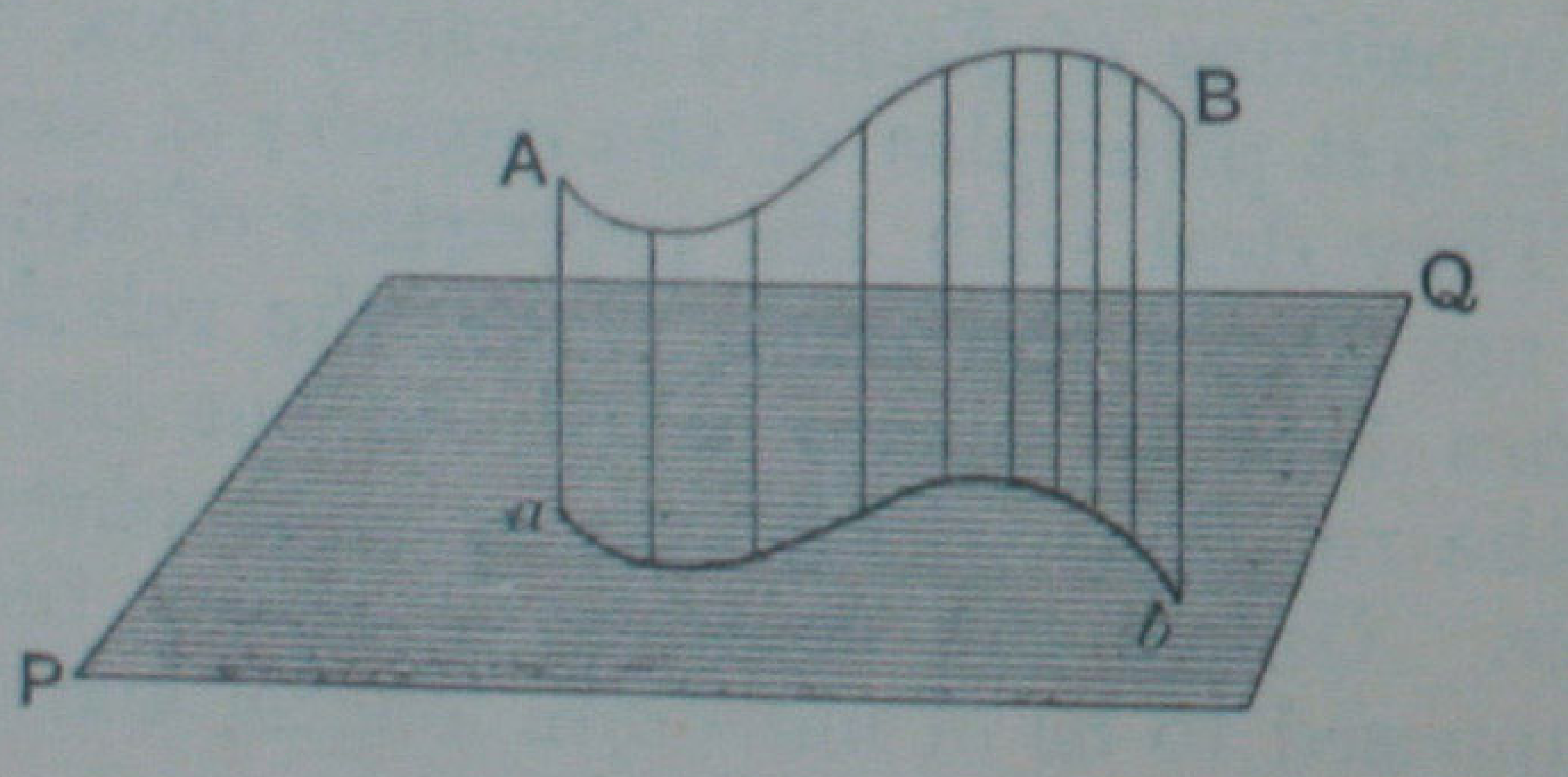


Note. It will be proved in Proposition 4 that if a straight line is perpendicular to *two* straight lines which meet it in a plane, it is also perpendicular to *every* straight line which meets it in that plane.

A straight line drawn perpendicular to a plane is said to be a **normal** to that plane.

2. The foot of the perpendicular let fall from a given point on a plane is called the **projection of that point** on the plane.

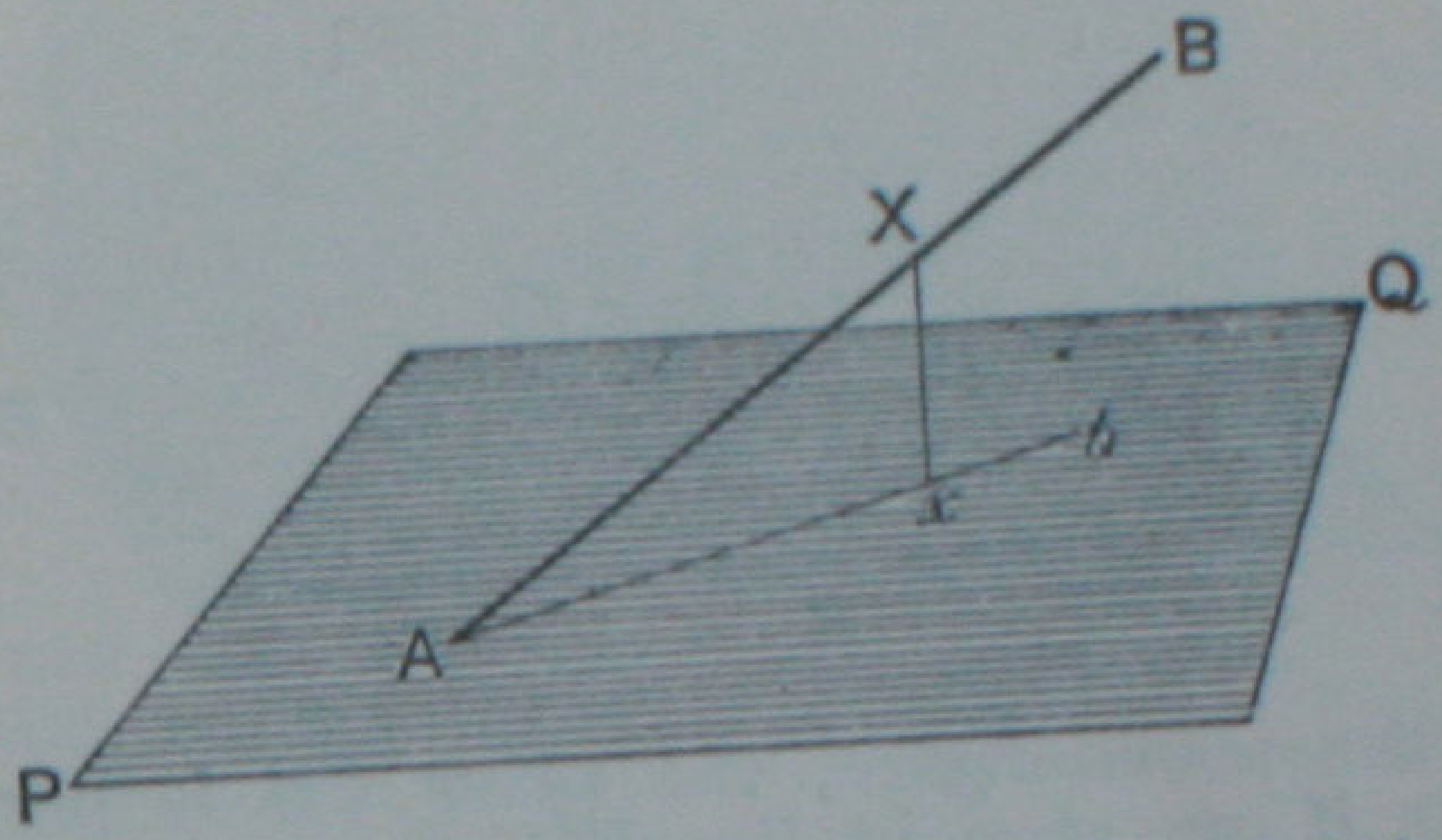
3. The **projection of a line** on a plane is the locus of the feet of perpendiculars drawn from all points in the given line to the plane.




Thus in the above figure the line *ab* is the projection of the line AB on the plane PQ.

Note. It will be proved hereafter (see page 446) that the projection of a straight line on a plane is also a straight line.

4. **The inclination of a straight line to a plane** is the acute angle contained by that line and another drawn from the point at which the first line meets the plane to the point at which a perpendicular to the plane let fall from any point of the first line meets the plane.

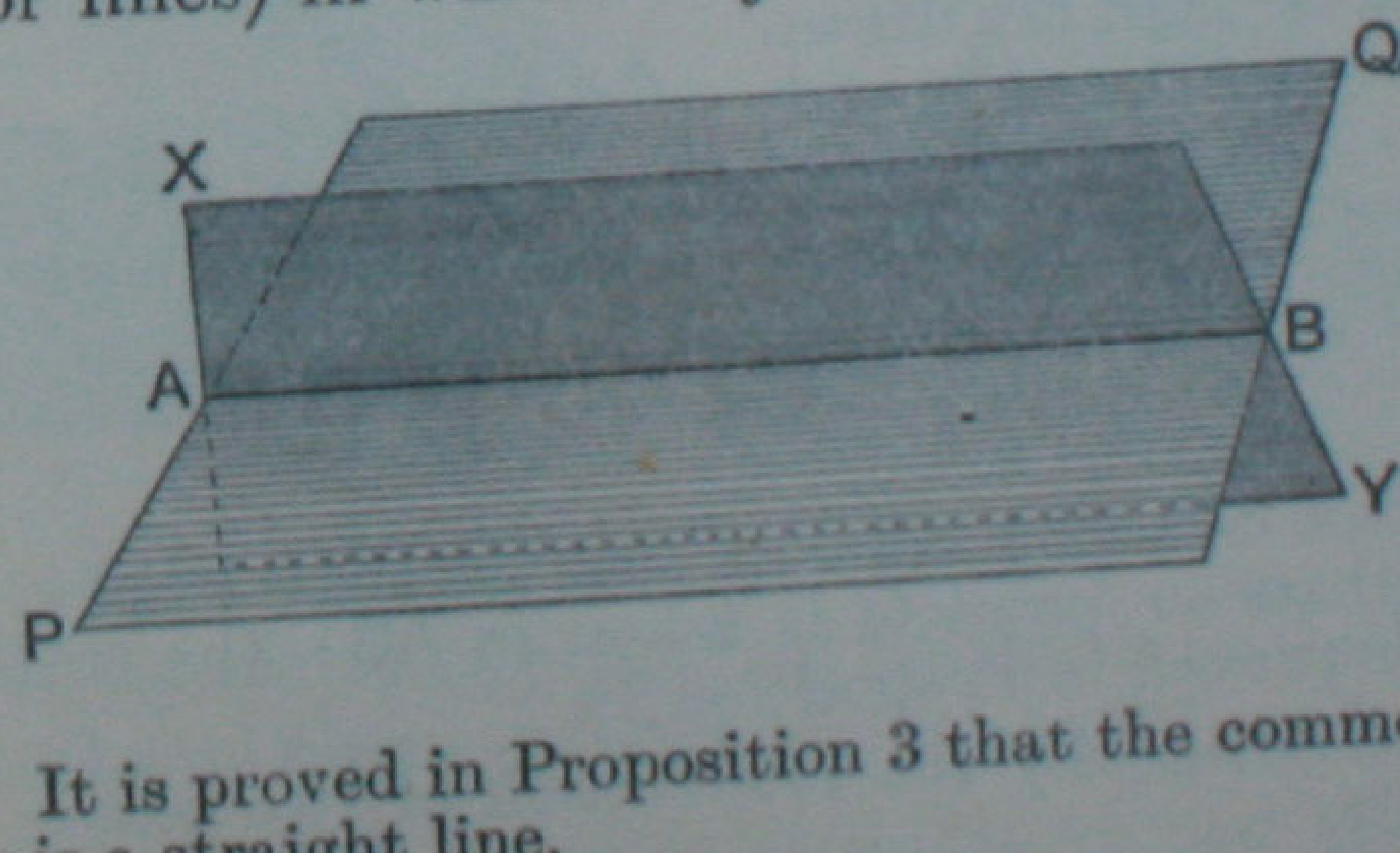



Thus in the above figure, if from any point X in the given straight line AB, which intersects the plane PQ at A, a perpendicular X$x$ is let fall on the plane, and the straight line A$xb$ is drawn from A through $x$, then the inclination of the straight line AB to the plane PQ is measured by the acute angle BA$b$. In other words :—

The **inclination of a straight line to a plane** is the acute angle contained by the given straight line and its *projection on the plane.*

Axiom. If two surfaces intersect one another, they meet in a *line or lines.*

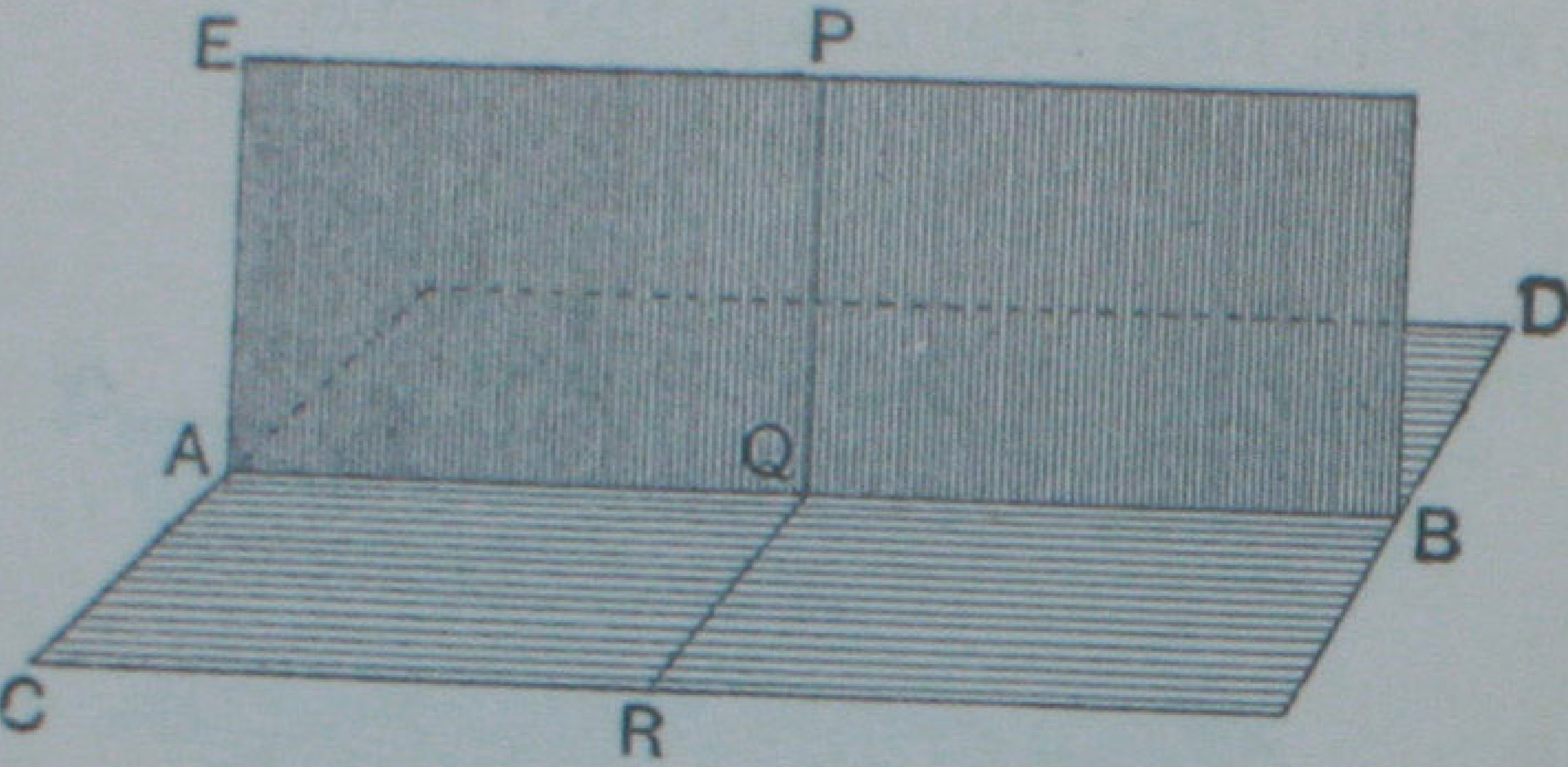
5. The **common section** of two intersecting surfaces is the line (or lines) in which they meet.




Note. It is proved in Proposition 3 that the common section of two planes is a straight line.

Thus AB, the common section of the two planes PQ, XY is proved to be a straight line.

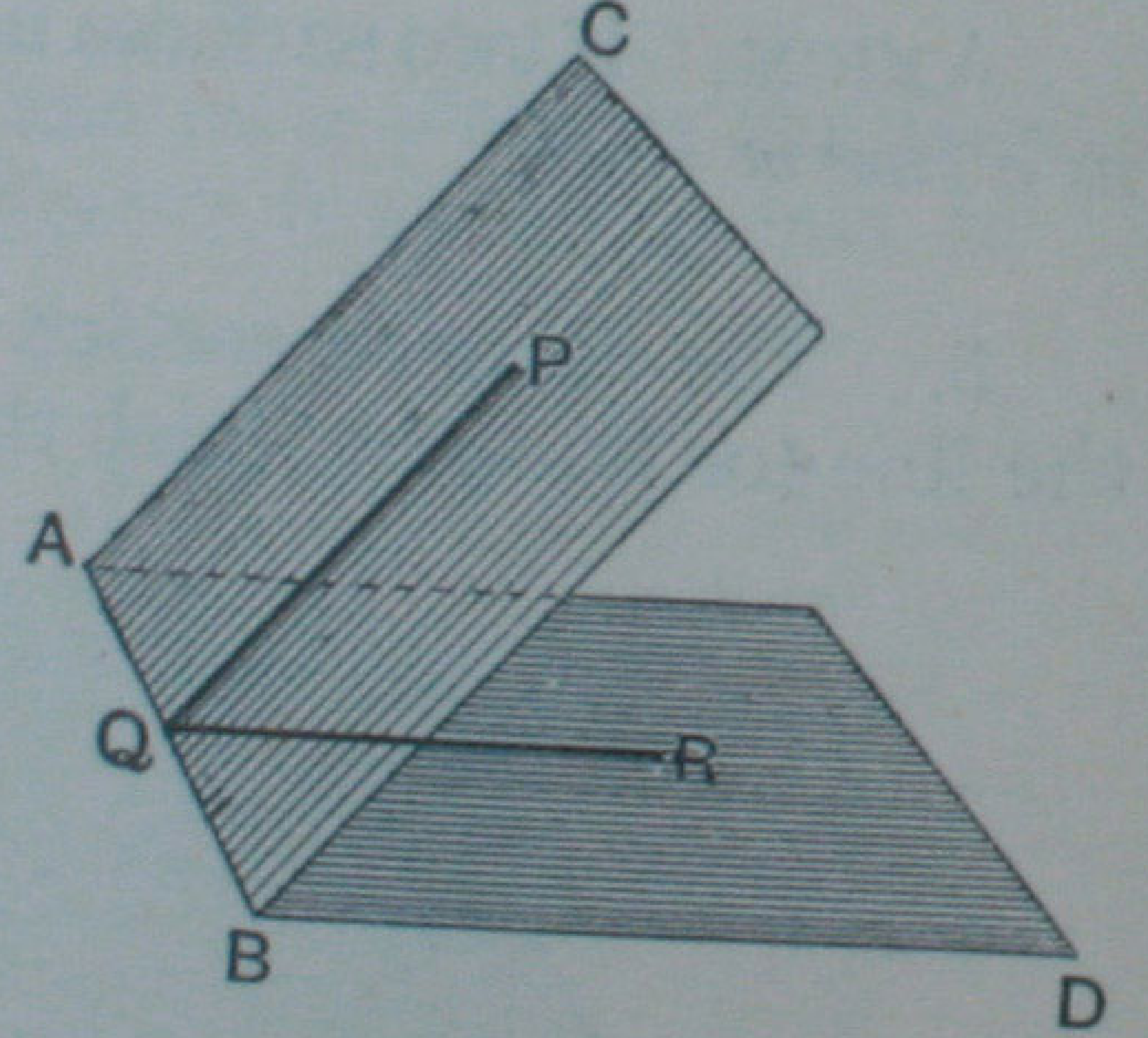
6. One plane is perpendicular to another plane when *any* straight line drawn in one of the planes perpendicular to the common section is also perpendicular to the other plane.




Thus in the above figure, the plane EB is perpendicular to the plane CD, if *any* straight line PQ, drawn in the plane EB at right angles to the common section AB, is also at right angles to the plane CD.

7. **The inclination of a plane to a plane** is the acute angle contained by two straight lines drawn from any point in the common section at right angles to it, one in one plane and one in the other.

Thus in the adjoining figure, the straight line AB is the common section of the two intersecting planes BC, AD; and from Q, *any point* in AB, two straight lines QP, QR are drawn perpendicular to AB, one in each plane: then the inclination of the two planes is measured by the acute angle PQR.




NOTE. This definition assumes that the angle PQR is of constant magnitude whatever point Q is taken in AB: the truth of which assumption is proved in Proposition 10.

The angle formed by the intersection of two planes is called a **dihedral angle**.

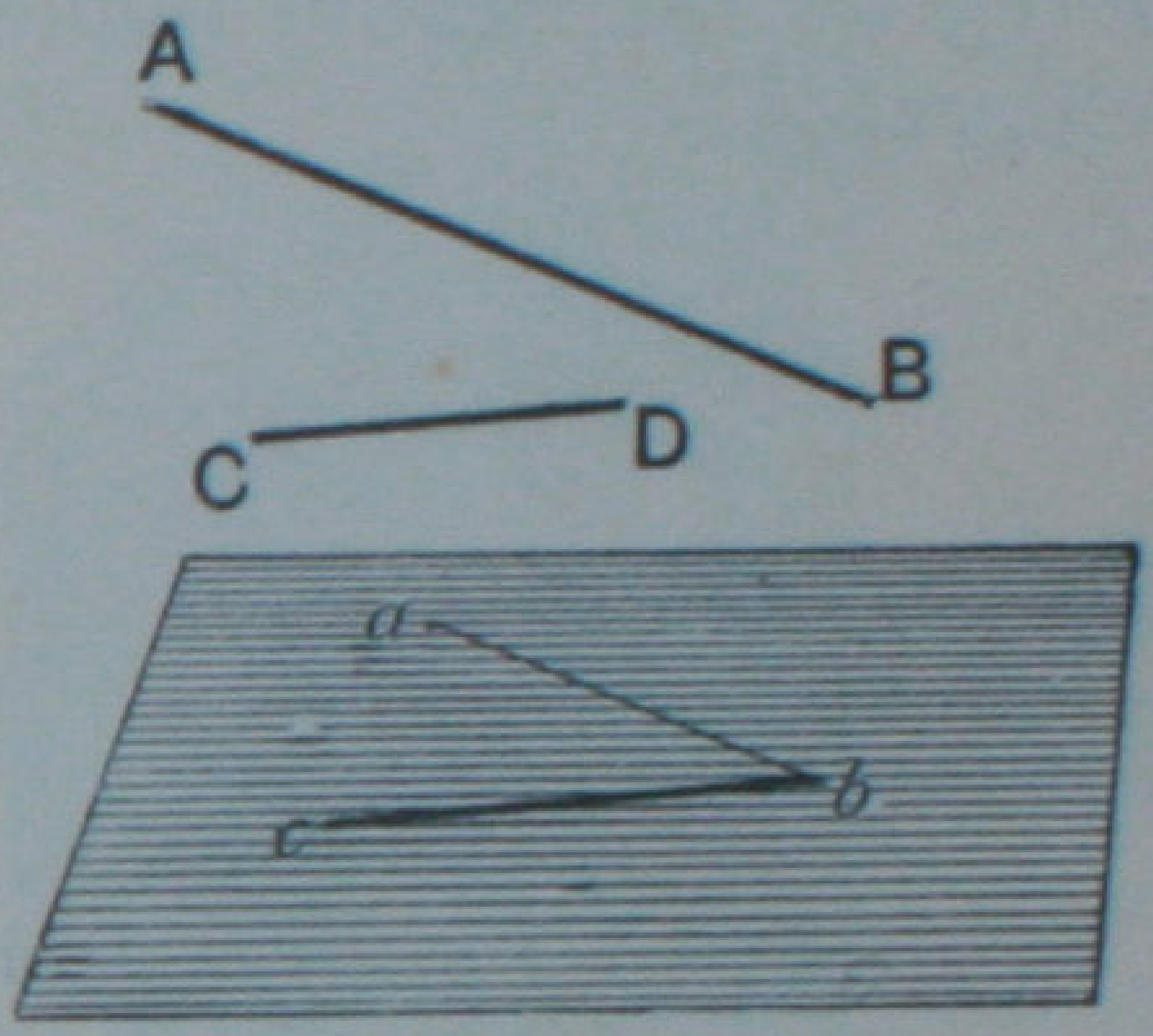
It may be proved that two planes are perpendicular to one another when the dihedral angle formed by them is a right angle.

8. **Parallel planes** are such as do not meet when produced.

9. A straight line is **parallel to a plane** if it does not meet the plane when produced.

10. The angle between two straight lines which do not meet is the angle contained by two *intersecting* straight lines respectively parallel to the two non-intersecting lines.

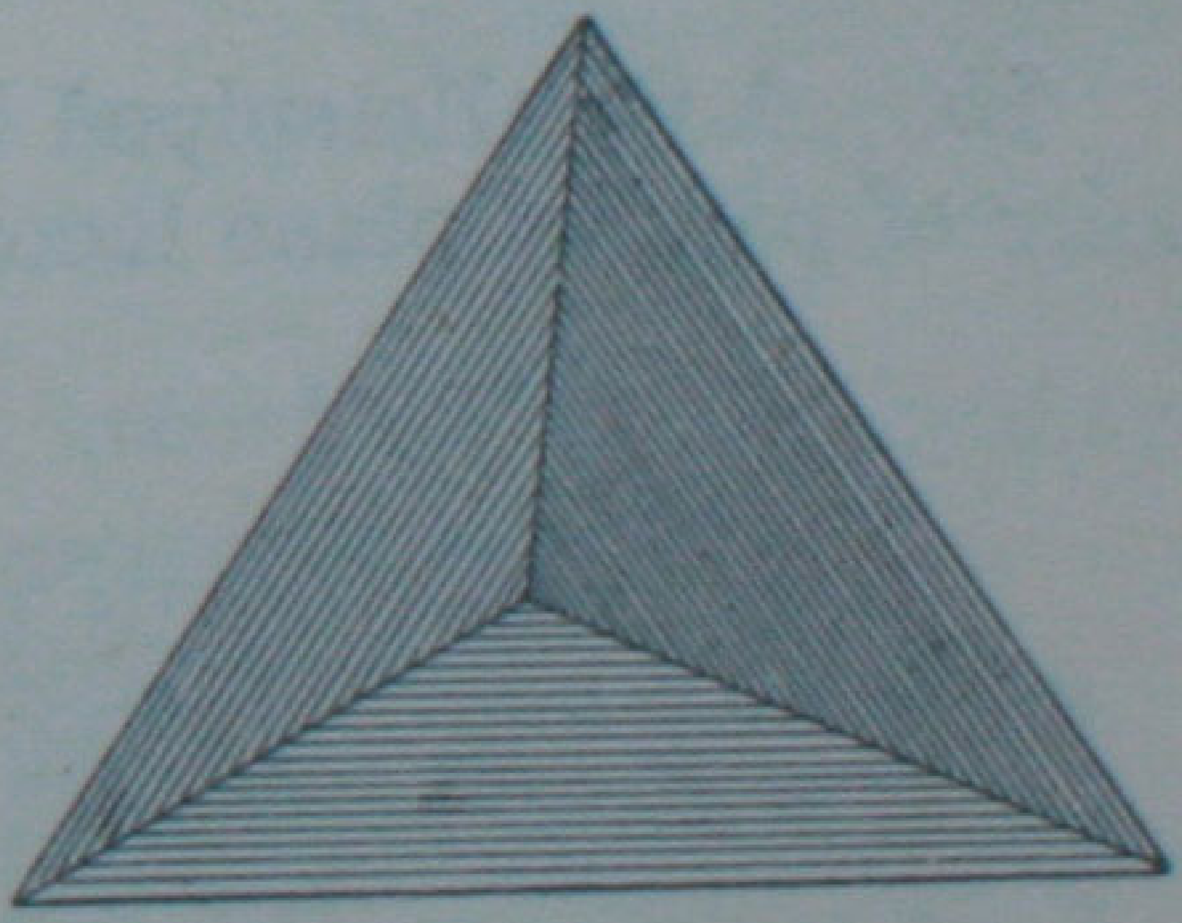
Thus if AB and CD are two straight lines which do not meet, and *ab*, *bc* are two intersecting lines parallel respectively to AB and CD; then the angle between AB and CD is measured by the angle *abc*.




11. A **solid angle** is that which is made by three or more plane angles which have a common vertex, but are not in the same plane.

A solid angle made by *three* plane angles is said to be **trihedral**; if made by more than three, it is said to be **polyhedral**.

A solid angle is sometimes called a **corner**.



