Proposition 14.

Ex Æquali. *If there are two sets of magnitudes, such that the first is to the second of the first set as the first to the second of the other set, and the second to the third of the first set as the second to the third of the other, and so on to the last magnitude: then the first is to the last of the first set as the first to the last of the other.*

First, let there be three magnitudes A, B, C of one set, and three, P, Q, R, of another set,

and let $A : B :: P : Q$,
and $B : C :: Q : R$;
then shall $A : C :: P : R$.

Because $A : B :: P : Q$,
$\therefore\ mA : mB :: mP : mQ$; v. 8, *Cor.*
and because $B : C :: Q : R$,
$\therefore\ mB : nC :: mQ : nR$, v. 9.
$\therefore$, *invertendo*, $nC : mB :: nR : mQ$. v. 3.

Now, if $mA > nC$,
then $mA : mB > nC : mB$; v. 5.
$\therefore\ mP : mQ > nR : mQ$,
and $\therefore\ mP > nR$. v. 7.

Similarly $mP =$ or $< nR$ according as $mA =$ or $< nC$.
$\therefore\ A : C :: P : R$. *Def.* 5.

Secondly, let there be any number of magnitudes, A, B, C, ... L, M, of one set, and the same number P, Q, R, ... Y, Z, of another set, such that

$A : B :: P : Q$,
$B : C :: Q : R$,
...... :: ......
$L : M :: Y : Z$;
then shall $A : M :: P : Z$.

For $A : C :: P : R$, *Proved.*
and $C : D :: R : S$; *Hyp.*
$\therefore$ by the first case $A : D :: P : S$,
and so on, until finally $A : M = P : Z$.

Corollary. If $A : B :: P : Q$,
and $B : C :: R : P$:
then $A : C :: R : Q$.

Proposition 15.

If $A : B :: X : Y$,
and $C : B :: Z : Y$;
then shall $A+C : B :: X+Z : Y$.

For since $C : B :: Z : Y$, *Hyp.*
∴, *invertendo*, $B : C :: Y : Z$. v. 3.
Also $A : B :: X : Y$,
∴, *ex æquali*, $A : C :: X : Z$, v. 14.
∴, *componendo*, $A+C : C :: X+Z : Z$. v. 13.
Again, $C : B :: Z : Y$, *Hyp.*
∴, *ex æquali*, $A+C : B :: X+Z : Y$. v. 14.

Proposition 16.

If two ratios are equal, their duplicate ratios are equal.

Let $A : B :: C : D$;
then shall the duplicate ratio of A to B be equal to that of C to D.

Let X be a third proportional to A and B, and Y a third proportional to C and D,

so that $A : B :: B : X$, and $C : D :: D : Y$;
then because $A : B :: C : D$,
∴ $B : X :: D : Y$;
∴, *ex æquali*, $A : X :: C : Y$.

But $A : X$ and $C : Y$ are respectively the duplicate ratios of
$A : B$ and $C : D$, *Def.* 13.
∴ the duplicate ratio of $A : B$ = that of $C : D$.

Note. The converse of this theorem may be readily proved; namely,

If the duplicates of two ratios are equal, the ratios themselves are equal.

ELEMENTARY PRINCIPLES OF PROPORTION.

INTRODUCTION TO BOOK VI.

1. The first four books of Euclid deal with the absolute equality or inequality of geometrical magnitudes. In Book VI. such magnitudes are compared by considering their *ratio* or *relative greatness.*

2. The meaning of the words *ratio* and *proportion* in their simplest arithmetical sense may be given as follows:

(i) *The ratio of one number to another is the multiple or fraction which the first is of the second.*

(ii) *Four numbers are in proportion when the ratio of the first to the second is equal to the ratio of the third to the fourth*

3. These definitions are however not strictly applicable to the purposes of Pure Geometry, for the following reasons:

(i) Pure Geometry deals only with magnitudes *as represented by diagrams,* without measuring them in terms of a common unit: in other words, it makes no use of *number* for the purpose of comparing magnitudes.

(ii) It commonly happens that Geometrical magnitudes of the same kind are *incommensurable,* that is, they are such that it is impossible to express them *exactly* in terms of some common unit. Nevertheless it is always possible to express the arithmetical ratio of two such magnitudes *within any required degree of accuracy.* [See Note, p. 131: also Hall and Knight's *Elementary Algebra,* Art. 289.]

4. Accordingly, the object of Euclid's Fifth Book is to establish the Theory of Proportion on a basis independent of *number*. But as Book V. is now very rarely read, we propose here merely to illustrate *algebraically* such principles of proportion as are required before proceeding to Book VI. The strict treatment of the subject given in Book V. may be studied at a later stage, if it is thought desirable.

Obs. In what follows the symbol $>$ will be used for the words *greater than*, and $<$ for *less than*.

5. The following definitions are selected from Book V.

Definition 1. One magnitude is said to be a **multiple** of another, when the first contains the second an *exact* number of times.

Thus ma is a multiple of a, if m is any whole number.

Definition 2. One magnitude is said to be a **submultiple** of another, when the first is contained in the second an *exact* number of times.

Thus $\frac{a}{m}$ is a submultiple of a, if m is any whole number.

Definition 3. The **ratio** of one magnitude to another *of the same kind* is the relation which the first bears to the second in regard to quantity; this is measured by the fraction which the first is of the second.

Thus if two such magnitudes contain a and b units respectively, the ratio of the first to the second is expressed by the fraction $\frac{a}{b}$.

The ratio of a to b is generally denoted thus, $a : b$; and a is called the **antecedent** and b the **consequent** of the ratio.

The two magnitudes compared in a ratio must be *of the same kind*; for example, both must be lines, or both angles, or both areas. It is clearly impossible to compare the *length* of a straight line with a magnitude of a different kind, such as the *area* of a triangle.

Definition 5. Four quantities are in **proportion**, when the ratio of the *first* to the *second* is equal to the ratio of the *third* to the *fourth.*

When the ratio of a to b is equal to that of x to y, the four magnitudes are called **proportionals**. This is expressed by saying "*a is to b as x is to y*," and the proportion is written

$$a : b :: x : y;$$

$$\text{or } a : b = x : y.$$

Here a and y are called the **extremes**, and b and x the **means**.

(i) **Algebraical Test of Proportion.** The ratios $a : b$ and $x : y$ may be expressed algebraically by the fractions $\frac{a}{b}$ and $\frac{x}{y}$; thus the four magnitudes a, b, x, y are in proportion if

$$\frac{a}{b} = \frac{x}{y}.$$

(ii) **Geometrical Test of Proportion.** The ratio of one magnitude to another is equal to that of a third magnitude to a fourth, when if any equimultiples whatever of the antecedents of the ratios are taken, and also any equimultiples whatever of the consequents, the multiple of one antecedent is greater than, equal to, or less than that of its consequent, according as the multiple of the other antecedent is greater than, equal to, or less than that of its consequent.

Thus the ratio of a to b is equal to that of x to y, that is to say,

$$a,\ b,\ x,\ y \text{ are in proportion,}$$

if $\quad mx >, =, \text{ or } < ny,$

according as $\quad ma >, =, \text{ or } < nb,$

whatever whole numbers m and n may be.

NOTE. The Algebraical and Geometrical Tests of Proportion, though differing widely in method, really determine the same property; for each may be deduced from the other. This is fully explained on the following page.

COMPARISON BETWEEN THE ALGEBRAICAL AND GEOMETRICAL TESTS OF PROPORTION.

(i) *If* a, b, x, y *satisfy the Algebraical test of proportion, to shew that they also satisfy the geometrical test.*

By hypothesis $\frac{a}{b}=\frac{x}{y}$;

and, multiplying both sides by $\frac{m}{n}$, where m and n are *any* whole numbers, we obtain $\frac{ma}{nb}=\frac{mx}{ny}$;

thus these fractions are *both improper*, or *both proper*, or *both equal to unity*;
hence $mx >, =,$ or $< ny$, according as $ma >, =,$ or $< nb$, which is the Geometrical test of proportion.

(ii) *If* a, b, x, y *satisfy the Geometrical test of proportion, to shew that they also satisfy the Algebraical test.*

By hypothesis $mx >, =,$ or $< ny$, according as $ma >, =,$ or $< nb$, it is required to prove that

$$\frac{a}{b}=\frac{x}{y}.$$

If $\frac{a}{b}$ is not equal to $\frac{x}{y}$, one of them must be the greater.

Suppose $\frac{a}{b}>\frac{x}{y}$; then it will be possible to find some fraction $\frac{n}{m}$ which lies between them, n and m being positive integers.

Hence $\frac{a}{b}>\frac{n}{m}$, ...(1)

and $\frac{x}{y}<\frac{n}{m}$. ....(2)

From (1), $ma > nb$;
from (2), $mx < ny$;
and these contradict the hypothesis.

Therefore $\frac{a}{b}$ and $\frac{x}{y}$ are not unequal; that is $\frac{a}{b}=\frac{x}{y}$.

Definition 6. Two terms in a proportion are said to be **homologous,** when they are *both antecedents* or *both consequents* of the ratios.

Thus if $a : b :: x : y$,
a and x are homologous; also b and y are homologous.

Definition 8. Two ratios are said to be **reciprocal**, when the antecedent and consequent of one are respectively the consequent and antecedent of the other.

Thus $b : a$ is the reciprocal of $a : b$.

Definition 9. Three magnitudes of the same kind are said to be proportionals, when the ratio of the *first* to the *second* is equal to that of the *second* to the *third*.

Thus a, b, c are proportionals if

$$a : b :: b : c.$$

Here b is called a **mean proportional** to a and c; and c is called a **third proportional** to a and b.

When *four* magnitudes are in proportion, namely when

$$a : b :: c : d,$$

then d is called a **fourth proportional** to a, b, and c.

Definition 10. A series of magnitudes of the same kind are said to be in **continued proportion**, when the ratios of the *first* to the *second*, of the *second* to the *third*, of the *third* to the *fourth*, and so on, are all equal.

Thus a, b, c, d, e are in continued proportion, if

$$a : b = b : c = c : d = d : e;$$

that is, if

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e}.$$

Definition 11. When there are any number of magnitudes of the same kind, the *first* is said to have to the *last* the **ratio compounded** of the ratios of the *first* to the *second*, of the *second* to the *third*, and so on up to the ratio of the *last but one* to the *last* magnitude.

Thus if a, b, c, d, e are magnitudes of the same kind, then $a : e$ is the ratio compounded of the ratios

$$a : b, \quad b : c, \quad c : d, \quad d : e.$$

NOTE. *Algebra* defines the *ratio compounded* of given ratios as that formed by *multiplying together* the fractions which represent the given ratios. In the above illustration it will be seen that on multiplying together the ratios $\frac{a}{b}$, $\frac{b}{c}$, $\frac{c}{d}$, $\frac{d}{e}$ we obtain the ratio $\frac{a}{e}$.

Definition 13. When three magnitudes are proportionals, the *first* is said to have to the *third* the **duplicate ratio** of that which it has to the *second*.

Thus if $a : b :: b : c$,

then $a : c$ is said to be the duplicate of the ratio $a : b$.

NOTE. In *Algebra* the duplicate of the ratio $a : b$ is defined as the ratio of a^2 to b^2.

It is easy to show that the two definitions are identical.

For if $a : b :: b : c$,

then $\frac{a}{b} = \frac{b}{c}$.

Now $\frac{a}{c} = \frac{a}{b} \cdot \frac{b}{c} = \frac{a}{b} \cdot \frac{a}{b} = \frac{a^2}{b^2}$;

that is, $a : c :: a^2 : b^2$.

6. The following theorems from Book V. are here proved algebraically. Reference is made to them in Book VI. under certain technical names.

THEOREM 1. **By Equal Ratios.** *Ratios which are equal to the same ratio are equal to one another.*

That is, if $a : b = x : y$, and $c : d = x : y$;

then shall $a : b = c : d$.

For, by hypothesis, $\frac{a}{b} = \frac{x}{y}$, and $\frac{c}{d} = \frac{x}{y}$;

hence $\frac{a}{b} = \frac{c}{d}$,

or $a : b = c : d$.

THEOREM 3. **Invertendo, or Inversely.** *If four magnitudes are proportionals, they are also proportionals taken inversely.*

That is, if $a : b = x : y$,

then shall $b : a = y : x$.

Since, by hypothesis, $\frac{a}{b} = \frac{x}{y}$, it follows that $\frac{b}{a} = \frac{y}{x}$;

or $b : a = y : x$.

THEOREM 11. **Alternando, or Alternately.** *If four magnitudes of the same kind are proportionals, they are also proportionals when taken alternately.*

That is, if $a:b=x:y$,

then shall $a:x=b:y$.

For, by hypothesis, $\frac{a}{b}=\frac{x}{y}$.

Multiplying both sides by $\frac{b}{x}$,

we have $\frac{a}{b}\cdot\frac{b}{x}=\frac{x}{y}\cdot\frac{b}{x}$;

that is, $\frac{a}{x}=\frac{b}{y}$,

or $a:x=b:y$.

NOTE. In this theorem the *hypothesis* requires that a and b shall be of the same kind, also that x and y shall be of the same kind; while the *conclusion* requires that a and x shall be of the same kind, and also b and y of the same kind.

THEOREM 12. **Addendo.** *In a series of equal ratios (the magnitudes being all of the same kind), as any antecedent is to its consequent so is the sum of the antecedents to the sum of the consequents.*

That is, if $a:x=b:y=c:z=\dots$;

then shall $a:x=a+b+c+\dots:x+y+z+\dots$.

Let each of the equal ratios $\frac{a}{x}, \frac{b}{y}, \frac{c}{z}\dots$ be equal to k.

Then $a=kx,\ b=ky,\ c=kz,\ \dots$;

$\therefore$, by addition,

$$a+b+c+\dots=k(x+y+z+\dots);$$

$$\therefore \frac{a+b+c+\dots}{x+y+z+\dots}=k=\frac{a}{x},$$

or $a:x=a+b+c+\dots:x+y+z+\dots$.

THEOREM 13. **Componendo.** *If four magnitudes are proportionals, the sum of the* first *and* second *is to the* second *as the sum of the* third *and* fourth *is to the* fourth.

That is, if $a : b = x : y$;

then shall $a + b : b = x + y : y$.

For, by hypothesis, $\frac{a}{b} = \frac{x}{y}$;

$$\therefore \quad \frac{a}{b} + 1 = \frac{x}{y} + 1, \text{ or } \frac{a+b}{b} = \frac{x+y}{y};$$

that is, $a + b : b = x + y : y$.

Dividendo. Similarly it may be shewn that $a - b : b = x - y : y$.

THEOREM 14. **Ex Æquali.** *If there are three magnitudes a, b, c of one set, and three magnitudes x, y, z of another set; and if these are so related that*

$$\left.\begin{array}{l} a : b = x : y, \\ b : c = y : z, \end{array}\right\}$$

and

then shall $a : c = x : z$.

For, by hypothesis, $\frac{a}{b} = \frac{x}{y}$, and $\frac{b}{c} = \frac{y}{z}$;

$\therefore$, by multiplication, $\frac{a}{b} \cdot \frac{b}{c} = \frac{x}{y} \cdot \frac{y}{z}$;

that is, $\frac{a}{c} = \frac{x}{z}$,

or $a : c = x : z$.

THEOREM 15. *If two proportions have the same consequents,*

that is, if
and

$$\left.\begin{array}{l} a : b = x : y, \\ c : b = z : y, \end{array}\right\}$$

then shall $a + c : b = x + z : y$.

For, by hypothesis, $\frac{a}{b} = \frac{x}{y}$, and $\frac{c}{b} = \frac{z}{y}$;

$\therefore$, by addition, $\frac{a+c}{b} = \frac{x+z}{y}$;

or $a + c : b = x + z : y$.

BOOK VI.

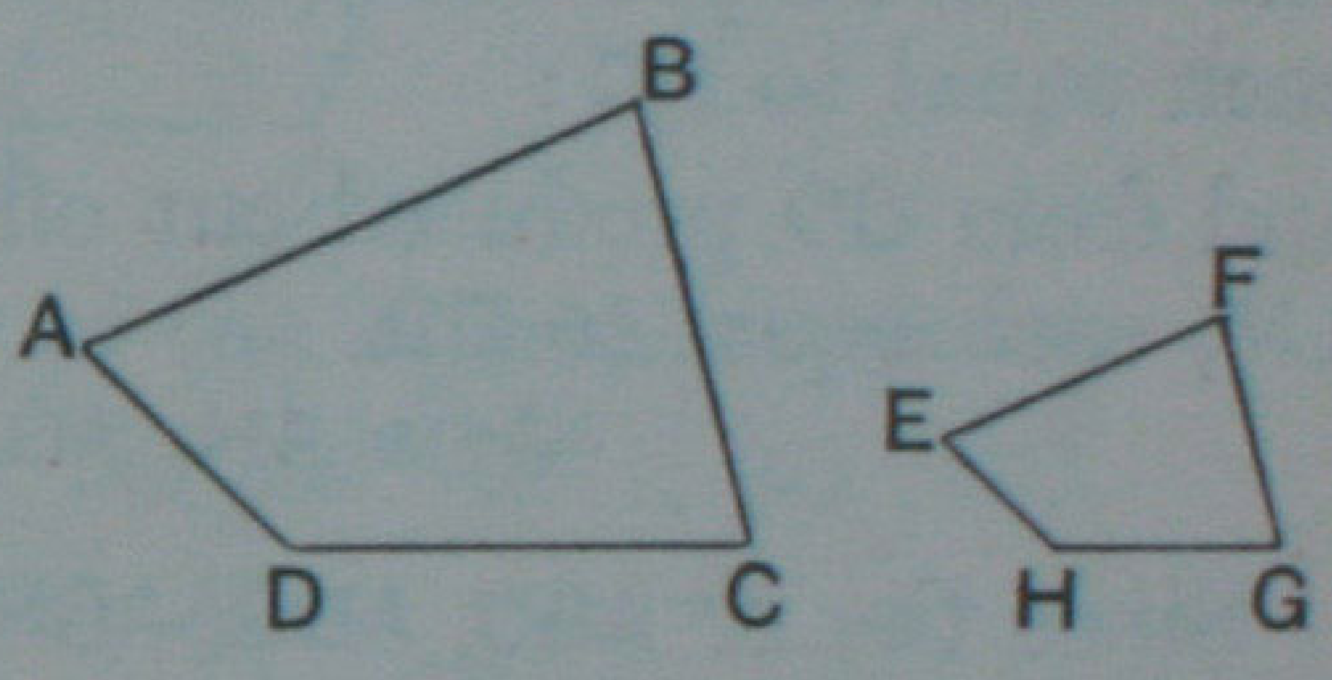
Definitions.

1. Two rectilineal figures are said to be **equiangular** to one another when the angles of the first, taken in order, are equal respectively to those of the second, taken in order.

2. Rectilineal figures are said to be **similar** when they are equiangular to one another, and also have the sides about the equal angles taken in order proportionals.

Thus the two quadrilaterals ABCD, EFGH are similar if the angles at A, B, C, D are respectively equal to those at E, F, G, H, and if the following proportions hold:

AB : BC :: EF : FG,
BC : CD :: FG : GH,
CD . DA :: GH : HE,
DA : AB :: HE : EF.



In these proportions, sides which are *both antecedents* or *both consequents* of the ratios are said to be *homologous* or *corresponding*. [Def. 6, p. 320.]

Thus AB and EF are homologous sides; so are BC and FG.

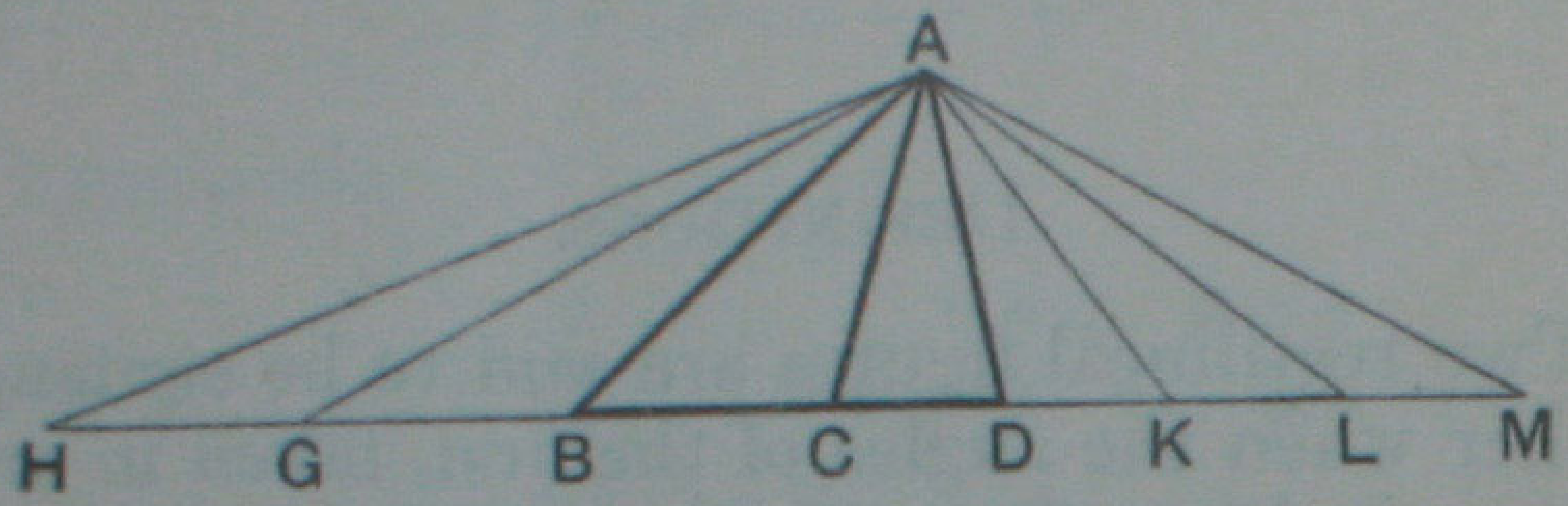
3. Two similar rectilineal figures are said to be **similarly situated** with respect to two of their sides when these sides are *homologous*.