12. A solid figure is any portion of space bounded by one or more surfaces, plane or curved.

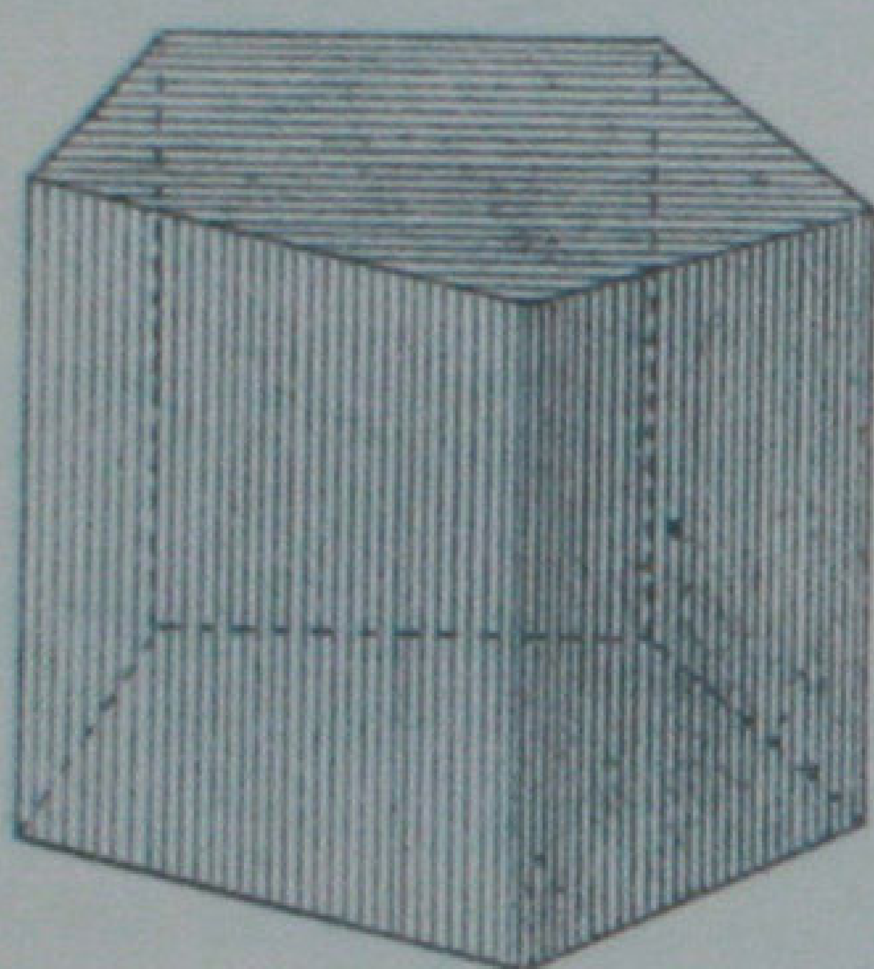
These surfaces are called the **faces** of the solid, and the intersections of adjacent faces are called **edges**.

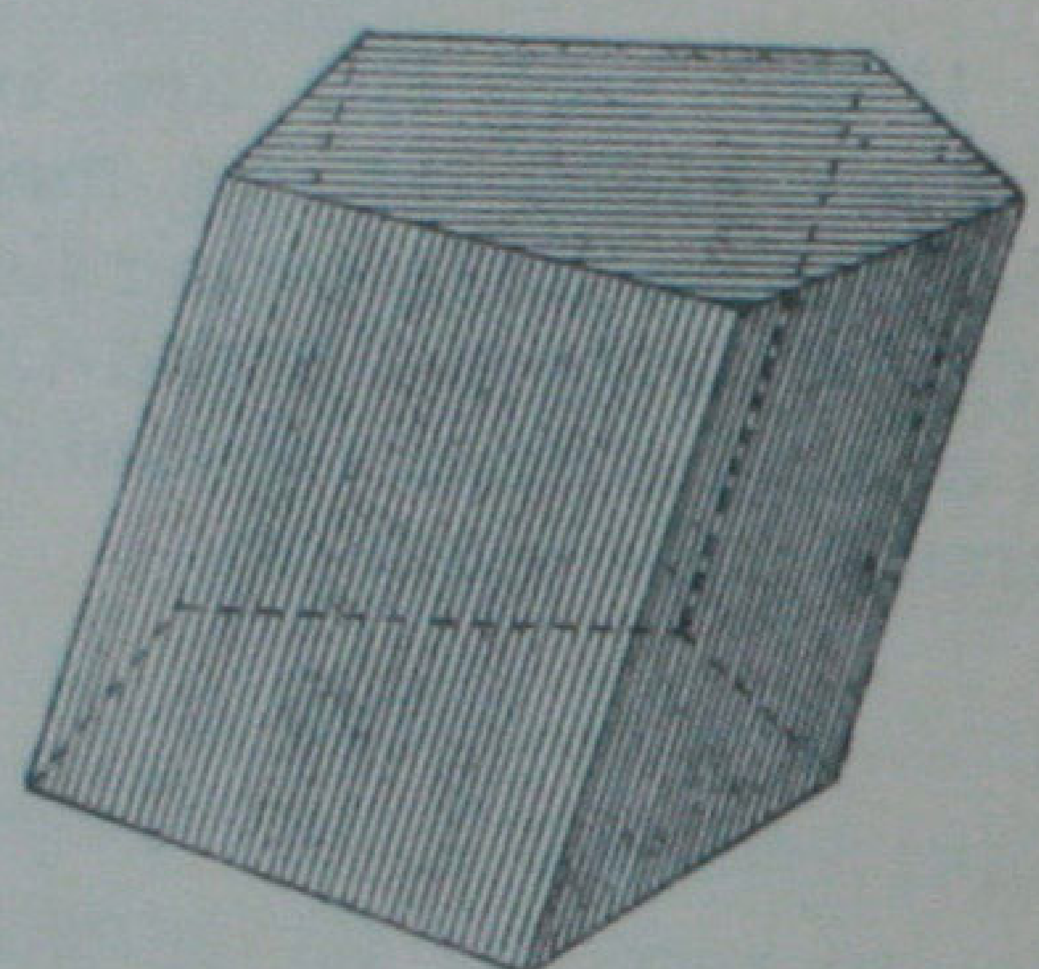
## POLYHEDRA.

13. A **polyhedron** is a solid figure bounded by plane faces.

NOTE. A plane rectilineal figure must at least have *three* sides; or *four*, if two of the sides are parallel. A polyhedron must at least have *four* faces; or, if two faces are parallel, it must at least have *five* faces.

14. A **prism** is a solid figure bounded by plane faces, of which two that are opposite are similar and equal polygons in parallel planes, and the other faces are parallelograms.





The polygons are called the **ends** of the prism. A prism is said to be **right** if the edges formed by each pair of adjacent parallelograms are perpendicular to the two ends; if otherwise the prism is **oblique**.

15. A **parallelepiped** is a solid figure bounded by three pairs of parallel plane faces.

Fig. 1.

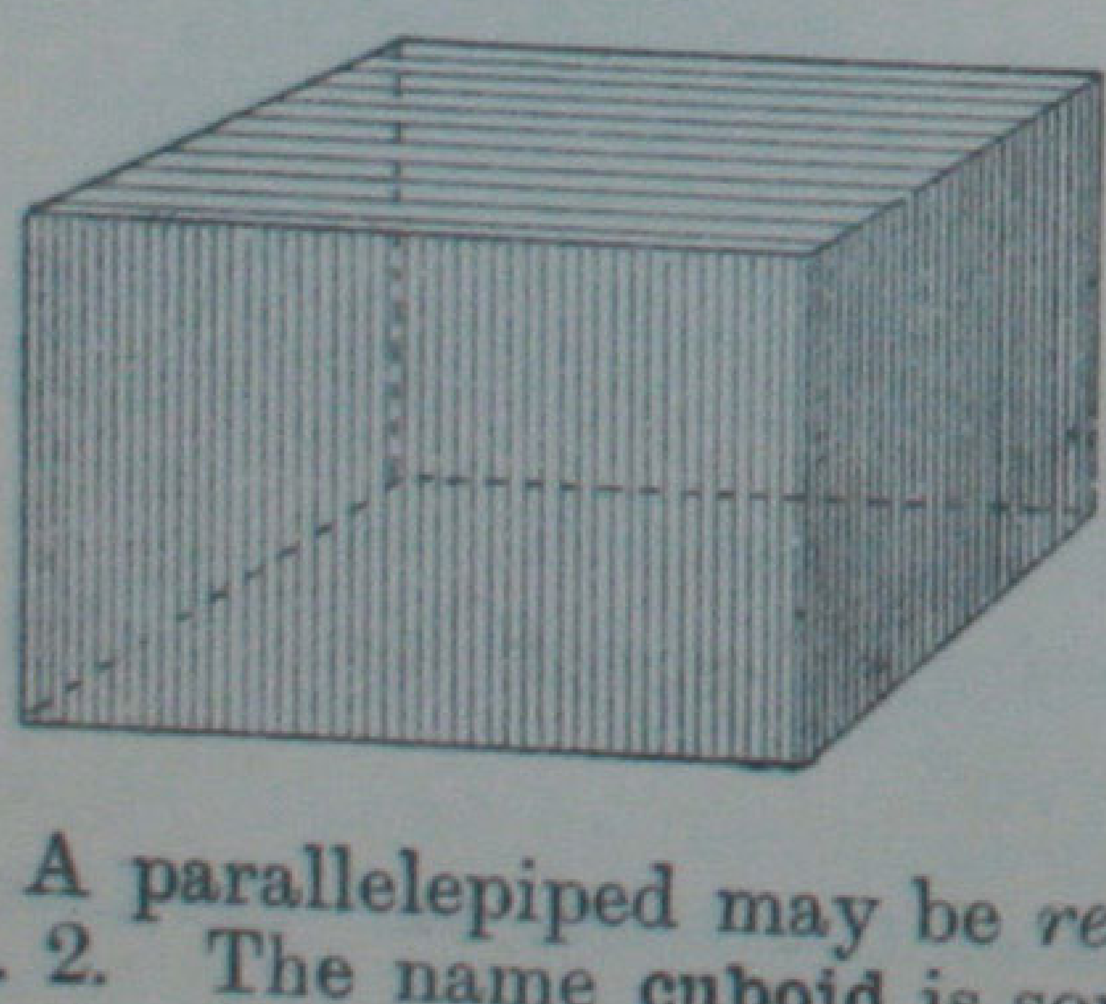
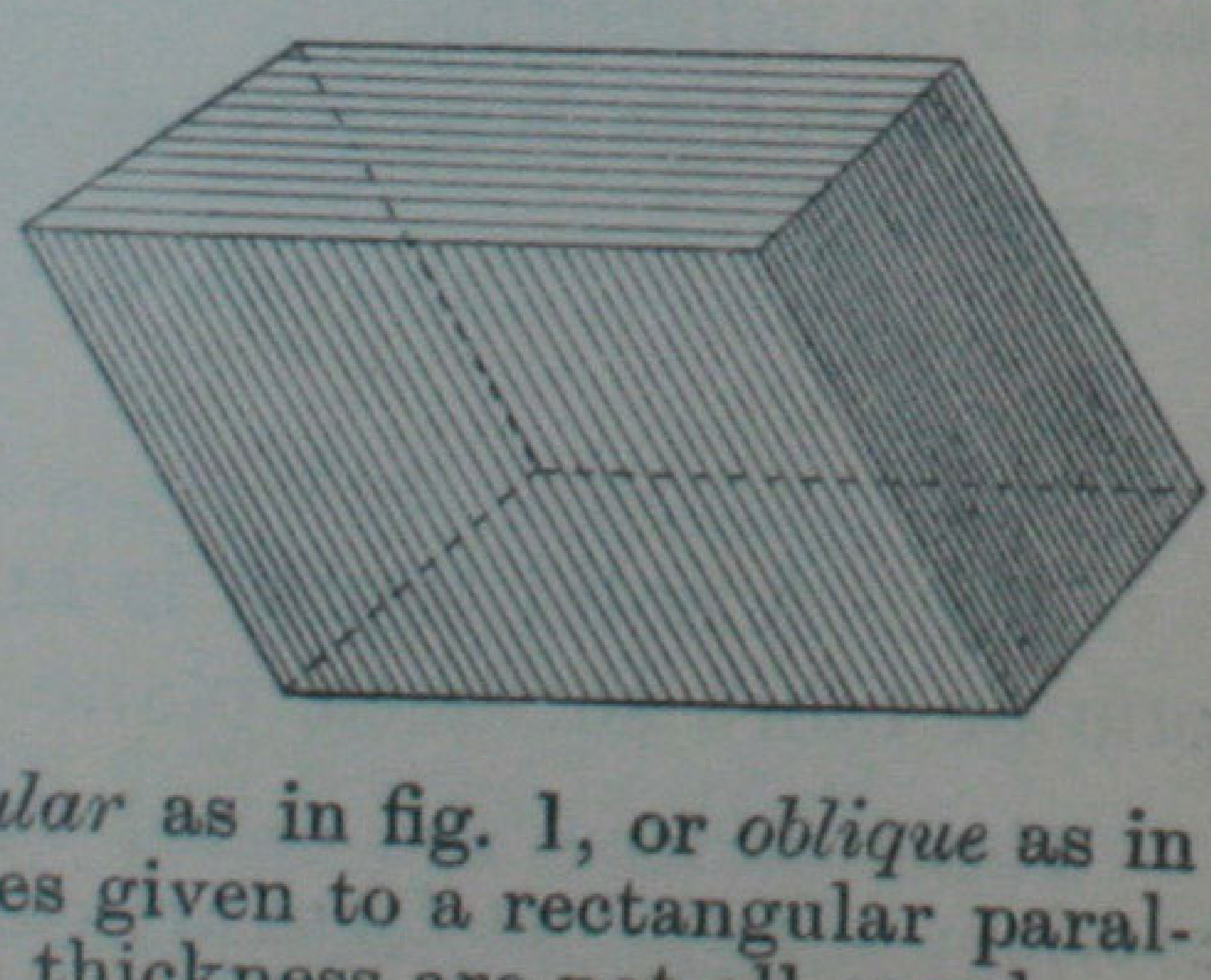
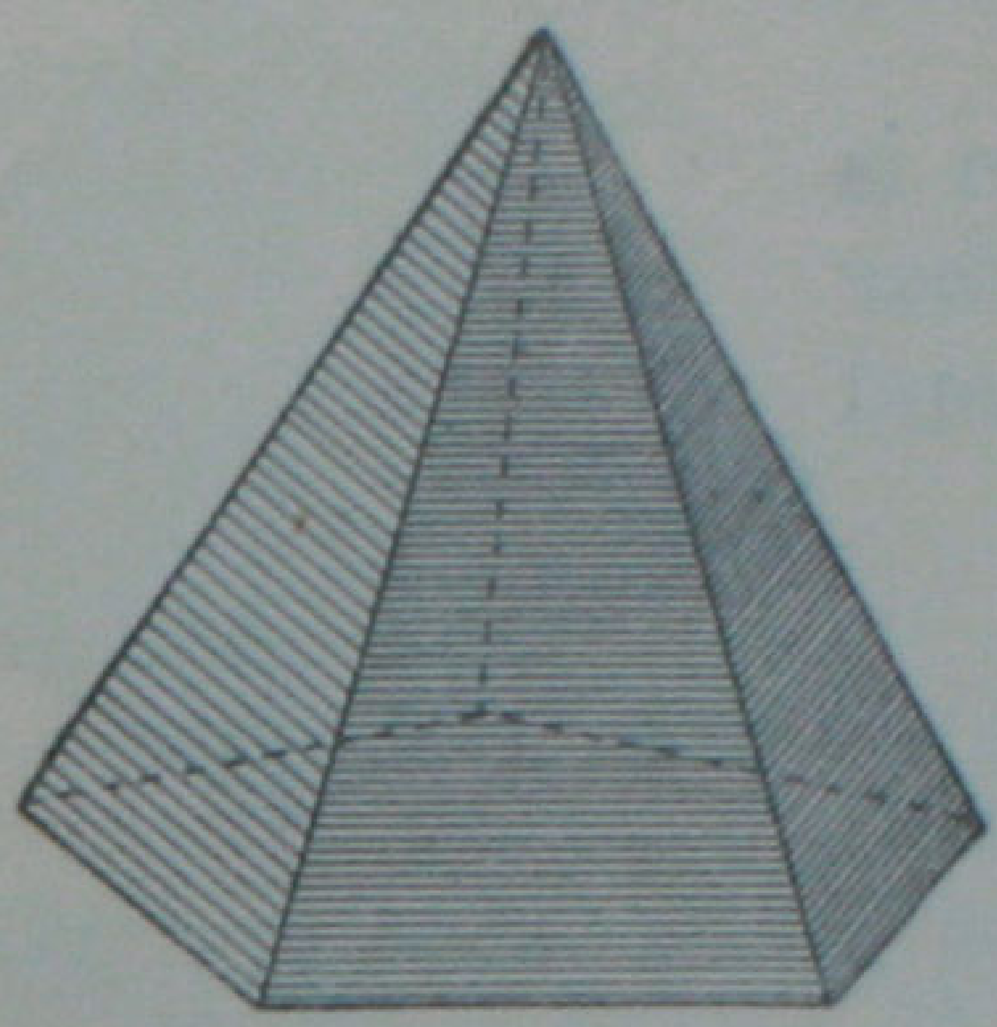


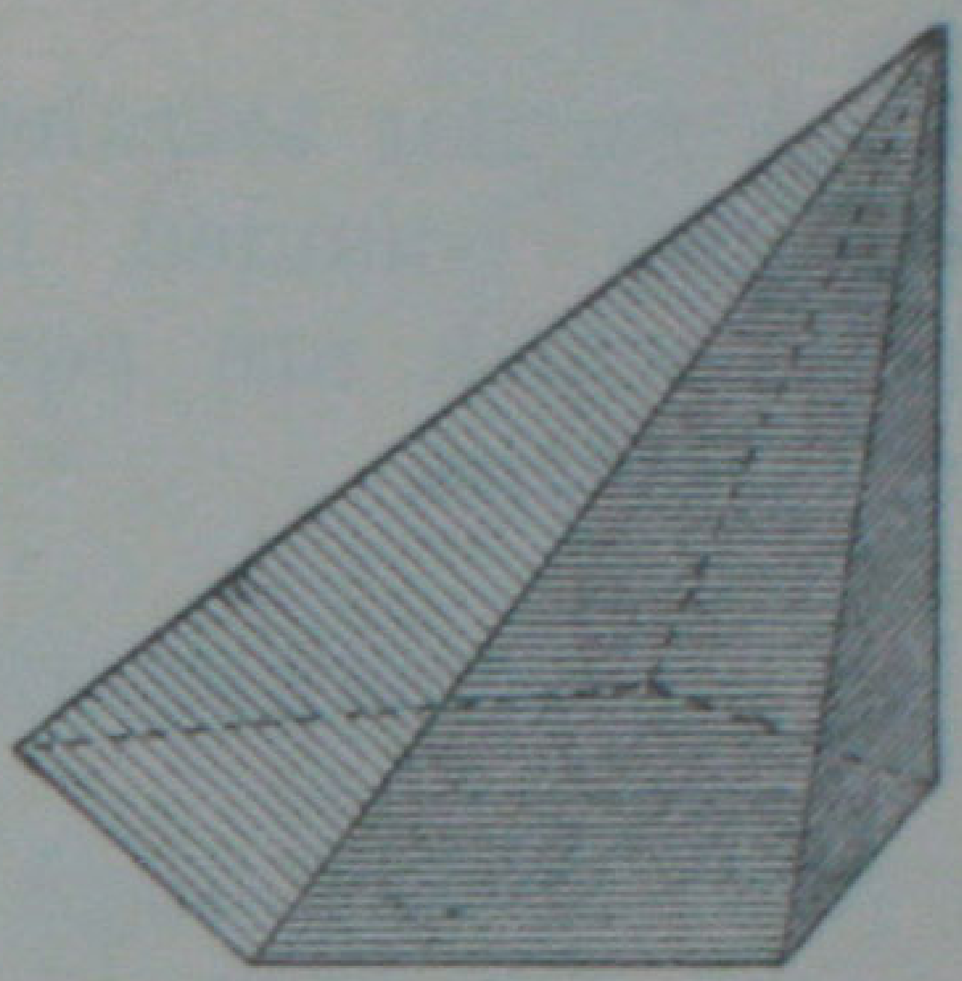
Fig. 2.



A parallelepiped may be *rectangular* as in fig. 1, or *oblique* as in fig. 2. The name **cuboid** is sometimes given to a rectangular parallelepiped whose length, breadth, and thickness are not all equal.

16. A **pyramid** is a solid figure bounded by plane faces, of which one is a polygon, and the rest are triangles having as bases the sides of the polygon, and as a common vertex some point not in the plane of the polygon.

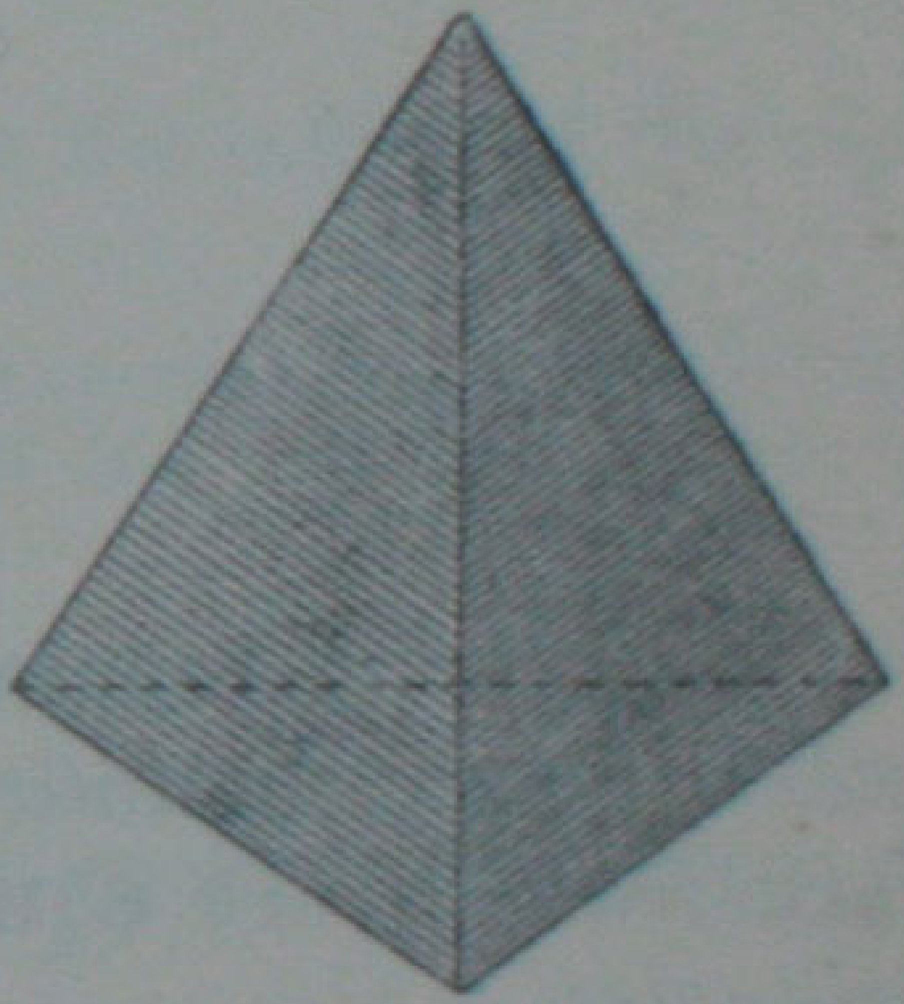




The polygon is called the **base** of the pyramid.

A pyramid having for its base a *regular* polygon is said to be **right** when the vertex lies in the straight line drawn perpendicular to the base from its central point (the centre of its inscribed or circumscribed circle).

17. A **tetrahedron** is a pyramid on a triangular base: it is thus contained by *four* triangular faces.



18. Polyhedra are classified according to the number of their *faces*:

thus a **hexahedron** has *six* faces;
an **octahedron** has *eight* faces;
a **dodecahedron** has *twelve* faces.

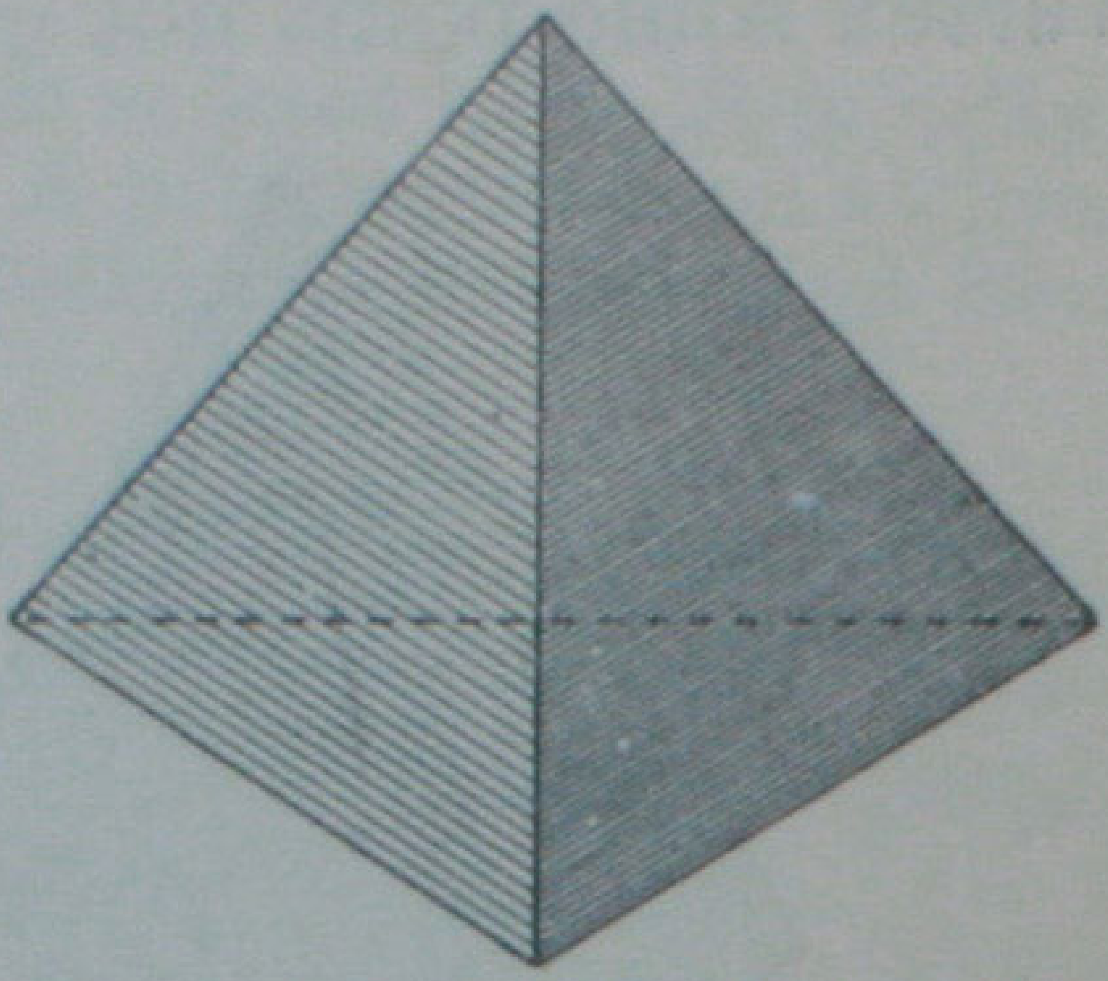
19. **Similar polyhedra** are such as have all their solid angles equal, each to each, and are bounded by the same number of similar faces.

20. A polyhedron is **regular** when its faces are similar and equal regular polygons.

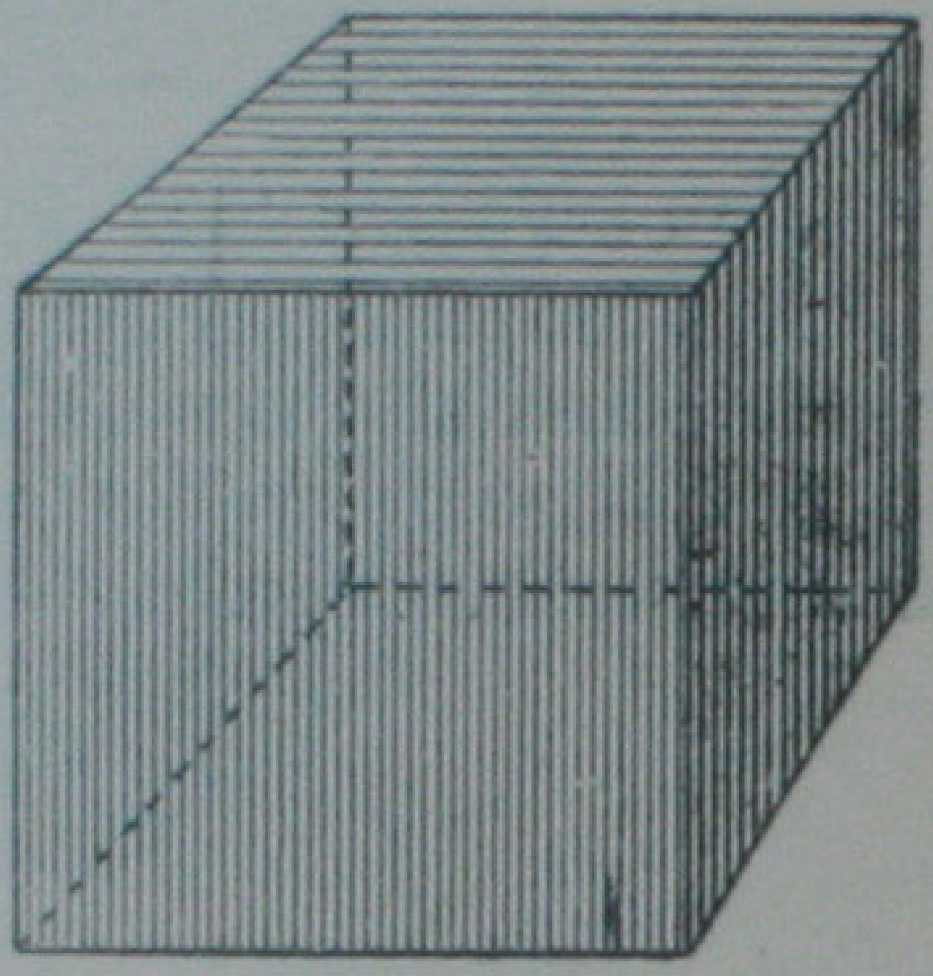
21. It will be proved (see page 451) that there can only be *five* regular polyhedra.

They are defined as follows:—

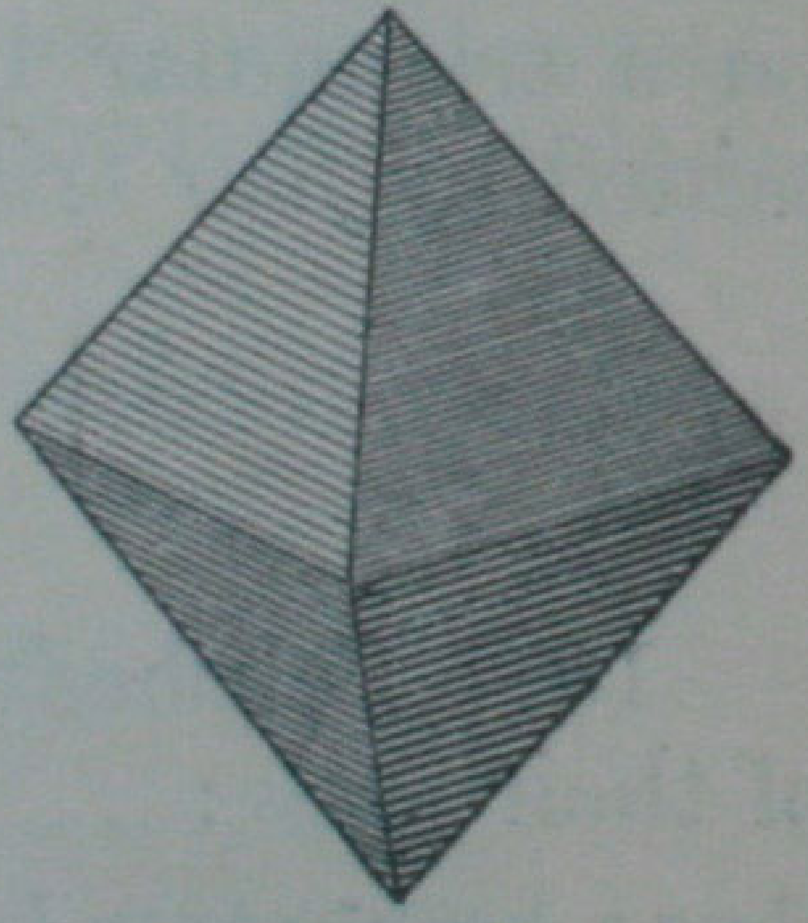
(i) A **regular tetrahedron** is a solid figure bounded by *four* plane faces, which are equal and equilateral *triangles*.



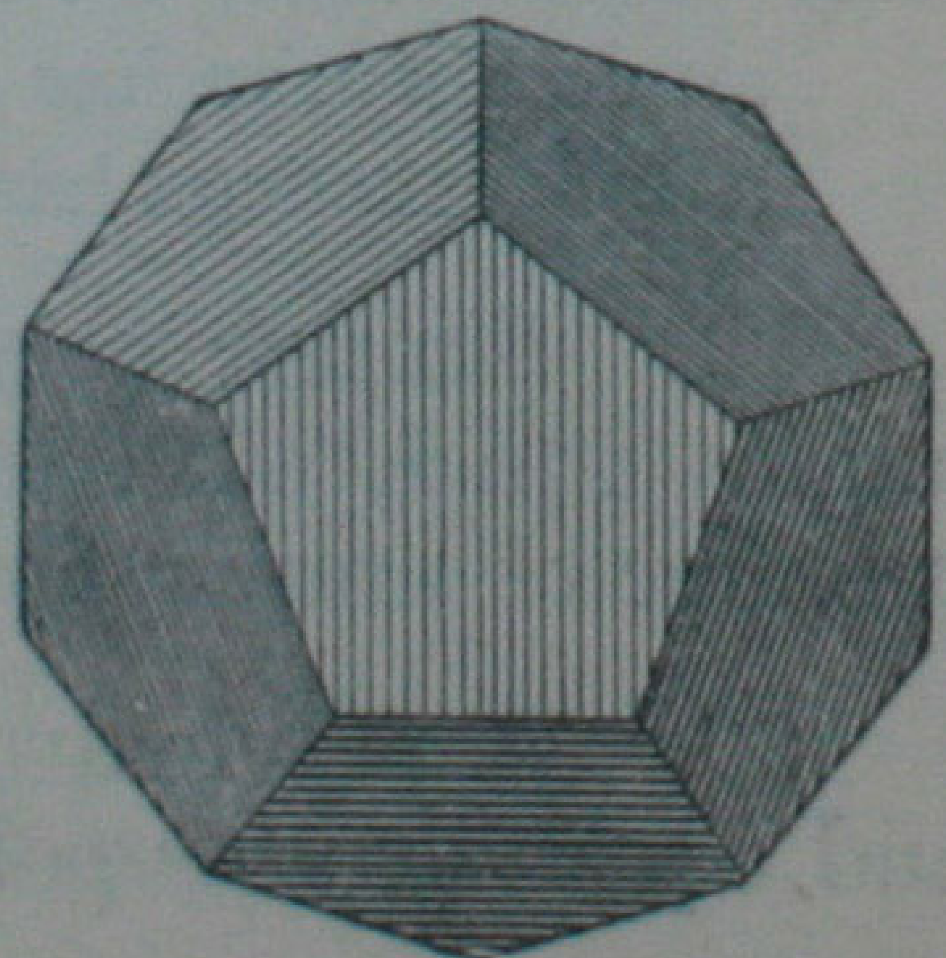
(ii) A **cube** is a solid figure bounded by *six* plane faces, which are equal *squares*.



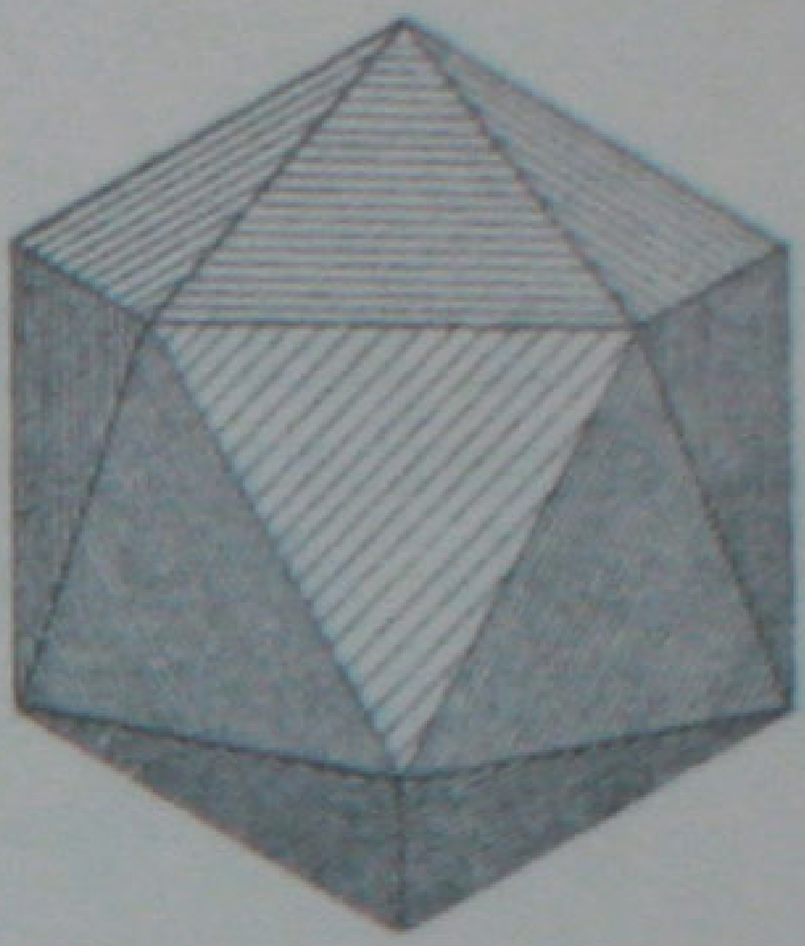
(iii) A **regular octahedron** is a solid figure bounded by *eight* plane faces, which are equal and equilateral *triangles*.



(iv) A **regular dodecahedron** is a solid figure bounded by *twelve* plane faces, which are equal and regular *pentagons*.



(v) A **regular icosahedron** is a solid figure bounded by *twenty* plane faces, which are equal and equilateral *triangles*.



## Solids of Revolution.

22*a*. A **sphere** is a solid figure described by the revolution of a semicircle about its diameter, which remains fixed.

The **axis** of the sphere is the fixed straight line about which the semicircle revolves.

The **centre** of the sphere is the same as the centre of the semicircle.

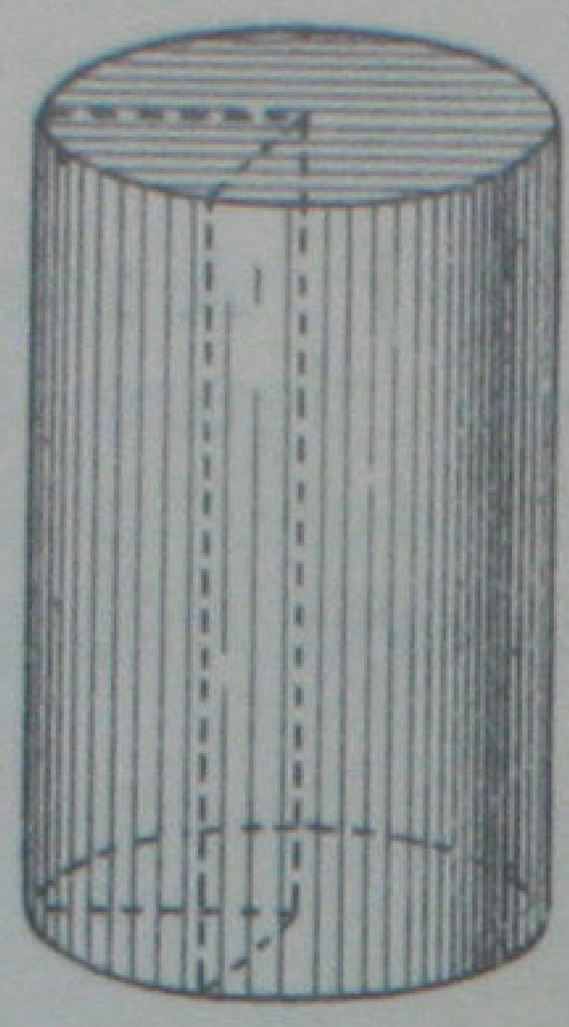
A **diameter** of a sphere is any straight line which passes through the centre, and is terminated both ways by the surface of the sphere.

The following definition of a sphere, analogous to that given for a circle (I. Def. 15), may also be noted:

22*b*. A **sphere** is a solid figure contained by one surface, which is such that all straight lines drawn from a certain point within it to the surface are equal: this point is called the centre of the sphere.

A **radius** of a sphere is a straight line drawn from the centre to the surface.

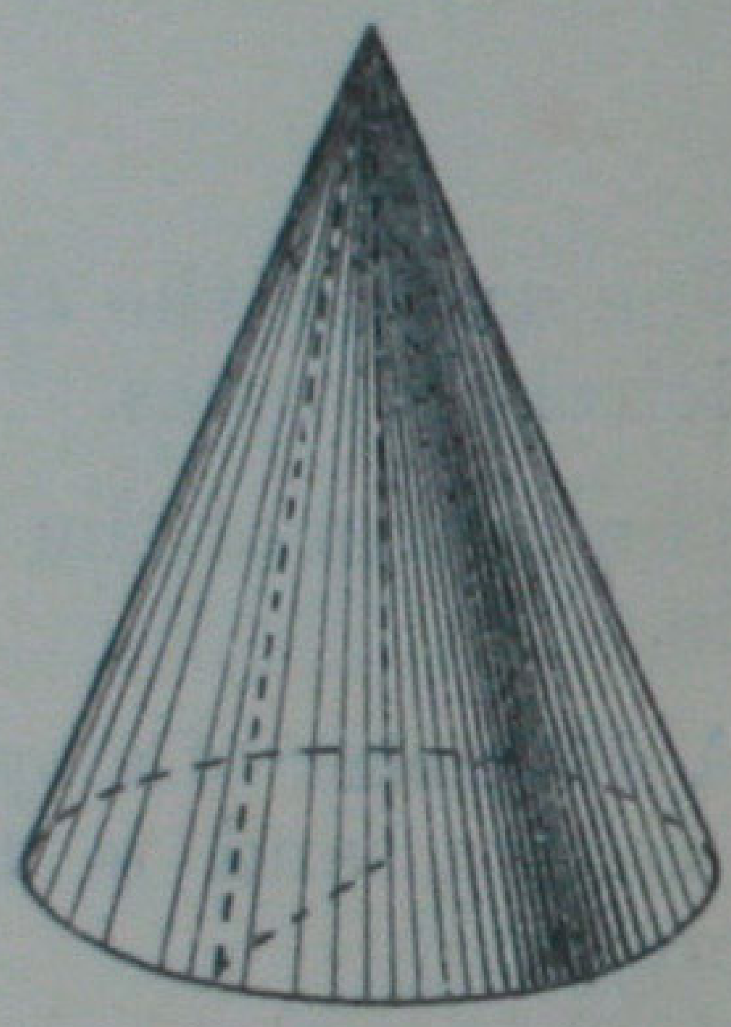
It will be seen that the surface of a sphere is the locus of a point which moves *in space* so that its distance from a certain fixed point (the centre) is constant.



23. A **right cylinder** is a solid figure described by the revolution of a rectangle about one of its sides which remains fixed.

The **axis** of the cylinder is the fixed straight line about which the rectangle revolves.

The **bases, or ends**, of the cylinder are the circular faces described by the two revolving opposite sides of the rectangle.



24. A **right cone** is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle which remains fixed.

The **axis** of the cone is the fixed straight line about which the triangle revolves.

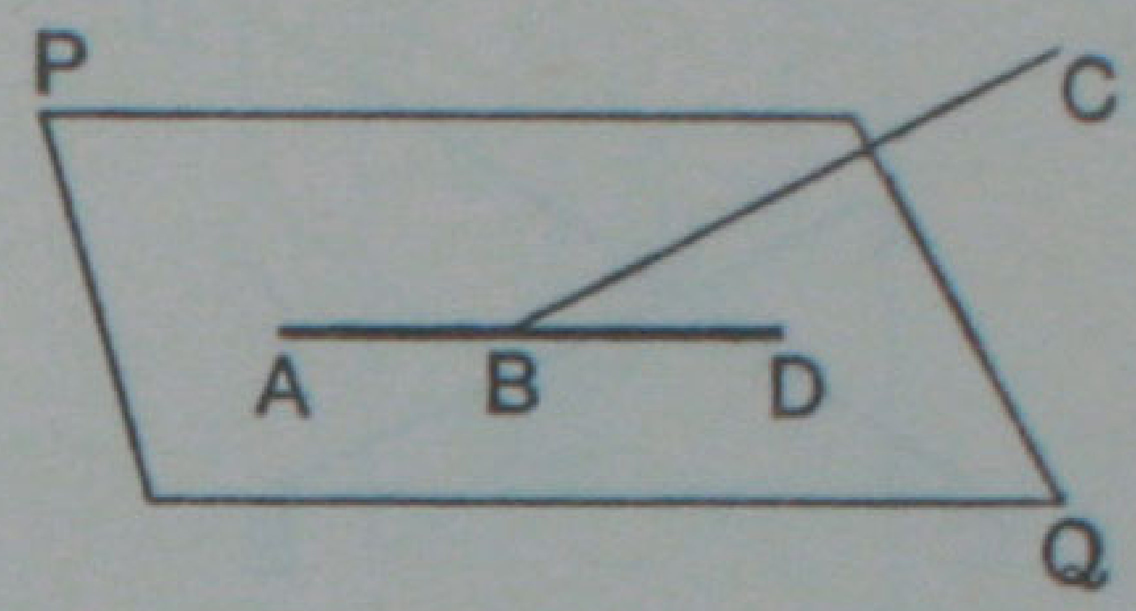
The **base** of the cone is the circular face described by that side which revolves.

The hypotenuse of the right-angled triangle in any one of its positions is called a **generating line** of the cone.

25. **Similar cones and cylinders** are those which have their axes and the diameters of their bases proportionals.

## Proposition 1. Theorem.

*One part of a straight line cannot be in a plane and another part outside it.*




If possible, let AB, part of the st. line ABC, be in the plane PQ, and the part BC outside it.

Then since the st. line AB is in the plane PQ,
∴ it can be produced in that plane. I. *Post.* 2.

Produce AB to D;
and let any other plane which passes through AD be turned about AD until it passes also through C.

Then because the points B and C are in this plane,
∴ the st. line BC is in it: I. *Def.* 7.

∴ ABC and ABD are in the same plane and are both st. lines; which is impossible. I. *Def.* 4.

∴ the st. line ABC has not one part AB in the plane PQ, and another part BC outside it. Q.E.D.

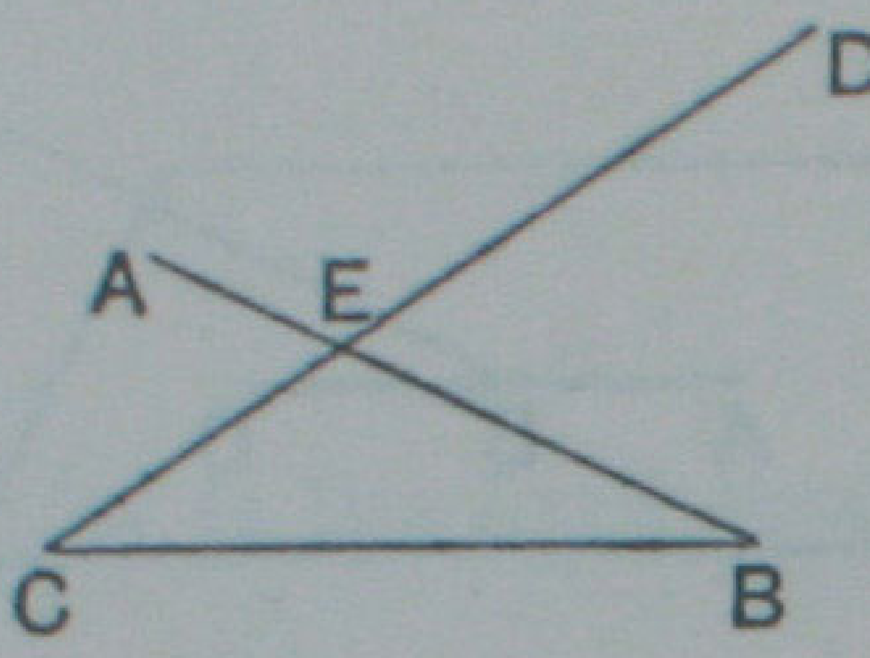
Note. This proposition scarcely needs proof, for the truth of it follows immediately from the definitions of a straight line and a plane.

It should be observed that the method of proof used in this and the next proposition rests upon the following axiom:

*If a plane of unlimited extent turns about a fixed straight line as an axis, it can be made to pass through any point in space.*

PROPOSITION 2. THEOREM.

*Any two intersecting straight lines are in one plane: and any three straight lines, of which each pair intersect one another, are in one plane.*




Let the two st. lines AB and CD intersect at E;
and let the st. line BC be drawn cutting AB and CD at B and C.

*Then* (i) AB *and* CD *shall lie in one plane.*
(ii) AB, BC, CD *shall lie in one plane.*

(i) Let any plane pass through AB;
and let this plane be turned about AB until it passes through C.

Then, since C and E are points in this plane,
∴ the whole st. line CED is in it. I. *Def.* 7 and XI. 1.
That is, AB and CD lie in one plane.

(ii) And since B and C are points in the plane which contains AB and CD,
∴ also the st. line BC lies in this plane. Q.E.D.

COROLLARY. *One, and only one, plane can be made to pass through two given intersecting straight lines.*

Hence the position of a plane is fixed,

(i) if it passes through a given straight line and a given point outside it; *Ax.* p. 419.

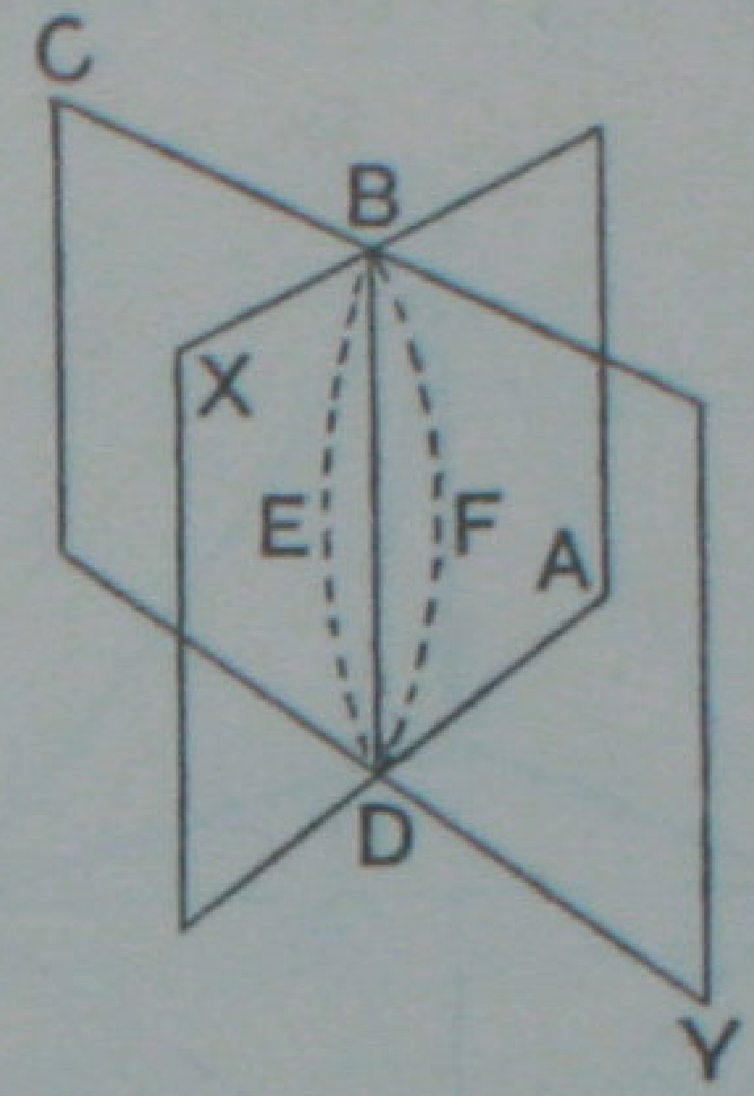
(ii) if it passes through two intersecting straight lines; XI. 2.

(iii) if it passes through three points not collinear; XI. 2.

(iv) if it passes through two parallel straight lines. I. *Def.* 35.

## Proposition 3. Theorem.

*If two planes cut one another, their common section is a straight line.*




Let the two planes XA, CY cut one another, and let BD be their common section.

*Then shall* BD *be a straight line.*

For if not, from B to D in the plane XA draw the st. line BED;

and in the plane CY draw the st. line BFD.

Then the st. lines BED, BFD have the same extremities;

∴ they include a space;

but this is impossible. *Ax.* 10.

∴ the common section BD cannot be otherwise than a st. line.

### Alternative Proof.

Let the planes XA, CY cut one another, and let B and D be two points in their common section.

Then because B and D are two points in the plane XA,

∴ the st. line joining B, D lies in that plane. I. *Def.* 7.

And because B and D are two points in the plane CY,

∴ the st. line joining B, D lies in that plane.

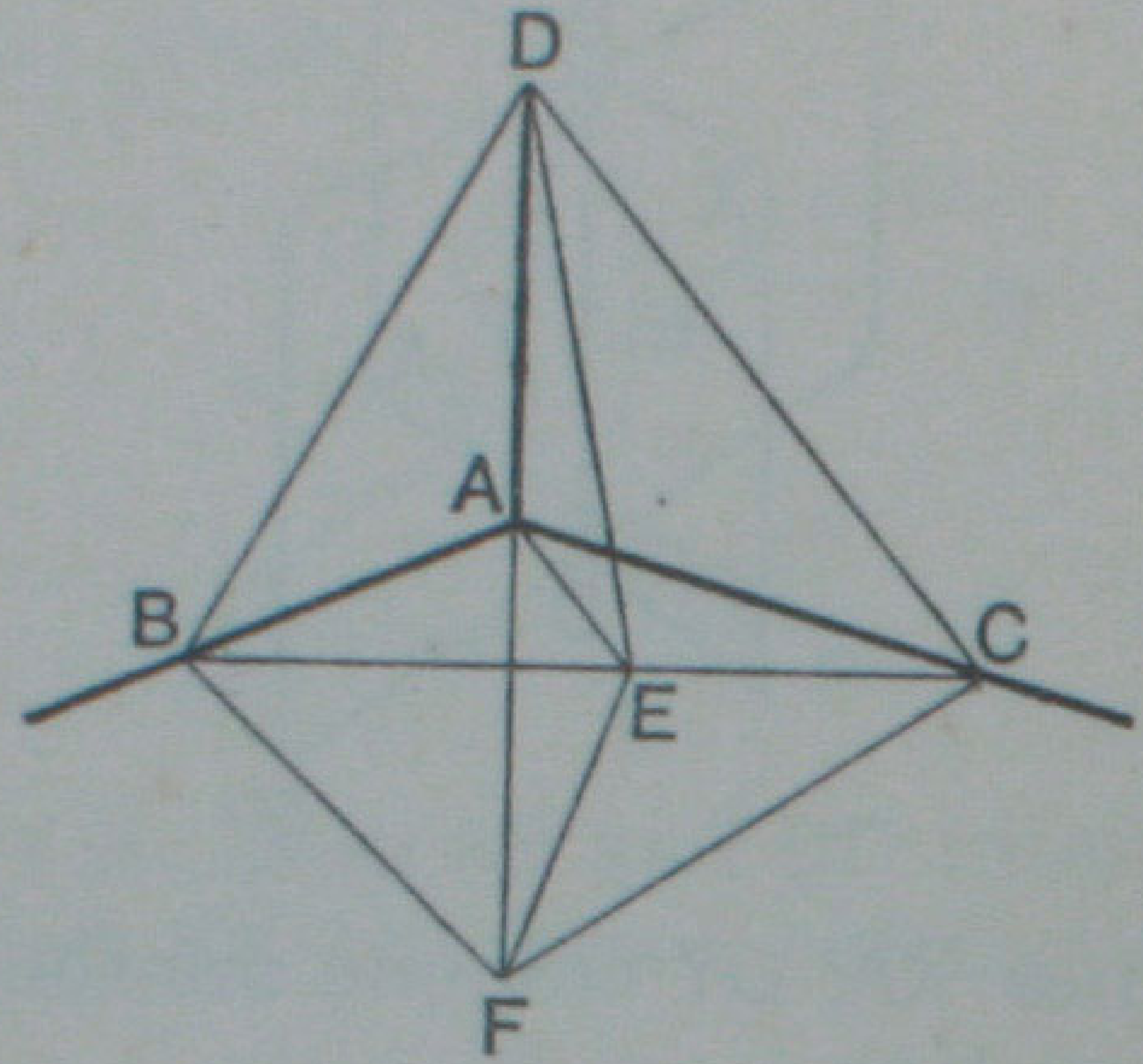
Hence the st. line BD lies in *both planes*,

and is therefore their common section.

That is, the common section of the two planes is a *straight* line. Q.E.D.

## PROPOSITION 4. THEOREM. [Alternative Proof.]

*If a straight line is perpendicular to each of two straight lines at their point of intersection, it shall also be perpendicular to the plane in which they lie.*




Let the straight line AD be perp. to each of the st. lines AB, AC at A their point of intersection.

*Then shall* AD *be perp. to the plane in which* AB *and* AC *lie.*

Produce DA to F, making AF equal to DA.

Draw any st. line BC in the plane of AB, AC, to cut AB, AC at B and C;
and in the same plane draw through A *any* st. line AE to cut BC at E.

It is required to prove that AD is perp. to AE. XI. *Def.* 1.

Join DB, DE, DC; and FB, FE, FC.

Then in the △ˢ BAD, BAF,
because DA = FA, *Constr.*
and the common side AB is perp. to DA, FA;
∴ BD = BF. I. 4.
Similarly CD = CF.

Now if the △ BFC be turned about its base BC until the vertex F comes into the plane of the △ BDC,
then F will coincide with D,
since the conterminous sides of the triangles are equal. I. 7
∴ EF will coincide with ED,
that is, EF = ED.

Hence in the $\triangle^s$ DAE, FAE,
since DA, AE, ED = FA, AE, EF respectively,
∴ the ∠ DAE = the ∠ FAE.

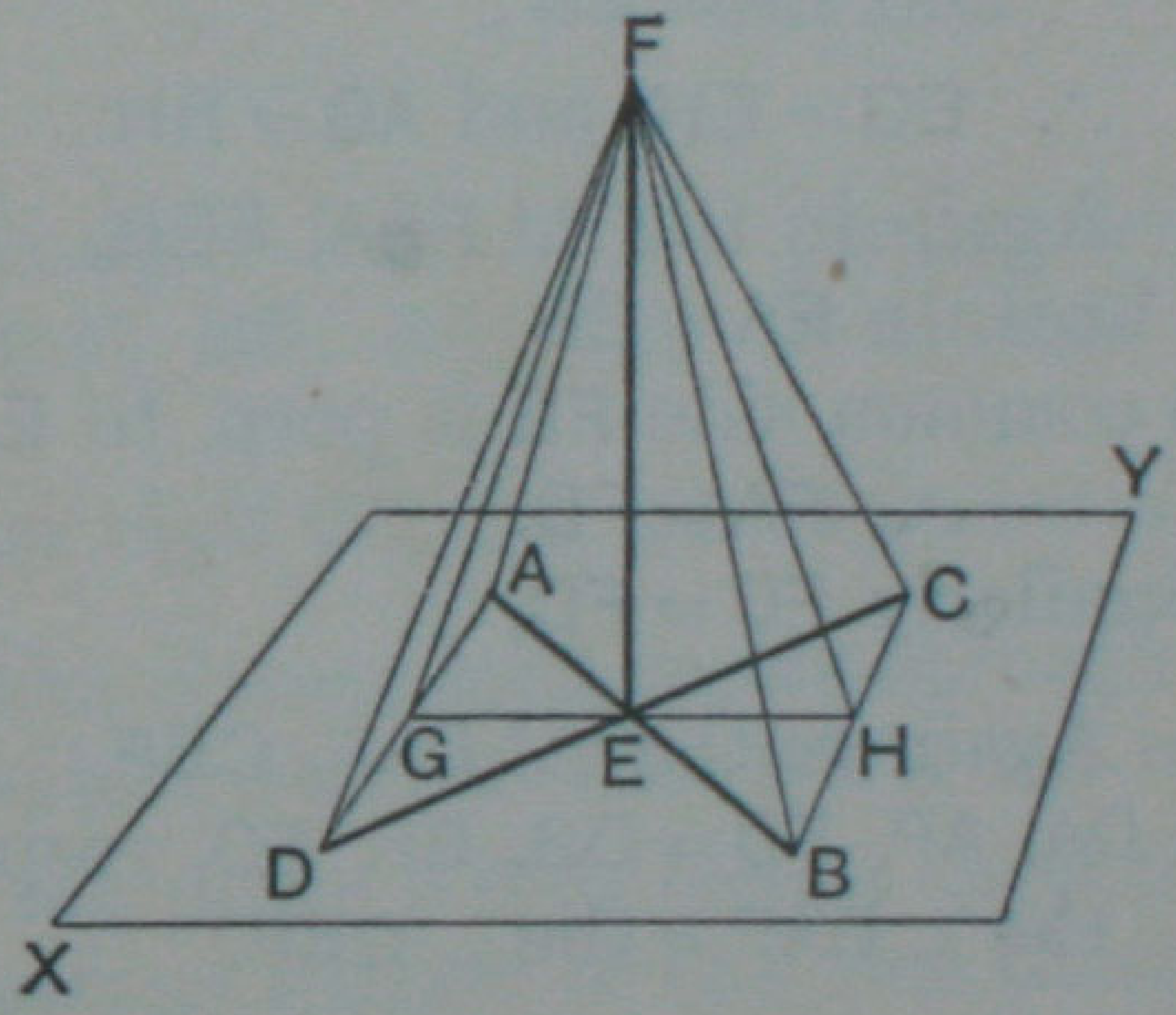
That is, DA is perp. to AE.