4. Two figures are said to have their sides about one angle in each **reciprocally proportional** when a side of the *first* figure is to a side of the *second* as the remaining side of the *second* figure is to the remaining side of the *first*.

5. A straight line is said to be divided **in extreme and mean ratio** when the whole is to the greater segment as the greater segment is to the less.

PROPOSITION 1. THEOREM. [EUCLID'S PROOF.]

The areas of triangles of the same altitude are to one another as their bases.



Let ABC, ACD be two triangles of the same altitude, namely the perpendicular from A to BD.

Then shall the △ ABC : *the* △ ACD :: BC : CD.

Produce BD both ways;
and from CB produced cut off *any* number of parts BG, GH, each equal to BC;
and from CD produced cut off *any* number of parts DK, KL, LM, each equal to CD.

Join AH, AG, AK, AL, AM.

P. Since the △^{s} ABC, ABG, AGH are of the same altitude, and stand on the equal bases CB, BG, GH,

∴ the △^{s} ABC, ABG, AGH are equal in area; I. 38.

∴ the △ AHC is the same multiple of the △ ABC that HC is of BC.

Similarly the △ ACM is the same multiple of the △ ACD that CM is of CD.

And if HC = CM,
the △ AHC = the △ ACM; I. 38.

and if HC is greater than CM,
the △ AHC is greater than the △ ACM; I. 38, *Cor.*

and if HC is less than CM,
the △ AHC is less than the △ ACM. I. 38, *Cor.*

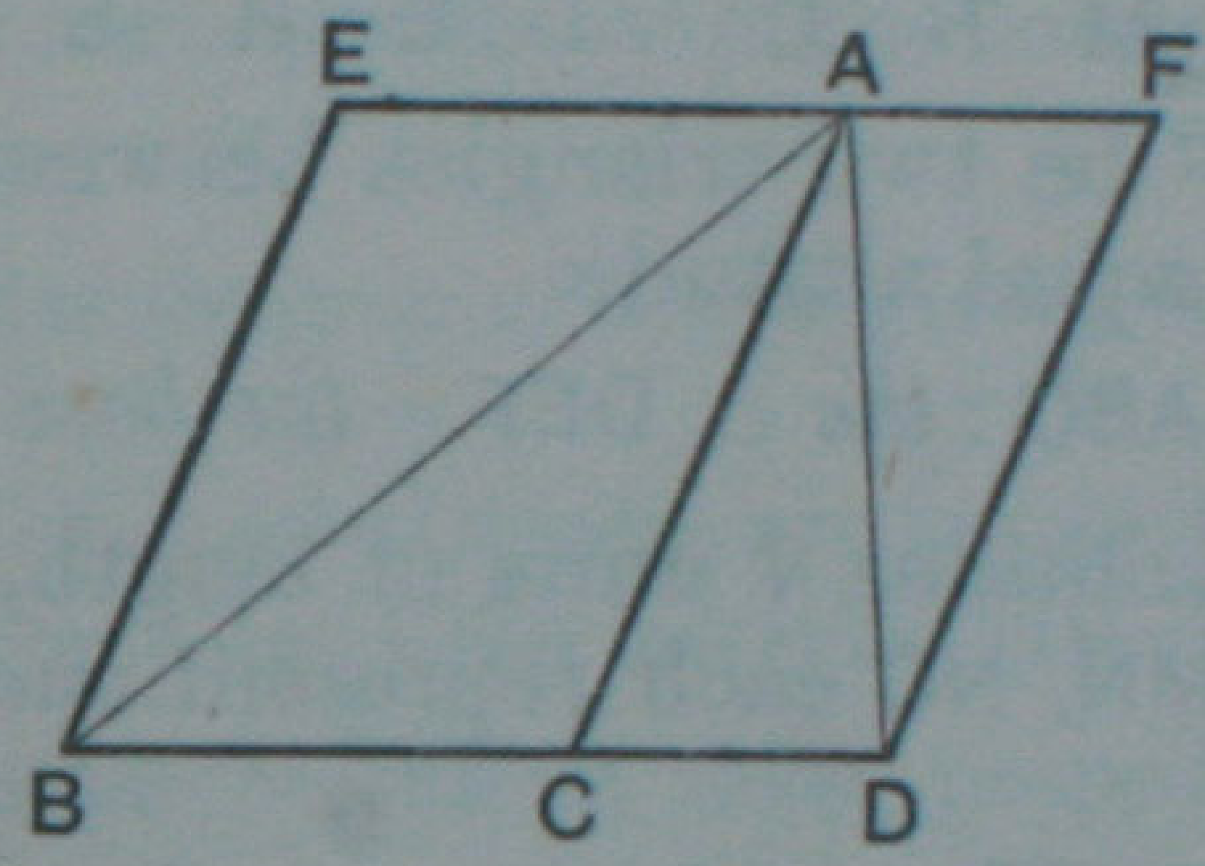
Now since there are four magnitudes, namely, the △^{s} ABC, ACD, and the bases BC, CD; and of the antecedents, any equimultiples have been taken, namely, the △ AHC and the base HC; and of the consequents, any equimultiples have been taken, namely the △ ACM and the base CM; and since it has been shewn that the △ AHC is greater than, equal to, or less than the △ ACM, according as HC is greater than, equal to, or less than CM;

∴ the four original magnitudes are proportionals; v. *Def.* 5.

that is,

the △ ABC : the △ ACD :: the base BC : the base CD. Q.E.D.

COROLLARY. *The areas of parallelograms of the same altitude are to one another as their bases.*



Let EC, CF be par^{ms} of the same altitude.

Then shall the par^{m} EC : *the par*^{m} CF :: BC : CD.

Join BA, AD.

Then the △ ABC : the △ ACD :: BC : CD; *Proved.*

but the par^{m} EC is double of the △ ABC, I. 34.

and the par^{m} CF is double of the △ ACD;

∴ the par^{m} EC : the par^{m} CF :: BC : CD. V. 8.

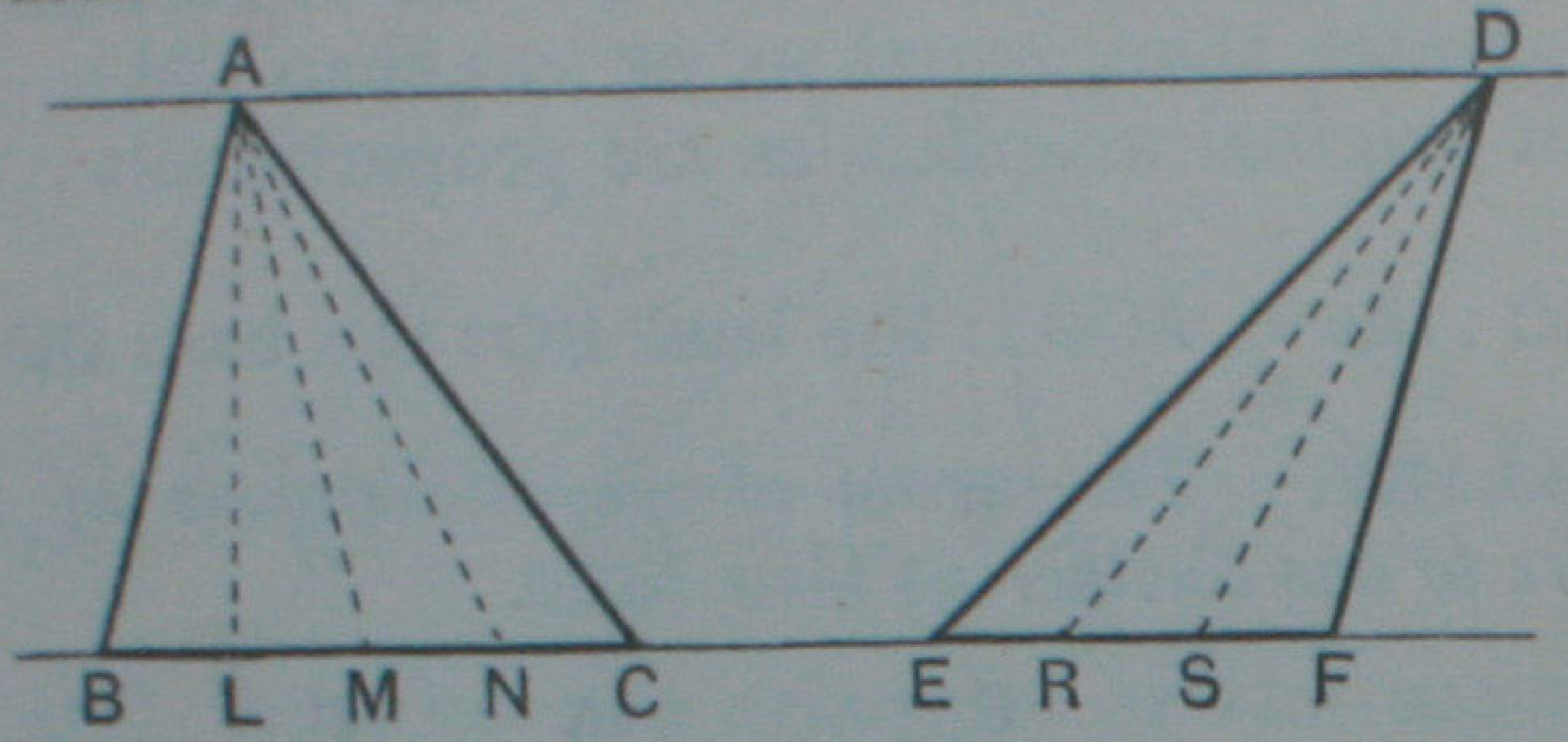
NOTE.

This proof of Proposition 1 is founded on Euclid's Test of Proportion, and therefore holds good whether the bases BC, CD are commensurable or otherwise.

The numerical treatment given on the following page applies in strict theory only to the former case; but the beginner would do well to accept it, at any rate provisionally, and thus postpone to a later reading the acknowledged difficulty of Euclid's Theory of Proportion.

PROPOSITION 1. [NUMERICAL ILLUSTRATION.]

The areas of triangles of equal altitude are to one another as their bases.



Let ABC, DEF be two triangles between the same par[ls], and therefore of equal altitude.

Then shall the △ ABC : *the* △ DEF = *the base* BC : *the base* EF.

Suppose BC contains 4 units of length, and EF 3 units; and let BL, LM, MN, NC each represent *one* unit, as also ER, RS, SF.

Then BC : EF = 4 : 3.

Join AL, AM, AN; also DR, DS.

Then the four △[s] ABL, ALM, AMN, ANC are all equal; for they stand on equal bases, and are of equal altitude.

∴ the △ ABC is *four* times the △ ABL.

Similarly, the △ DEF is *three* times the △ DER.

But the △[s] ABL and DER are equal, for they are on equal bases BL, ER, and of equal altitude;

hence the △ ABC : the △ DEF = 4 : 3
= BC : EF.

This reasoning holds good however many units of length the bases BC, EF contain.

Thus if BC = m units, and EF = n units, then, whatever whole numbers m and n represent,

the △ ABC : the △ DEF = $m : n$
= BC : EF.

The corollary should then be proved as on page 327.

Exercises on Proposition 1.

1. Two triangles of equal altitude stand on bases of 6·3 inches and 5·4 inches respectively; if the area of the first triangle is $12\frac{1}{4}$ square inches, find the area of the other. [$10\frac{1}{2}$ sq. in.]

2. The areas of two triangles of equal altitude have the ratio 24 : 17; if the base of the first is 4·2 centimetres, find the base of the second to the nearest millimetre. [3·0 c.m.]

3. Two triangles lying between the same parallels have bases of 16·20 metres and 20·70 metres; find to the nearest square centimetre the area of the second triangle, if that of the first is 50·1204 sq. metres. [60·0427 sq. m.]

4. Assuming that *the area of a triangle* $= \frac{1}{2}$ *base* $\times$ *altitude*, prove algebraically that

(i) Triangles of equal altitudes are proportional to their bases;

(ii) Triangles on equal bases are proportional to their altitudes.

Also deduce the second of these propositions *geometrically* from the first.

5. Two triangular fields lie on opposite sides of a common base; and their altitudes with respect to it are 4·20 chains and 3·71 chains. If the first field contains 18 acres, find the acreage of the whole quadrilateral. [33·9 acres.]

Definition.

Two straight lines are **cut proportionally** when the segments of one line are in the same ratio as the corresponding segments of the other. [See definition, page 139.]

Fig. 1.

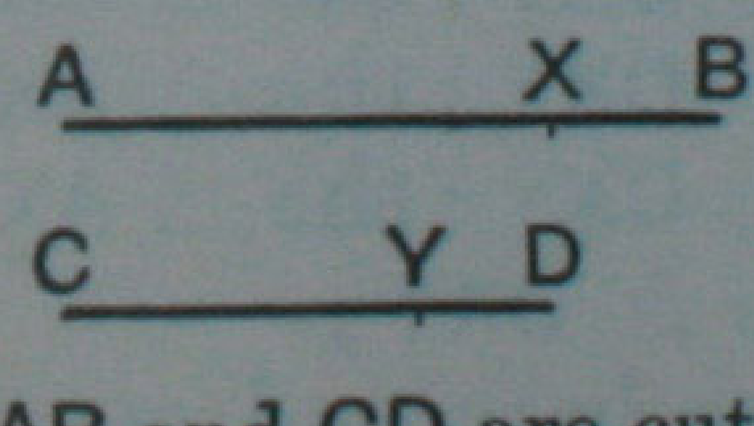
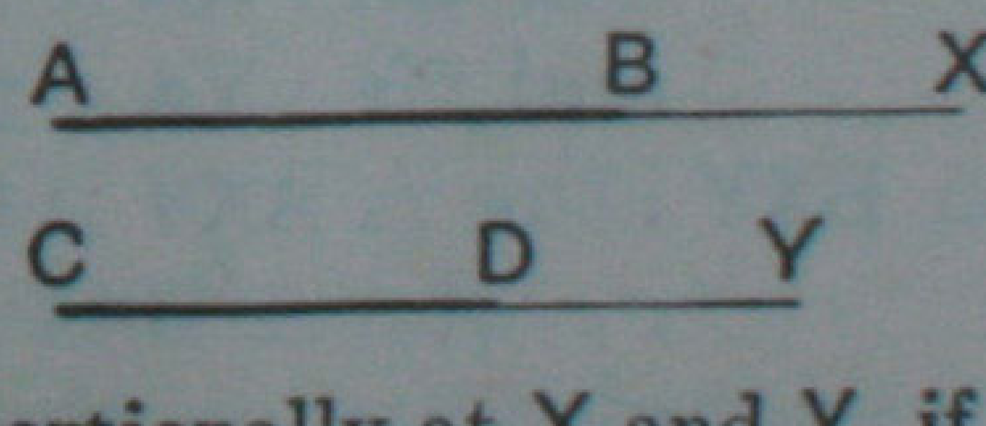


Fig. 2.



Thus AB and CD are cut proportionally at X and Y, if

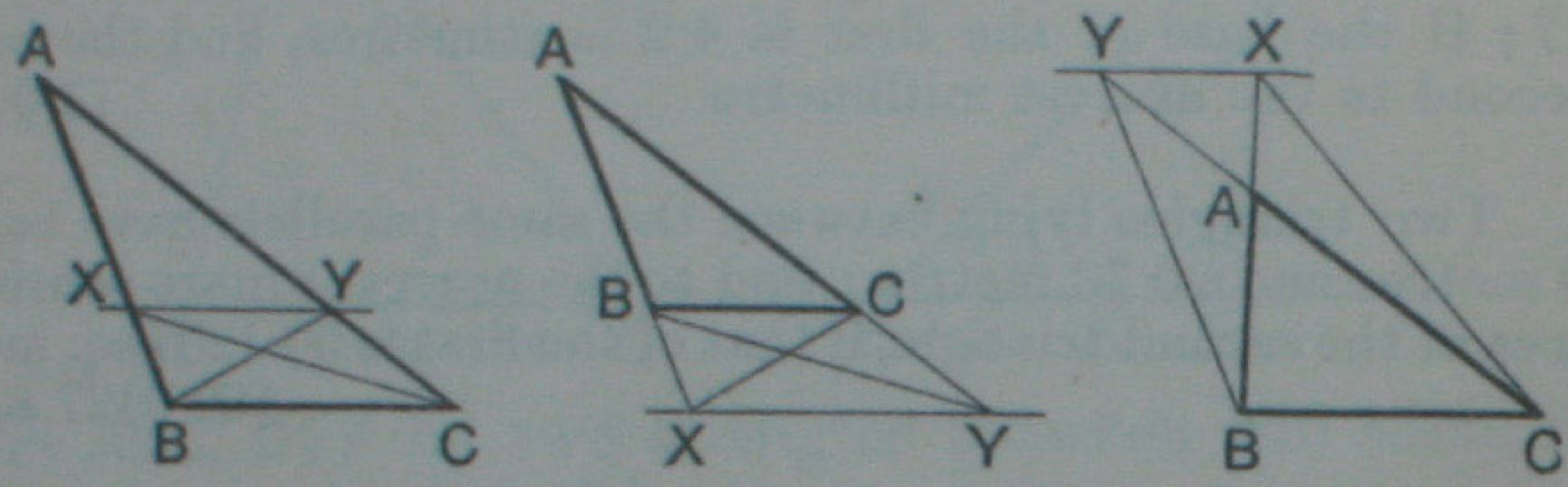
AX : XB :: CY : YD.

And the same definition applies equally whether X and Y divide AB and CD internally as in Fig. 1 or externally as in Fig. 2.

Proposition 2. Theorem.

If a straight line is drawn parallel to one side of a triangle, it cuts the other sides, or those sides produced, proportionally.

Conversely, if the sides, or the sides produced, are cut proportionally, the straight line which joins the points of section, is parallel to the remaining side of the triangle.



Let XY be drawn par[1] to BC, one of the sides of the △ ABC.

Then shall BX : XA :: CY : YA.

Join BY, CX.

Now the △[s] BXY, CXY are on the same base XY and between the same par[ls] XY, BC;

∴ the △ BXY = the △ CXY; I. 37.

and AXY is another triangle;

∴ the △ BXY : the △ AXY :: the △ CXY : the △ AXY. V. 4.

But the △ BXY : the △ AXY :: BX : XA, VI. 1.

and the △ CXY : the △ AXY :: CY : YA;

∴ BX : XA :: CY : YA. V. 1.

Conversely. Let BX : XA :: CY : YA, and let XY be joined.

Then shall XY *be par[1] to* BC.

As before, join BY, CX.

By hypothesis, BX : XA :: CY : YA;

but BX : XA :: the △ BXY : the △ AXY, VI. 1.

and CY : YA :: the △ CXY : the △ AXY;

∴ the △ BXY : the △ AXY :: the △ CXY : the △ AXY. V. 1.

∴ the △ BXY = the △ CXY; V. 6.

and these triangles are on the same base and on the same side of it;

∴ XY is par[1] to BC. I. 39.

Q.E.D.

EXERCISES.

1. Shew that every quadrilateral is divided by its diagonals into four triangles whose areas are proportionals.

2. *If any two straight lines are cut by three parallel straight lines, they are cut proportionally.*

3. From the point E in the common base of two triangles ACB, ADB, straight lines are drawn parallel to AC, AD, meeting BC, BD at F, G: shew that FG is parallel to CD.

4. In a triangle ABC the straight line DEF meets the sides BC, CA, AB at the points D, E, F respectively, and it makes equal angles with AB and AC: prove that

$$BD : CD :: BF : CE.$$

5. In a triangle ABC, AD is drawn perpendicular to BD, the bisector of the angle at B: shew that a straight line through D parallel to BC will bisect AC.

6. From B and C, the extremities of the base of a triangle ABC, straight lines BE, CF are drawn to the opposite sides so as to intersect on the median from A: shew that EF is parallel to BC.

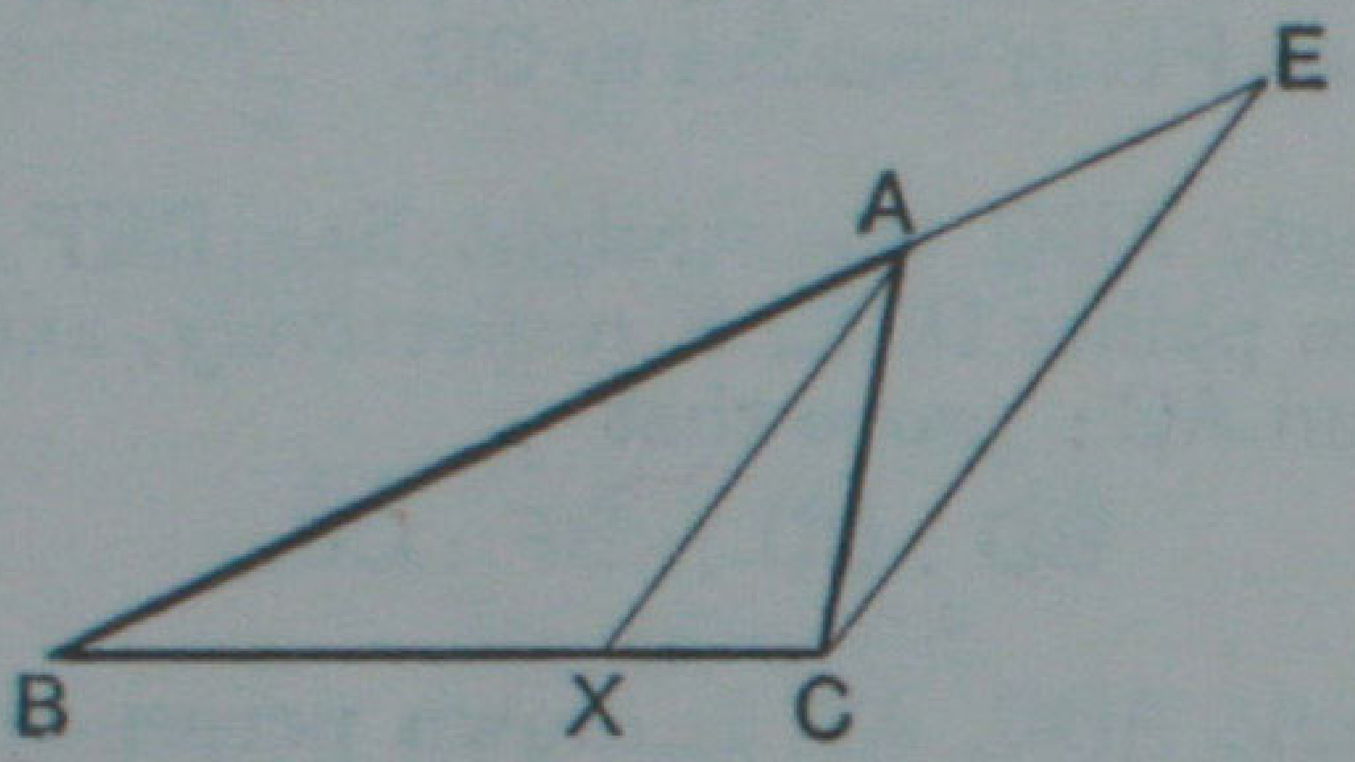
7. From P, a given point in the side AB of a triangle ABC, draw a straight line to AC produced, so that it will be bisected by BC.

8. Find a point within a triangle such that, if straight lines be drawn from it to the three angular points, the triangle will be divided into three equal triangles.

Proposition 3. Theorem.

If the vertical angle of a triangle be bisected by a straight line which cuts the base, the segments of the base shall have to one another the same ratio as the remaining sides of the triangle.

Conversely, if the base be divided so that its segments have to one another the same ratio as the remaining sides of the triangle, the straight line drawn from the vertex to the point of section shall bisect the vertical angle.



In the △ ABC, let the ∠ BAC be bisected by AX, which meets the base at X.

Then shall BX : XC :: BA : AC.

Through C draw CE par[l] to XA, to meet BA produced at E. I. 31.

Then because XA and CE are par[l],
∴ the ∠ BAX = the int. opp. ∠ AEC, I. 29.
and the ∠ XAC = the alt. ∠ ACE. I. 29.
But the ∠ BAX = the ∠ XAC; *Hyp.*
∴ the ∠ AEC = the ∠ ACE;
∴ AC = AE. I. 6.

Again, because XA is par[l] to CE, a side of the △ BCE,
∴ BX : XC :: BA : AE; VI. 2.
that is, BX : XC :: BA : AC.

Conversely. Let BX : XC :: BA : AC ; and let AX be joined.

Then shall the ∠ BAX = *the* ∠ XAC.

For, with the same construction as before,
because XA is par[l] to CE, a side of the △ BCE,
∴ BX : XC :: BA : AE. VI. 2.
But, *by hypothesis*, BX : XC :: BA : AC ;
∴ BA : AE :: BA : AC ; V. 1.
∴ AE = AC ;
∴ the ∠ ACE = the ∠ AEC. I. 5.
But because XA is par[l] to CE,
∴ the ∠ XAC = the alt. ∠ ACE. I. 29.
and the ext. ∠ BAX = the int. opp. ∠ AEC ; I. 29.
∴ the ∠ BAX = the ∠ XAC.

Q.E.D.

EXERCISES.

1. The side BC of a triangle ABC is bisected at D, and the angles ADB, ADC are bisected by the straight lines DE, DF, meeting AB, AC at E, F respectively: shew that EF is parallel to BC.

2. Apply Proposition 3 to trisect a given finite straight line.

3. If the line bisecting the vertical angle of a triangle is divided into parts which are to one another as the base to the sum of the sides, the point of division is the centre of the inscribed circle.

4. ABCD is a quadrilateral: shew that if the bisectors of the angles A and C meet in the diagonal BD, the bisectors of the angles B and D will meet on AC.

5. *Construct a triangle having given the base, the vertical angle, and the ratio of the remaining sides.*

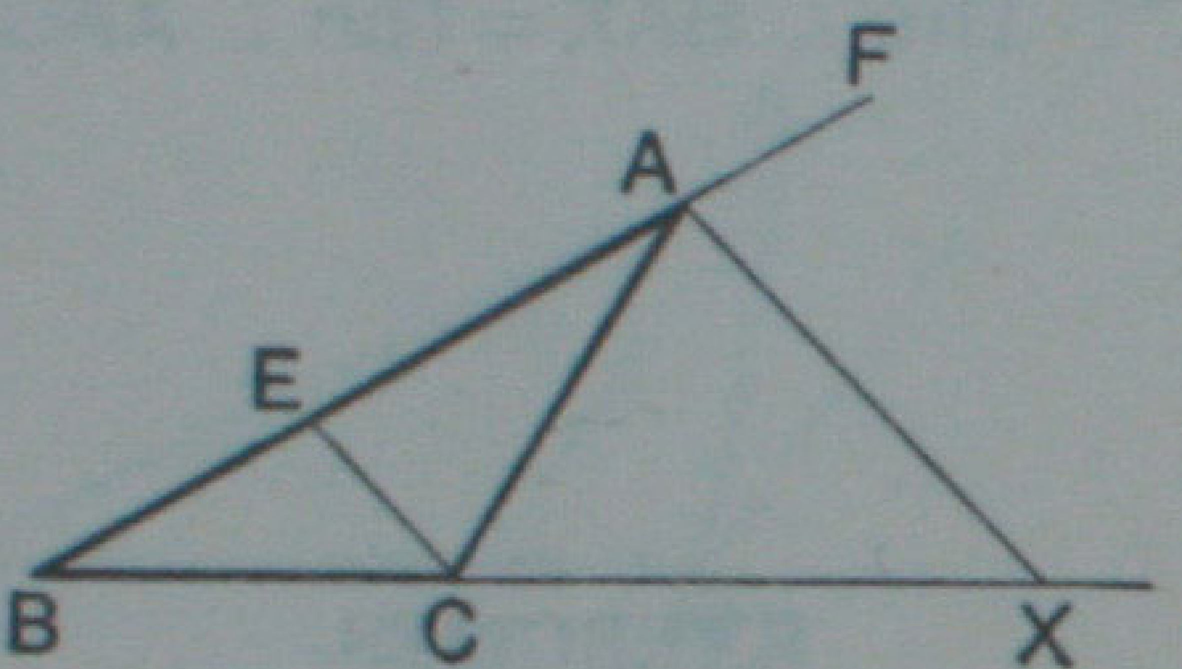
6. Employ Proposition 3 to shew that the bisectors of the angles of a triangle are concurrent.

7. AB is a diameter of a circle, CD is a chord at right angles to it, and E any point in CD; AE and BE are drawn and produced to cut the circle in F and G: shew that the quadrilateral CFDG has any two of its adjacent sides in the same ratio as the remaining two.

PROPOSITION A. THEOREM.

If one side of a triangle be produced, and the exterior angle so formed be bisected by a straight line which cuts the base produced, the segments between the point of section and the extremities of the base shall have to one another the same ratio as the remaining sides of the triangle.

Conversely, if the segments of the base produced have to one another the same ratio as the remaining sides of the triangle, the straight line drawn from the vertex to the point of section shall bisect the exterior vertical angle.



In the △ ABC let BA be produced to F, and let the exterior ∠ CAF be bisected by AX which meets the base produced at X.

Then shall BX : XC :: BA : AC.

Through C draw CE par[l] to XA, — I. 31.
and let CE meet BA at E.

Then because AX and CE are par[l],
∴ the ext. ∠ FAX = the int. opp. ∠ AEC,
and the ∠ XAC = the alt. ∠ ACE. — I. 29.
But the ∠ FAX = the ∠ XAC; — *Hyp.*
∴ the ∠ AEC = the ∠ ACE;
∴ AC = AE. — I. 6.

Again, because XA is par[l] to CE, a side of the △ BCE, — *Constr.*

∴ BX : XC :: BA : AE; — VI. 2.

that is, BX : XC :: BA : AC.

Conversely. Let BX : XC :: BA : AC, and let AX be joined.

Then shall the ∠ FAX = *the* ∠ XAC.

For, with the same construction as before,

because AX is par[l] to CE, a side of the △ BCE,

∴ BX : XC :: BA : AE. VI. 2.

But, *by hypothesis,* BX : XC :: BA : AC;

∴ BA : AE :: BA : AC; V. 1.

∴ AE = AC;

∴ the ∠ ACE = the ∠ AEC. I. 5.

But because AX is par[l] to CE,

∴ the ∠ XAC = the alt. ∠ ACE,

and the ext. ∠ FAX = the int. opp. ∠ AEC; I. 29.

∴ the ∠ FAX = the ∠ XAC. Q.E.D.

Propositions 3 and A may be both included in one enunciation as follows:

If the interior or exterior vertical angle of a triangle be bisected by a straight line which also cuts the base, the base shall be divided internally or externally into segments which have the same ratio as the other sides of the triangle.

Conversely, if the base be divided internally or externally into segments which have the same ratio as the other sides of the triangle, the straight line drawn from the point of division to the vertex will bisect the interior or exterior vertical angle.

EXERCISES.

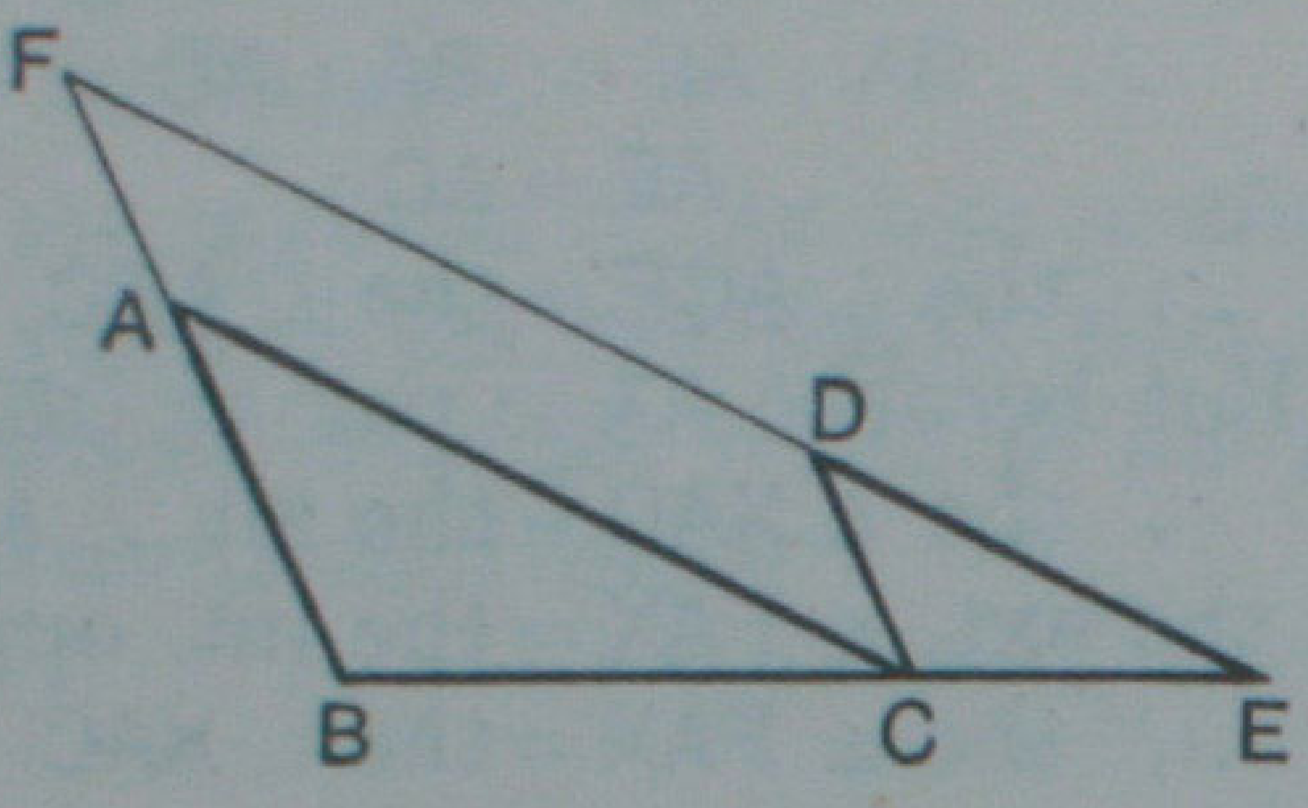
1. In the circumference of a circle of which AB is a diameter, a point P is taken; straight lines PC, PD are drawn equally inclined to AP and on opposite sides of it, meeting AB in C and D; shew that AC : CB :: AD : DB.

2. From a point A straight lines are drawn making the angles BAC, CAD, DAE, each equal to half a right angle, and they are cut by a straight line BCDE, which makes BAE an isosceles triangle: shew that BC or DE is a mean proportional between BE and CD.

3. By means of Propositions 3 and A, prove that the straight lines bisecting one angle of a triangle internally, and the other two externally, are concurrent.

PROPOSITION 4. THEOREM.

If two triangles be equiangular to one another, the sides about the equal angles shall be proportionals, those sides which are opposite to equal angles being homologous.



Let the △ ABC be equiangular to the △ DCE,
having the ∠ ABC equal to the ∠ DCE,
the ∠ BCA equal to the ∠ CED,
and consequently the ∠ CAB equal to the ∠ EDC. I. 32.

Then shall the sides about these equal angles be proportionals, namely

AB : BC :: DC : CE,
BC : CA :: CE : ED,
and AB : AC :: DC : DE.

Let the △ DCE be placed so that its side CE may be contiguous to BC, and in the same straight line with it.

Then because the ∠ˢ ABC, ACB are together less than two rt. angles, I. 17.
and the ∠ ACB = the ∠ DEC; *Hyp.*
∴ the ∠ˢ ABC, DEC are together less than two rt. angles;
∴ BA and ED will meet if produced. *Ax.* 12.
Let them be produced and meet at F.
Then because the ∠ ABC = the ∠ DCE, *Hyp.*
∴ BF is parˡ to CD; I. 28.
and because the ∠ ACB = the ∠ DEC, *Hyp.*
∴ AC is parˡ to FE; I. 28.
∴ FACD is a parᵐ;
∴ AF = CD, and AC = FD. I. 34.

Again, because CD is par[l] to BF, a side of the △ EBF,

∴ BC : CE :: FD : DE ; VI. 2.

but FD = AC ;

∴ BC : CE :: AC : DE ;

and, *alternately*, BC : CA :: CE : ED. V. 11.

Again, because AC is par[l] to FE, a side of the △ FBE,

∴ BA : AF :: BC : CE ; VI. 2.

but AF = CD ;

∴ BA : CD :: BC : CE ;

and, *alternately*, AB : BC :: DC : CE. V. 11.

Also BC : CA :: CE : ED ; *Proved.*

∴, *ex æquali*, AB : AC :: DC : DE. V. 14.

Q.E.D.

[For Alternative Proof see Page 342.]

EXERCISES.

1. If one of the parallel sides of a trapezium is double the other, shew that the diagonals intersect one another at a point of trisection.

2. In the side AC of a triangle ABC any point D is taken : shew that if AD, DC, AB, BC are bisected in E, F, G, H respectively, then EG is equal to HF.

3. AB and CD are two parallel straight lines ; E is the middle point of CD ; AC and BE meet at F, and AE and BD meet at G : shew that FG is parallel to AB.

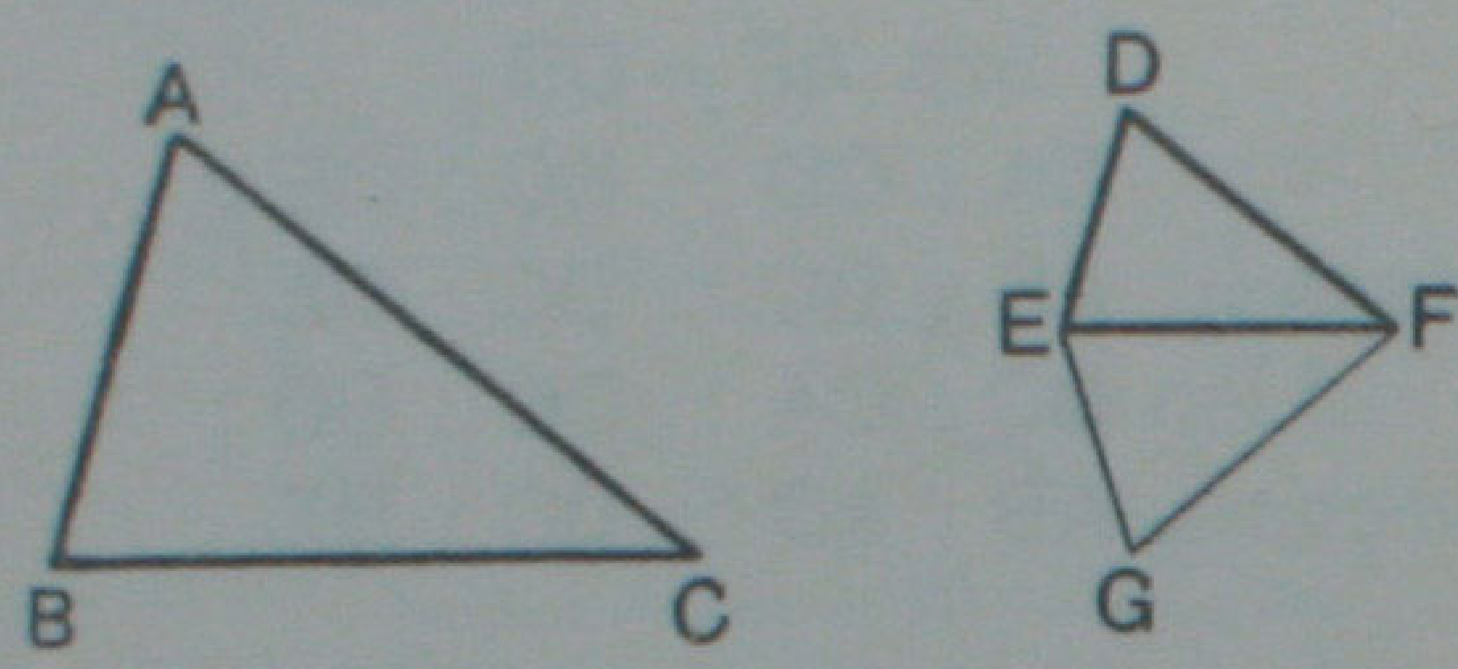
4. ABCDE is a regular pentagon, and AD and BE intersect in F : shew that AF : AE :: AE : AD.

5. In the figure of I. 43 shew that EH and GF are parallel, and that FH and GE will meet on CA produced.

6. Chords AB and CD of a circle are produced towards B and D respectively to meet in the point E, and through E, the line EF is drawn parallel to AD to meet CB produced in F. Prove that EF is a mean proportional between FB and FC.

PROPOSITION 5. THEOREM.

If the sides of two triangles, taken in order about each of their angles, be proportionals, the triangles shall be equiangular to one another, having those angles equal which are opposite to the homologous sides.



Let the △s ABC, DEF have their sides proportionals, so that

AB : BC :: DE : EF,

BC : CA :: EF : FD,

and consequently, *ex æquali*,

AB : AC :: DE : DF.