Similarly it may be shewn that DA is perp. to *every* st. line which meets it in the plane of AB, AC;

∴ DA is perp. to this plane. Q.E.D.

## PROPOSITION 4. THEOREM. [Euclid's Proof.]

*If a straight line is perpendicular to each of two straight lines at their point of intersection, it shall also be perpendicular to the plane in which they lie.*




Let the st. line EF be perp. to each of the st. lines AB, DC at E their point of intersection.

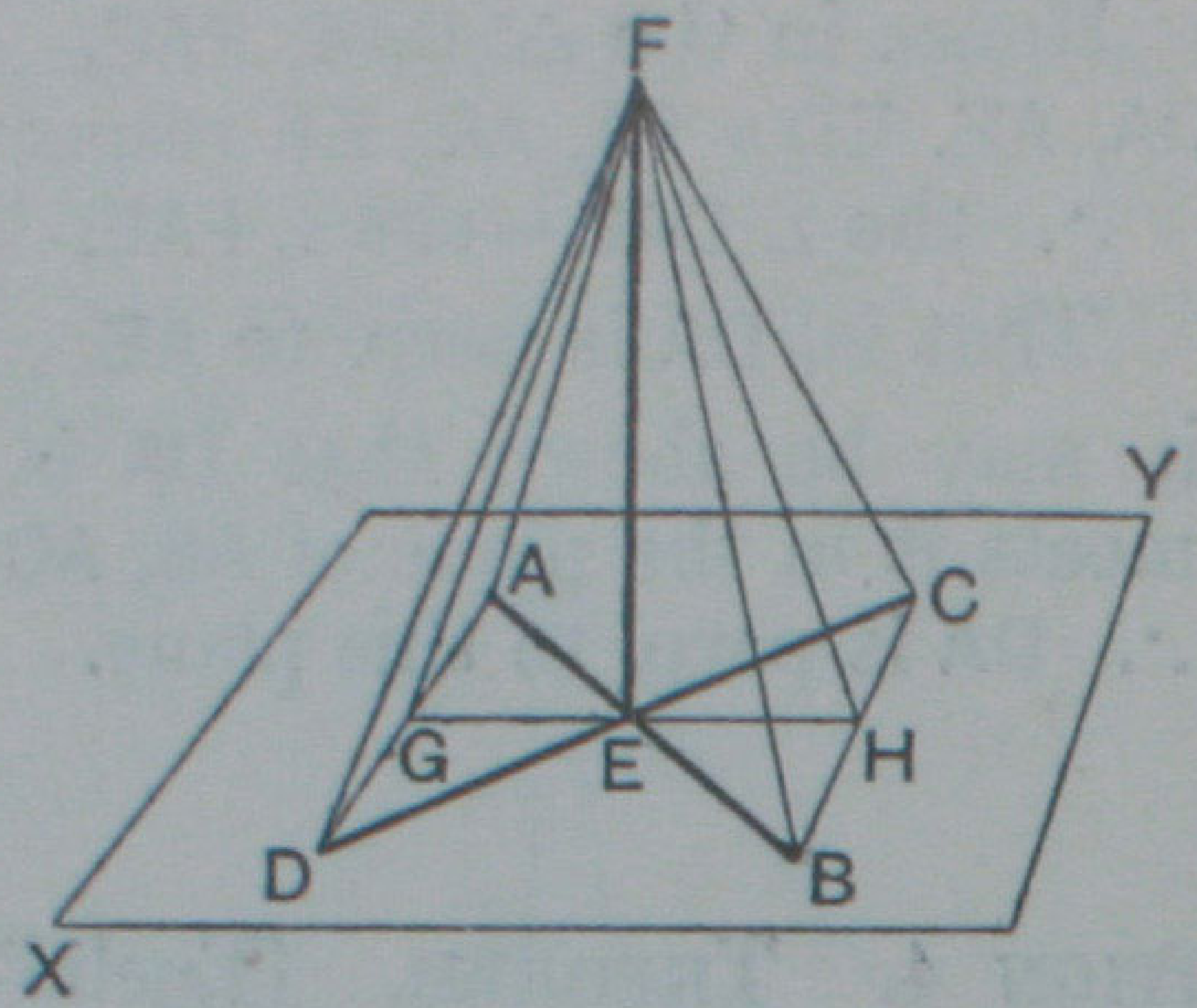
*Then shall* EF *be also perp. to the plane* XY, *in which* AB *and* DC *lie.*

Make EA, EC, EB, ED all equal, and join AD, BC.

Through E in the plane XY draw *any* st. line cutting AD and BC in G and H.

Take any pt. F in EF; and join FA, FG, FD, FB, FH, FC.

Then in the $\triangle^s$ AED, BEC,
because AE, ED = BE, EC respectively, *Constr.*
and the ∠ AED = the ∠ BEC; I. 15.
∴ AD = BC, and the ∠ DAE = the ∠ CBE. I. 4.




In the $\triangle^s$ AEG, BEH,
because the ∠ GAE = the ∠ HBE, *Proved.*
and the ∠ AEG = the ∠ BEH, I. 15.
and EA = EB; *Constr.*
∴ EG = EH, and AG = BH. I. 26.

Again in the $\triangle^s$ FEA, FEB,
because EA = EB,
and the common side FE is perp. to EA, EB; *Hyp.*
∴ FA = FB. I. 4.
Similarly FC = FD,

Again in the $\triangle^s$ DAF, CBF,
because DA, AF, FD = CB, BF, FC, respectively,
∴ the ∠ DAF = the ∠ CBF. I. 8.

And in the $\triangle^s$ FAG, FBH,
because FA, AG = FB, BH, respectively,
and the ∠ FAG = the ∠ FBH, *Proved.*
∴ FG = FH. I. 4.

Lastly in the $\triangle^s$ FEG, FEH,
because FE, EG, GF = FE, EH, HF, respectively,
∴ the ∠ FEG = the ∠ FEH; I. 8.
that is, FE is perp. to GH.

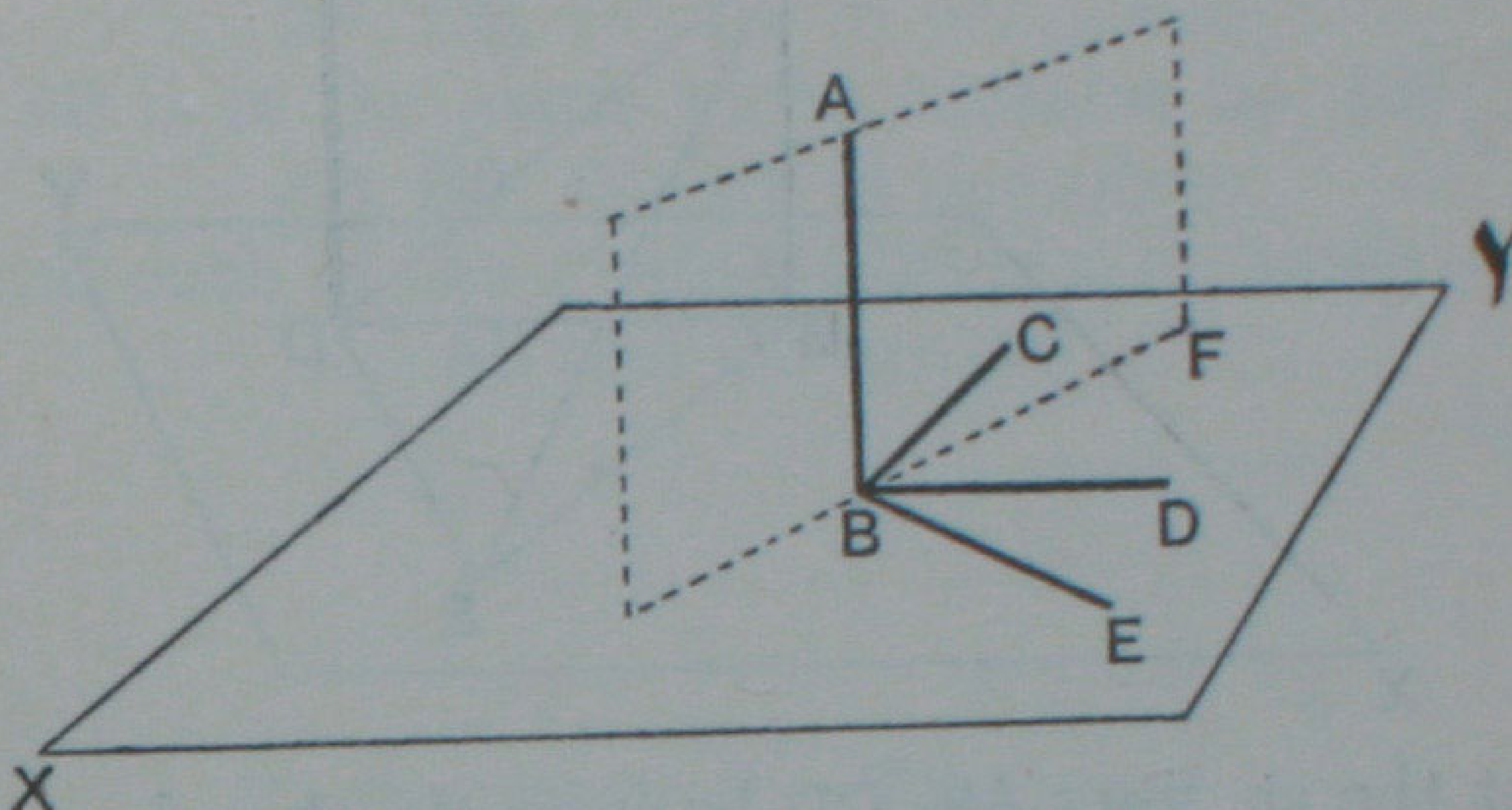
Similarly it may be shewn that FE is perp. to *every* st. line which meets it in the plane XY,

∴ FE is perp. to this plane. XI. *Def.* 1.

Q.E.D.

## Proposition 5 Theorem.

*If a straight line is perpendicular to each of three concurrent straight lines at their point of intersection, these three straight lines shall be in one plane.*




Let the straight line AB be perpendicular to each of the straight lines BC, BD, BE, at B their point of intersection.

*Then shall* BC, BD, BE *be in one plane.*

Let XY be the plane which passes through BE, BD; XI. 2.
and, if possible, suppose that BC is not in this plane.

Let AF be the plane which passes through AB, BC; and let the common section of the two planes XY, AF be the st. line BF. XI. 3.

Then since AB is perp. to BE and BD,
∴ AB is perp. to the plane containing BE, BD, namely the plane XY; XI. 4.

and since BF is in this plane,
∴ AB is also perp. to BF. XI. *Def.* 1.

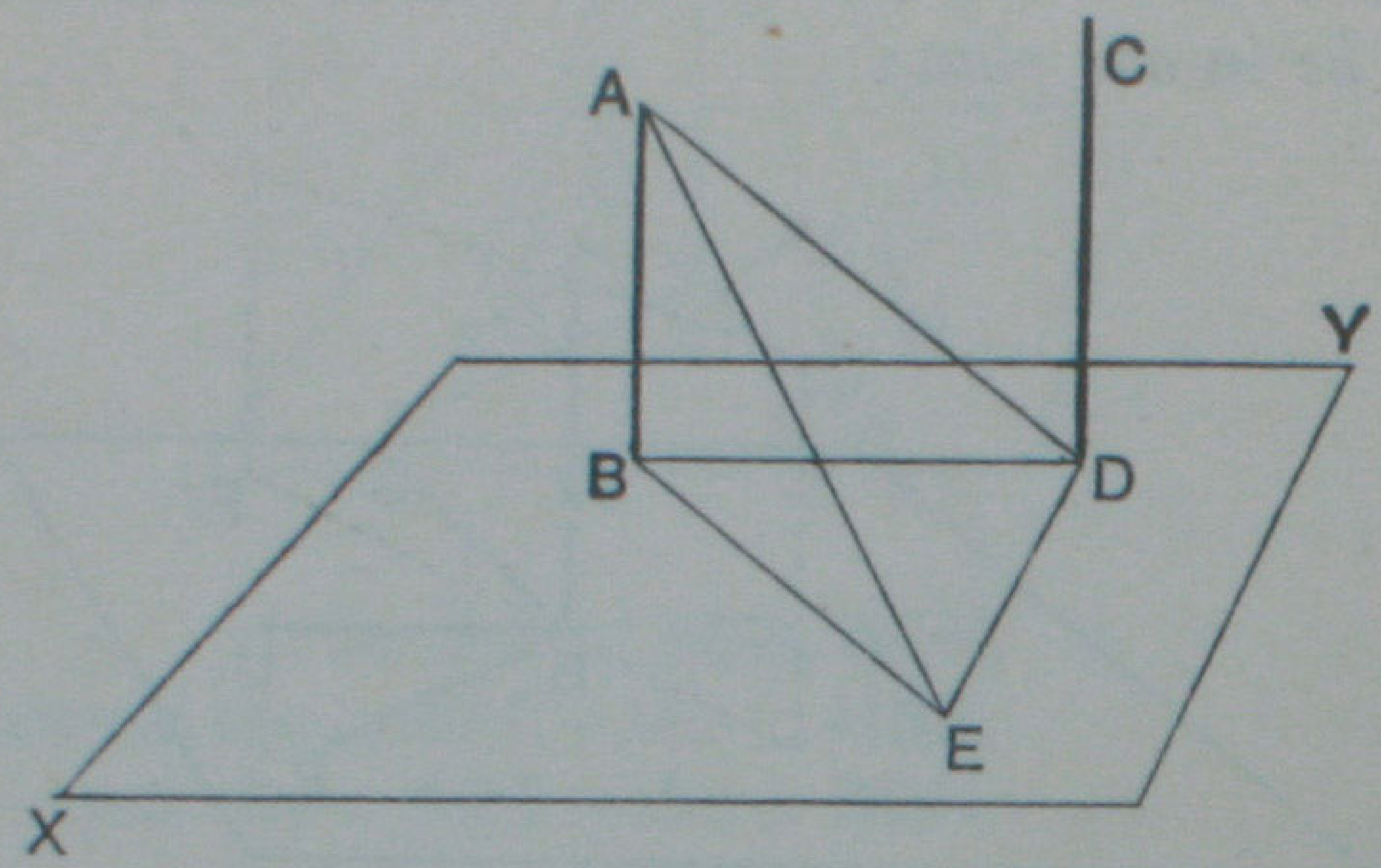
But AB is perp. to BC; *Hyp.*
∴ the ∠ˢ ABF, ABC, which are in the same plane AF, are both rt. angles; which is impossible.

∴ BC is not outside the plane of BD, BE:
that is, BC, BD, BE are in one plane.

Q.E.D.

## PROPOSITION 6. THEOREM.

*If two straight lines are perpendicular to the same plane, they shall be parallel to one another.*




Let the st. lines AB, CD be perp. to the plane XY.

*Then shall* AB *and* CD *be par*[l].*

Let AB and CD meet the plane XY at B and D.

Join BD;

and in the plane XY draw DE perp. to BD, making DE equal to AB.

Join BE, AE, AD.

Then since AB is perp. to the plane XY, *Hyp.*

∴ AB is also perp. to BD and BE, which meet it in that plane; XI. *Def.* 1.

that is, the ∠s ABD, ABE are rt. angles.

Similarly the ∠s CDB, CDE are rt. angles.

Now in the △s ABD, EDB,

because AB, BD = ED, DB, respectively, *Constr.*

and the ∠ ABD = the ∠ EDB, being rt. angles;

∴ AD = EB. I. 4.

Again in the △s ABE, EDA,

because AB, BE = ED, DA, respectively,

and AE is common;

∴ the ∠ ABE = the ∠ EDA. I. 8.

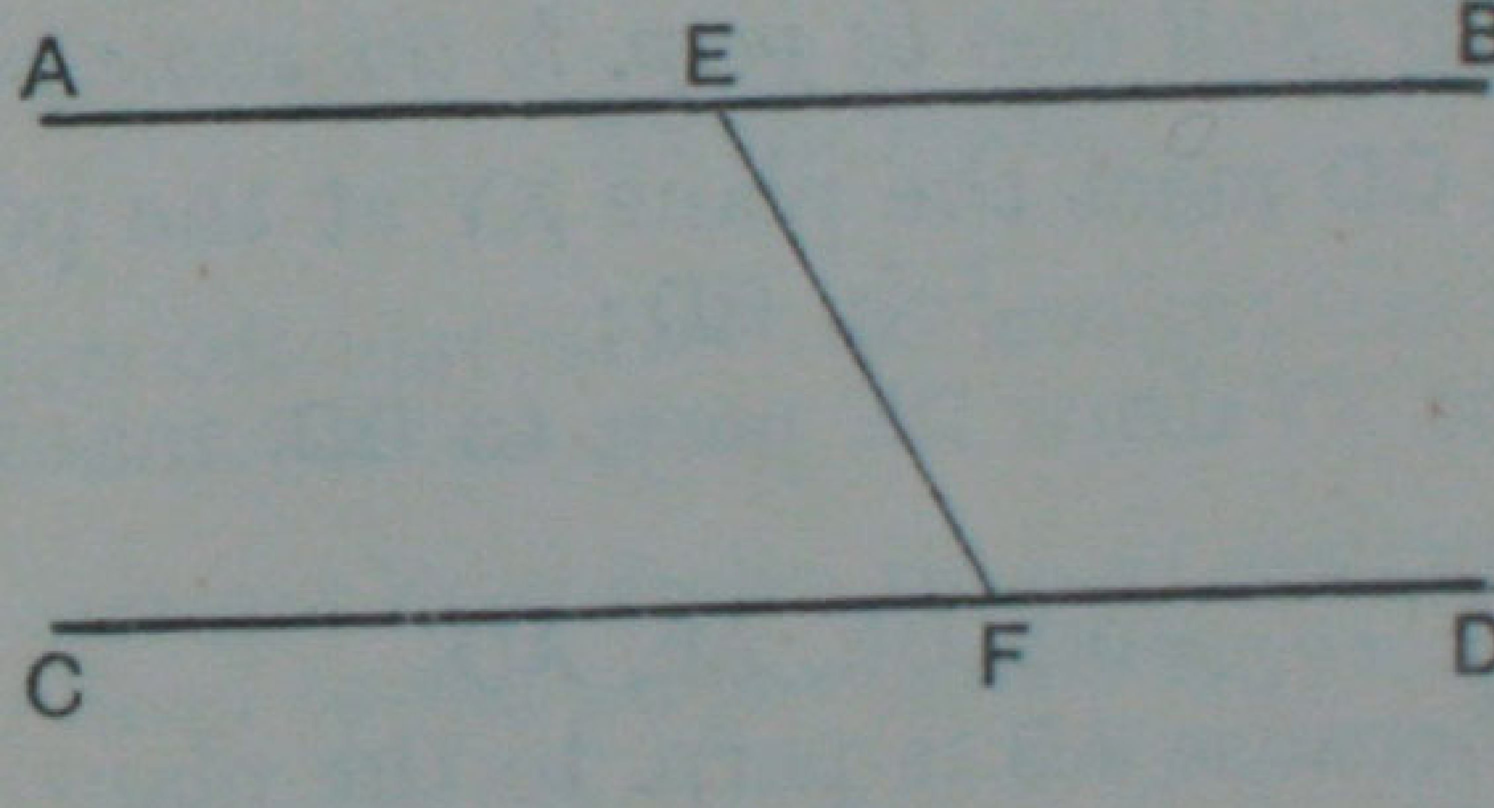
*NOTE. In order to shew that AB and CD are parallel, it is necessary to prove that (i) they are *in the same plane*, (ii) the angles ABD, CDB, are supplementary.

But the ∠ ABE is a rt. angle; *Proved.*
∴ the ∠ EDA is a rt. angle.
But the ∠ EDB is a rt. angle by construction,
and the ∠ EDC is a rt. angle, since CD is perp. to the plane XY. *Hyp.*
Hence ED is perp. to the three lines DA, DB, and DC;
∴ DA, DB, DC are in one plane. XI. 5.
But AB is in the plane which contains DA, DB; XI. 2.
∴ AB, BD, DC are in one plane.
And each of the ∠s ABD, CDB is a rt. angle; *Hyp.*
∴ AB and CD are par[l]. I. 28.

Q.E.D.

## PROPOSITION 7. THEOREM.

*If two straight lines are parallel, the straight line which joins any point in one to any point in the other is in the same plane as the parallels.*




Let AB and CD be two par[l] st. lines,
and let E, F be any two points, one in each st. line.
*Then shall the st. line which joins* E, F *be in the same plane as* AB, CD.

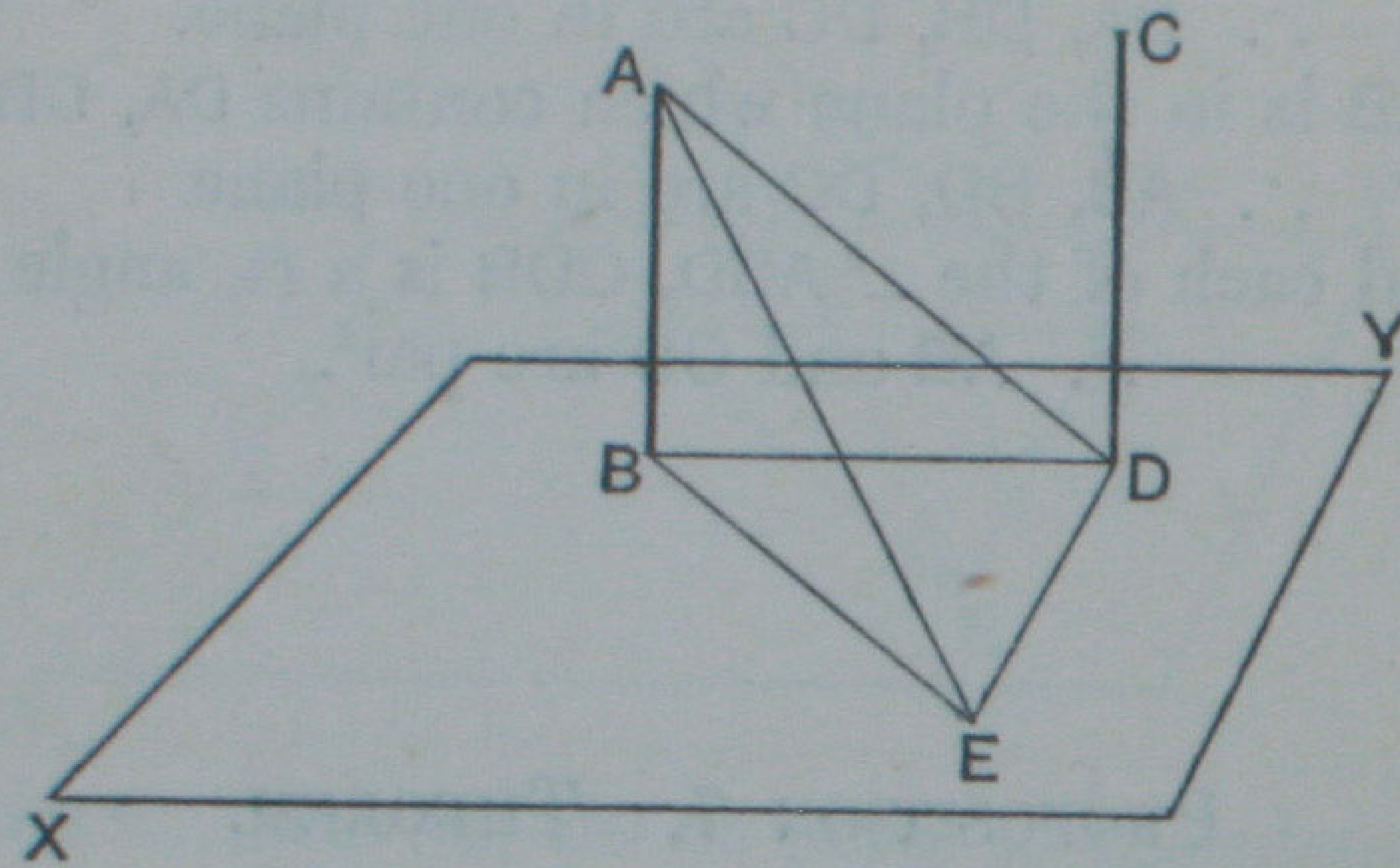
For since AB and CD are par[l],
∴ they are in one plane. I. *Def.* 35.
And since the points E and F are in this plane,
∴ the st. line which joins them lies wholly in this plane. I. *Def.* 7.

That is, EF is in the plane of the par[ls] AB, CD.

Q E.D.

## PROPOSITION 8. THEOREM.

*If two straight lines are parallel, and if one of them is perpendicular to a plane, then the other shall also be perpendicular to the same plane.*




Let AB, CD be two par[l] st. lines, of which AB is perp. to the plane XY.

*Then* CD *shall also be perp. to the same plane.*

Let AB and CD meet the plane XY at the points B, D.

Join BD;

and in the plane XY draw DE perp. to BD, making DE equal to AB.

Join BE, AE, AD.

Then because AB is perp. to the plane XY, *Hyp.*

∴ AB is also perp. to BD and BE, which meet it in that plane; XI. *Def.* 1.

that is, the ∠[s] ABD, ABE are rt. angles.

Now in the △[s] ABD, EDB,

because AB, BD = ED, DB, respectively, *Constr.*

and the ∠ ABD = the ∠ EDB, being rt. angles;

∴ AD = EB. I. 4.

Again in the △[s] ABE, EDA,

because AB, BE = ED, DA, respectively,

and AE is common;

∴ the ∠ ABE = the ∠ EDA. I. 8.

But the ∠ ABE is a rt. angle ; *Proved.*
∴ the ∠ EDA is a rt. angle :
that is, ED is perp. to DA.

But ED is also perp. to DB : *Constr.*
∴ ED is perp. to the plane containing DB, DA. XI. 4.
And DC is in this plane ;
for both DB and DA are in the plane of the par[ls] AB, CD.
XI. 7.

∴ ED is also perp. to DC ; XI. *Def.* 1.
that is, the ∠ CDE is a rt. angle.

Again since AB and CD are par[l], *Hyp.*
and since the ∠ ABD is a rt. angle,
∴ the ∠ CDB is also a rt. angle. I. 29.
∴ CD is perp. both to DB and DE ;
∴ CD is also perp. to the plane XY, which contains DB, DE. XI. 4.

Q.E.D.

## EXERCISES.

1. The perpendicular is the least straight line that can be drawn from an external point to a plane.

2. Equal straight lines drawn from an external point to a plane are equally inclined to the perpendicular drawn from that point to the plane.

3. Shew that two observations with a spirit-level are sufficient to determine if a plane is horizontal: and prove that for this purpose the two positions of the level must not be parallel.

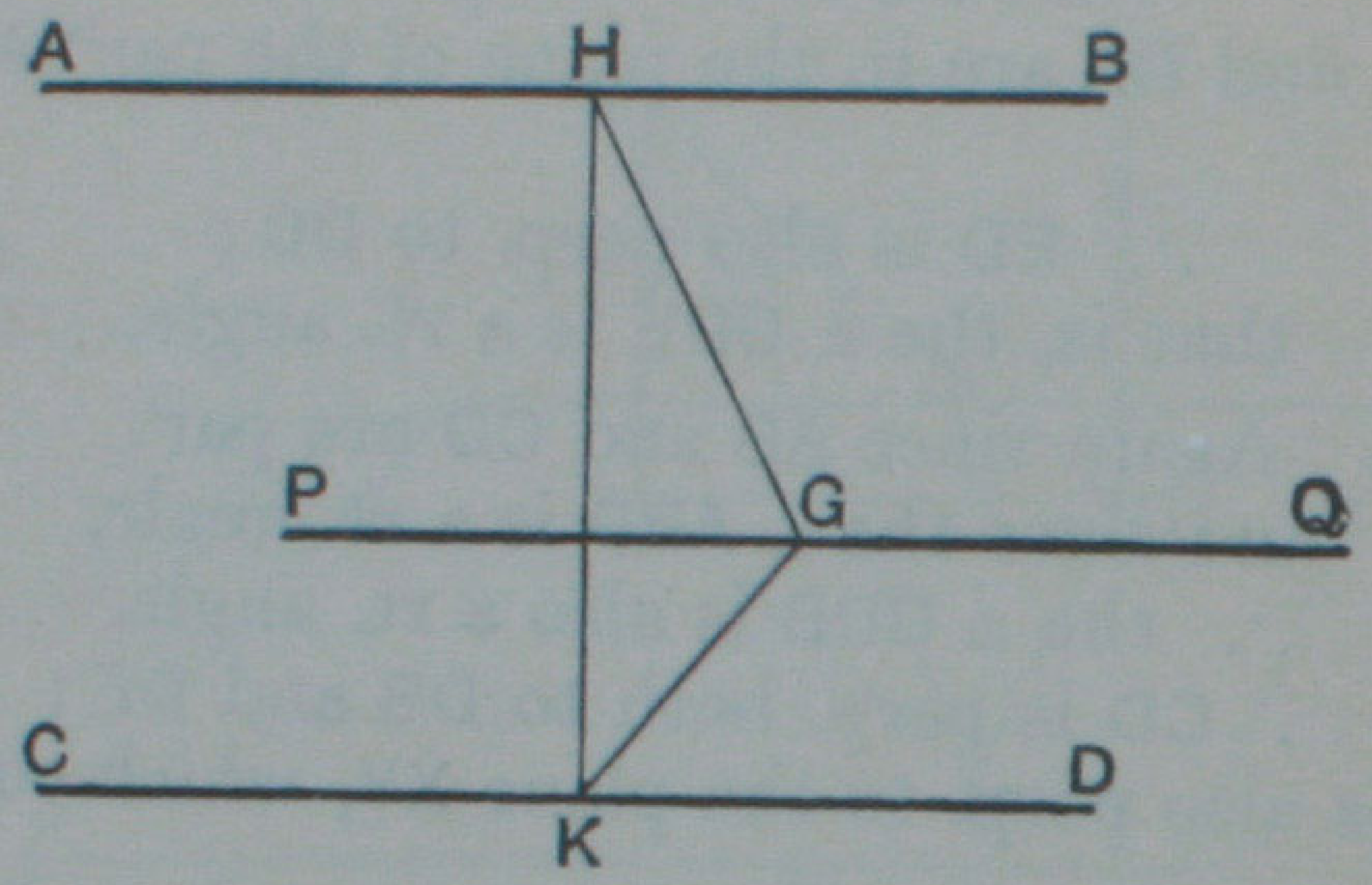
4. What is the locus of points in space which are equidistant from two fixed points?