Then shall the △s ABC, DEF *be equiangular to one another.*

At E in FE make the ∠ FEG equal to the ∠ ABC; I. 23.
and at F in EF make the ∠ EFG equal to the ∠ BCA;
∴ the remaining ∠ EGF = the remaining ∠ BAC. I. 32.

Then the △s ABC, GEF are equiangular to one another;

∴ AB : BC :: GE : EF. VI. 4.

But, *by hypothesis*, AB : BC :: DE : EF;

∴ GE : EF :: DE : EF; V. 1.

∴ GE = DE.

Similarly GF = DF.

Then in the △s GEF, DEF,

Because { GE = DE,
GF = DF,
and EF is common;

∴ the ∠ GEF = the ∠ DEF, I. 8.
and the ∠ GFE = the ∠ DFE,
and the ∠ EGF = the ∠ EDF.

But the ∠ GEF = the ∠ ABC ; *Constr.*

∴ the ∠ DEF = the ∠ ABC.

Similarly, the ∠ EFD = the ∠ BCA ;

∴ the remaining ∠ FDE = the remaining ∠ CAB ; I. 32.

that is, the △ DEF is equiangular to the △ ABC.

Q.E.D.

NOTE ON SIMILAR FIGURES.

Similar figures may be described as those which have the *same shape.*

For this, *two* conditions are necessary [see VI., *Def.* 2] ;

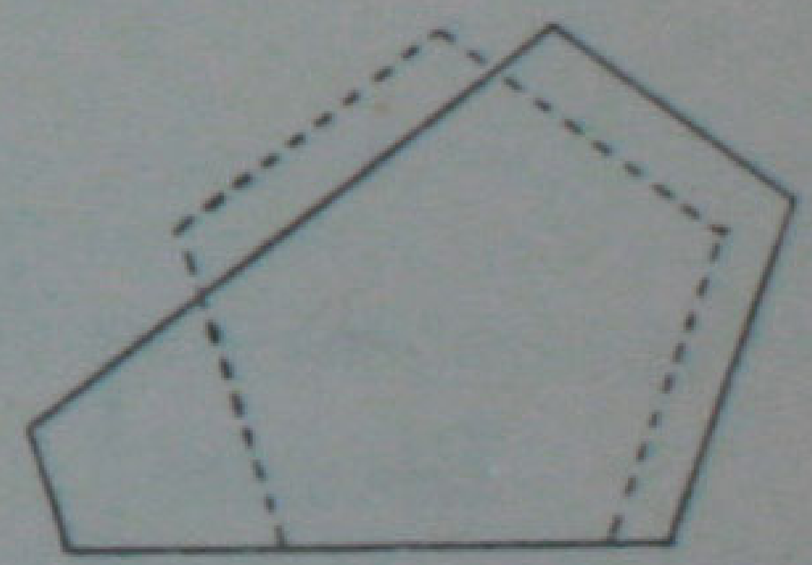
(i) *the figures must have their angles equal each to each ;*

(ii) *their sides about the equal angles taken in order must be proportional.*

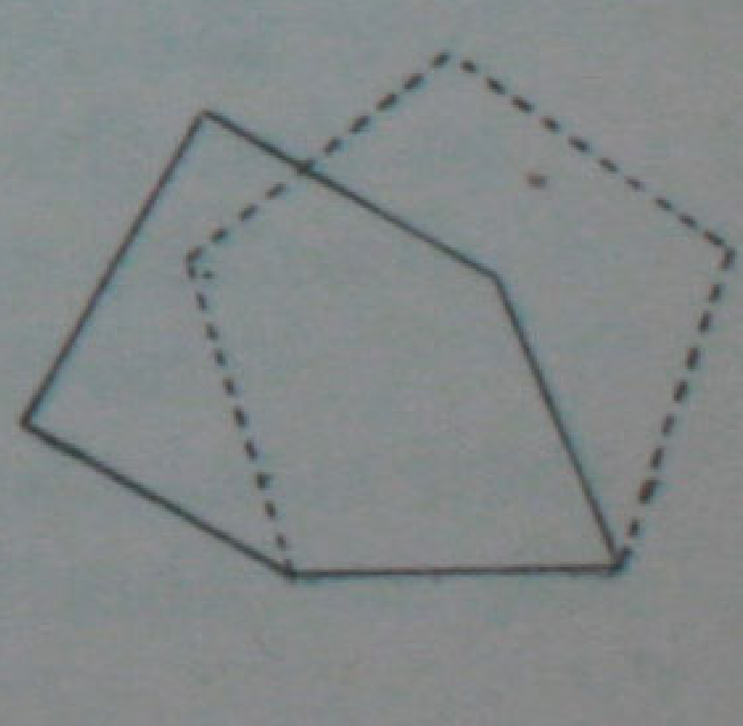
In the case of *triangles* we have learned that these conditions are not independent, for each follows from the other : thus

(i) if the triangles are *equiangular,* Proposition 4 proves the *proportionality of their sides ;*

(ii) if the triangles have their *sides proportional,* Proposition 5 proves their *equiangularity.*

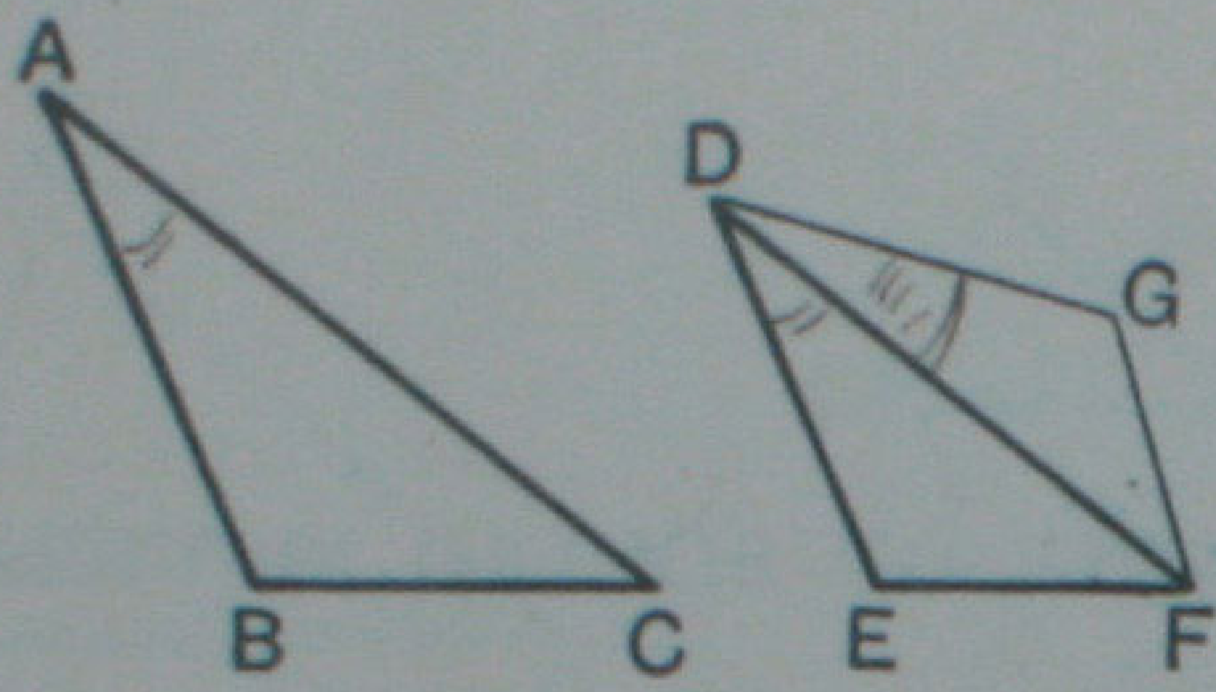


This, however, is not necessarily the case with rectilineal figures of more than three sides. For example, the first diagram in the margin shews two figures which are equiangular to one another, but which clearly have not their sides proportional ; while the figures in the second diagram have their sides proportional, but are not equiangular to one another.



PROPOSITION 6. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be similar.



In the $\triangle^s$ BAC, EDF, let the $\angle$ BAC = the $\angle$ EDF,
and let BA : AC :: ED : DF.

Then shall the $\triangle^s$ BAC, EDF *be similar.*

At D in FD make the $\angle$ FDG equal to the $\angle$ CAB: I. 23.
at F in DF make the $\angle$ DFG equal to the $\angle$ ACB;
∴ the remaining $\angle$ DGF = the remaining $\angle$ ABC. I. 32.

Then the $\triangle^s$ BAC, GDF are equiangular to one another;
∴ BA : AC :: GD : DF. VI. 4.
But, *by hypothesis*, BA : AC :: ED : DF;
∴ GD : DF :: ED : DF,
∴ GD = ED.

Then in the $\triangle^s$ GDF, EDF,
Because { GD = ED,
and DF is common;
and the $\angle$ GDF = the $\angle$ EDF; } *Constr.*
∴ the $\triangle^s$ GDF, EDF are equal in all respects; I. 4.
so that the $\triangle$ EDF is equiangular to the $\triangle$ GDF;
but the $\triangle$ GDF is equiangular to the $\triangle$ BAC; *Constr.*
∴ the $\triangle$ EDF is equiangular to the $\triangle$ BAC;
∴ their sides about the equal angles are proportionals; VI. 4.
that is, the $\triangle^s$ BAC, EDF are similar.

Q.E.D.

EXERCISES.

ON PROPOSITIONS 1 TO 6.

1. Shew that the diagonals of a trapezium cut one another in the same ratio.

2. If three straight lines drawn from a point cut two parallel straight lines in A, B, C and P, Q, R respectively, prove that

AB : BC :: PQ : QR.

3. From a point O, a tangent OP is drawn to a given circle, and a secant OQR is drawn cutting it in Q and R; shew that

OQ : OP :: OP : OR.

4. *If two triangles are on equal bases and between the same parallels, any straight line parallel to their bases will cut off equal areas from the two triangles.*

5. *If two straight lines* PQ, XY *intersect in a point* O, *so that* PO : OX :: YO : OQ, *prove that* P, X, Q, Y *are concyclic.*

6. On the same base and on the same side of it two equal triangles ACB, ADB are described; AC and BD intersect in O, and through O lines parallel to DA and CB are drawn meeting the base in E and F. Shew that AE = BF.

7. BD, CD are perpendicular to the sides AB, AC of a triangle ABC, and CE is drawn perpendicular to AD, meeting AB in E: shew that the triangles ABC, ACE are similar.

8. AC and BD are drawn perpendicular to a given straight line CD from two given points A and B; AD and BC intersect in E, and EF is perpendicular to CD: shew that AF and BF make equal angles with CD.

9. ABCD is a parallelogram; P and Q are points in a straight line parallel to AB; PA and QB meet at R, and PD and QC meet at S: shew that RS is parallel to AD.

10. In the sides AB, AC of a triangle ABC two points D, E are taken such that BD is equal to CE; if DE, BC produced meet at F, shew that AB : AC :: EF : DF.

11. Find a point the perpendiculars from which on the sides of a given triangle shall be in a given ratio.

PROPOSITION 7. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about one other angle in each proportional, so that the sides opposite to the equal angles are homologous, then the third angles are either equal or supplementary; and in the former case the triangles are similar.

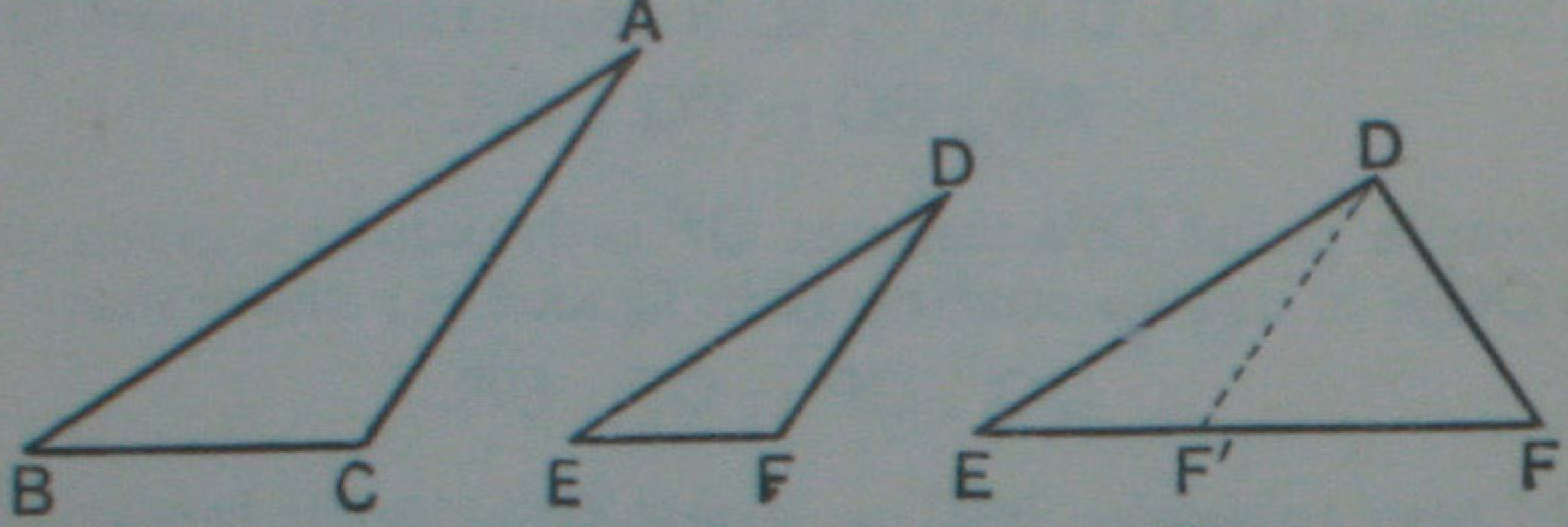


Fig. 1. Fig. 2. Fig. 3.

Let ABC, DEF be two triangles having the ∠ ABC equal to the ∠ DEF, and the sides about the angles at A and D proportional, namely

BA : AC :: ED : DF.

Then shall the ∠s ACB, DFE *be either equal* (*as in Figs.* 1 *and* 2) *or supplementary* (*as in Figs.* 1 *and* 3), *and in the former case the triangles shall be similar.*

If the ∠ BAC = the ∠ EDF, [*Figs.* 1 *and* 2.]
then the ∠ ACB = the ∠ DFE; I. 32.
and the △s are equiangular, and therefore similar. VI. 4.

But if the ∠ BAC is not equal to the ∠ EDF, [*Figs.* 1 *and* 3.]
one of them must be the greater.
Let the ∠ EDF be greater than the ∠ BAC.
At D in ED make the ∠ EDF′ equal to the ∠ BAC. [*Fig.* 3.]
Then the △s BAC, EDF′ are equiangular, I. 32.
∴ BA : AC :: ED : DF′; VI. 4.
but, *by hypothesis*, BA : AC :: ED : DF;
∴ ED : DF :: ED : DF′, V. 1.
∴ DF = DF′,
∴ the ∠ DFF′ = the ∠ DF′F. I. 5.
But the ∠s DF′F, DF′E are supplementary, I. 13.
∴ the ∠s DFF′, DF′E are supplementary:
that is, the ∠s DFE, ACB are supplementary. Q.E.D.

Corollaries to Proposition 7.

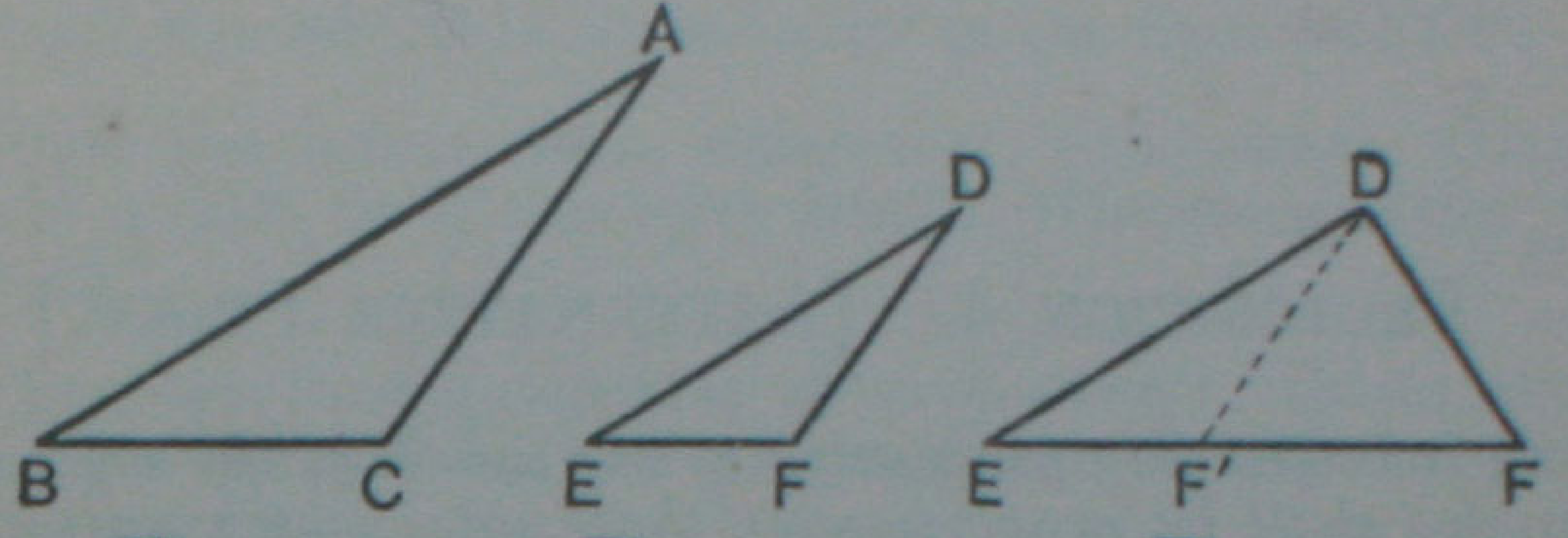


Fig. 1. Fig. 2. Fig. 3.

Three cases of this theorem deserve special attention.

It has been proved that if the angles ACB, DFE are not *supplementary*, they are *equal*.

Hence, in addition to the hypothesis of this theorem,

(i) If the angles ACB, DFE, opposite to the two homologous sides AB, DE are both acute or both obtuse, they cannot be supplementary, and are therefore equal: or if one of them is a right angle, the other must also be a right angle (whether considered as supplementary or equal to it):

in either case the triangles are similar.

(ii) If the two given angles at B and E are right angles or obtuse angles, it follows that the angles ACB, DFE must be both acute, and therefore equal, by (i):

so that the triangles are similar.

(iii) If in each triangle the side opposite the *given* angle is not less than the other given side; that is, if AC and DF are not less than AB and DE respectively, then the angles ACB, DFE cannot be greater than the angles ABC, DEF, respectively;

therefore the angles ACB, DFE, are both acute;

hence, as above, they are equal;

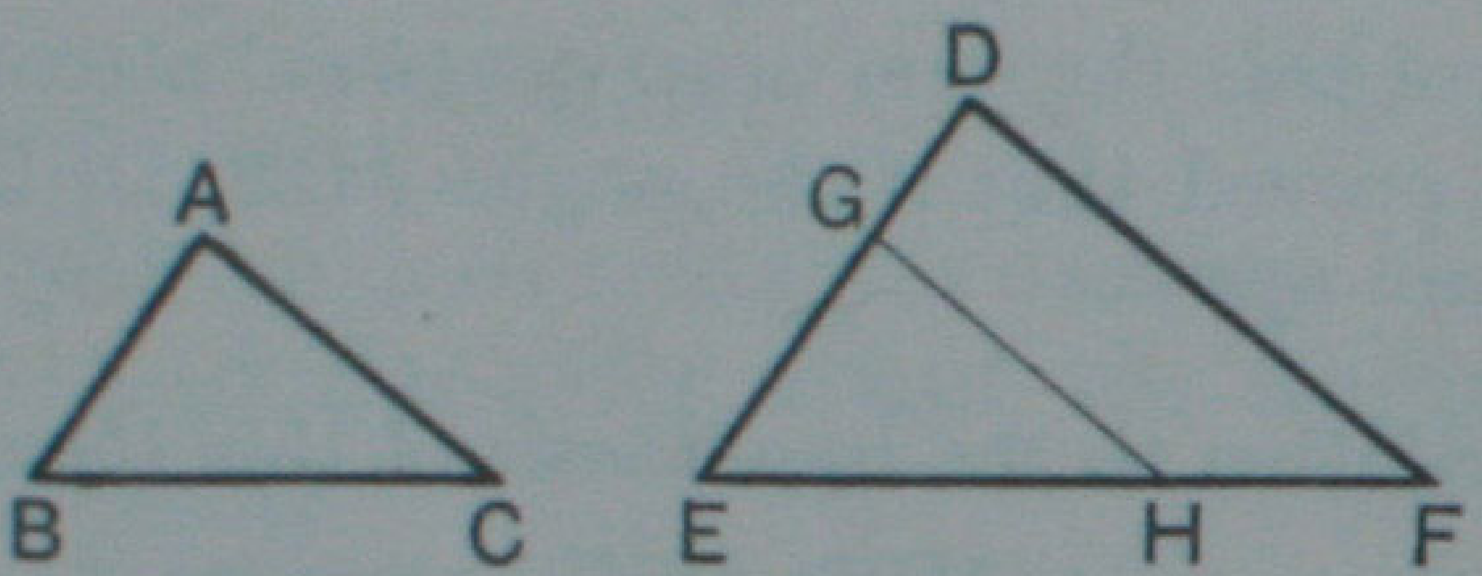
and the triangles ABC, DEF are similar.