5. Shew how to determine in a given straight line the point which is equidistant from two fixed points. When is this impossible?

6. If a straight line is parallel to a plane, shew that any plane passing through the given straight line will have with the given plane a common section which is parallel to the given straight line.

## Proposition 9. Theorem.

*Two straight lines which are parallel to a third straight line are parallel to one another.*




Let the st. lines AB, CD be each par[l] to the st. line PQ.

*Then shall* AB *be par[l] to* CD.

Case I. If AB, CD and PQ are *in one plane*, the proposition has already been proved. I. 30.

Case II. But if AB, CD and PQ are not in one plane, in PQ take any point G;

and from G, in the plane of the par[ls] AB, PQ, draw GH perp. to PQ; I. 11.

also from G, in the plane of the par[ls] CD, PQ, draw GK perp. to PQ. I. 11.

Then because PQ is perp. to GH and GK, *Constr.*

∴ PQ is perp. to the plane HGK, which contains them. XI. 4.

But AB is par[l] to PQ; *Hyp.*

∴ AB is also perp. to the plane HGK. XI. 8.

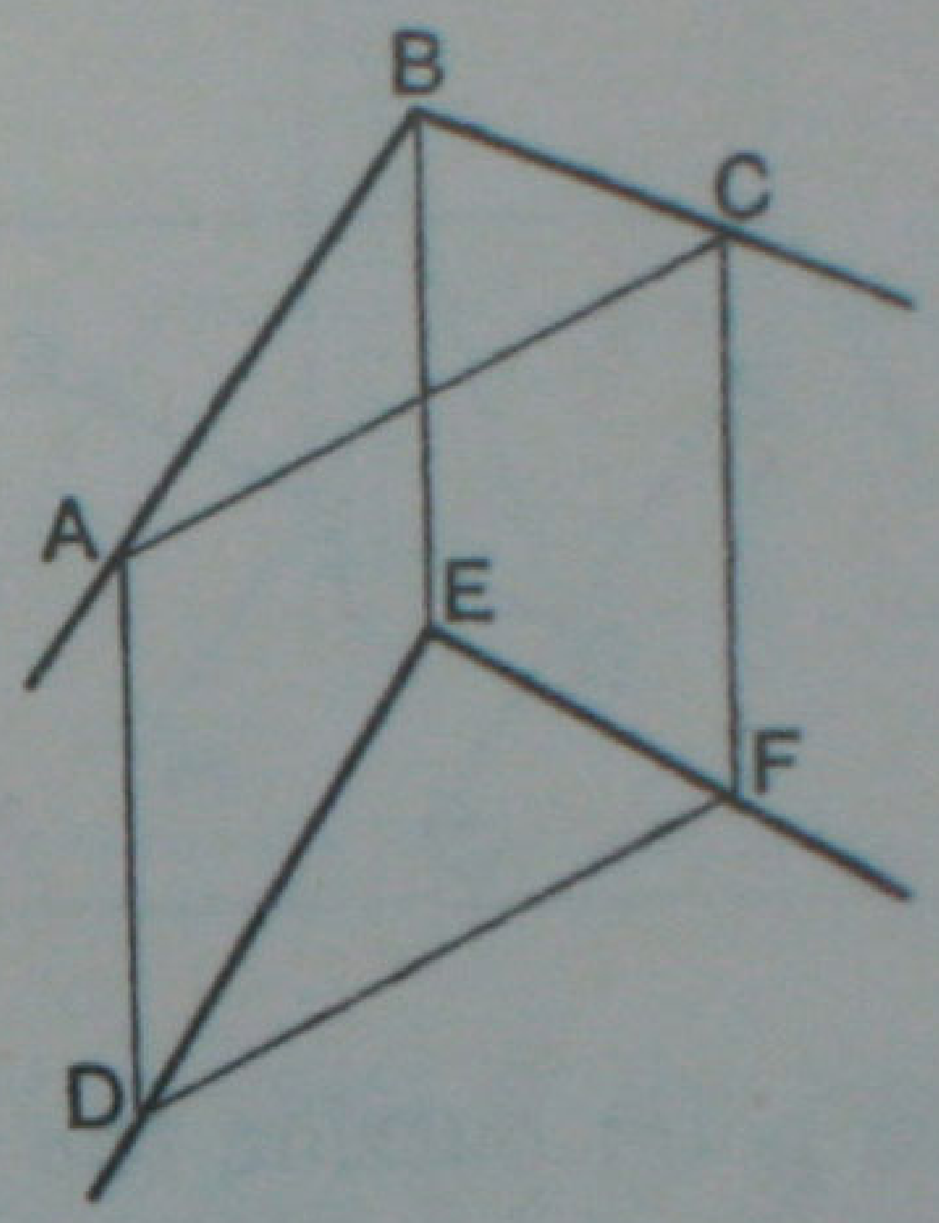
Similarly, CD is perp. to the plane HGK.

Hence AB and CD, being perp. to the same plane, are par[l] to one another. XI. 6.

Q.E.D.

## Proposition 10. Theorem.

*If two intersecting straight lines are respectively parallel to two other intersecting straight lines not in the same plane with them, then the first pair and the second pair shall contain equal angles.*




Let the st. lines AB, BC be respectively par[l] to the st. lines DE, EF, which are not in the same plane with them.

*Then shall the* ∠ ABC = *the* ∠ DEF.

In BA and ED, make BA equal to ED;
and in BC and EF, make BC equal to EF.
Join AD, BE, CF, AC, DF.

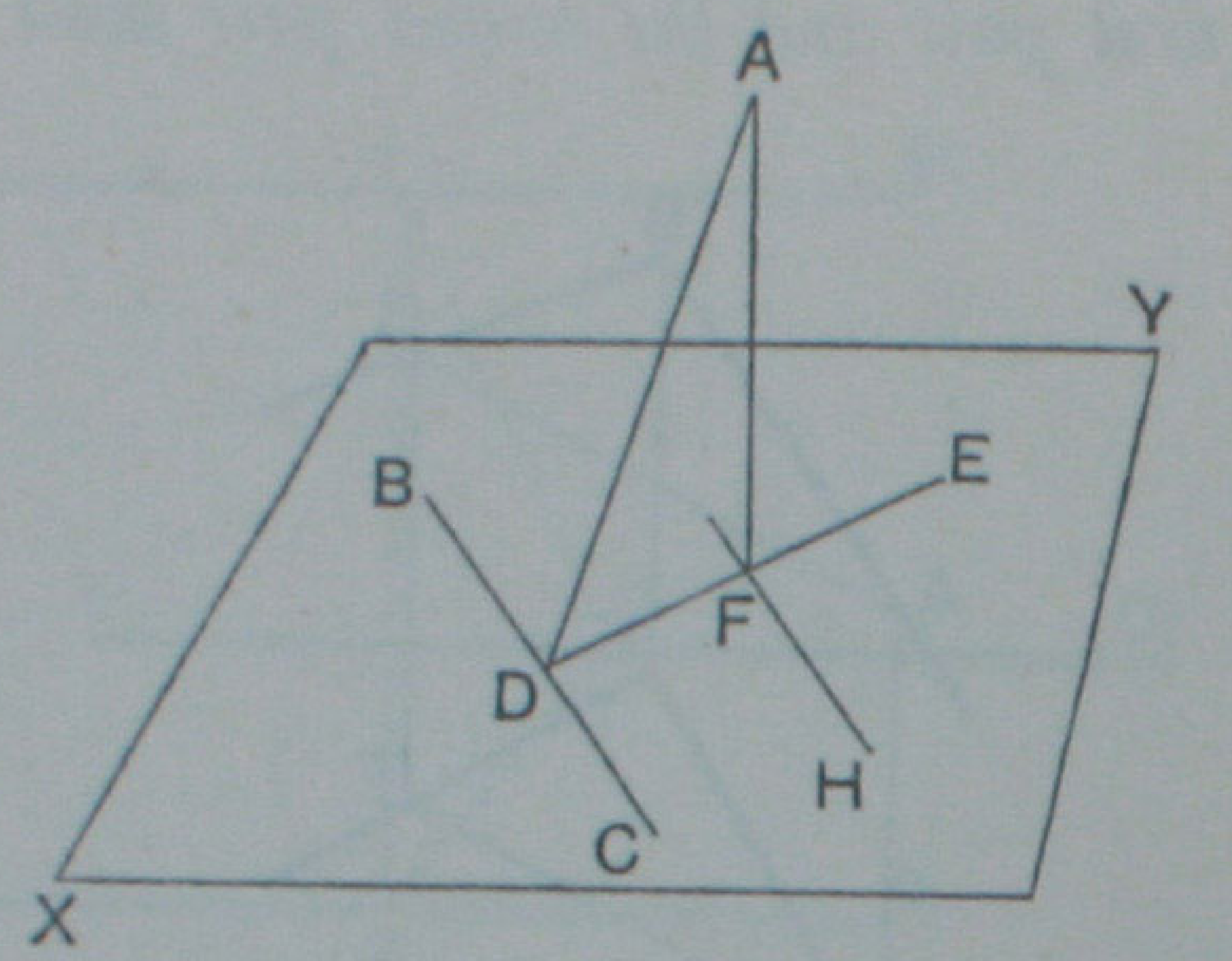
Then because BA is equal and par[l] to ED, *Hyp. and Constr.*
∴ AD is equal and par[l] to BE. I. 33.
And because BC is equal and par[l] to EF,
∴ CF is equal and par[l] to BE. I. 33.
Hence AD and CF, being each equal and par[l] to BE, are equal and par[l] to one another; *Ax.* 1 and XI. 9.
hence it follows that AC is equal and par[l] to DF. I. 33.

Then in the △[s] ABC, DEF,
because AB, BC, AC = DE, EF, DF, respectively,
∴ the ∠ ABC = the ∠ DEF. I. 8.

Q.E.D.

## PROPOSITION 11. PROBLEM.

*To draw a straight line perpendicular to a given plane from a given point outside it.*




Let A be the given point outside the plane XY.

*It is required to draw from* A *a st. line perp. to the plane* XY.

Draw any st. line BC in the plane XY;
and from A draw AD perp. to BC. I. 12.

Then if AD is also perp. to the plane XY, what was required is done.

But if not, from D draw DE in the plane XY perp. to BC; I. 11.
and from A draw AF perp. to DE. I. 12.

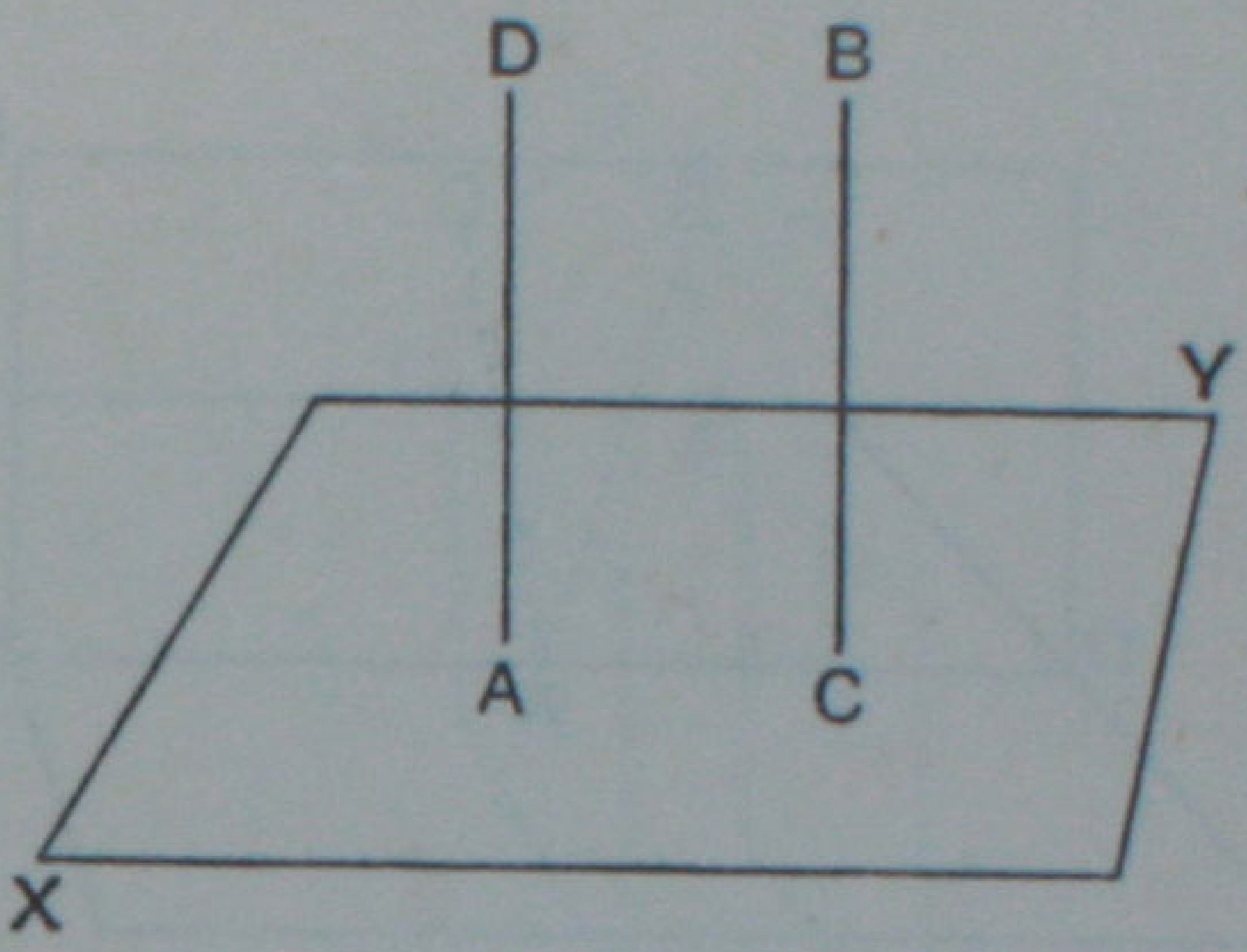
*Then* AF *shall be perp. to the plane* XY.

Through F draw FH par[l] to BC. I. 31.

Now because CD is perp. to DA and DE, *Constr.*
∴ CD is perp. to the plane containing DA, DE. XI. 4.
And HF is par[l] to CD;
∴ HF is also perp. to the plane containing DA, DE. XI. 8.
And since FA meets HF in this plane,
∴ the ∠ HFA is a rt. angle; XI. *Def.* 1.
that is, AF is perp. to FH.
And AF is also perp. to DE; *Constr.*
∴ AF is perp. to the plane containing FH, DE;
that is, AF is perp. to the plane XY. Q.E.F.

## PROPOSITION 12. PROBLEM.

*To draw a straight line perpendicular to a given plane from a given point in the plane.*




Let A be the given point in the plane XY.

*It is required to draw from* A *a st. line perp. to the plane* XY.

From any point B outside the plane XY draw BC perp. to the plane. XI. 11.

Then if BC passes through A, what was required is done.

But if not, from A draw AD par[1] to BC. I. 31.

*Then* AD *shall be the perpendicular required.*

For since BC is perp. to the plane XY, *Constr.*
and since AD is par[1] to BC, *Constr.*
∴ AD is also perp. to the plane XY. XI. 8.

Q.E.F.

### EXERCISES.

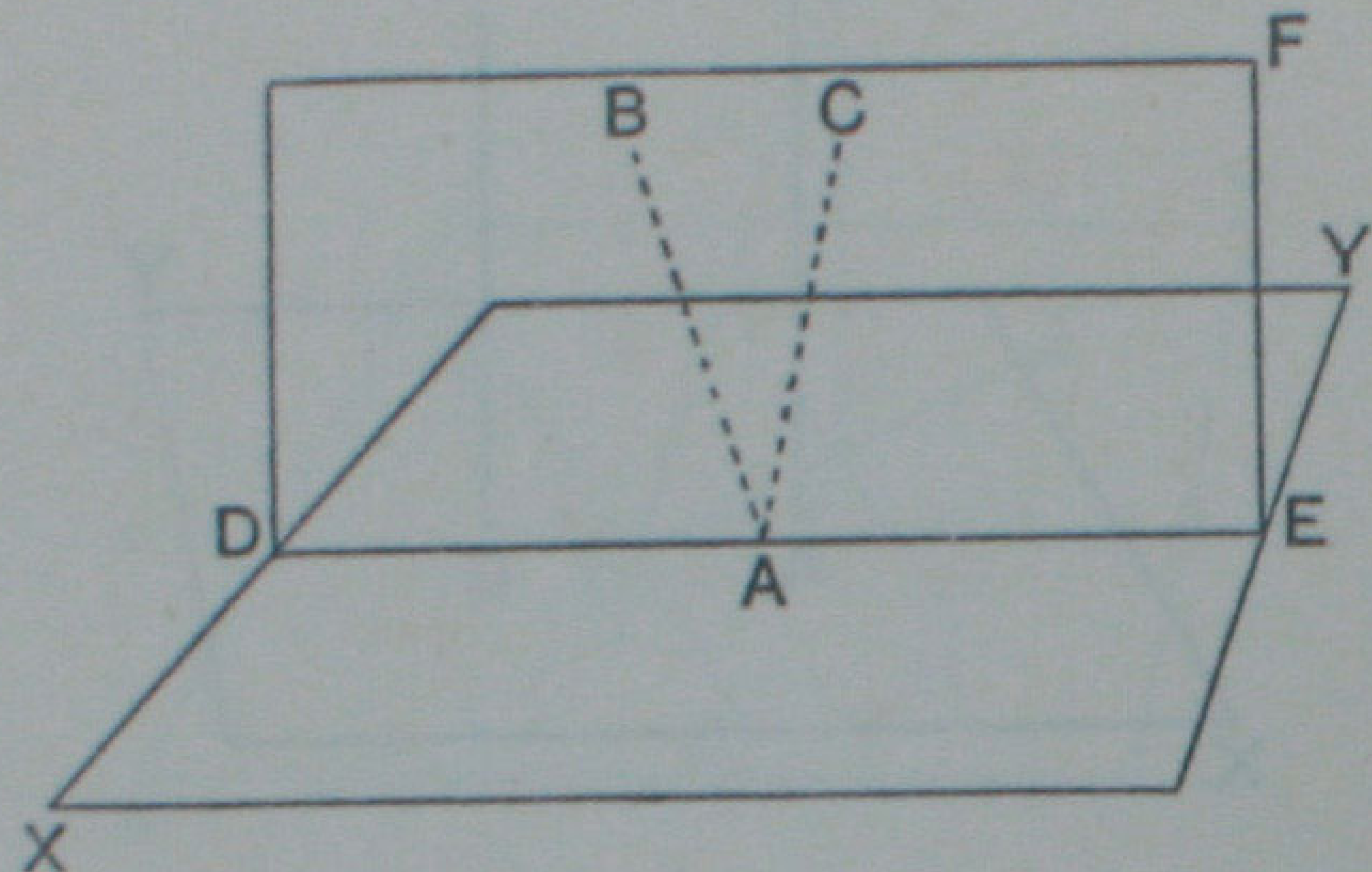
1. Equal straight lines drawn to meet a plane from a point without it are equally inclined to the plane.

2. Find the locus of the foot of the perpendicular drawn from a given point upon any plane which passes through a given straight line.

3. From a given point A a perpendicular AF is drawn to a plane XY; and from F, FD is drawn perpendicular to BC, any line in that plane: shew that AD is also perpendicular to BC.

## Proposition 13. Theorem.

*Only one perpendicular can be drawn to a given plane from a given point either in the plane or outside it.*




Case I. Let the given point A be *in* the given plane XY; and, if possible, let two perps. AB, AC be drawn from A to the plane XY.

Let DF be the plane which contains AB and AC; and let the st. line DE be the common section of the planes DF and XY. XI. 3

Then the st. lines AB, AC, AE are in one plane.

And because BA is perp. to the plane XY, *Hyp.*

∴ BA is also perp. to AE, which meets it in this plane; XI. *Def.* 1.

that is, the ∠ BAE is a rt. angle.

Similarly, the ∠ CAE is a rt. angle.

∴ the ∠s BAE, CAE, which are in the same plane, are equal to one another; which is impossible.

∴ two perpendiculars cannot be drawn to the plane XY from the point A in that plane.

Case II. Let the given point A be *outside* the plane XY.

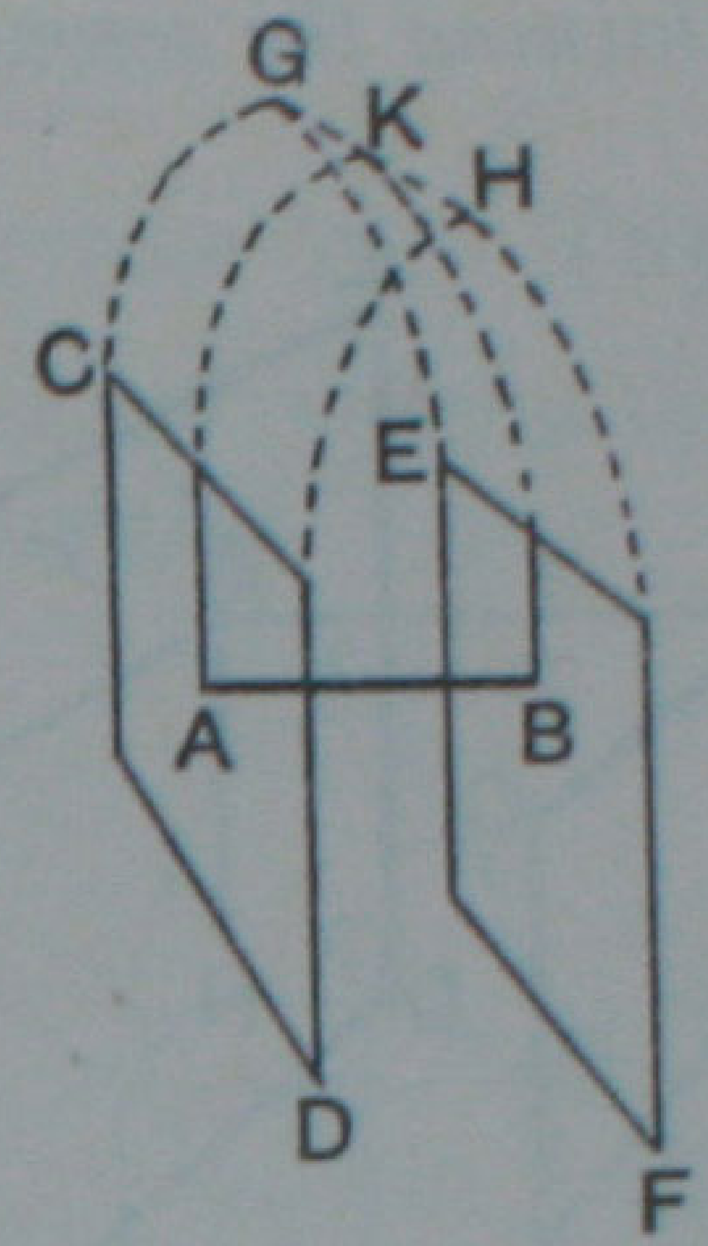
Then two perps cannot be drawn from A to the plane;

for if there could be two, they would be parl, XI. 6.

which is absurd. Q.E.D.

## PROPOSITION 14. THEOREM.

*Planes to which the same straight line is perpendicular are parallel to one another.*




Let the st. line AB be perp. to each of the planes CD, EF.

*Then shall the planes* CD, EF *be par[l].*

For if not, they will meet when produced.

If possible, let the two planes meet, and let the st. line GH be their common section. XI. 3.

In GH take any point K;

and join AK, BK.

Then because AB is perp. to the plane EF,

∴ AB is also perp. to BK, which meets it in this plane; XI. *Def.* 1.

that is, the ∠ ABK is a rt. angle.

Similarly, the ∠ BAK is a rt. angle.

∴ in the △ KAB, the two ∠[s] ABK, BAK are together equal to two rt. angles;

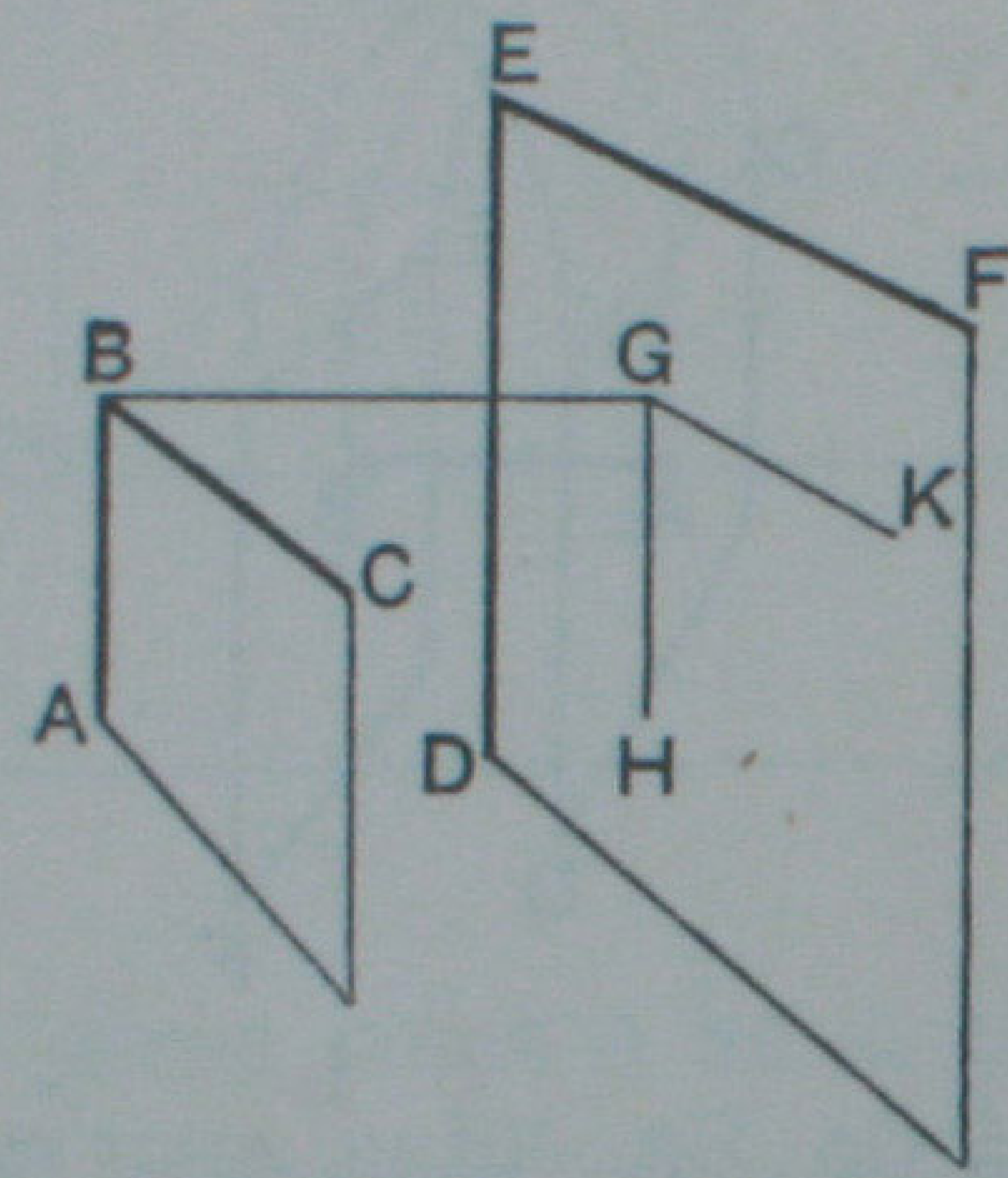
which is impossible. I. 17.

∴ the planes CD, EF, though produced, do not meet:

that is, they are par[l]. Q.E.D.

## PROPOSITION 15. THEOREM.

*If two intersecting straight lines are parallel respectively to two other intersecting straight lines which are not in the same plane with them, then the plane containing the first pair shall be parallel to the plane containing the second pair.*




Let the st. lines AB, BC be respectively par[l] to the st. lines DE, EF, which are not in the same plane as AB, BC.

*Then shall the plane containing* AB, BC *be par[l] to the plane containing* DE, EF.

From B draw BG perp. to the plane of DE, EF; XI. 11.
and let it meet that plane at G.
Through G draw GH, GK par[l] respectively to DE, EF. I. 31.

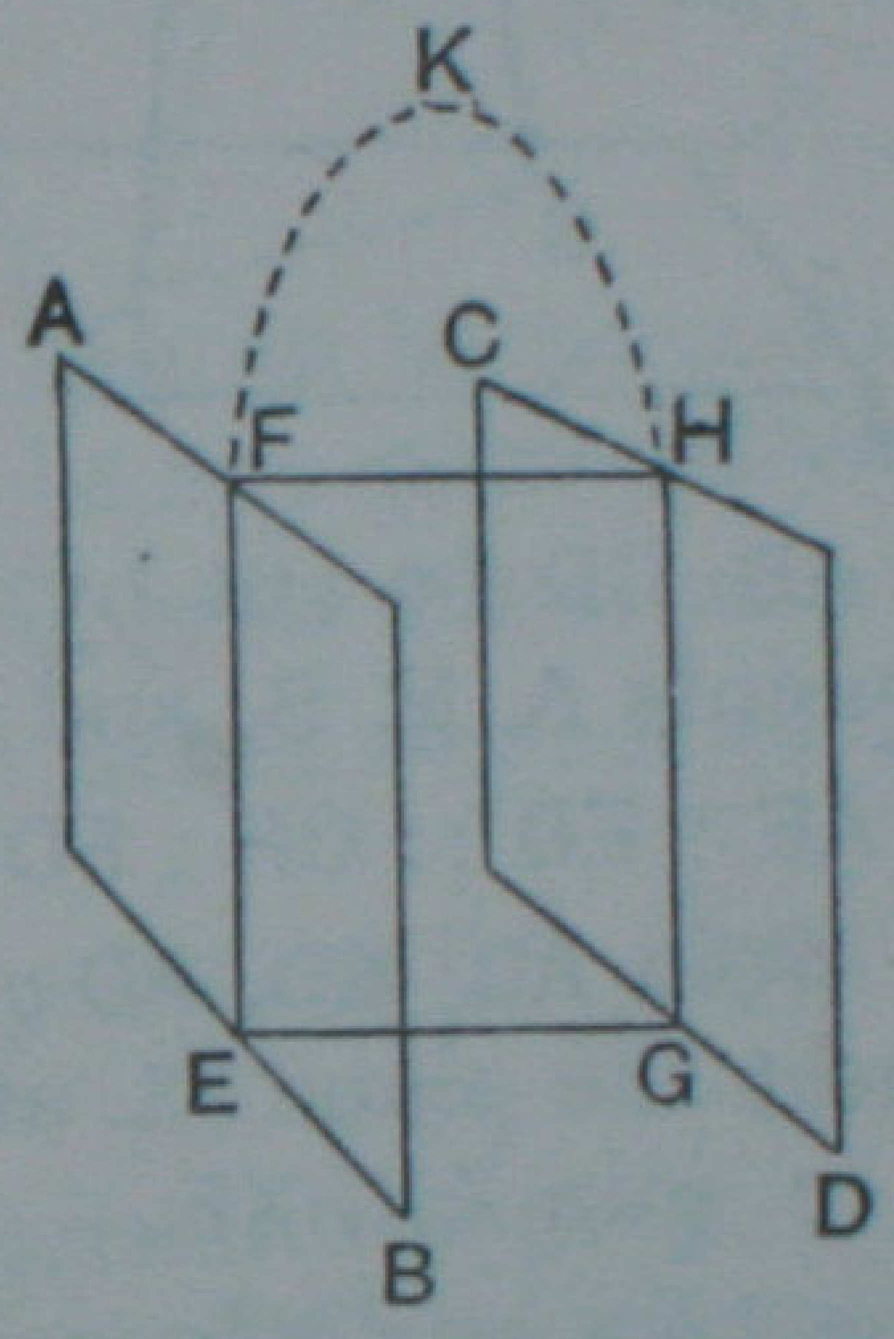
Then because BG is perp. to the plane of DE, EF,
∴ BG is also perp. to GH and GK, which meet it in that plane: XI. *Def.* 1.
that is, each of the ∠[s] BGH, BGK is a rt. angle.

Now *by hypothesis* BA is par[l] to ED,
and *by construction* GH is par[l] to ED;
∴ BA is par[l] to GH. XI. 9.
And since the ∠ BGH is a rt. angle; *Proved.*
∴ the ∠ ABG is a rt. angle. I. 29.
Similarly the ∠ CBG is a rt. angle.

Then since BG is perp. to each of the st. lines BA, BC,
∴ BG is perp. to the plane containing them. XI. 4.
But BG is also perp. to the plane of ED, EF; *Constr.*
that is, BG is perp. to the two planes AC, DF;
∴ these planes are par[l]. XI. 14.
Q.E.D.

## Proposition 16. Theorem.

*If two parallel planes are cut by a third plane, their common sections with it shall be parallel.*



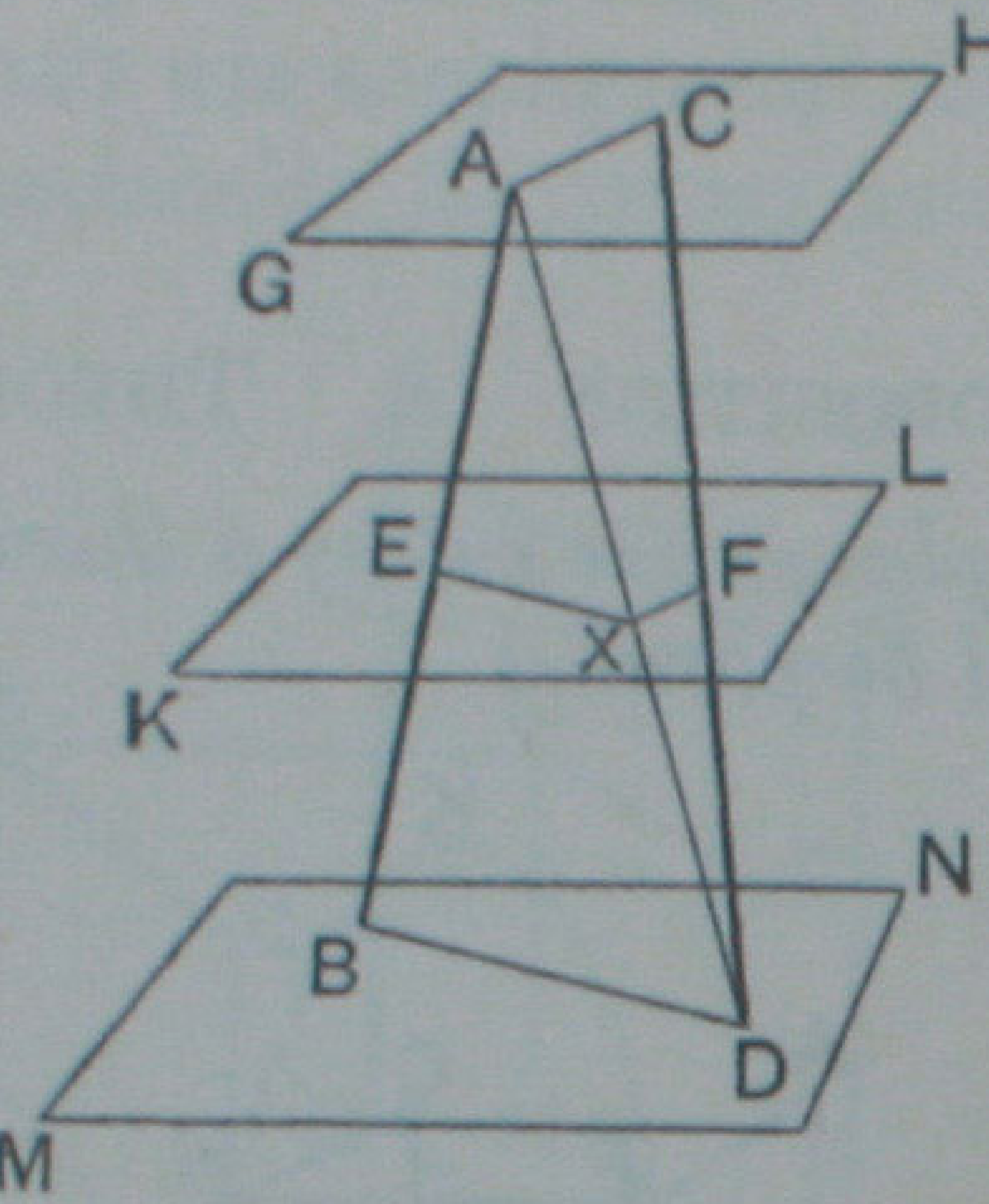

Let the par[l] planes AB, CD be cut by the plane EFHG, and let the st. lines EF, GH be their common sections with it.

*Then shall* EF, GH *be par[l].*

For if not, EF and GH will meet if produced.
If possible, let them meet at K.
Then since the whole st. line EFK is in the plane AB, XI. 1.
and K is a point in that line,
∴ the point K is in the plane AB.
Similarly the point K is in the plane CD.
Hence the planes AB, CD when produced meet at K;
which is impossible, since they are par[l]. *Hyp.*
∴ the st. lines EF and GH do not meet;
and they are in the same plane EFHG;
∴ they are par[l]. I. *Def.* 35.
Q.E.D.

## PROPOSITION 17. THEOREM.

*Straight lines which are cut by parallel planes are cut proportionally.*




Let the st. lines AB, CD be cut by the three par[1] planes GH, KL, MN at the points A, E, B, and C, F, D.

*Then shall* AE : EB :: CF : FD.

Join AC, BD, AD;
and let AD meet the plane KL at the point X:
join EX, XF.

Then because the two par[1] planes KL, MN are cut by the plane ABD,

∴ the common sections EX, BD are par[1]. XI. 16.

And because the two par[1] planes GH, KL are cut by the plane DAC,

∴ the common sections XF, AC are par[1]. XI. 16.

Now since EX is par[1] to BD, a side of the △ ABD,

∴ AE : EB :: AX : XD. VI. 2.

Again because XF is par[1] to AC, a side of the △ DAC,

∴ AX : XD :: CF : FD. VI. 2.

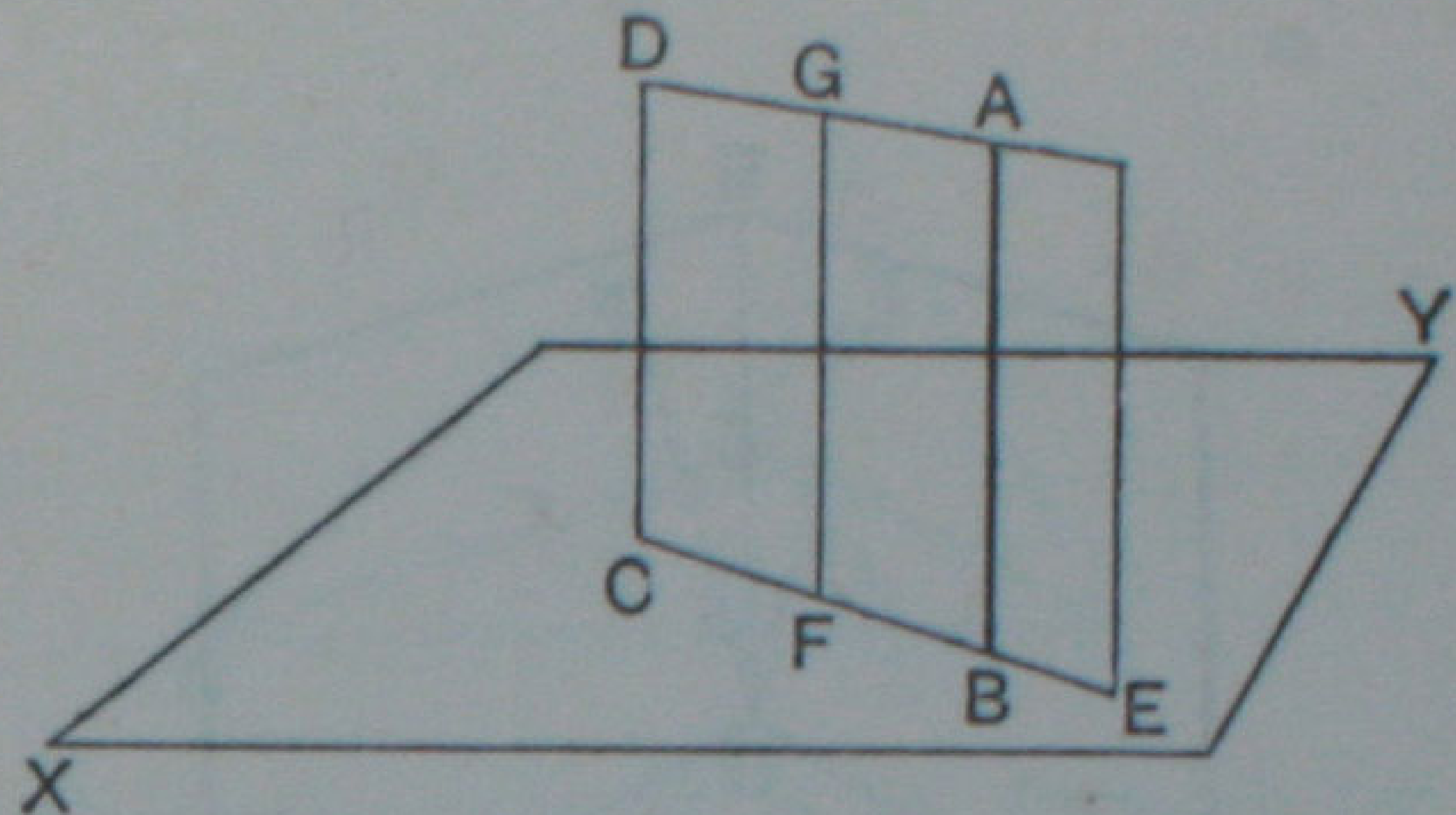
Hence AE : EB :: CF : FD. V. 1.

Q.E.D.

DEFINITION. One plane is perpendicular to another plane, when *any* straight line drawn in one of the planes perpendicular to their common section is also perpendicular to the other plane. [Book XI. *Def.* 6.]

## PROPOSITION 18. THEOREM.

*If a straight line is perpendicular to a plane, then every plane which passes through the straight line is also perpendicular to the given plane.*




Let the st. line AB be perp. to the plane XY;
and let DE be any plane passing through AB.
*Then shall the plane* DE *be perp. to the plane* XY.

Let the st. line CE be the common section of the planes XY, DE. XI. 3.

From F, *any* point in CE, draw FG in the plane DE perp. to CE. I. 11.

Then because AB is perp. to the plane XY, *Hyp.*
∴ AB is also perp. to CE, which meets it in that plane, XI. *Def.* 1.
that is, the ∠ ABF is a rt. angle.
But the ∠ GFB is also a rt. angle; *Constr.*
∴ GF is par[l] to AB. I. 28.
And AB is perp. to the plane XY, *Hyp.*
∴ GF is also perp. to the plane XY. XI. 8.

Hence it has been shewn that *any* st. line GF drawn in the plane DE perp. to the common section CE is also perp. to the plane XY.

∴ the plane DE is perp. to the plane XY. XI. *Def.* 6.

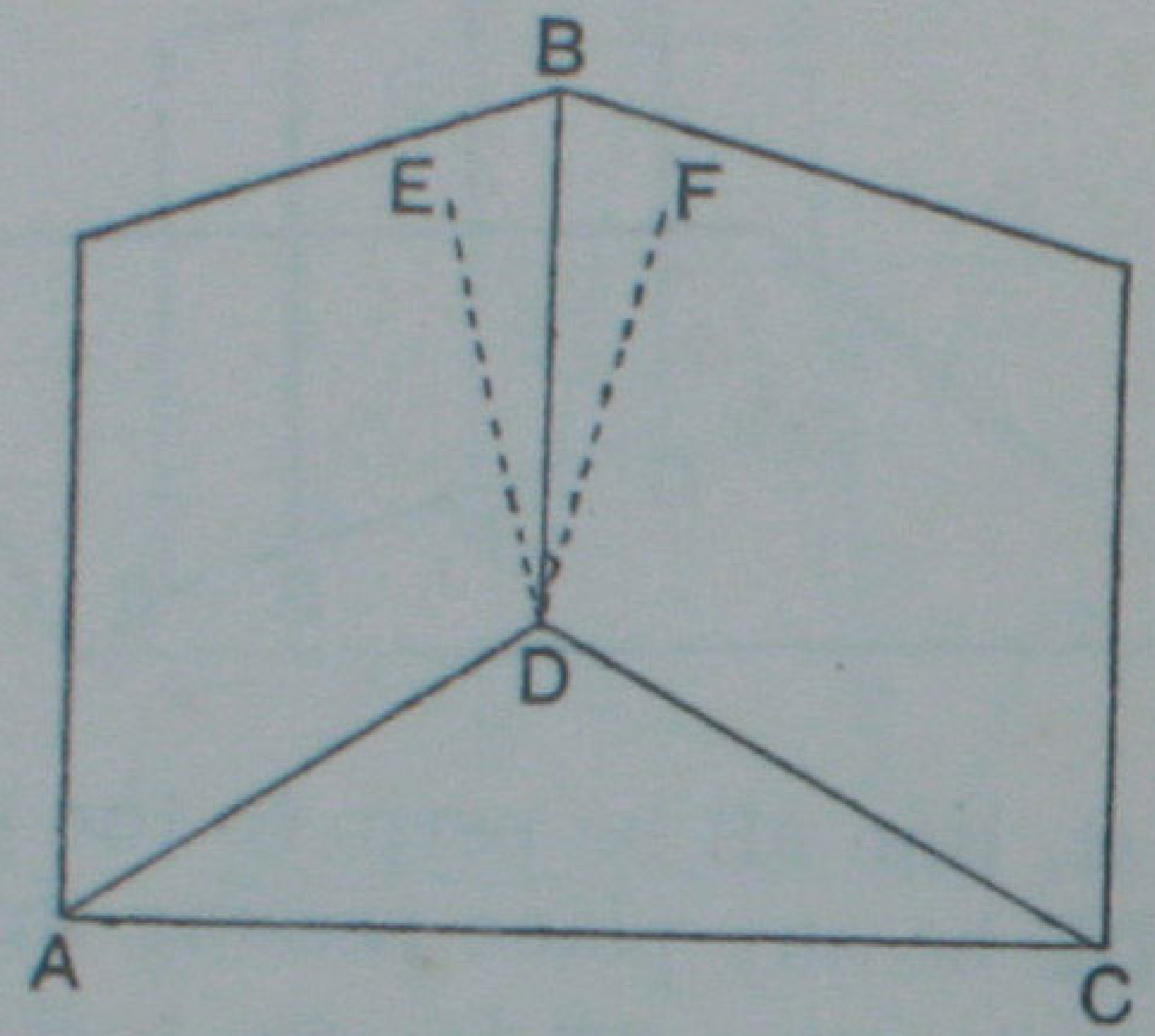
Q.E.D.

EXERCISE.

*Shew that two planes are perpendicular to one another when the dihedral angle* [see XI. *Def.* 7] *formed by them is a right angle.*

## PROPOSITION 19. THEOREM.

*If two intersecting planes are each perpendicular to a third plane, their common section shall also be perpendicular to that plane.*




Let each of the planes AB, BC be perp. to the plane ADC, and let BD be their common section.

*Then shall* BD *be perp. to the plane* ADC.

For if not, from D draw in the plane AB the st. line DE perp. to AD, the common section of the planes ADB, ADC : I. 11.

and from D draw in the plane BC the st. line DF perp. to DC, the common section of the planes BDC, ADC.

Then because the plane BA is perp. to the plane ADC, *Hyp.*

and DE is drawn in the plane BA perp. to AD the common section of these planes, *Constr.*

∴ DE is perp. to the plane ADC. XI. *Def.* 6.

Similarly DF is perp. to the plane ADC.

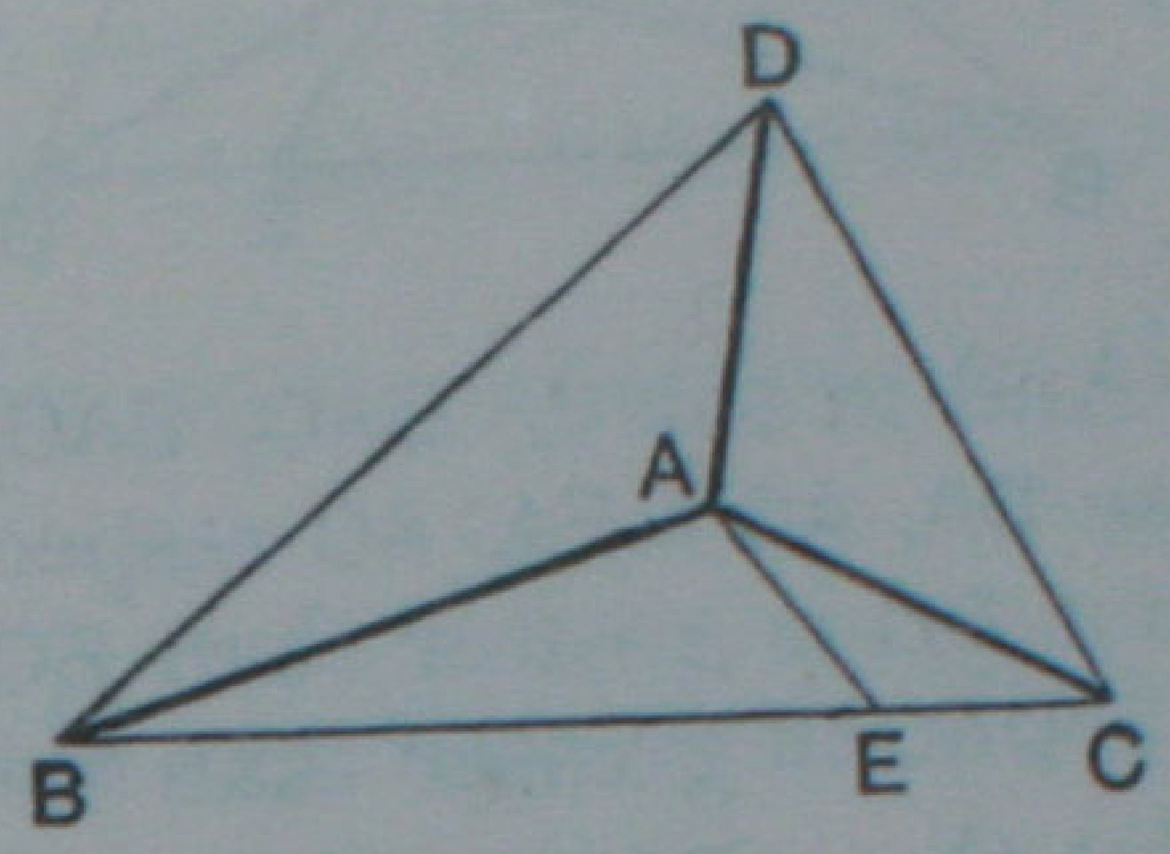
∴ from the point D two st. lines are drawn perp. to the plane ADC ; which is impossible. XI. 13.

Hence DB cannot be otherwise than perp. to the plane ADC.

Q.E.D.

## PROPOSITION 20. THEOREM.

*Of the three plane angles which form a trihedral angle, any two are together greater than the third.*




Let the trihedral angle at A be formed by the three plane ∠ˢ BAD, DAC, BAC.

*Then shall any two of them, such as the* ∠ˢ BAD, DAC, *be together greater than the third, the* ∠ BAC.

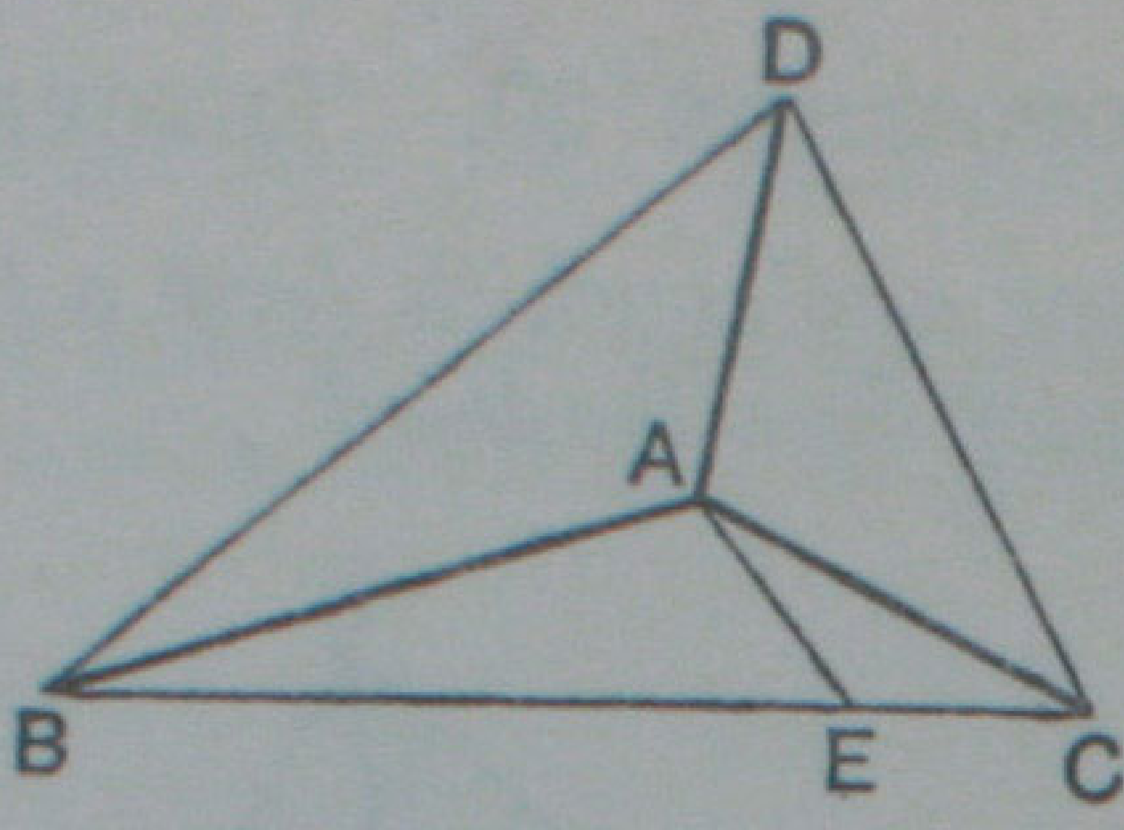
CASE I. If the ∠ BAC is less than, or equal to, either of the ∠ˢ BAD, DAC;
it is evident that the ∠ˢ BAD, DAC are together greater than the ∠ BAC.

CASE II. But if the ∠ BAC is greater than either of the ∠ˢ BAD, DAC;
then at the point A in the plane BAC make the ∠ BAE equal to the ∠ BAD;
and cut off AE equal to AD.

Through E, and in the plane BAC, draw the st. line BEC cutting AB, AC at B and C:
join DB, DC.

Then in the △ˢ BAD, BAE,
since BA, AD = BA, AE, respectively, *Constr.*
and the ∠ BAD = the ∠ BAE; *Constr.*
∴ BD = BE. I. 4.

Again in the △ BDC, since BD, DC are together greater than BC, I. 20.
and BD = BE, *Proved.*
∴ DC is greater than EC.

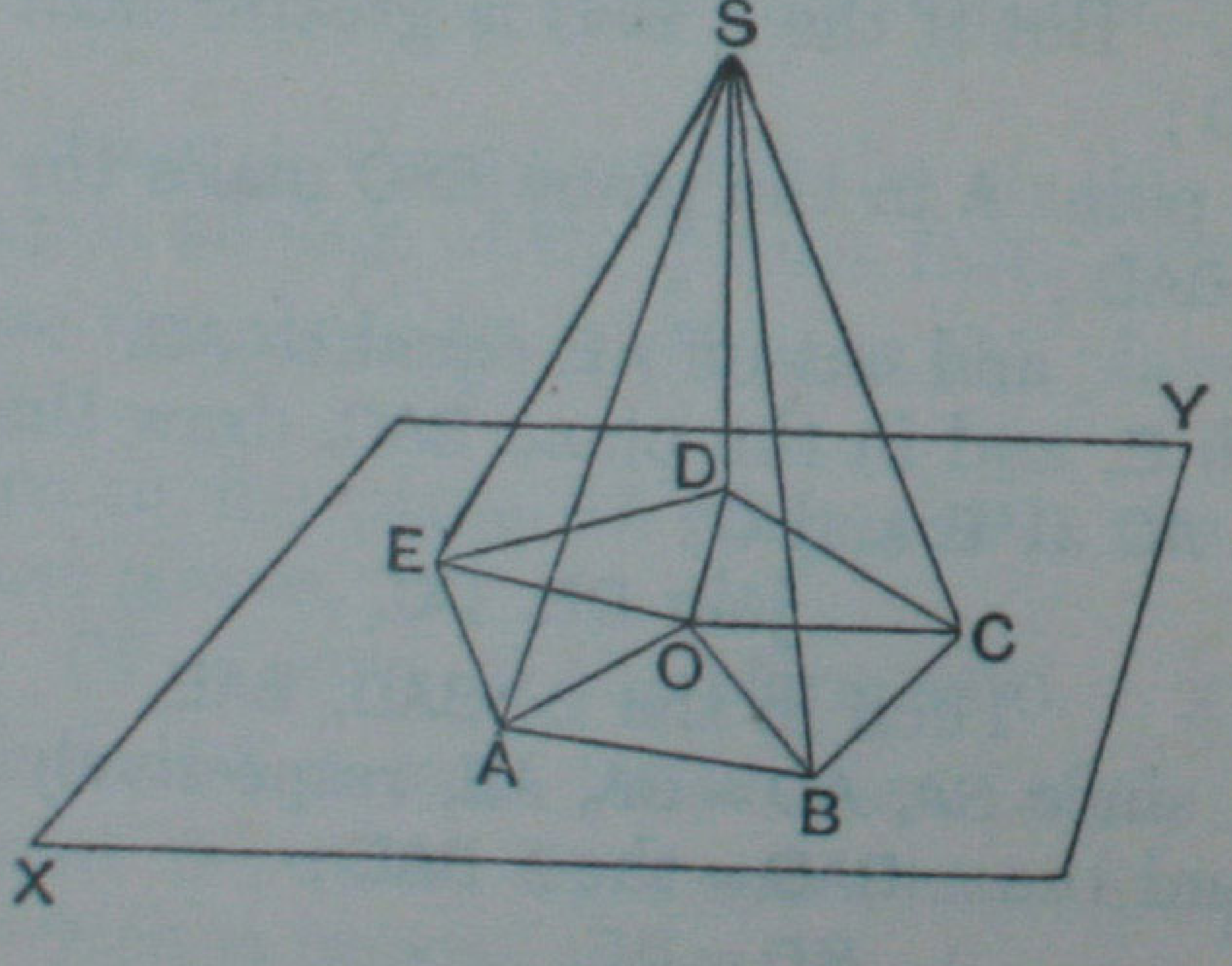



And in the $\triangle^s$ DAC, EAC,
because DA, AC = EA, AC respectively, *Constr.*
but DC is greater than EC; *Proved.*
∴ the ∠ DAC is greater than the ∠ EAC. I. 25.
But the ∠ BAD = the ∠ BAE; *Constr.*
∴ the two ∠ˢ BAD, DAC are together greater than the ∠ BAC.