Obs. We have given Euclid's demonstrations of Propositions 4, 5, 6; but these propositions also admit of easy proof by the method of superposition.

As an illustration, we will apply this method to Proposition 4.

PROPOSITION 4. [ALTERNATIVE PROOF.]

If two triangles be equiangular to one another, the sides about the equal angles shall be proportionals, those sides which are opposite to equal angles being homologous.



Let the △ ABC be equiangular to the △ DEF,
having the ∠ ABC equal to the ∠ DEF,
the ∠ BCA equal to the ∠ EFD,
and consequently the ∠ CAB equal to the ∠ FDE. I. 32.

Then shall the sides about these equal angles be proportionals.

Apply the △ ABC to the △ DEF, so that B falls on E, and BA along ED:
then BC will fall along EF, since the ∠ ABC = the ∠ DEF. *Hyp.*
Let G and H be the points in ED and EF, on which A and C fall;
then GH represents AC in its new position.

Then because the ∠ EGH (*i.e.* the ∠ BAC) = the ∠ EDF, *Hyp.*
∴ GH is par[l] to DF : VI. 2.
∴ DG : GE :: FH : HE;
∴, *componendo*, DE : GE :: FE : HE, V. 13.
∴, *alternately*, DE : FE :: GE : HE, V. 11.
that is, DE : EF :: AB : BC.

Similarly by applying the △ ABC to the △ DEF, so that the point C may fall on F, it may be proved that
EF : FD :: BC : CA.
∴, *ex æquali*, DE : DF :: AB : AC.

Q.E.D.

QUESTIONS FOR REVISION, AND NUMERICAL ILLUSTRATIONS.

1. Distinguish between the use of the word *equiangular* in the following cases:

   (i) the figure ABCD is *equiangular*;

   (ii) the figure ABCD is *equiangular* to the figure EFGH.

2. Define the terms *ratio, antecedent, consequent.* Why must the terms of a ratio be of the *same kind*? When are ratios said to be reciprocal?

3. When are four quantities *in proportion*? Quote the algebraical and geometrical tests of proportion; and deduce the latter from the former.

4. What is meant by *homologous terms* in a proportion? In the enunciation of Prop. 4, why is it necessary to add—*those sides which are opposite to equal angles being homologous*?

5. Quote the enunciation of the theorem known as *alternando* or *alternately*; and explain why the terms of a proportion to which this theorem is applied must be all of the same kind.

6. In the Particular Enunciation of Proposition 5 it is given that
$$AB : BC :: DE : EF,$$
and
$$BC : CA :: EF : FD;$$
Why do we add "*and consequently,*"
$$AB : CA :: DE : FD?$$

7. Define *similar figures.* In what way do the conditions of similarity in *triangles* differ from those in figures of more than three sides?

8. Two parallelograms whose areas are in the ratio 2·1 : 3·5 lie between the same parallels. If the base of the first is 6·6 inches in length, shew that the base of the second is 11 inches.

9. ABC is a triangle, and XY is drawn parallel to BC, cutting the other sides at X and Y:

   (i) If AB = 1 foot, AC = 8 inches, and AX = 7 inches; shew that AY = $4\frac{2}{3}$ inches.

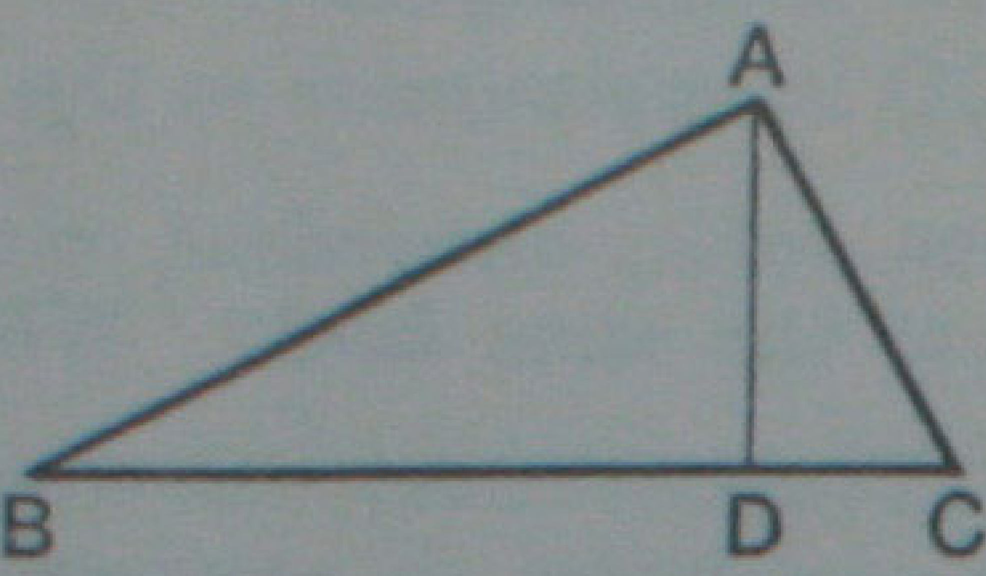
   (ii) If AB = 20 inches, AC = 15 inches, and AY = 9 inches, shew that BX = 8 inches.

   (iii) If X divides AB in the ratio 8 : 3, and if AC = 2·2 inches, shew that AY, YC measure respectively 1·6 and ·6 inches.

10. The vertical angle A of a triangle ABC is bisected by a line which cuts BC at X; if BC = 25 inches in length, and if the sides BA, AC are in the ratio 7 : 3, shew that the segments of the base are 17·5 and 7·5 inches respectively.

PROPOSITION 8. THEOREM.

In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the triangles on each side of it are similar to the whole triangle and to one another.



Let BAC be a triangle right-angled at A, and let AD be drawn perp. to BC.

Then shall the △ˢ BDA, ADC *be similar to the* △ BAC *and to one another.*

In the △ˢ BDA, BAC,
the ∠ BDA = the ∠ BAC, being rt. angles,
and the angle at B is common to both;
∴ the remaining ∠ BAD = the remaining ∠ BCA, I. 32.
that is, the △ BDA is equiangular to the △ BAC;
∴ their sides about the equal angles are proportionals; VI. 4.
∴ the △ˢ BDA, BAC are similar.

In the same way it may be proved that the △ˢ ADC, BAC are similar.

Hence the △ˢ BDA, ADC, having their angles severally equal to those of the △ BAC, are equiangular to one another;
∴ they are similar. VI. 4.

Q.E.D.

COROLLARY. Because the △ˢ BDA, ADC are similar,
∴ BD : DA :: DA : DC;
and because the △ˢ CBA, ABD are similar,
∴ CB : BA :: BA : BD;
and because the △ˢ BCA, ACD are similar,
∴ BC : CA :: CA : CD.

EXERCISES.

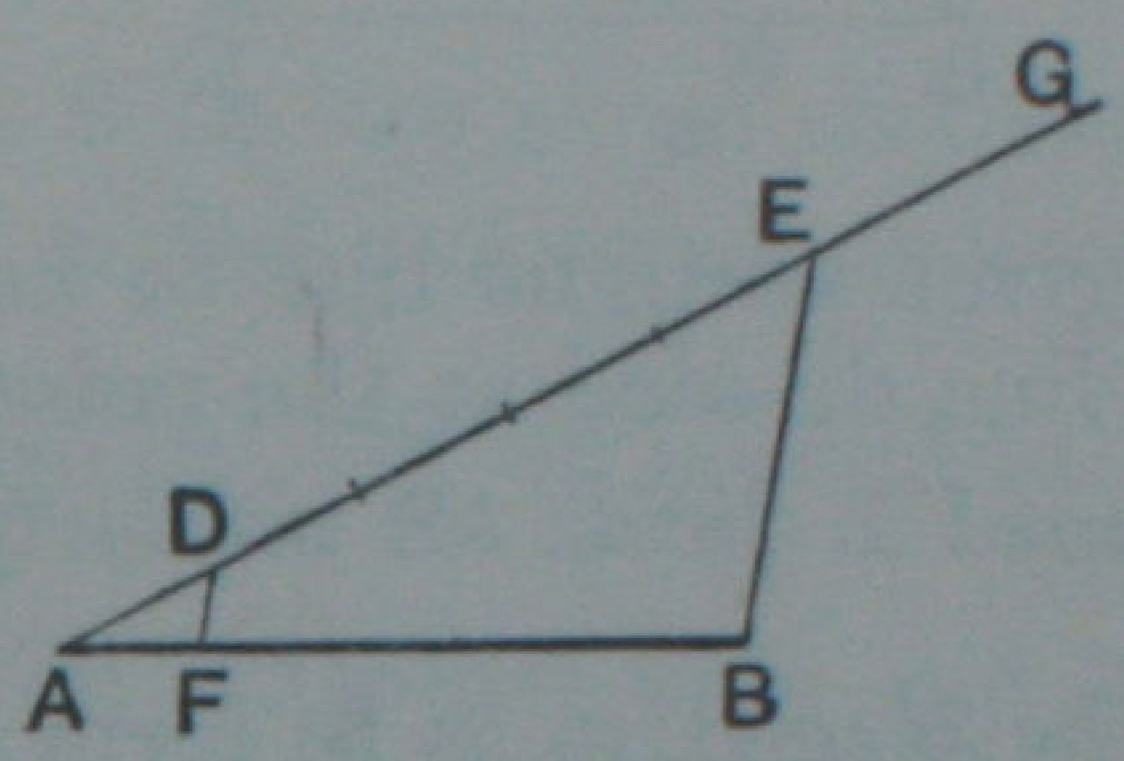
1. In the figure of Prop. 8 prove that the hypotenuse is to one side as the second side is to the perpendicular.

2. *Shew that the radius of a circle is a mean proportional between the segments of any tangent between its point of contact and a pair of parallel tangents.*

DEFINITION. One magnitude is said to be a **submultiple** of another, when the first is contained an *exact* number of times in the second. [Book V. Def. 2.]

PROPOSITION 9. PROBLEM.

From a given straight line to cut off any required submultiple.



Let AB be the given straight line.

It is required to cut off a certain submultiple from AB.

From A draw a straight line AG of indefinite length, making any angle with AB.

In AG take *any* point D; and, by cutting off successive parts each equal to AD, make AE to contain AD as many times as AB contains the required submultiple.

Join EB.

Through D draw DF par' to EB, meeting AB in F.

Then shall AF *be the required submultiple.*

Because DF is par' to EB, a side of the △ AEB,

∴ BF : FA :: ED : DA; VI. 2.

∴, *componendo*, BA : AF :: EA : AD. V. 13.

But AE contains AD the required number of times; *Constr.*

∴ AB contains AF the required number of times;

that is, AF is the required submultiple. Q.E.F.

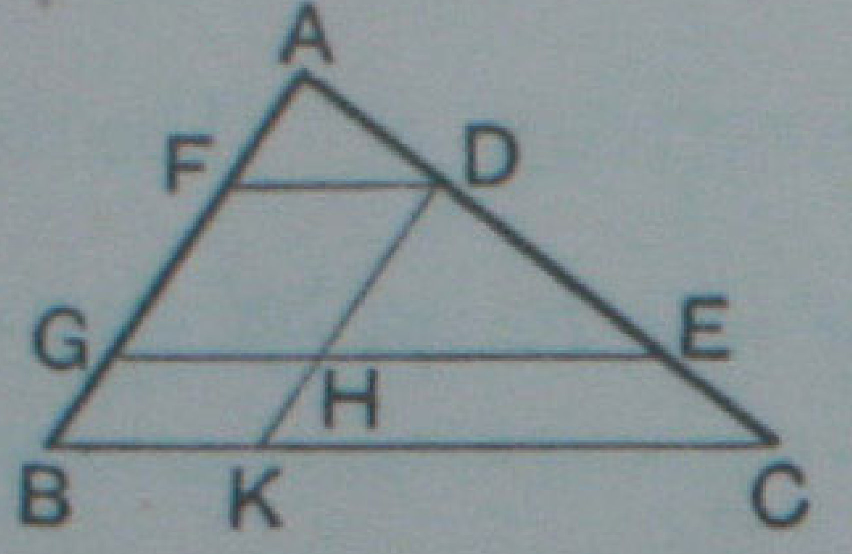
EXERCISES.

1. Divide a straight line into five equal parts.

2. Give a geometrical construction for cutting off two-sevenths of a given straight line.

PROPOSITION 10. PROBLEM.

To divide a straight line similarly to a given divided straight line.



Let AB be the given straight line to be divided, and AC the given straight line divided at the points D and E.

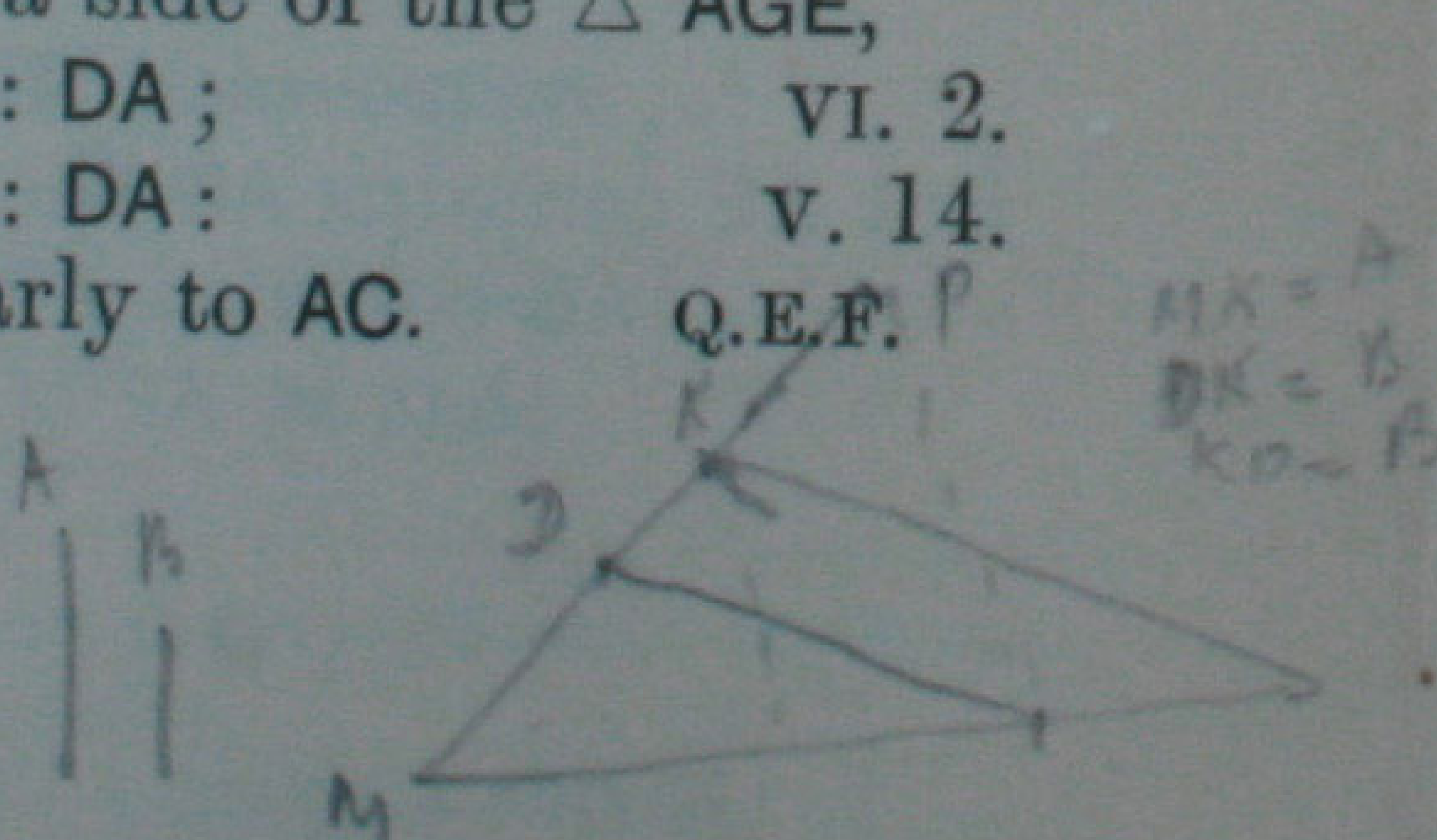
It is required to divide AB *similarly to* AC.

Let AB, AC be placed so as to form any angle.
Join CB.
Through D draw DF par^{l} to CB, I. 31.
and through E draw EG par^{l} to CB.

Then AB *shall be divided at* F *and* G *similarly to* AC.

Through D draw DHK par^{l} to AB.
Now by construction each of the figs. FH, HB is a par^{m};
∴ DH = FG, and HK = GB. I. 34.
Now since HE is par^{l} to KC, a side of the △ DKC,
∴ KH : HD :: CE : ED. VI. 2.
But KH = BG, and HD = GF;
∴ BG : GF :: CE : ED. V. 1.

Again, because FD is par^{l} to GE, a side of the △ AGE,
∴ GF : FA :: ED : DA; VI. 2.
∴, *ex æquali*, BG : FA :: CE : DA: V. 14.
∴ AB is divided similarly to AC. Q.E.F.

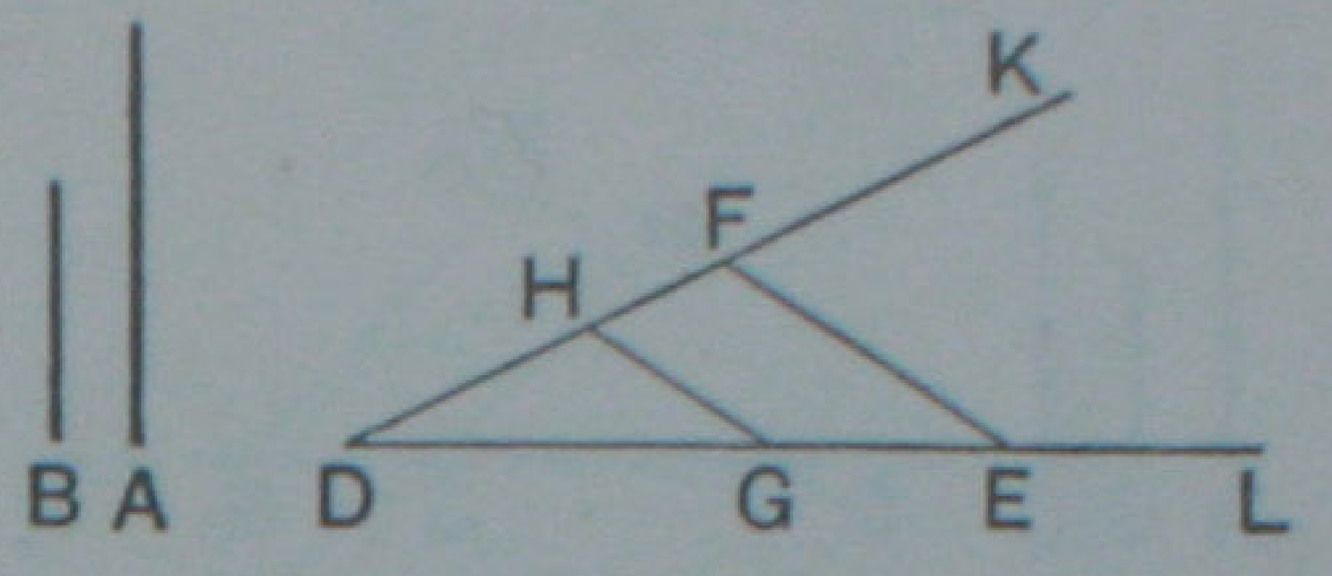


EXERCISE.

Divide a straight line internally and externally in a given ratio. Is this always possible?

Proposition 11. Problem.

To find a third proportional to two given straight lines.



Let A, B be two given straight lines.

It is required to find a third proportional to A *and* B.

Take two st. lines DL, DK of indefinite length, containing any angle.

From DL cut off DG equal to A, and GE equal to B;
and from DK cut off DH also equal to B. I. 3.
Join GH.
Through E draw EF par[l] to GH, meeting DK in F. I. 31.

Then shall HF *be a third proportional to* A *and* B.

Because GH is par[l] to EF, a side of the △ DEF;
∴ DG : GE :: DH : HF. VI. 2.
But DG = A; and GE, DH each = B; *Constr.*
∴ A : B :: B : HF;
that is, HF is a third proportional to A and B.

Q.E.F.

EXERCISES.