Q.E.D.

## PROPOSITION 21. THEOREM.

*Every (convex) solid angle is formed by plane angles which are together less than four right angles.*




Let the solid angle at S be formed by the plane ∠ˢ ASB, BSC, CSD, DSE, ESA.

*Then shall the sum of these plane angles be less than four rt. angles.*

For let a plane XY intersect all the arms of the plane angles on the same side of the vertex at the points A, B, C, D, E: and let AB, BC, CD, DE, EA be the common sections of the plane XY with the planes of the several angles.

Within the polygon ABCDE take any point O;
and join O to each of the vertices of the polygon.

Then since the ∠s SAE, SAB, EAB form the trihedral angle A,
∴ the ∠s SAE, SAB are together greater than the ∠ EAB; XI. 20.

that is,
the ∠s SAE, SAB are together greater than the ∠s OAE, OAB.

Similarly,
the ∠s SBA, SBC are together greater than the ∠s OBA, OBC:
and so on, for each of the angular points of the polygon.

Thus by addition,
the sum of the base angles of the triangles whose vertices are at S, is greater than the sum of the base angles of the triangles whose vertices are at O.

But these two systems of triangles are equal in number;
∴ the sum of all the angles of the one system is equal to the sum of all the angles of the other.

It follows that the sum of the vertical angles at S *is less* than the sum of the vertical angles at O.

But the sum of the angles at O is four rt. angles;
∴ the sum of the angles at S is less than four rt. angles.
Q.E.D.

NOTE. This proposition was not given in this form by Euclid, who established its truth only in the case of *trihedral* angles. The above demonstration, however, applies to all cases in which the polygon ABCDE is *convex*, but it must be observed that without this condition the proposition is not necessarily true.

A solid angle is **convex** when it lies entirely on one side of each of the infinite planes which pass through its plane angles. If this is the case, the polygon ABCDE will have no *re-entrant* angle. And it is clear that it would not be possible to apply XI. 20 to a vertex at which a re-entrant angle existed.

## Exercises on Book XI.

1. Equal straight lines drawn to a plane from a point without it have equal projections on that plane.

2. If S is the centre of the circle circumscribed about the triangle ABC, and if SP is drawn perpendicular to the plane of the triangle, shew that any point in SP is equidistant from the vertices of the triangle.

3. Find the locus of points in space equidistant from three given points.

4. From Example 2 deduce a practical method of drawing a perpendicular from a given point to a plane, having given ruler, compasses, and a straight rod longer than the required perpendicular.

5. Give a geometrical construction for drawing a straight line equally inclined to three straight lines which meet in a point, but are not in the same plane.

6. In a *gauche* quadrilateral (that is, a quadrilateral whose sides are not in the same plane) if the middle points of adjacent sides are joined, the figure thus formed is a parallelogram.

7. AB and AC are two straight lines intersecting at right angles, and from B a perpendicular BD is drawn to the plane in which they are: shew that AD is perpendicular to AC.

8. If two intersecting planes are cut by two parallel planes, the lines of section of the first pair with each of the second pair contain equal angles.

9. If a straight line is parallel to a plane, shew that any plane passing through the given straight line will intersect the given plane in a line of section which is parallel to the given line.

10. Two intersecting planes pass one through each of two parallel straight lines; shew that the common section of the planes is parallel to the given lines.

11. If a straight line is parallel to each of two intersecting planes, it is also parallel to the common section of the planes.

12. Through a given point in space draw a straight line to intersect each of two given straight lines which are not in the same plane.

13. If AB, BC, CD are straight lines not all in one plane, shew that a plane which passes through the middle point of each one of them is parallel both to AC and BD.

14. From a given point A a perpendicular AB is drawn to a plane XY; and a second perpendicular AE is drawn to a straight line CD in the plane XY: shew that EB is perpendicular to CD.

15. From a point A two perpendiculars AP, AQ are drawn one to each of two intersecting planes: shew that the common section of these planes is perpendicular to the plane of AP, AQ.

16. From A, a point in one of two given intersecting planes, AP is drawn perpendicular to the first plane, and AQ perpendicular to the second: if these perpendiculars meet the second plane at P and Q, shew that PQ is perpendicular to the common section of the two planes.

17. A, B, C, D are four points not in one plane, shew that the four angles of the *gauche* quadrilateral ABCD [see Ex. 6, p. 444] are together less than four right angles.

18. OA, OB, OC are three straight lines drawn from a given point O not in the same plane, and OX is another straight line within the solid angle formed by OA, OB, OC: shew that

(i) the sum of the angles AOX, BOX, COX is greater than half the sum of the angles AOB, BOC, COA.

(ii) the sum of the angles AOX, COX is less than the sum of the angles AOB, COB.

(iii) the sum of the angles AOX, BOX, COX is less than the sum of the angles AOB, BOC, COA.

19. OA, OB, OC are three straight lines forming a solid angle at O, and OX bisects the plane angle AOB; shew that the angle XOC is less than half the sum of the angles AOC, BOC.

20. If a point is equidistant from the angles of a right-angled triangle and not in the plane of the triangle, the line joining it with the middle point of the hypotenuse is perpendicular to the plane of the triangle.

21. The angle which a straight line makes with its projection on a plane is less than that which it makes with any other straight line which meets it in that plane.

22. Find a point in a given plane such that the sum of its distances from two given points (not in the plane but on the same side of it) may be a minimum.

23. If two straight lines in one plane are equally inclined to another plane, they will be equally inclined to the common section of these planes.

24. PA, PB, PC are three concurrent straight lines, each of which is at right angles to the other two: PX, PY, PZ are perpendiculars drawn from P to BC, CA, AB respectively. Shew that XYZ is the pedal triangle of the triangle ABC.

25. PA, PB, PC are three concurrent straight lines, each of which is at right angles to the other two, and from P a perpendicular PO is drawn to the plane of ABC: shew that O is the orthocentre of the triangle ABC.

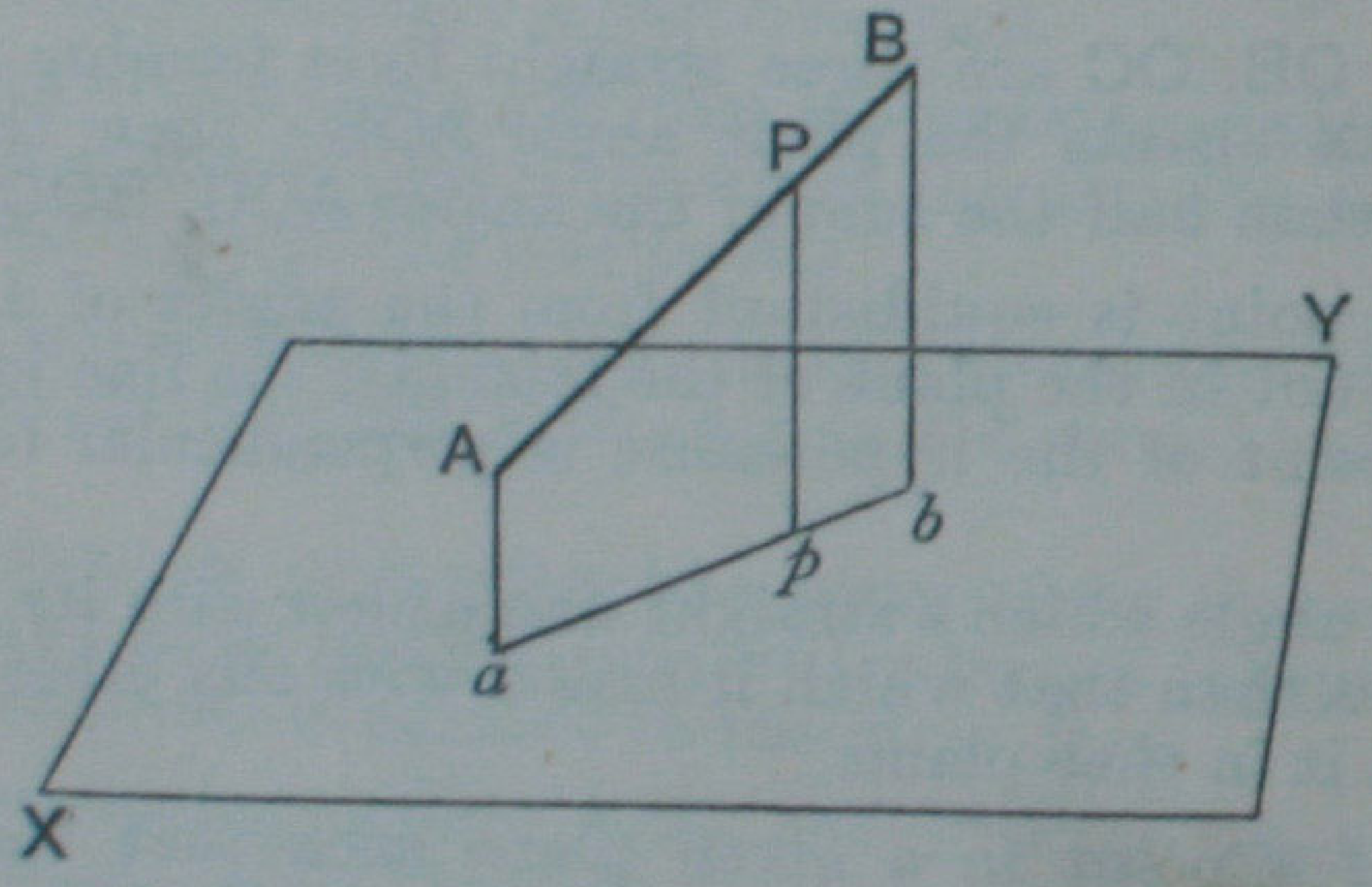
# THEOREMS AND EXAMPLES ON BOOK XI.

## DEFINITIONS.

(i) Lines which are drawn on a plane, or through which a plane may be made to pass, are said to be **co-planar.**

(ii) The **projection of a line** on a plane is the locus of the feet of perpendiculars drawn from all points in the given line to the plane.

THEOREM 1. *The projection of a straight line on a plane is itself a straight line.*




Let AB be the given st. line, and XY the given plane.

From P, *any* point in AB, draw P$p$ perp. to the plane XY.

*It is required to shew that the locus of* p *is a st. line.*

From A and B draw A$a$, B$b$ perp. to the plane XY.

Now since A$a$, P$p$, B$b$ are all perp. to the plane XY,

∴ they are par[l]. XI. 6.

And since these par[ls] all intersect AB,

∴ they are co-planar. XI. 7.

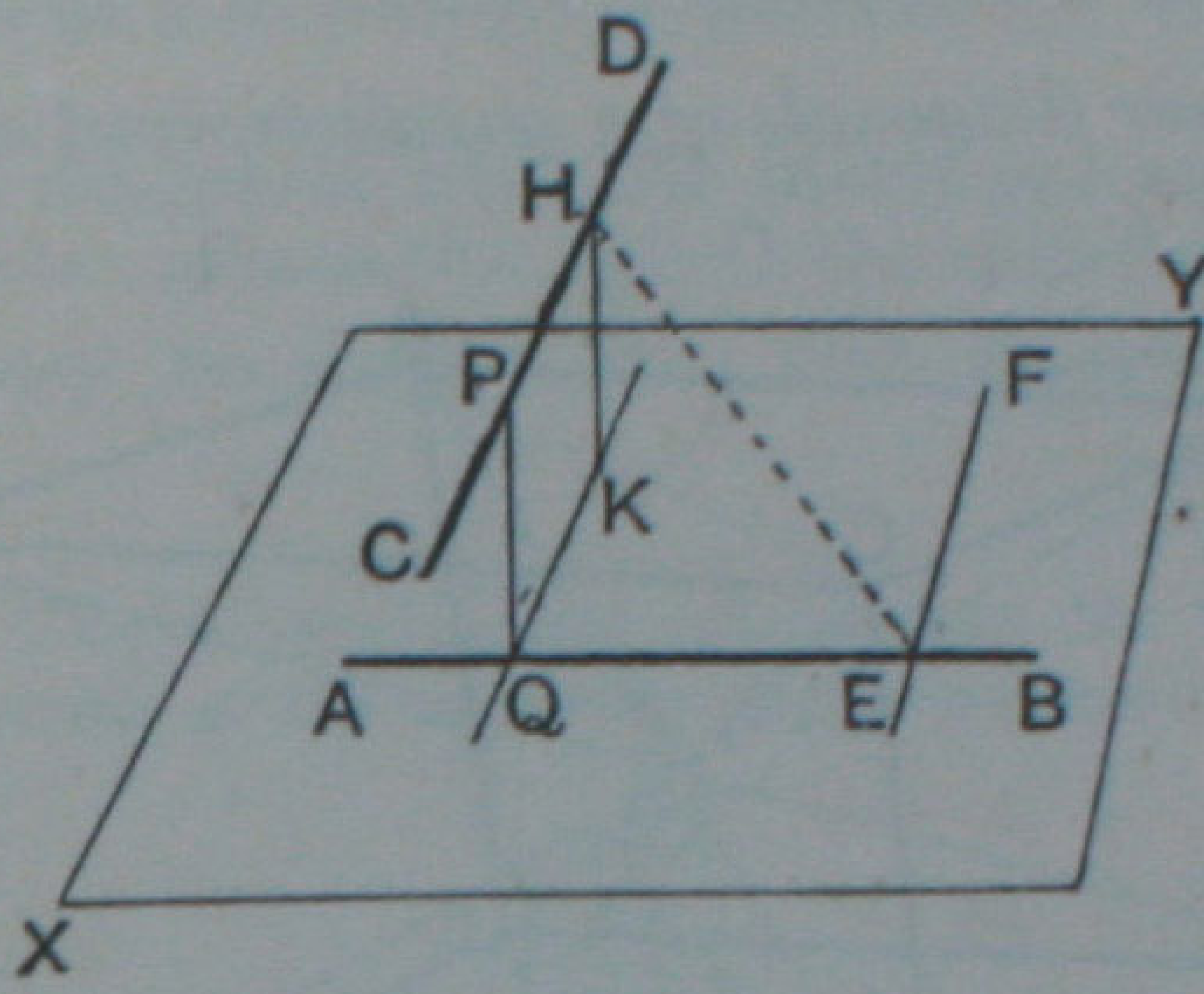
∴ the point $p$ is in the common section of the planes A$b$, XY;

that is, $p$ is in the st. line $ab$.

But $p$ is *any* point in the projection of AB,

∴ the projection of AB is the st. line $ab$. Q.E.D.

THEOREM 2. *Draw a perpendicular to each of two straight lines which are not in the same plane. Prove that this perpendicular is the shortest distance between the two lines.*




Let AB and CD be the two straight lines, not in the same plane.

(i) *It is required to draw a st. line perp. to each of them.*

Through E, any point in AB, draw EF par[l] to CD.

Let XY be the plane which passes through AB, EF.

From H, any point in CD, draw HK perp. to the plane XY. XI. 11.

And through K, draw KQ par[l] to EF, cutting AB at Q.

Then KQ is also par[l] to CD; XI. 9.

and CD, HK, KQ are in one plane. XI. 7.

From Q, draw QP par[l] to HK to meet CD at P.

Then shall PQ be perp. to both AB and CD.

For, since HK is perp. to the plane XY, and PQ is par[l] to HK, *Constr.*

∴ PQ is perp. to the plane XY; XI. 8.

∴ PQ is perp. to AB, which meets it in that plane. XI. *Def.* 1.

For a similar reason PQ is perp. to QK,

∴ PQ is also perp. to CD, which is par[l] to QK.

(ii) *It is required to shew that* PQ *is the least of all st. lines drawn from* AB *to* CD.

Take HE, any other st. line drawn from AB to CD.

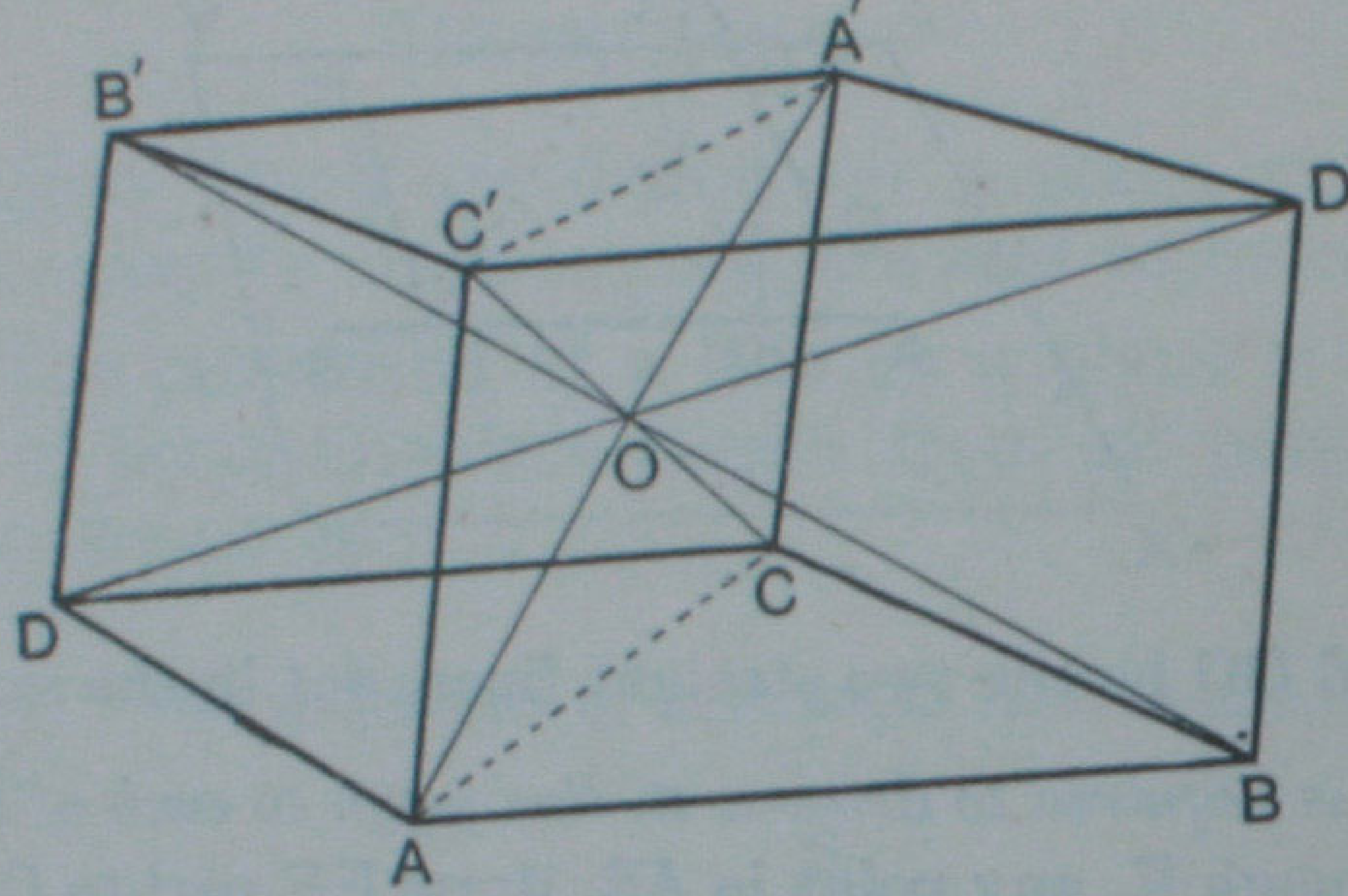
Then HE, being oblique to the plane XY, is greater than the perp. HK. Ex. 1, p. 429.

∴ HE is also greater than PQ. Q.E.D.

DEFINITION. A parallelepiped is a solid figure bounded by three pairs of parallel plane faces.

THEOREM 3. (i) *The faces of a parallelepiped are parallelograms, of which those which are opposite are identically equal.*

(ii) *The four diagonals of a parallelepiped are concurrent and bisect one another.*




Let ABA'B' be a par$^{ped}$, of which ABCD, C'D'A'B' are opposite faces.

(i) *Then all the aces shall be par$^{ms}$, and the opposite faces shall be identically equal.*

For since the planes DA', AD' are par$^{l}$, XI. *Def.* 15.
and the plane DB meets them,
∴ the common sections AB and DC are par$^{l}$. XI. 16.
Similarly AD and BC are par$^{l}$.
∴ the fig. ABCD is a par$^{m}$,
and AB = DC; also AD = BC. I. 34.

Similarly each of the faces of the par$^{ped}$ is a par$^{m}$;
so that the edges AB, C'D', B'A', DC are equal and par$^{l}$:
so also are the edges AD, C'B', D'A', BC; and likewise AC', BD', CA', DB'.

Then in the opp. faces ABCD, C'D'A'B',
we have AB = C'D' and BC = D'A'; *Proved.*
and since AB, BC are respectively par$^{l}$ to C'D', D'A',
∴ the ∠ ABC = the ∠ C'D'A'; XI. 10.
∴ the par$^{m}$ ABCD = the par$^{m}$ C'D'A'B' identically. Ex. 11, p. 70.

(ii) *The diagonals* AA', BB', CC', DD' *shall be concurrent and bisect one another.*

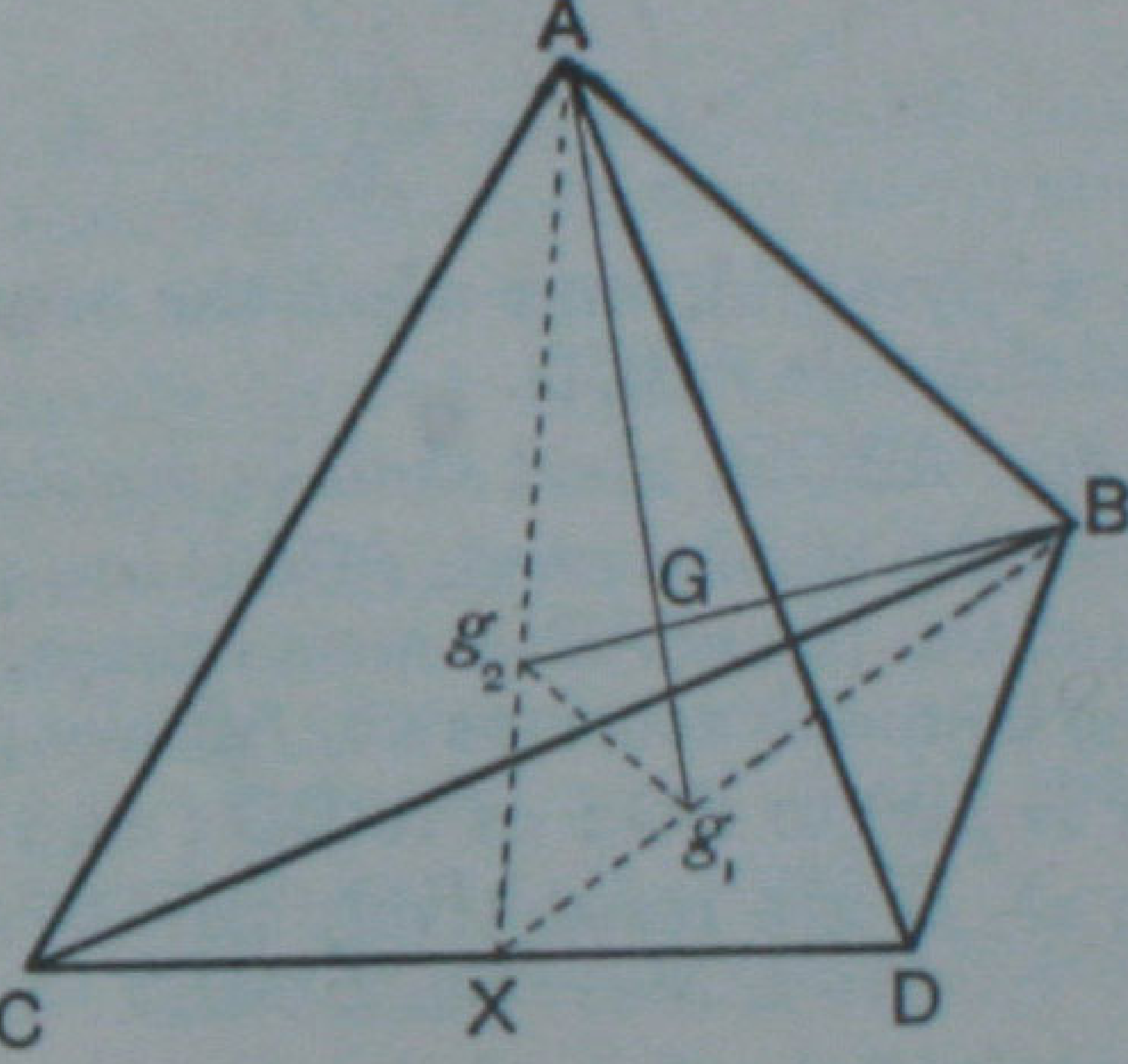
Join AC and A'C'.

Then since AC′ is equal and par[l] to A′C,
∴ the fig. ACA′C′ is a par[m];
∴ its diagonals AA′, CC′ bisect one another. Ex. 5, p. 70.
That is, AA′ passes through O, the middle point of CC′.

Similarly if BC′ and B′C were joined, the fig. BCB′C′ would be a par[m];
∴ the diagonals BB′, CC′ bisect one another.
That is, BB′ also passes through O the middle point of CC′.

Similarly it may be shewn that DD′ passes through, and is bisected at, O. Q.E.D.

THEOREM 4. *The straight lines which join the vertices of a tetrahedron to the centroids of the opposite faces are concurrent.*




Let ABCD be a tetrahedron, and let $g_1, g_2, g_3, g_4$ be the centroids of the faces opposite respectively to A, B, C, D.

*Then shall* $Ag_1$, $Bg_2$, $Cg_3$, $Dg_4$ *be concurrent.*

Take X the middle point of the edge CD;
then $g_1$ and $g_2$ must lie respectively in BX and AX,
so that $BX = 3 . Xg_1$, Ex. 4, p. 113.
and $AX = 3 . Xg_2$;
∴ $g_1g_2$ is par[l] to AB.

And $Ag_1$, $Bg_2$ must intersect one another, since they are both in the plane of the △ AXB:
let them intersect at the point G.

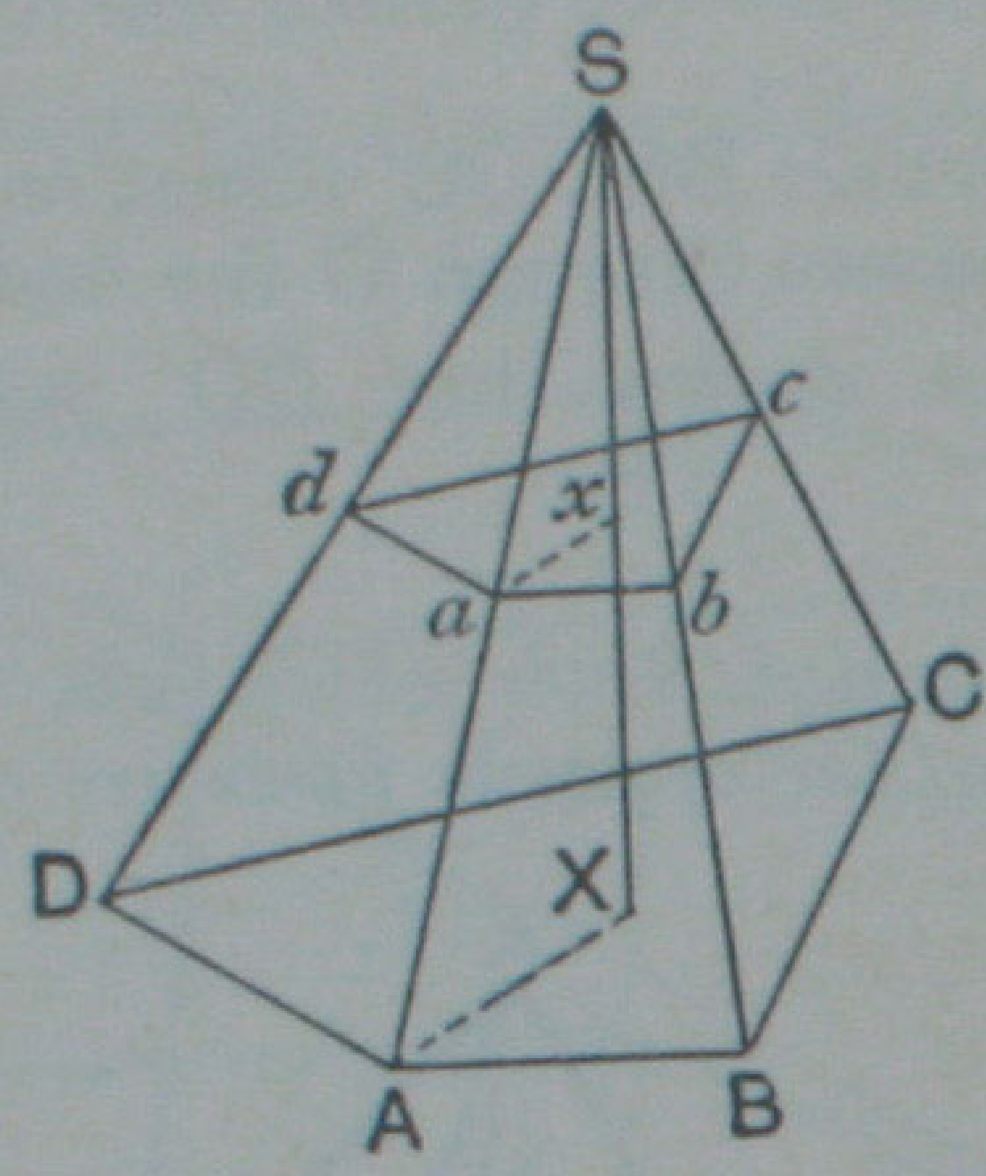
Then by similar △[s], $AG : Gg_1 = AB : g_1g_2$
$= AX : Xg_2$
$= 3 : 1.$

∴ $Bg_2$ cuts $Ag_1$ at a point G whose distance from $g_1 = \frac{1}{4} . Ag_1$.

Similarly it may be shewn that $Cg_3$ and $Dg_4$ cut $Ag_1$ at the same point;
∴ these lines are concurrent. Q.E.D.

THEOREM 5. (i) *If a pyramid is cut by planes drawn parallel to its base, the sections are similar to the base.*

(ii) *The areas of such sections are in the duplicate ratio of their perpendicular distances from the vertex.*




Let SABCD be a pyramid, and *abcd* the section formed by a plane drawn par[l] to the base ABCD.

(i) *Then the figs.* ABCD, abcd *shall be similar.*

Because the planes *abcd*, ABCD are par[l],
and the plane AB*ba* meets them,
∴ the common sections *ab*, AB are par[l].

Similarly *bc* is par[l] to BC; *cd* to CD; and *da* to DA.

And since *ab*, *bc* are respectively par[l] to AB, BC,
∴ the ∠ *abc*=the ∠ ABC. XI. 10.

Similarly the remaining angles of the fig. *abcd* are equal to the corresponding angles of the fig. ABCD.

And since the △s S*ab*, SAB are similar,
∴ *ab* : AB=S*b* : SB
= *bc* : BC, for the △s S*bc*, SBC are similar.

Or, *ab* : *bc*=AB : BC.
In like manner, *bc* : *cd*=BC : CD; and so on.

∴ the figs. *abcd*, ABCD are equiangular to one another, and have their sides about the equal angles proportional;
∴ they are similar.

(ii) From S draw S*x*X perp. to the par[l] planes *abcd*, ABCD and meeting them at *x* and X.

*Then shall fig.* abcd : *fig.* ABCD=Sx² : SX².

Join *ax*, AX.

Then it is clear that the △s S*ax*, SAX are similar.

And the fig. *abcd* : fig. ABCD $= ab^2 : AB^2$ VI. 20.
$= aS^2 : AS^2$,
$= Sx^2 : SX^2$. Q.E.D.

DEFINITION. A polyhedron is *regular* when its faces are similar and equal regular polygons.

THEOREM 6. *There cannot be more than five regular polyhedra.*

This is proved by examining the number of ways in which it is possible to form a solid angle out of the plane angles of various regular polygons; bearing in mind that *three* plane angles at least are required to form a solid angle, and the sum the plane angles forming a solid angle *is less than four right angles.* XI. 21.

Suppose the faces of the regular polyhedron to be *equilateral triangles.*

Then since each angle of an equilateral triangle is $\frac{2}{3}$ of a right angle, it follows that a solid angle may be formed (i) by *three*, (ii) by *four*, or (iii) by *five* such faces; for the sums of the plane angles would be respectively (i) two right angles, (ii) $\frac{8}{3}$ of a right angle, (iii) $\frac{10}{3}$ of a right angle;
that is, in all three cases the sum of the plane angles would be less than four right angles.

But it is impossible to form a solid angle of *six* or more equilateral triangles, for then the sum of the plane angles would be equal to, or greater than four right angles.

Again, suppose that the faces of the polyhedron are *squares.*

(iv) Then it is clear that a solid angle could be formed of *three*, but not more than three, of such faces.

Lastly, suppose the faces are *regular pentagons.*

(v) Then, since each angle of a regular pentagon is $\frac{6}{5}$ of a right angle, it follows that a solid angle may be formed of *three* such faces; but the sum of more than three angles of a regular pentagon is greater than four right angles.

Further, since each angle of a *regular hexagon* is equal to $\frac{4}{3}$ of a right angle, it follows that no solid angle could be formed of such faces; for the sum of three angles of a hexagon is equal to four right angles.

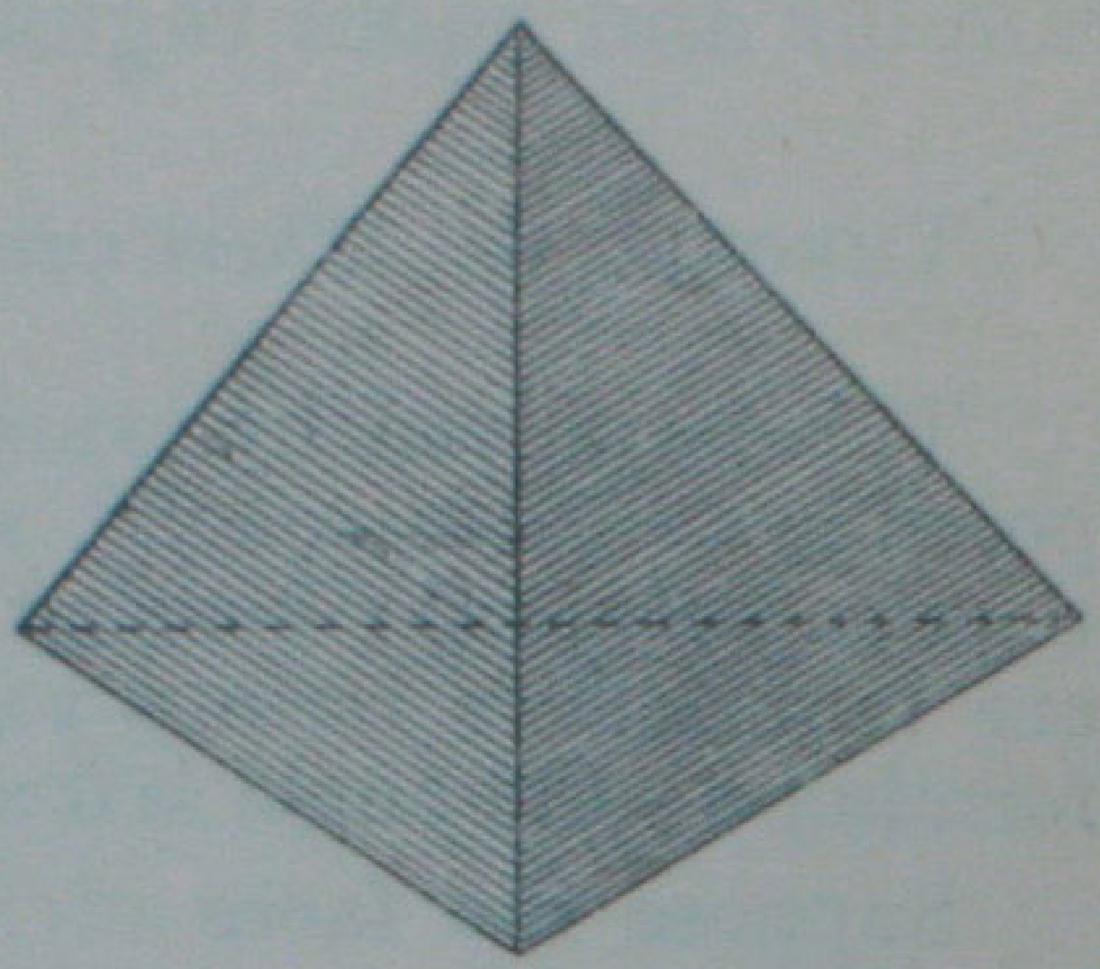
Similarly, no solid angle can be formed of the angles of a polygon of more sides than six.

Thus there can be no more than *five* regular polyhedra.

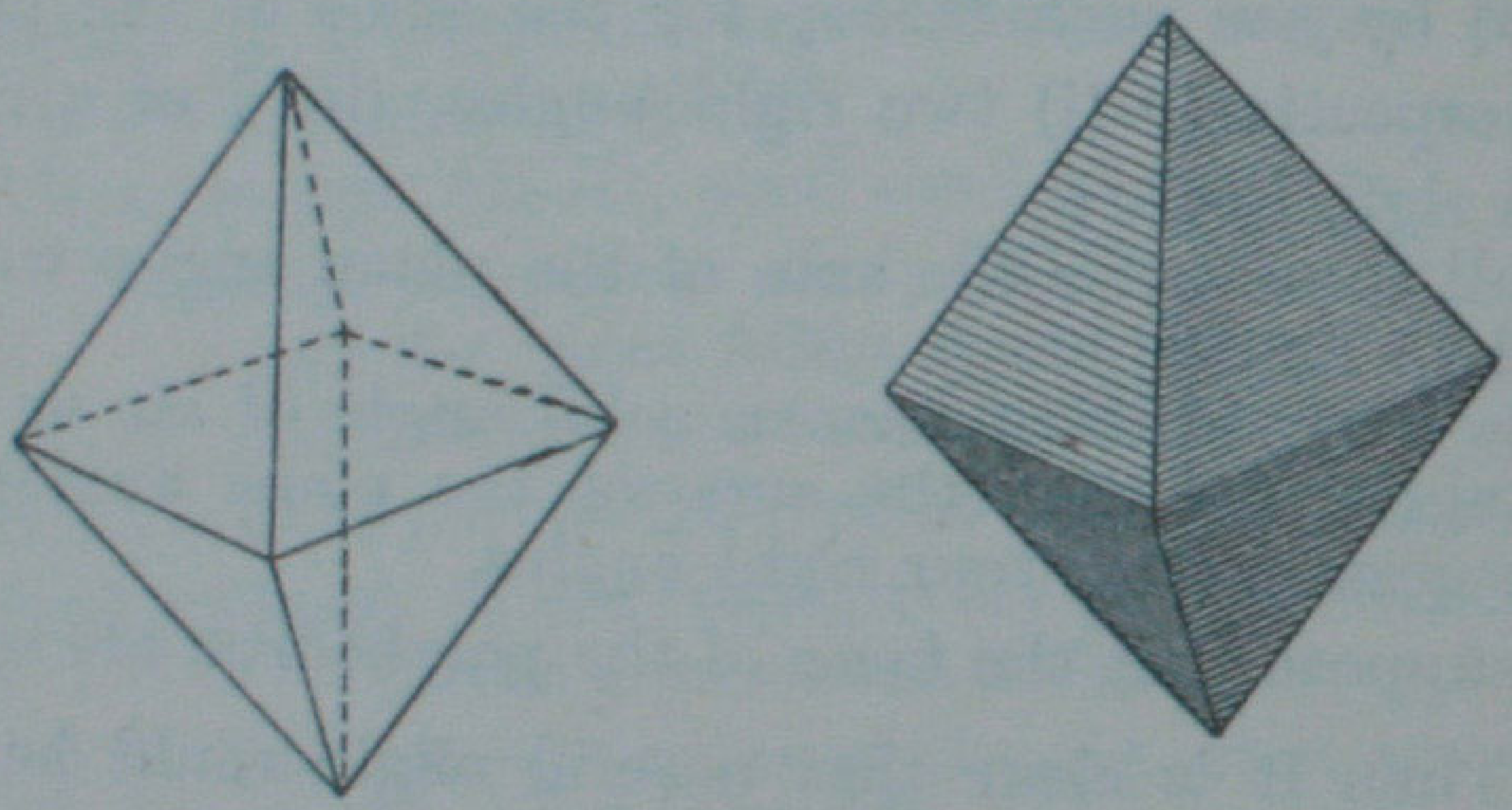
## NOTE ON THE REGULAR POLYHEDRA.

(i) The polyhedron of which each solid angle is formed by *three equilateral triangles* is called a regular **tetrahedron**.

It has *four* faces,
*four* vertices,
*six* edges.

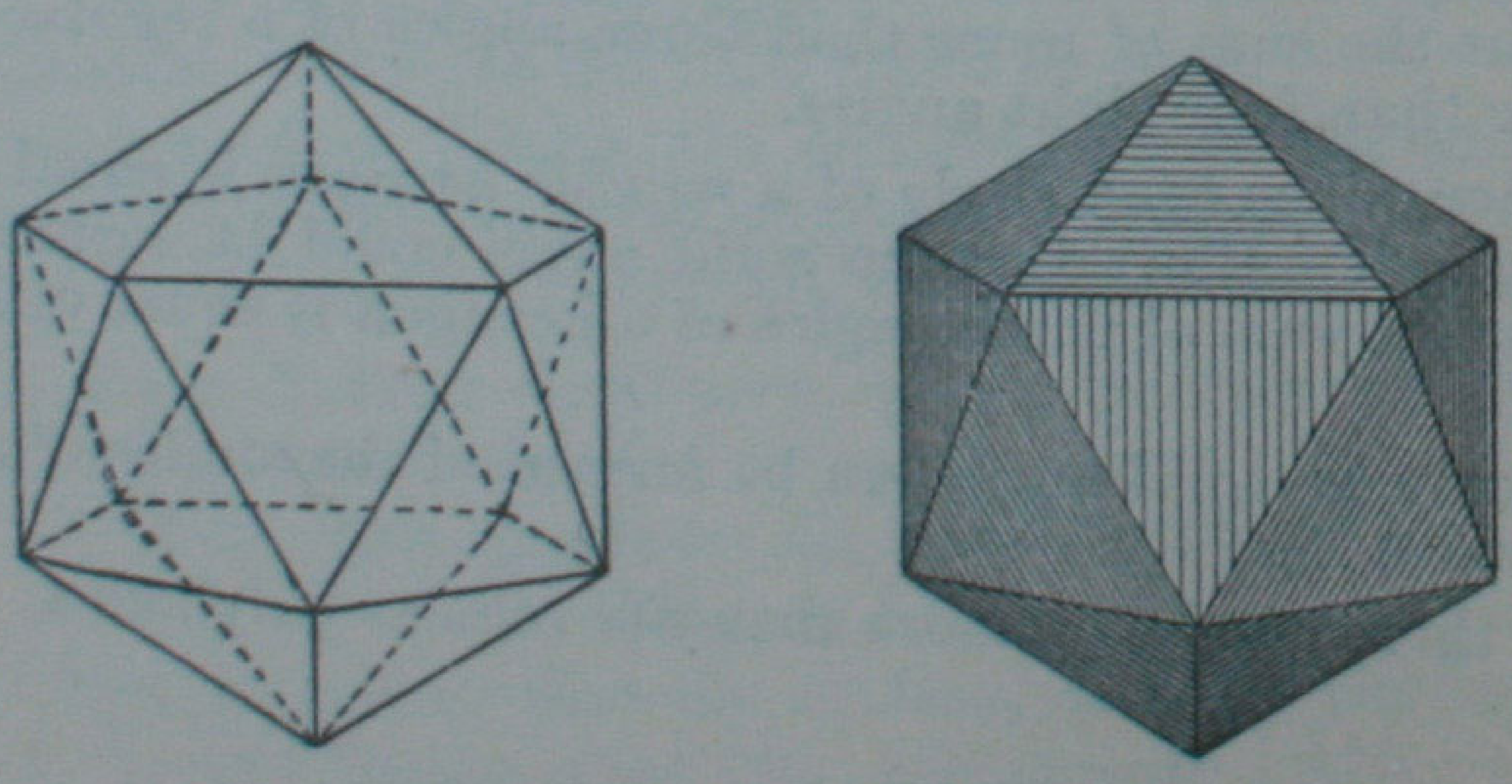


(ii) The polyhedron of which each solid angle is formed by *four equilateral triangles* is called a regular **octahedron**.



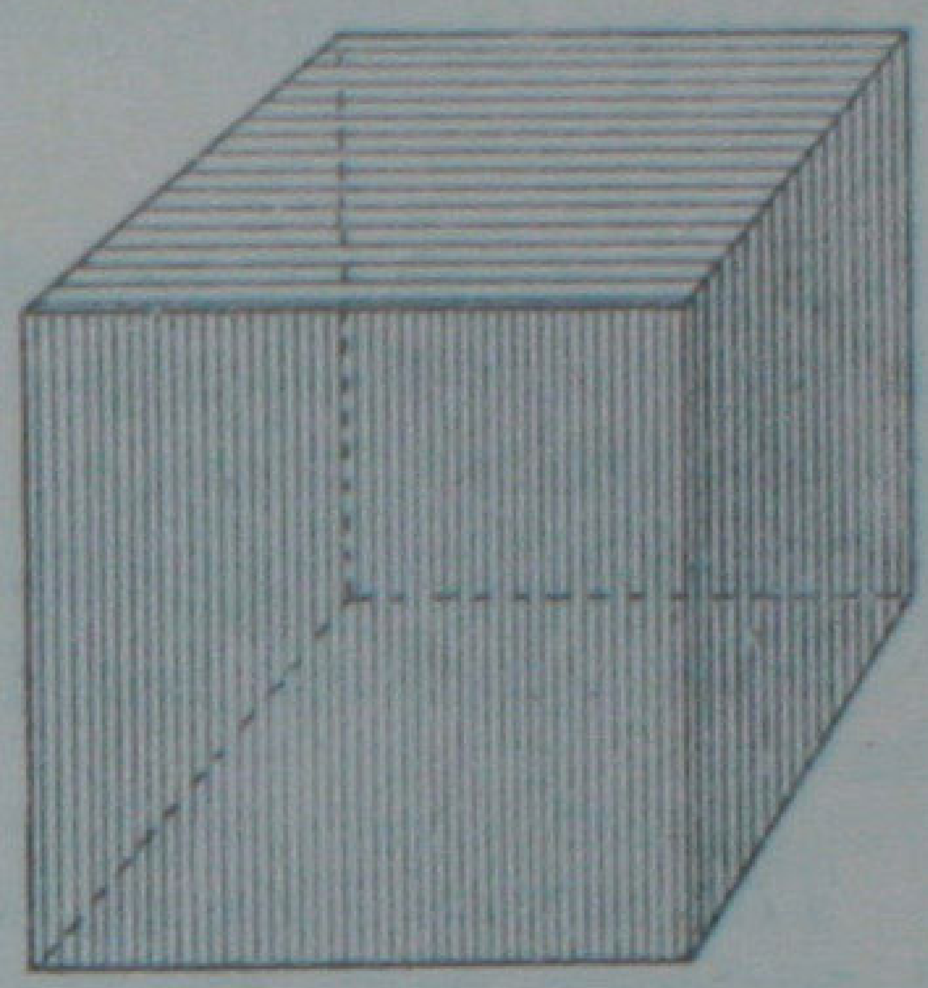
It has *eight* faces, *six* vertices, *twelve* edges.

(iii) The polyhedron of which each solid angle is formed by *five equilateral triangles* is called a regular **icosahedron**.



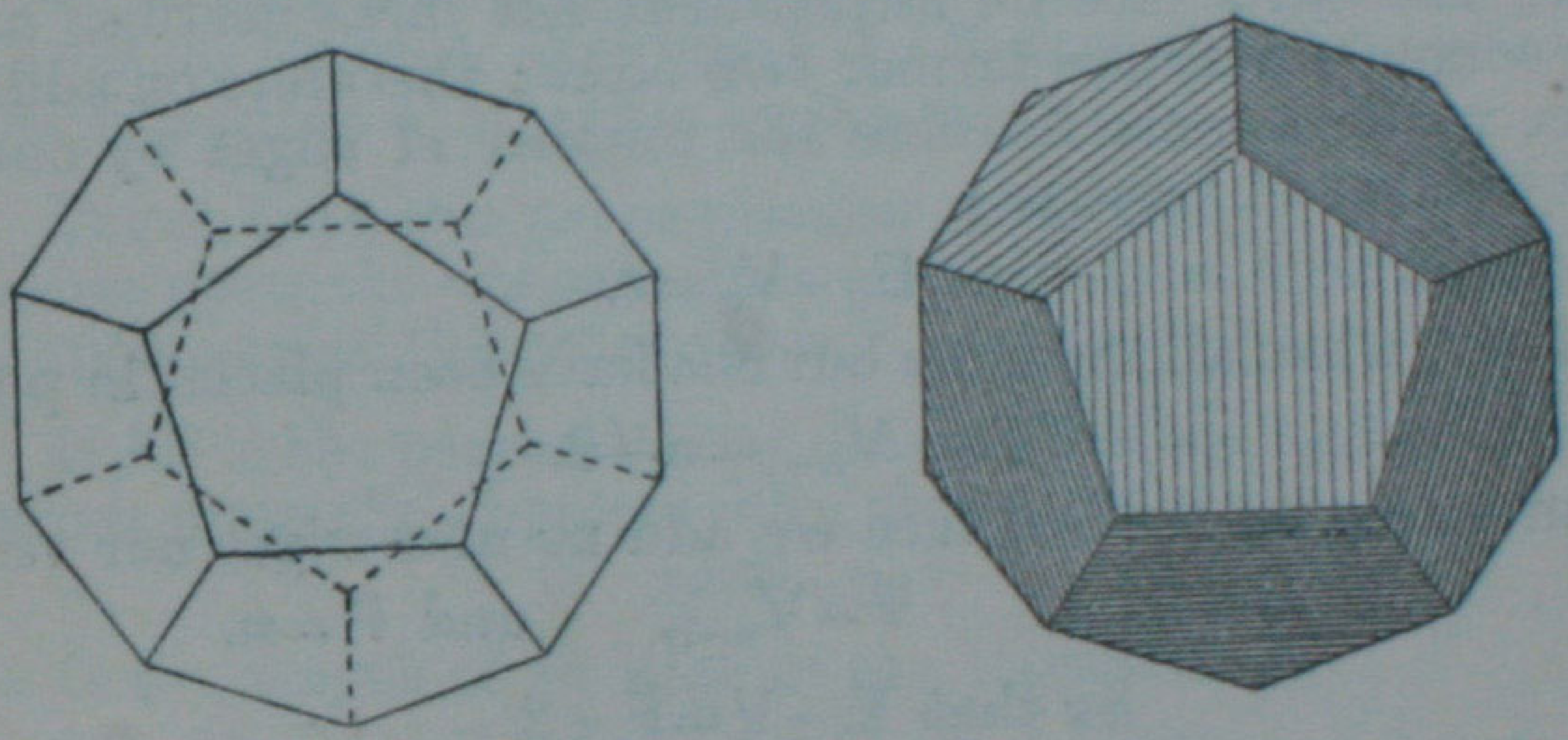
It has *twenty* faces, *twelve* vertices, *thirty* edges.

(iv) The regular polyhedron of which each solid angle is formed by *three squares* is called a **cube**.



It has *six* faces,
*eight* vertices,
*twelve* edges.

(v) The polyhedron of which each solid angle is formed by *three regular pentagons* is called a regular **dodecahedron**.



It has *twelve* faces, *twenty* vertices, *thirty* edges.

THEOREM 7. *If* F *denote the number of faces,* E *of edges, and* V *of vertices in any polyhedron, then will*

$$E + 2 = F + V.$$

Suppose the polyhedron to be formed by fitting together the faces in succession: suppose also that $E_r$ denotes the number of edges, and $V_r$ of vertices, when $r$ faces have been placed in position, and that the polyhedron has $n$ faces when complete.

Now when *one* face is taken there are as many vertices as edges, that is,

$$E_1 = V_1.$$

The *second* face on being adjusted has *two* vertices and *one* edge in common with the first; therefore by adding the second face we increase the number of edges by one more than the number of vertices;

$$\therefore\ E_2 - V_2 = 1.$$

Again, the *third* face on adjustment has *three* vertices and *two* edges in common with the former two faces; therefore on adding the third face we once more increase the number of edges by one more than the number of vertices;

$$\therefore\ E_3 - V_3 = 2.$$

Similarly, when all the faces but one have been placed in position,

$$E_{n-1} - V_{n-1} = n - 2.$$

But in fitting on the last face we add no new edges nor vertices;

$$\therefore\ E = E_{n-1}, \qquad V = V_{n-1}, \qquad \text{and } F = n.$$

So that $E - V = F - 2$,

or, $E + 2 = F + V$.

This is known as *Euler's Theorem.*

## MISCELLANEOUS EXAMPLES ON SOLID GEOMETRY.

1. The projections of parallel straight lines on any plane are parallel.

2. If $ab$ and $cd$ are the projections of two parallel straight lines AB, CD on any plane, shew that $AB : CD = ab : cd$.

3. Draw two parallel planes one through each of two straight lines which do not intersect and are not parallel.

4. If two straight lines do not intersect and are not parallel, on what planes will their projections be parallel?

5. Find the locus of the middle point of a straight line of constant length whose extremities lie one on each of two non-intersecting straight lines.

6. Three points A, B, C are taken one on each of the conterminous edges of a cube: prove that the angles of the triangle ABC are all acute.

7. If a parallelepiped is cut by a plane which intersects two pairs of opposite faces, the common sections form a parallelogram.

8. The square on the diagonal of a rectangular parallelepiped is equal to the sum of the squares on the three edges conterminous with the diagonal.

9. The square on the diagonal of a cube is three times the square on one of its edges.

10. The sum of the squares on the four diagonals of a parallelepiped is equal to the sum of the squares on the twelve edges.

11. If a perpendicular is drawn from a vertex of a regular tetrahedron on its base, shew that the foot of the perpendicular will divide each median of the base in the ratio 2 : 1.

12. Prove that the perpendicular from the vertex of a regular tetrahedron upon the opposite face is three times that dropped from its foot upon any of the other faces.

13. If AP is the perpendicular drawn from the vertex of a regular tetrahedron upon the opposite face, shew that

$$3AP^2 = 2a^2,$$

where $a$ is the length of an edge of the tetrahedron.

14. The straight lines which join the middle points of opposite edges of a tetrahedron are concurrent.

15. If a tetrahedron is cut by any plane parallel to two opposite edges, the section will be a parallelogram.

16. Prove that the shortest distance between two opposite edges of a regular tetrahedron is one half of the diagonal of the square on an edge.

17. In a tetrahedron if two pairs of opposite edges are at right angles, then the third pair will also be at right angles.

18. In a tetrahedron whose opposite edges are at right angles in pairs, the four perpendiculars drawn from the vertices to the opposite faces and the three shortest distances between opposite edges are concurrent.

19. In a tetrahedron whose opposite edges are at right angles, the sum of the squares on each pair of opposite edges is the same.

20. The sum of the squares on the edges of any tetrahedron is four times the sum of the squares on the straight lines which join the middle points of opposite edges.

21. In any tetrahedron the plane which bisects a dihedral angle divides the opposite edge into segments which are proportional to the areas of the faces meeting at that edge.

22. If the angles at one vertex of a tetrahedron are all right angles, and the opposite face is equilateral, shew that the sum of the perpendiculars dropped from any point in this face upon the other three faces is constant.

23. Shew that the polygons formed by cutting a prism by parallel planes are equal.

24. Three straight lines in space OA, OB, OC, are mutually at right angles, and their lengths are $a$, $b$, $c$: express the area of the triangle ABC in its simplest form.

25. Find the diagonal of a regular octahedron in terms of one of its edges.

26. Shew how to cut a cube by a plane so that the lines of section may form a regular hexagon.

27. Shew that every section of a sphere by a plane is a circle.

28. Find in terms of the length of an edge the radius of a sphere inscribed in a regular tetrahedron.

29. Find the locus of points in a given plane at which a straight line of fixed length and position subtends a right angle.

30. A fixed point O is joined to any point P in a given plane which does not contain O; on OP a point Q is taken such that the rectangle OP, OQ is constant: shew that Q lies on a fixed sphere.

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1. A mathematica é a sciencia das relações [entre as dimensões das cousas]

2. A geometria é aquella parte da mathematica que trata das relações das cousas consideradas como existindo no espaço.

3.

---

Def. Sciencia.

— Sc. do concreto, do abstracto, et mixtae.

abstractas: mathematica, {Algebra. arithmetica. Geometria.}

ab-conc: mecanica

concreto: physico-chimica.

---

A sciencia trata das relações [constantes] entre as cousas

uma lei é uma relação constante (ou tida por constante) entre as cousas.

Abstracta é toda a sc.ª que inclue elementos ~~todo~~ que não são da experiencia real e directa —

A 1ª sc. concreta (é a mechanica).

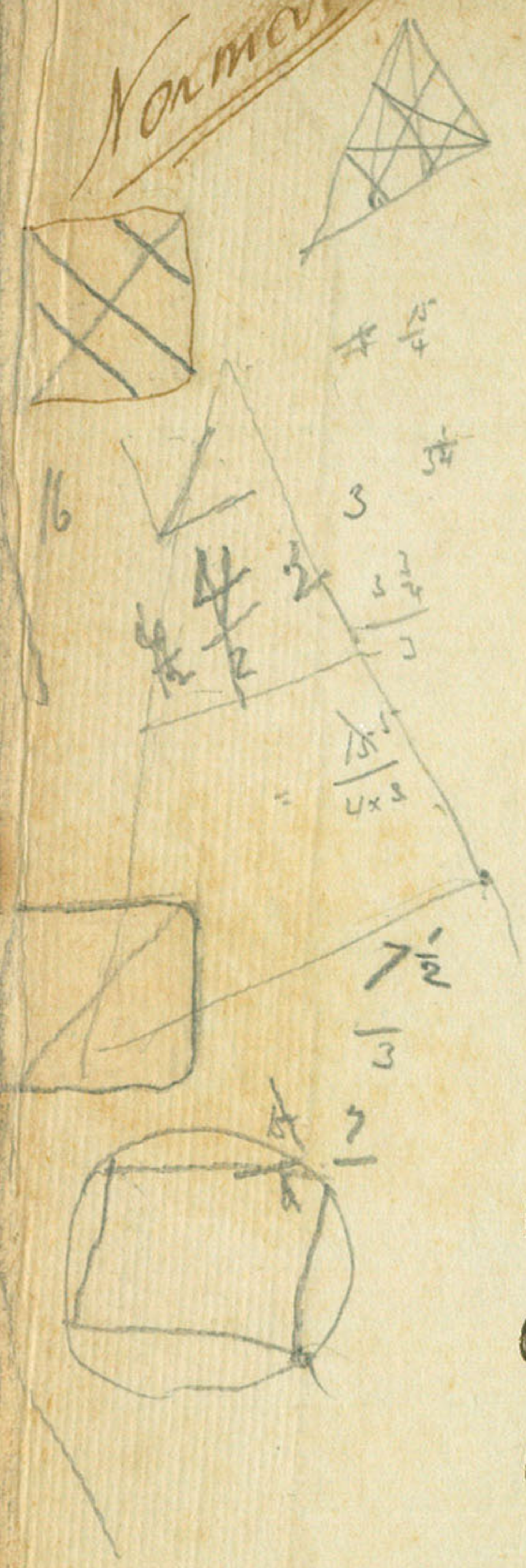
~~As~~ dimensões ~~de~~ ~~certo~~ Polar solido não ~~são~~ um solido ~~são~~ a expressão do solido quê causa visto (á vista). Polar 15 3¾ 90 1⅞

Quando 2 rectas se cortam de modo que os quatro angulos formados são [todos] eguaes, cada angulo d'esses chama-se um angulo recto; e diz-se de cada uma das rectas, que forma ~~esse~~ cada angulo d'esses, que é perpendicular á outra.

Como a geometria é uma sciencia abstracta, temos, para a tornar comprehensivel, que mostrar em quê e como ella é abstracta.

Tome-se um ~~cubo~~ solido regular qualquer.

Normal





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Tudo quanto existe no espaço tem um volume, isto é, um modo de ser pelo qual é accessivel á vista,x quer á vista nua, quer á vista armada, isto é, ajudada de microscopio ou telescopio.

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