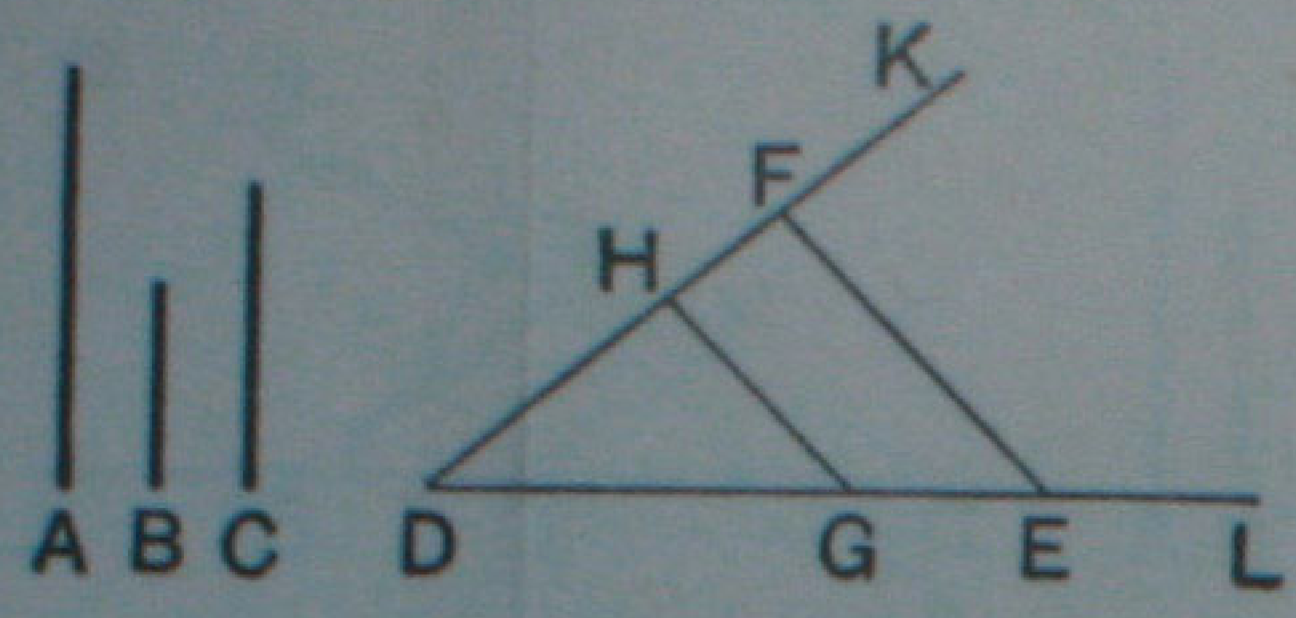
1. AB is a diameter of a circle, and through A any straight line is drawn to cut the circumference in C and the tangent at B in D: shew that AC is a third proportional to AD and AB.

2. ABC is an isosceles triangle having each of the angles at the base double of the vertical angle BAC; the bisector of the angle BCA meets AB at D. Shew that AB, BC, BD are three proportionals.

3. Two circles intersect at A and B; and at A tangents are drawn, one to each circle, to meet the circumferences at C and D: shew that if CB, BD are joined, BD is a third proportional to CB, BA.

PROPOSITION 12. PROBLEM.

To find a fourth proportional to three given straight lines.



Let A, B, C be the three given straight lines.

It is required to find a fourth proportional to A, B, C.

Take two straight lines DL, DK of indefinite length, containing any angle.

From DL cut off DG equal to A, and GE equal to B;

and from DK cut off DH equal to C. I. 3.

Join GH.

Through E draw EF par[l] to GH. I. 31.

Then shall HF *be a fourth proportional to* A, B, C.

Because GH is par[l] to EF, a side of the △ DEF;

∴ DG : GE :: DH : HF. VI. 2.

But DG = A, GE = B, and DH = C; *Constr.*

∴ A : B :: C : HF;

that is, HF is a fourth proportional to A, B, C.

Q.E.F.

EXERCISES.

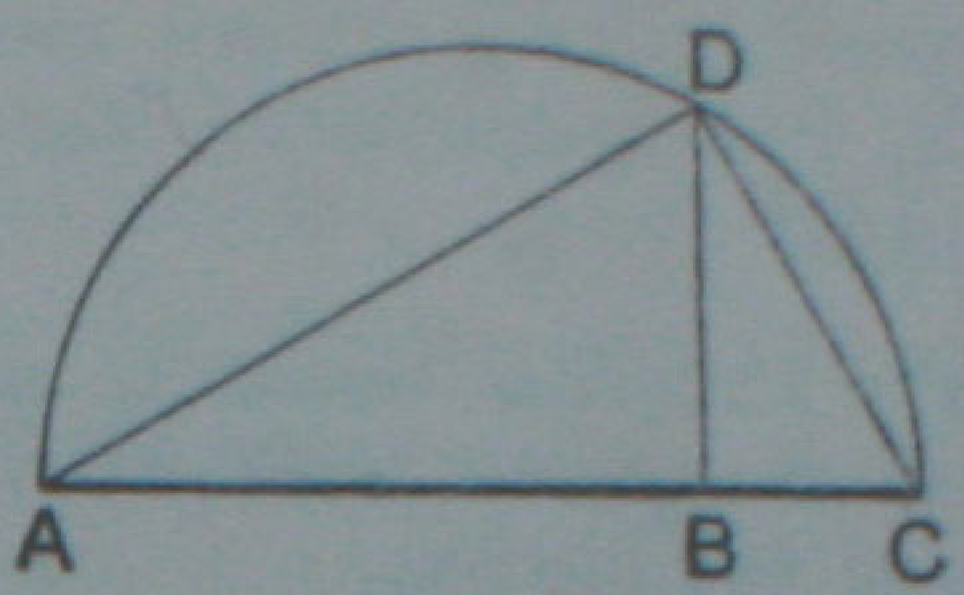
1. If from D, one of the angular points of a parallelogram ABCD, a straight line is drawn meeting AB at E and CB at F; shew that CF is a fourth proportional to EA, AD, and AB.

2. In a triangle ABC the bisector of the vertical angle BAC meets the base at D and the circumference of the circumscribed circle at E: shew that BA, AD, EA, AC are four proportionals.

3. From a point P tangents PQ, PR are drawn to a circle whose centre is C, and QT is drawn perpendicular to RC produced: shew that QT is a fourth proportional to PR, RC, and RT.

Proposition 13. Problem.

To find a mean proportional between two given straight lines.



Let AB, BC be the two given straight lines.

It is required to find a mean proportional between AB *and* BC.

Place AB, BC in a straight line, and on AC describe the semicircle ADC.

From B draw BD at rt. angles to AC. I. 11.

Then shall BD *be a mean proportional between* AB *and* BC.

Join AD, DC.

Now the ∠ ADC, being in a semicircle, is a rt. angle; III. 31.
and because in the right-angled △ ADC, DB is drawn from the rt. angle perp. to the hypotenuse,

∴ the △s ABD, DBC are similar; VI. 8.

∴ AB : BD :: BD : BC;

that is, BD is a mean proportional between AB and BC.

Q.E.F.

EXERCISES.

1. If from one angle A of a parallelogram a straight line is drawn cutting the diagonal in E and the sides in P, Q, shew that AE is a mean proportional between PE and EQ.

2. A, B, C are three points in order in a straight line: find a point P in the straight line so that PB may be a mean proportional between PA and PC.

3. The diameter AB of a semicircle is divided at any point C, and CD is drawn at right angles to AB meeting the circumference in D; DO is drawn to O the centre, and CE is perpendicular to OD: shew that DE is a third proportional to AO and DC.

4. AC is the diameter of a semicircle on which a point B is taken so that BC is equal to the radius: shew that AB is a mean proportional between BC and the sum of BC, CA.

5. A is any point in a semicircle on BC as diameter; from D any point in BC a perpendicular is drawn meeting AB, AC, and the circumference in E, G, F respectively; shew that DG is a third proportional to DE and DF.

6. Two circles have external contact, and a common tangent touches them at A and B: prove that AB is a mean proportional between the diameters of the circles. [See Ex. 21, p. 237.]

7. If a straight line is divided at two given points, determine a third point such that its distances from the extremities may be proportional to its distances from the given points.

8. AB is a straight line divided at C and D so that AB, AC, AD are in continued proportion; from A a line AE is drawn in any direction and equal to AC; shew that BC and CD subtend equal angles at E.

9. In a given triangle draw a straight line parallel to one of the sides, so that it may be a mean proportional between the segments of the base.

10. On the radius OA of a quadrant OAB, a semicircle ODA is described, and at A a tangent AE is drawn; from O any line ODFE is drawn meeting the circumferences in D and F and the tangent in E: if DG is drawn perpendicular to OA, shew that OE, OF, OD, and OG are in continued proportion.

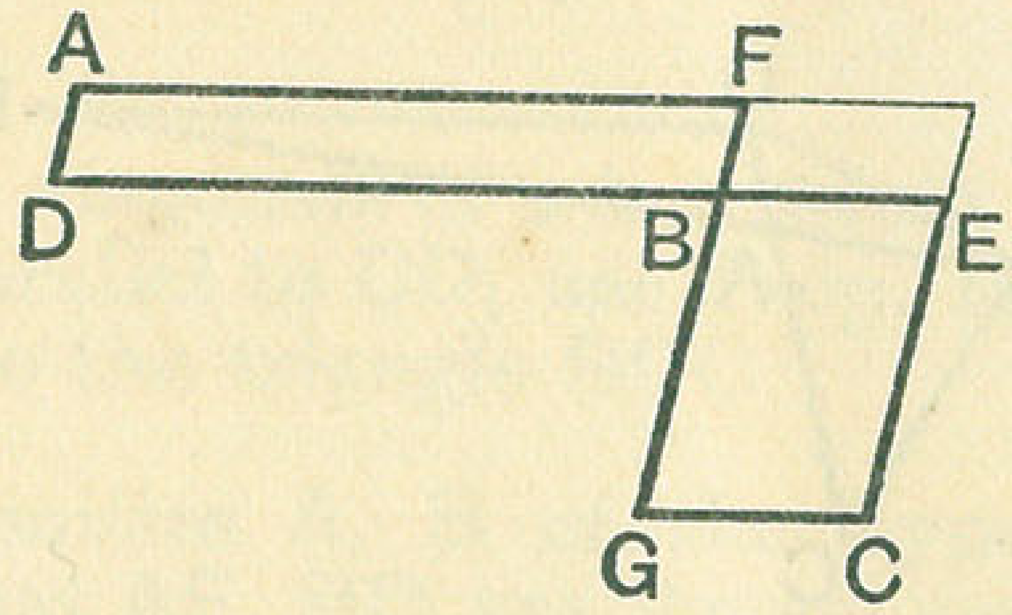
11. From any point A, on the circumference of the circle ABE, as centre, and with any radius, a circle BDC is described cutting the former circle in B and C; from A any line AFE is drawn meeting the chord BC in F, and the circumferences BDC, ABE in D, E respectively: shew that AD is a mean proportional between AF and AE.

DEFINITION. Two figures are said to have their sides about one angle in each **reciprocally proportional**, when a side of the *first* figure is to a side of the *second* as the remaining side of the *second* figure is to the remaining side of the *first*. [Book VI. Def. 4.]

Proposition 14. Theorem.

Parallelograms which are equal in area, and which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional.

Conversely, parallelograms which have one angle of the one equal to one angle of the other, and the sides about these angles reciprocally proportional, are equal in area.



Let the par^{ms} AB, BC be of equal area, and have the ∠ DBF equal to the ∠ GBE.

Then shall the sides about the ∠^{s} DBF, GBE *be reciprocally proportional,*

namely, DB : BE :: GB : BF.

Place the par^{ms} so that DB, BE may be in the same straight line;

∴ FB, BG are also in one straight line. I. 14.

Complete the par^m FE.

Then because the par^m AB = the par^m BC, *Hyp.*

and FE is another par^m,

∴ the par^m AB : the par^m FE :: the par^m BC : the par^m FE ;

but the par^m AB : the par^m FE :: DB : BE, VI. 1. *Cor.*

and the par^m BC : the par^m FE :: GB : BF ;

∴ DB : BE :: GB : BF. V. 1.

Conversely. Let the ∠ DBF be equal to the ∠ GBE,

and let DB : BE :: GB : BF.

Then shall the par^m AB *be equal in area to the par^m* BC.

For, with the same construction as before,

by hypothesis, DB : BE :: GB : BF ;

but DB : BE :: the par^m AB : the par^m FE, VI. 1.

and GB : BF :: the par^m BC : the par^m FE,

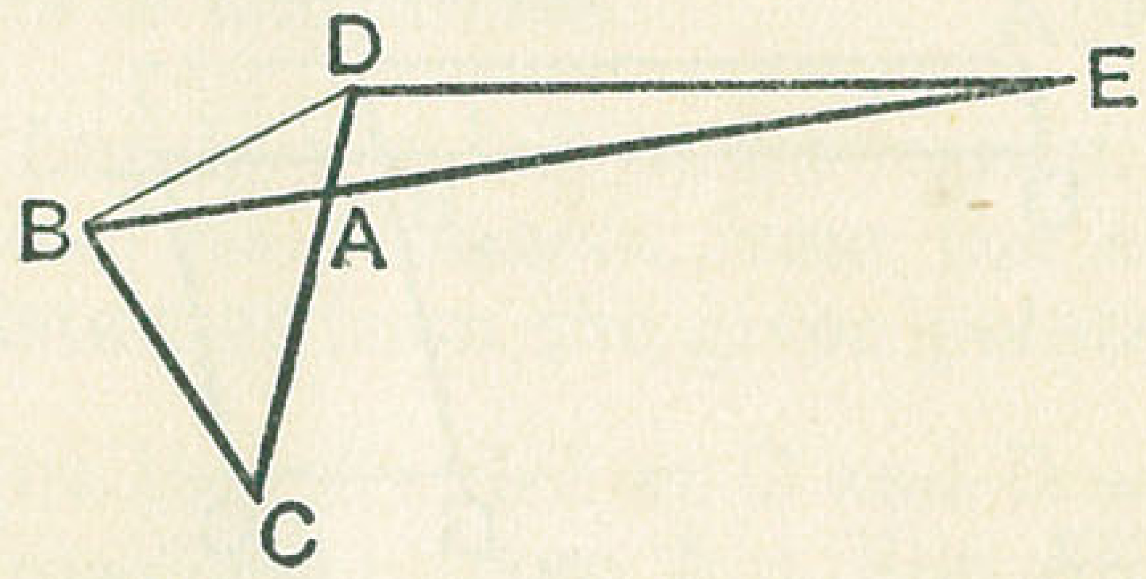
∴ the par^m AB : the par^m FE :: the par^m BC : the par^m FE ; V. 1.

∴ the par^m AB = the par^m BC. Q.E.D.

Proposition 15. Theorem.

Triangles which are equal in area, and which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional.

Conversely, triangles which have one angle of the one equal to one angle of the other, and the sides about these angles reciprocally proportional, are equal in area.



Let the △ˢ CAB, EAD be of equal area, and have the ∠ CAB equal to the ∠ EAD.

Then shall the sides about the ∠ˢ CAB, EAD *be reciprocally proportional,*
namely, CA : AD :: EA : AB.

Place the △ˢ so that CA and AD may be in the same st. line;
∴ BA, AE are also in one st. line. I. 14.
Join BD.

Then because the △ CAB = the △ EAD, *Hyp.*
and ABD is another triangle,
∴ the △ CAB : the △ ABD :: the △ EAD : the △ ABD;
but the △ CAB : the △ ABD :: CA : AD, VI. 1.
and the △ EAD : the △ ABD :: EA : AB;
∴ CA : AD :: EA : AB. V. 1.

Conversely. Let the ∠ CAB be equal to the ∠ EAD,
and let CA : AD :: EA : AB.
Then shall the △ CAB = *the* △ EAD.

For, with the same construction as before,
by hypothesis, CA : AD :: EA : AB;
but CA : AD :: the △ CAB : the △ ABD, VI. 1.
and EA : AB :: the △ EAD : the △ ABD;
∴ the △ CAB : the △ ABD :: the △ EAD : the △ ABD; V. 1.
∴ the △ CAB = the △ EAD. Q.E.D.

EXERCISES.

ON PROPOSITIONS 14 AND 15.

1. *Parallelograms which are equal in area and which have their sides reciprocally proportional, have their angles respectively equal.*

2. *Triangles which are equal in area, and which have the sides about a pair of angles reciprocally proportional, have those angles equal or supplementary.*

3. AC, BD are the diagonals of a trapezium which intersect in O; if the side AB is parallel to CD, use Prop. 15 to prove that the triangle AOD is equal to the triangle BOC. ✓

4. From the extremities A, B of the hypotenuse of a right-angled triangle ABC lines AE, BD are drawn perpendicular to AB, and meeting BC and AC produced in E and D respectively: employ Prop. 15 to shew that the triangles ABC, ECD are equal in area. ✓

5. On AB, AC, two sides of any triangle, squares are described externally to the triangle. If the squares are ABDE, ACFG, shew that the triangles DAG, FAE are equal in area.

6. ABCD is a parallelogram; from A and C any two parallel straight lines are drawn meeting DC and AB in E and F respectively; EG, which is parallel to the diagonal AC, meets AD in G: shew that the triangles DAF, GAB are equal in area.

7. Describe an isosceles triangle equal in area to a given triangle and having its vertical angle equal to one of the angles of the given triangle.

8. Prove that the equilateral triangle described on the hypotenuse of a right-angled triangle is equal to the sum of the equilateral triangles described on the sides containing the right angle.

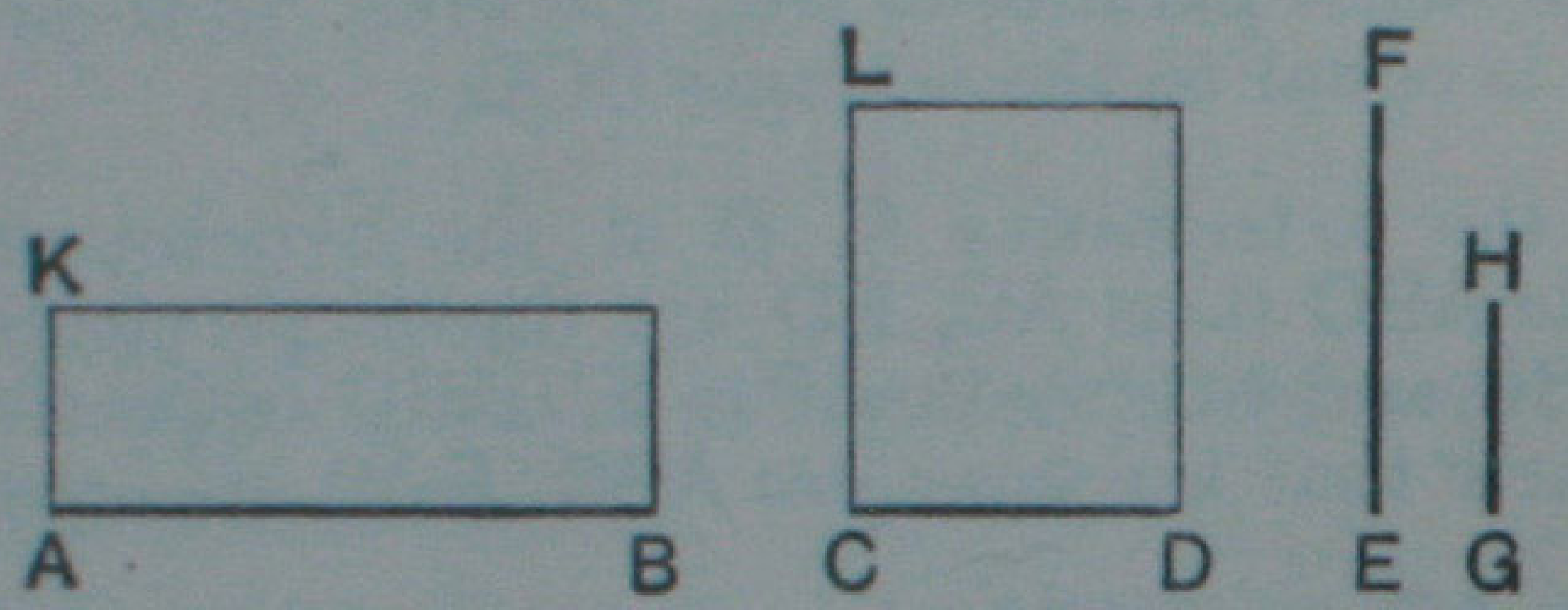
[Let ABC be the triangle right-angled at C; and let BXC, CYA, AZB be the equilateral triangles. Draw CD perpendicular to AB; and join DZ. Then shew by Prop. 15 that the △ AYC = the △ DAZ; and similarly that the △ BXC = the △ BDZ.]

Proposition 16. Theorem.

If four straight lines are proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means.

Conversely, if the rectangle contained by the extremes is equal to the rectangle contained by the means, the four straight lines are proportional.

If a:b::c:d, ad = bc



Let the st. lines AB, CD, EF, GH be proportional, so that

AB : CD :: EF : GH.

Then shall the rect. AB, GH = *the rect.* CD, EF.

From A draw AK perp. to AB, and equal to GH. I. 11, 3.

From C draw CL perp. to CD, and equal to EF.

Complete the par^ms KB, LD.

Then because AB : CD :: EF : GH ; *Hyp.*

and EF = CL, and GH = AK ; *Constr.*

∴ AB : CD :: CL : AK ;

that is, the par^ms KB, LD have their sides about the equal angles at A and C reciprocally proportional ;

∴ KB = LD. VI. 14.

But KB is the rect. AB, GH, for AK = GH, *Constr.*

and LD is the rect. CD, EF, for CL = EF ;

∴ the rect. AB, GH = the rect. CD, EF.

Conversely. Let the rect. AB, GH = the rect. CD, EF.

Then shall AB : CD :: EF : GH.

For, with the same construction as before,

because the rect. AB, GH = the rect. CD, EF; *Hyp.*

and the rect. AB, GH = KB, for GH = AK, *Constr.*

and the rect. CD, EF = LD, for EF = CL;

∴ KB = LD;

that is, the par^ms KB, LD, which have the angle at A equal to the angle at C, are equal in area;

∴ the sides about the equal angles are reciprocally proportional;

that is, AB : CD :: CL : AK;

∴ AB : CD :: EF : GH.

Q.E.D.

QUESTIONS FOR REVISION.

1. State and prove the *algebraical* theorem corresponding to Proposition 16.

2. Define the terms: *multiple, submultiple, fourth proportional, third proportional, mean proportional.*

3. ABC is a triangle right-angled at A, and AD is drawn perpendicular to BC: if AB, AC measure respectively 12 and 5 inches, shew that the segments of the hypotenuse are $11\frac{1}{13}$ and $1\frac{12}{13}$ inches.

4. Find in inches the length of the mean proportional between 1 inch and 3 inches. Hence give a geometrical construction for drawing a line $\sqrt{3}$ inches in length: and extend the method to finding a line $\sqrt{n}$ inches long.

5. A straight line AB, 21 inches in length, is divided at F and G into parts of 5, 7, 9 inches respectively. If a second line AC, 35 inches long, is similarly divided by the method of Proposition 10, shew that the lengths of the parts are $8\frac{1}{3}$, $11\frac{2}{3}$ and 15 inches respectively.

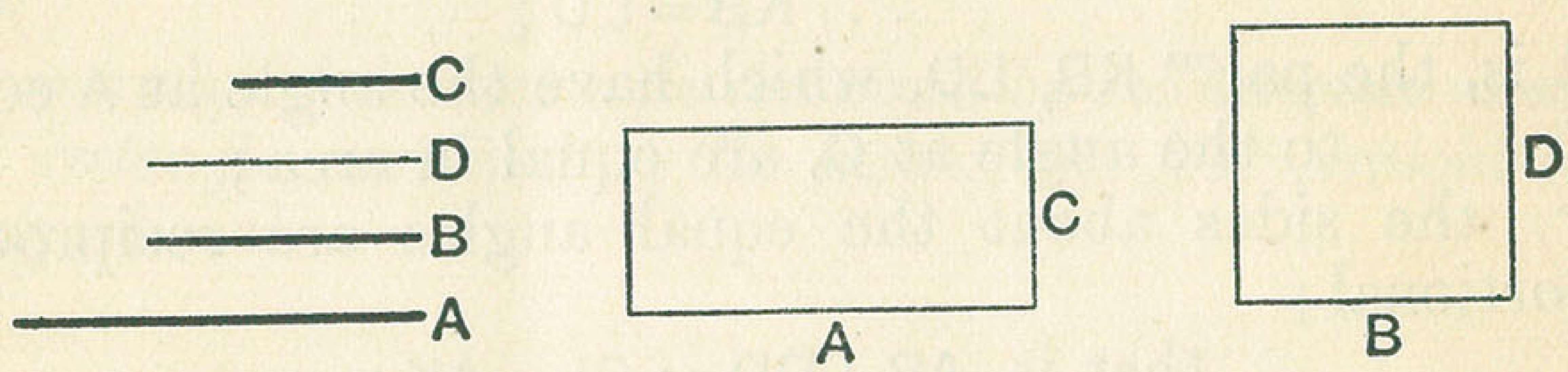
6. When are figures said to have their sides about one angle in each *reciprocally proportional?* Two equal parallelograms ABCD, EFGH have their angles at B and F equal: if AB = 2 inches, BC = 10 inches, and EF = 5 inches; find the length of FG.

If a:b::b:c then ac=b²

Proposition 17. Theorem.

If three straight lines are proportional the rectangle contained by the extremes is equal to the square on the mean.

Conversely, if the rectangle contained by the extremes is equal to the square on the mean, the three straight lines are proportional.



Let the three st. lines A, B, C be proportional, so that

A : B :: B : C.

Then shall the rect. A, C *be equal to the sq. on* B.

Take D equal to B.

Then because A : B :: B : C, and D = B;

∴ A : B :: D : C;

∴ the rect. A, C = the rect. B, D; VI. 16.

but the rect. B, D = the sq. on B, for D = B;

∴ the rect. A, C = the sq. on B.

Conversely. Let the rect. A, C = the sq. on B.

Then shall A : B :: B : C.

For, with the same construction as before,

because the rect. A, C = the sq. on B, *Hyp.*

and the sq. on B = the rect. B, D, for D = B;

∴ the rect. A, C = the rect. B, D;

∴ A : B :: D : C, VI. 16.

that is, A : B :: B : C. Q.E.D.

QUESTIONS FOR REVISION.

1. State and prove the *algebraical* theorem corresponding to Proposition 17.

2. Two adjacent sides of a rectangle measure 12·1 and ·9 inches in length; shew that the side of an equal square is 3·3 inches.

3. ABC is an isosceles triangle, the equal sides each measuring 12 inches. DAE is a triangle of equal area, having the angle DAE equal to the angle CAB. If AD = 36 inches, find the length of AE.

EXERCISES.

ON PROPOSITIONS 16 AND 17.

1. Apply Proposition 16 to prove that if two chords of a circle intersect, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

2. Prove that the rectangle contained by the sides of a right-angled triangle is equal to the rectangle contained by the hypotenuse and the perpendicular drawn to it from the right angle.

3. On a given straight line construct a rectangle equal to a given rectangle.

4. ABCD is a parallelogram ; from B any straight line is drawn cutting the diagonal AC at F, the side DC at G, and the side AD produced at E : shew that the rectangle EF, FG is equal to the square on BF.

5. On a given straight line as base describe an isosceles triangle equal to a given triangle.

6. AB is a diameter of a circle, and any line ACD cuts the circle in C and the tangent at B in D ; shew by Prop. 17 that the rectangle AC, AD is constant.

7. The exterior angle at A of a triangle ABC is bisected by a straight line which meets the base in D and the circumscribed circle in E : shew that the rectangle BA, AC is equal to the rectangle EA, AD.

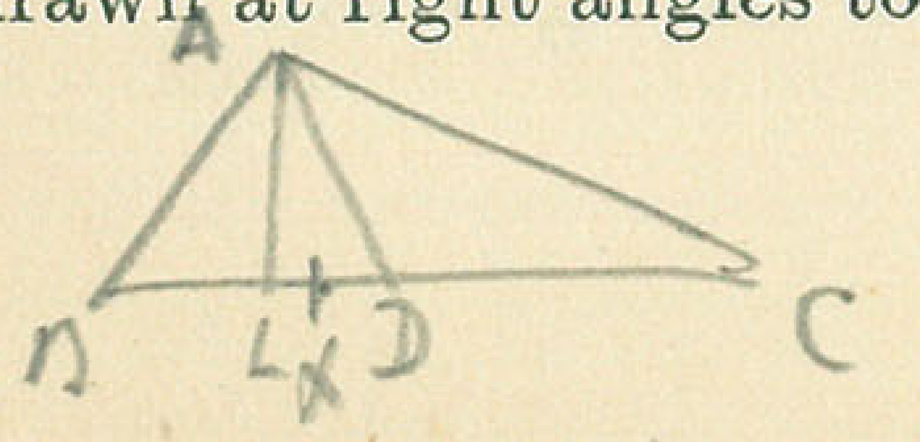
8. If two chords AB, AC drawn from any point A in the circumference of the circle ABC are produced to meet the tangent at the other extremity of the diameter through A in D and E, shew that the triangle AED is similar to the triangle ABC.

9. At the extremities of a diameter of a circle tangents are drawn ; these meet the tangent at a point P in Q and R : shew that the rectangle QP, PR is constant for all positions of P.

10. A is the vertex of an isosceles triangle ABC inscribed in a circle, and ADE is a straight line which cuts the base in D and the circle in E ; shew that the rectangle EA, AD is equal to the square on AB.

11. Two circles touch one another externally at A ; a straight line touches the circles at B and C, and is produced to meet the straight line joining the centres at S : shew that the rectangle SB, SC is equal to the square on SA.

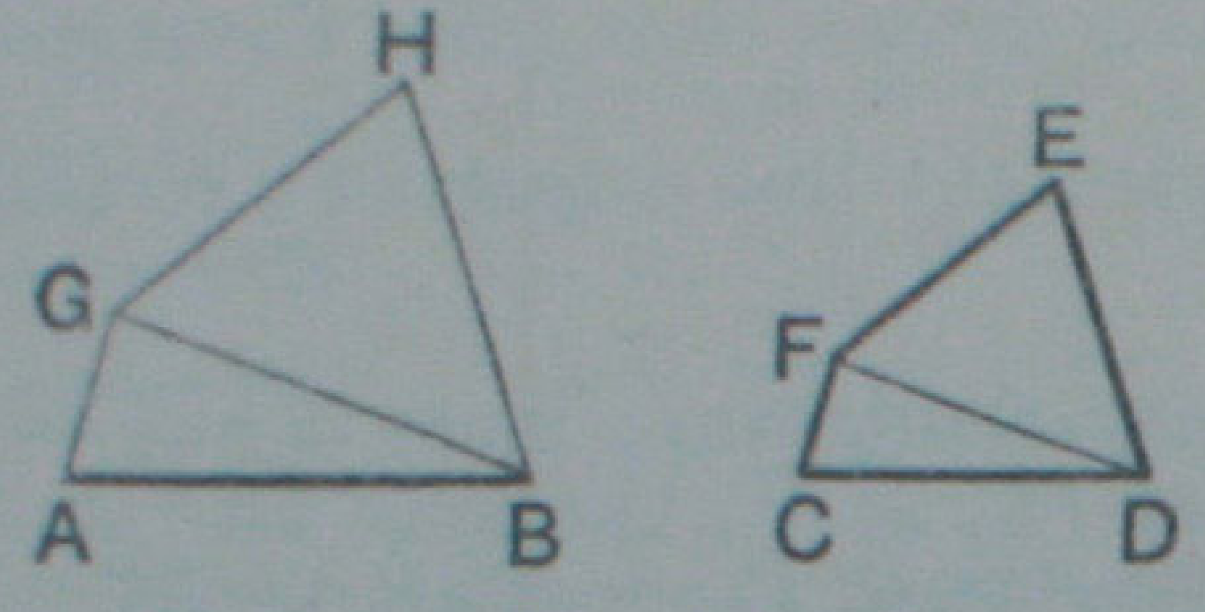
12. Divide a triangle into two equal parts by a straight line drawn at right angles to one of the sides.



DEFINITION. Two similar rectilineal figures are said to be **similarly situated** with respect to two of their sides when these sides are *homologous*. [Book VI. Def. 3.]

PROPOSITION 18. PROBLEM.

On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure.



Let AB be the given st. line, and CDEF the given rectilineal figure.

It is required to describe on the st. line AB *a rectilineal figure similar and similarly situated to* CDEF.

First suppose CDEF to be a quadrilateral.

Join DF.

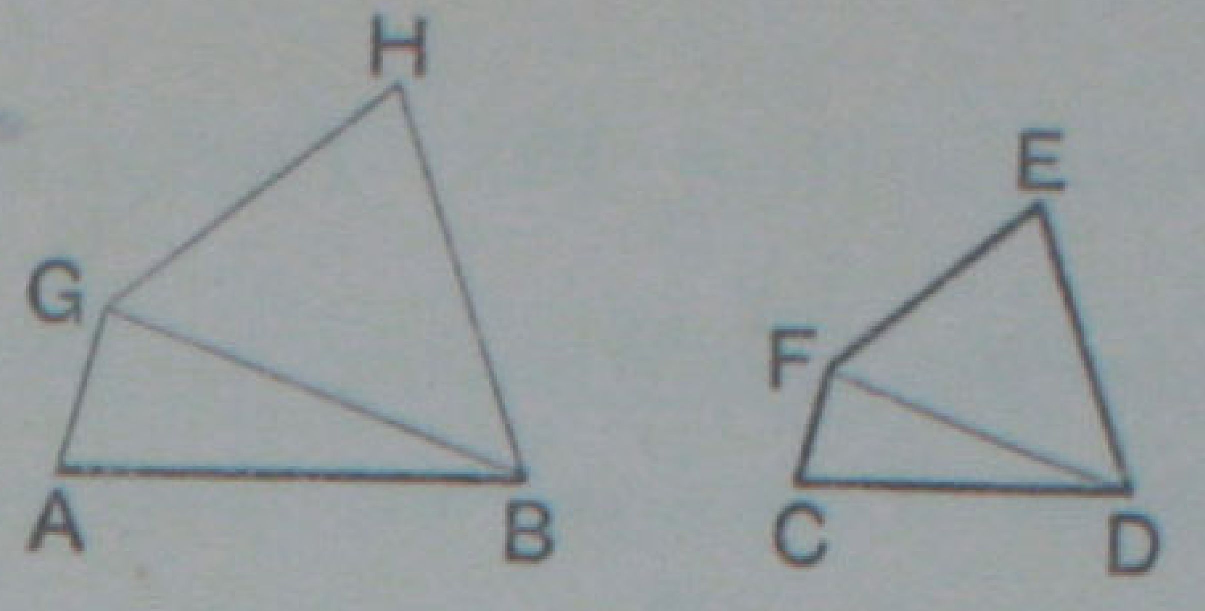
At A in BA make the ∠ BAG equal to the ∠ DCF, I. 23.
and at B in AB make the ∠ ABG equal to the ∠ CDF;
∴ the remaining ∠ AGB = the remaining ∠ CFD; I. 32.
and the △ AGB is equiangular to the △ CFD.

Again at B in GB make the ∠ GBH equal to the ∠ FDE,
and at G in BG make the ∠ BGH equal to the ∠ DFE; I. 23.
∴ the remaining ∠ BHG = the remaining ∠ DEF; I. 32.
and the △ BHG is equiangular to the △ DEF.

Then shall ABHG *be the required figure.*

(i) To prove that the fig. ABHG is equiangular to the fig. CDEF.

Because the ∠ AGB = the ∠ CFD, *Proved.*
and the ∠ BGH = the ∠ DFE; *Constr.*
∴ the whole ∠ AGH = the whole ∠ CFE.
Similarly the ∠ ABH = the ∠ CDE;
and the angles at A and H are respectively equal to the angles at C and E; *Constr. and proof.*
∴ the fig. ABHG is equiangular to the fig. CDEF.



(ii) To prove that the figs. ABHG, CDEF have the sides about their equal angles proportional.

Because the △ AGB is equiangular to the △ CFD,

∴ AG : GB :: CF : FD. VI. 4.

And because the △ BGH is equiangular to the △ DFE,

∴ BG : GH :: DF : FE;

∴, *ex æquali*, AG : GH :: CF : FE. V. 14.

Similarly it may be shewn that

AB : BH :: CD : DE.

Also BA : AG :: DC : CF, VI. 4.

and GH : HB :: FE : ED.

∴ the figs. ABHG, CDEF are equiangular and have their sides about the equal angles proportional;

that is, ABHG is similar to CDEF. VI. *Def* 2.

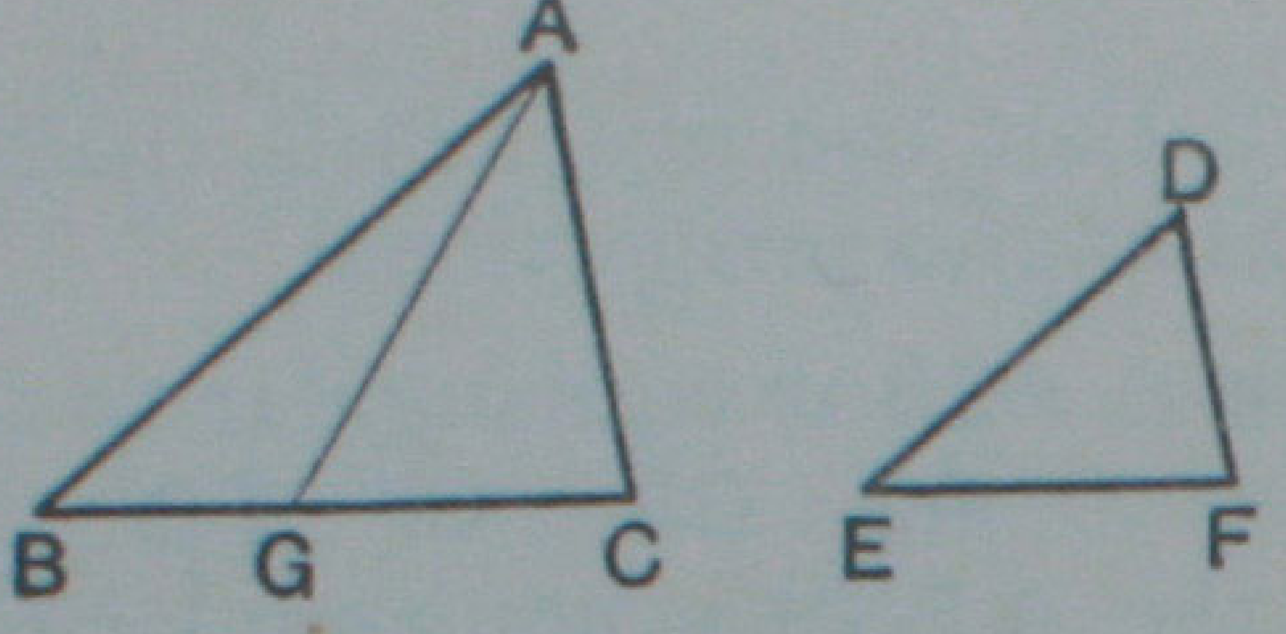
In like manner the process of construction may be extended to a figure of five or more sides.

Q.E.F.

DEFINITION. When three magnitudes are proportionals the *first* is said to have to the *third* the **duplicate ratio** of that which it has to the *second*. [Book V. Def. 13.]

Proposition 19. Theorem.

Similar triangles are to one another in the duplicate ratio of their homologous sides.



Let ABC, DEF be similar triangles, having the ∠ ABC equal to the ∠ DEF, and let BC and EF be homologous sides.

Then shall the △ ABC *be to the* △ DEF *in the duplicate ratio of* BC *to* EF.

To BC and EF take a *third* proportional BG,
so that BC : EF :: EF : BG. VI. 11.
Join AG.

Then because the △s ABC, DEF are similar, *Hyp.*
∴ AB : BC :: DE : EF ;
∴, *alternately*, AB : DE :: BC : EF ; V. 11.
but BC : EF :: EF : BG ; *Constr.*
∴ AB : DE :: EF : BG ; V. 1.
that is, the sides of the △s ABG, DEF about the equal angles at B and E are reciprocally proportional ;
∴ the △ ABG = the △ DEF. VI. 15.

Again, because BC : EF :: EF : BG, *Constr.*
∴ BC : BG in the duplicate ratio of BC to EF. V. *Def.* 13.
But the △ ABC : the △ ABG :: BC : BG ; VI. 1.
∴ the △ ABC : the △ ABG in the duplicate ratio of BC to EF : V. 1.
and the △ ABG = the △ DEF ; *Proved.*
∴ the △ ABC : the △ DEF in the duplicate ratio of BC : EF.

Q.E.D.

QUESTIONS FOR REVISION, AND NUMERICAL ILLUSTRATIONS.

1. Quote the Geometrical and Algebraical definitions of the *duplicate of the ratio* $a : b$; and deduce the latter from the former. Estimate numerically the duplicate of the ratio 36 : 21.

2. The smaller of two similar triangles has an area of 20 square feet, and two corresponding sides are 3 ft. 6 in. and 2 ft. 4 in. respectively: shew that the area of the greater triangle is 45 square feet.

3. XY is drawn parallel to BC, the base of a triangle ABC, to meet the other sides at X and Y: if AX and XB measure respectively 3 inches and 7 inches, shew that the areas of the triangles AXY, ABC are in the ratio 9 : 100.

4. Two similar triangles have areas in the ratio 529 : 361; shew that any pair of homologous sides are to one another as 23 : 19.

5. When are similar figures said to be *similarly situated*? Shew that similar and similarly situated triangles are to one another in the duplicate ratio of their altitudes.

6. Two similar and similarly situated triangles have areas in the ratio 1369 : 1681; if the altitude of the greater is 10 ft. 3 in., shew that the altitude of the other is 1 foot less.

7. The sides of a triangle are 11, 23, 29; find the sides of a similar triangle whose area is 289 times that of the former.

8. Shew how to draw a straight line XY parallel to BC the base of a triangle ABC, so that the area of the triangle AXY may be *nine-sixteenths* of that of the triangle ABC.

9. XY is drawn parallel to the base BC of a triangle ABC, so that the triangle AXY has to the figure XBCY the ratio 4 : 5; shew that AB and AC are cut by XY in the ratio 2 : 1.

10. A triangle ABC is bisected by a straight line XY drawn parallel to the base BC. In what ratio is AB divided at X?

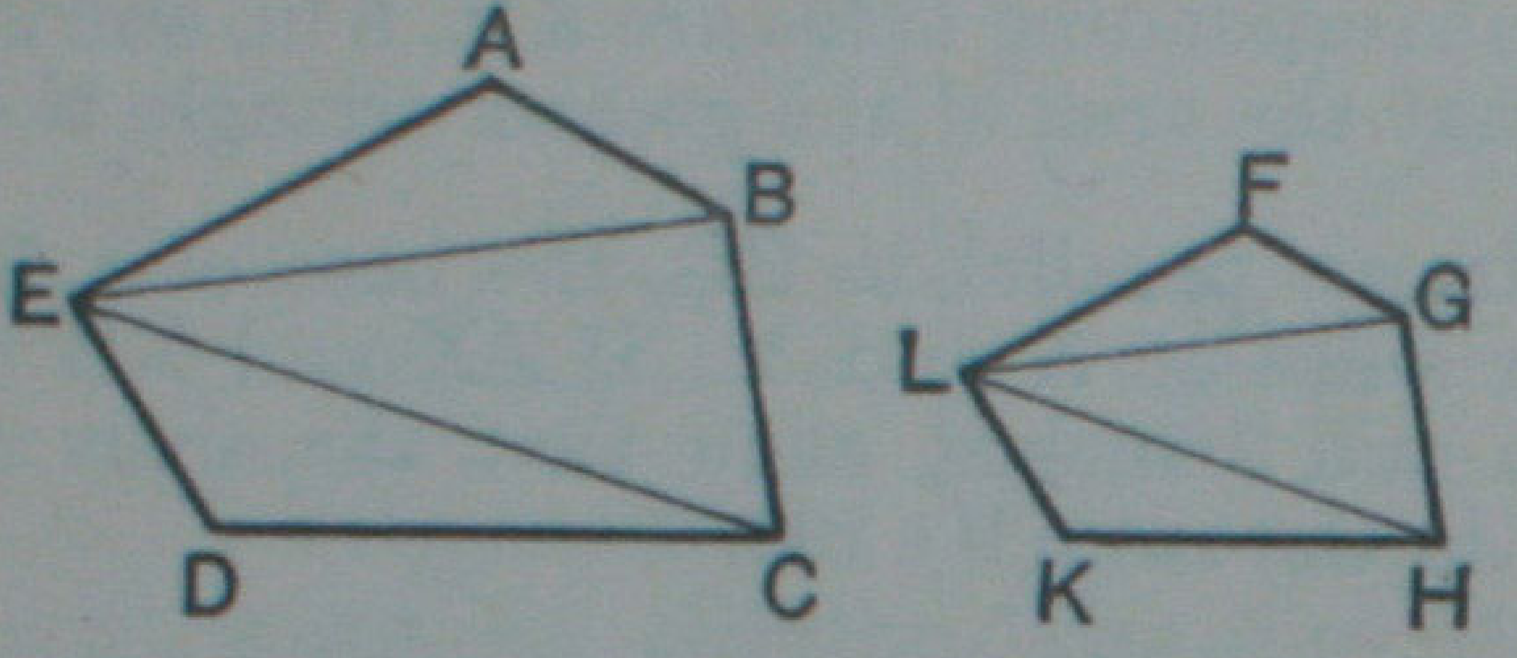
Hence shew how to bisect a triangle by a straight line drawn parallel to the base.

11. ABC is a triangle whose area is 16 square feet; and XY is drawn parallel to BC, dividing AB in the ratio 3 : 5; shew that if BY is joined, the area of the triangle BXY is 3 sq. ft. 108 sq. in.

12. ABC is a triangle right-angled at A, and AD is the perpendicular drawn from A to the hypotenuse: if the area of the triangle ABC is 54 square inches and AB is 1 foot, shew that the area of the triangle ADC is 19·44 square inches.

Proposition 20. Theorem.

Similar polygons may be divided into the same number of similar triangles, having the same ratio each to each that the polygons have; and the polygons are to one another in the duplicate ratio of their homologous sides.



Let ABCDE, FGHKL be similar polygons, and let AB and FG be homologous sides.

Then (i) *the polygons may be divided into the same number of similar triangles;*

(ii) *these triangles shall have each to each the same ratio that the polygons have;*

(iii) *the polygon* ABCDE *shall be to the polygon* FGHKL *in the duplicate ratio of* AB *to* FG.

Join EB, EC, LG, LH.

(i) Then because the polygon ABCDE is similar to the polygon FGHKL, *Hyp.*

∴ the ∠ EAB = the ∠ LFG,

and EA : AB :: LF : FG; VI. *Def.* 2.

∴ the △ EAB is similar to the △ LFG; VI. 6.

∴ the ∠ ABE = the ∠ FGL.

But because the polygons are similar, *Hyp.*

∴ the ∠ ABC = the ∠ FGH; VI. *Def.* 2.

∴ the remaining ∠ EBC = the remaining ∠ LGH.

And because the △s EAB, LFG are similar, *Proved.*

∴ EB : BA :: LG : GF;

and because the polygons are similar, *Hyp.*

∴ AB : BC :: FG : GH; VI. *Def.* 2.

∴, *ex æquali*, EB : BC :: LG : GH; V. 14.

that is, the sides about the equal ∠s EBC, LGH are proportionals;

∴ the △ EBC is similar to the △ LGH. VI. 6.

In the same way it may be proved that the △ ECD is similar to the △ LHK.

∴ the polygons have been divided into the same number of similar triangles.

(ii) Again, because the △ EAB is similar to the △ LFG,
∴ the △ EAB is to the △ LFG in the duplicate ratio of EB : LG; VI. 19.

and, in like manner,
the △ EBC is to the △ LGH in the duplicate ratio of EB to LG;

∴ the △ EAB : the △ LFG :: the △ EBC : the △ LGH. V. 1.
In like manner it can be shewn that
the △ EBC : the △ LGH :: the △ ECD : the △ LHK;
∴ the △ EAB : the △ LFG :: the △ EBC : the △ LGH
:: the △ ECD : the △ LHK.

But in a series of equal ratios, as each antecedent is to its consequent so is the sum of the antecedents to the sum of the consequents; [*Addendo.* V. 12.]

∴ the △ EAB : the △ LFG :: the fig. ABCDE : the fig. FGHKL.

(iii) Now the △ EAB : the △ LFG in the duplicate ratio of AB : FG, VI. 19.

and the △ EAB : the △ LFG :: the fig. ABCDE : the fig. FGHKL;
∴ the fig. ABCDE : the fig. FGHKL in the duplicate ratio of AB : FG. Q.E.D.

Corollary 1. Let a third proportional X be taken to AB and FG,
then AB is to X in the duplicate ratio of AB : FG;
but the fig. ABCDE : the fig. FGHKL in the duplicate ratio of AB : FG; *Proved.*
∴ AB : X :: the fig. ABCDE : the fig. FGHKL.

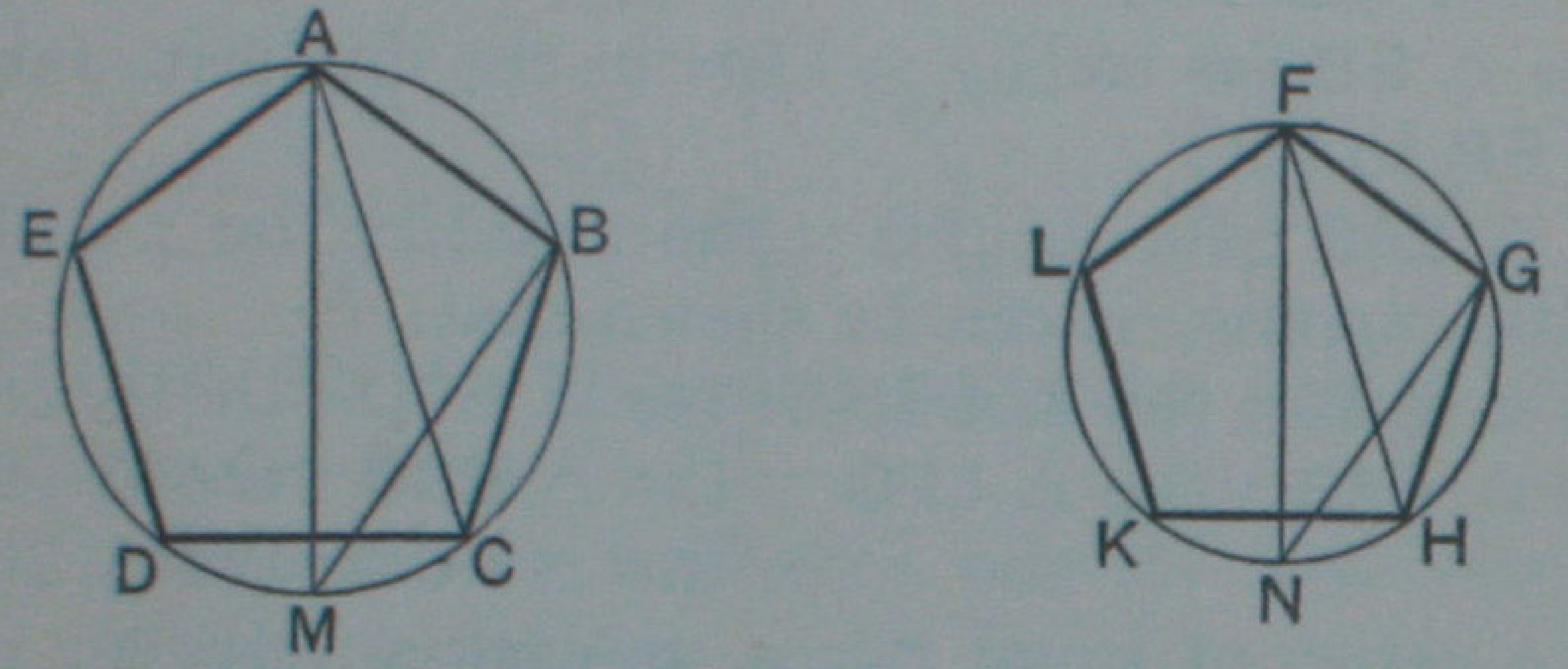
Hence, if three straight lines are proportionals, as the first *is to the* third, *so is any rectilineal figure described on the* first *to a similar and similarly described rectilineal figure on the* second.

Corollary 2. It follows that *similar rectilineal figures are to one another as the squares on their homologous sides.* For squares are similar figures and therefore are to one another in the duplicate ratio of their sides.

Obs. The following theorem, taken from Euclid's Twelfth Book, is given here as an important application of the preceding proposition.

Book XII. Proposition 1.

The areas of similar polygons inscribed in circles are to one another as the squares on the diameters.



Let ABCDE and FGHKL be two similar polygons, inscribed in the circles ACE, FHL, of which AM, FN are diameters.

Then shall

the fig. ABCDE : *the fig.* FGHKL :: *the sq. on* AM : *the sq. on* FN.

Join BM, AC and GN, FH.

Then since the polygon ABCDE is similar to the polygon FGHKL,

∴ the ∠ ABC = the ∠ FGH,

and AB : BC :: FG : GH; VI. *Def.* 2.

∴ the △ ABC is similar to the △ FGH; VI. 6.

∴ the ∠ ACB = the ∠ FHG.

But the ∠ ACB = the ∠ AMB; III. 21.

and the ∠ FHG = the ∠ FNG;

∴ the ∠ AMB = the ∠ FNG.

Also in the △s ABM, FGN, the ∠s ABM, FGN are equal, being rt. angles; III. 31.

hence the remaining ∠s BAM, GFN are equal; I. 32.

and the △s ABM, FGN are similar: VI. 4.

∴ AB : FG :: AM : FN.

But the fig. ABCDE : the fig. FGHKL in the duplicate ratio of AB : FG, VI. 20.

that is, in the duplicate ratio of AM : FN. V. 16.

Hence

the fig. ABCDE : the fig. FGHKL :: the sq. on AM : the sq. on FN.

VI. 20, *Cor.* 2.

Obs. The following theorem, which forms Proposition 3 of Euclid's Twelfth Book, may be derived as a corollary from the preceding proof.

COROLLARY. *The areas of circles are to one another as the squares on their diameters.*

It has been shewn that

the fig. ABCDE : the fig. FGHKL :: the sq. on AM : the sq. on FN : and this is true however many sides the two polygons may have.

Suppose the polygons are *regular*; then by sufficiently increasing the number of their sides, we may make their areas differ from the areas of their circumscribed circles by quantities smaller than any that can be named; hence ultimately,

the ⊙ ACE : the ⊙ FHL :: the sq. on AM : the sq. on FN.

EXERCISES ON PROPOSITIONS 19, 20.

1. If ABC is a triangle right-angled at A, and AD is drawn perpendicular to BC, shew that

(i) CB : BD in the duplicate ratio of CB to BA;

(ii) The square on CB : the square on BA :: CB : BD;

(iii) The △ ABD : the △ CAD in the duplicate ratio of BA to AC.

2. In any triangle ABC, the sides AB, AC are cut by a line XY drawn parallel to BC. If AX is one-third of AB, what part is the triangle AXY of the triangle ABC?

3. A trapezium ABCD has its sides AB, CD parallel, and its diagonals intersect at O. If AB is double of CD, find the ratio of the triangle AOB to the triangle COD.

4. ABC and XYZ are two similar triangles whose areas are respectively 245 and 5 square inches. If AB is 21 inches in length, find XY.

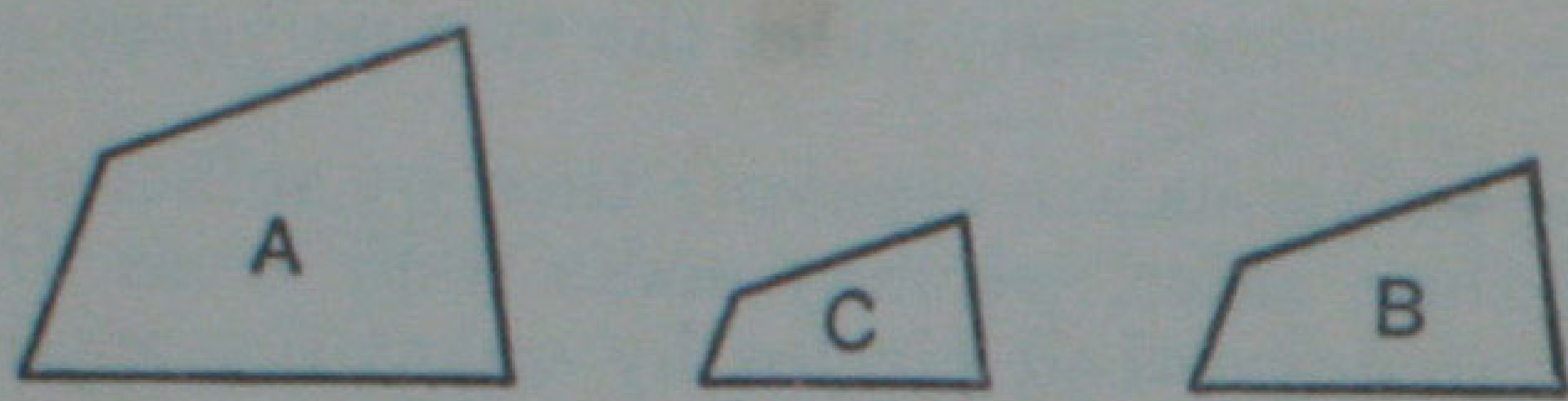
5. Shew how to draw a straight line XY parallel to the base BC of a triangle ABC, so that the area of the triangle AXY may be four-ninths of the triangle ABC.

6. Two circles intersect at A and B, and at A tangents are drawn, one to each circle, meeting the circumferences at C and D. If AB, CB and BD are joined, shew that

the △ CBA : the △ ABD :: CB : BD.

PROPOSITION 21. THEOREM.

Rectilineal figures which are similar to the same rectilineal figure, are also similar to each other.



Let each of the rectilineal figures A and B be similar to C.

Then shall A *be similar to* B.

For because A is similar to C, *Hyp.*

∴ A is equiangular to C,

and the sides about their equal angles are proportionals. VI. *Def.* 2.

Again, because B is similar to C, *Hyp.*

∴ B is equiangular to C,

and the sides about their equal angles are proportionals. VI. *Def.* 2.

∴ A and B are each of them equiangular to C, and have their sides about the equal angles proportional to the corresponding sides of C;

∴ A is equiangular to B, *Ax.* 1.

and the sides of A and B about their equal angles are proportionals; V. 1.

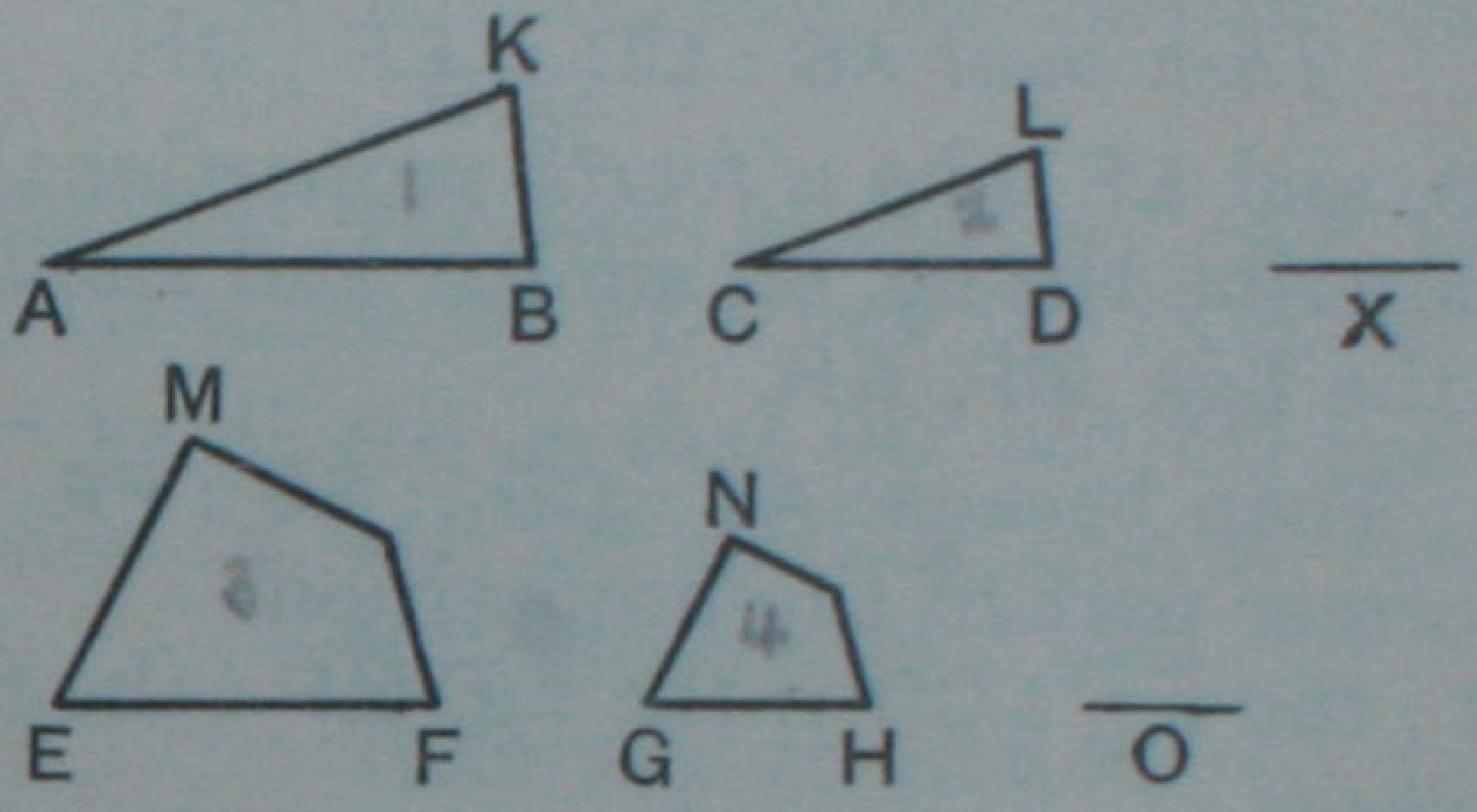
∴ A is similar to B.

Q.E.D.

PROPOSITION 22. THEOREM.

If four straight lines be proportional and a pair of similar rectilineal figures be similarly described on the first and second, and also a pair on the third and fourth, these figures shall be proportional.

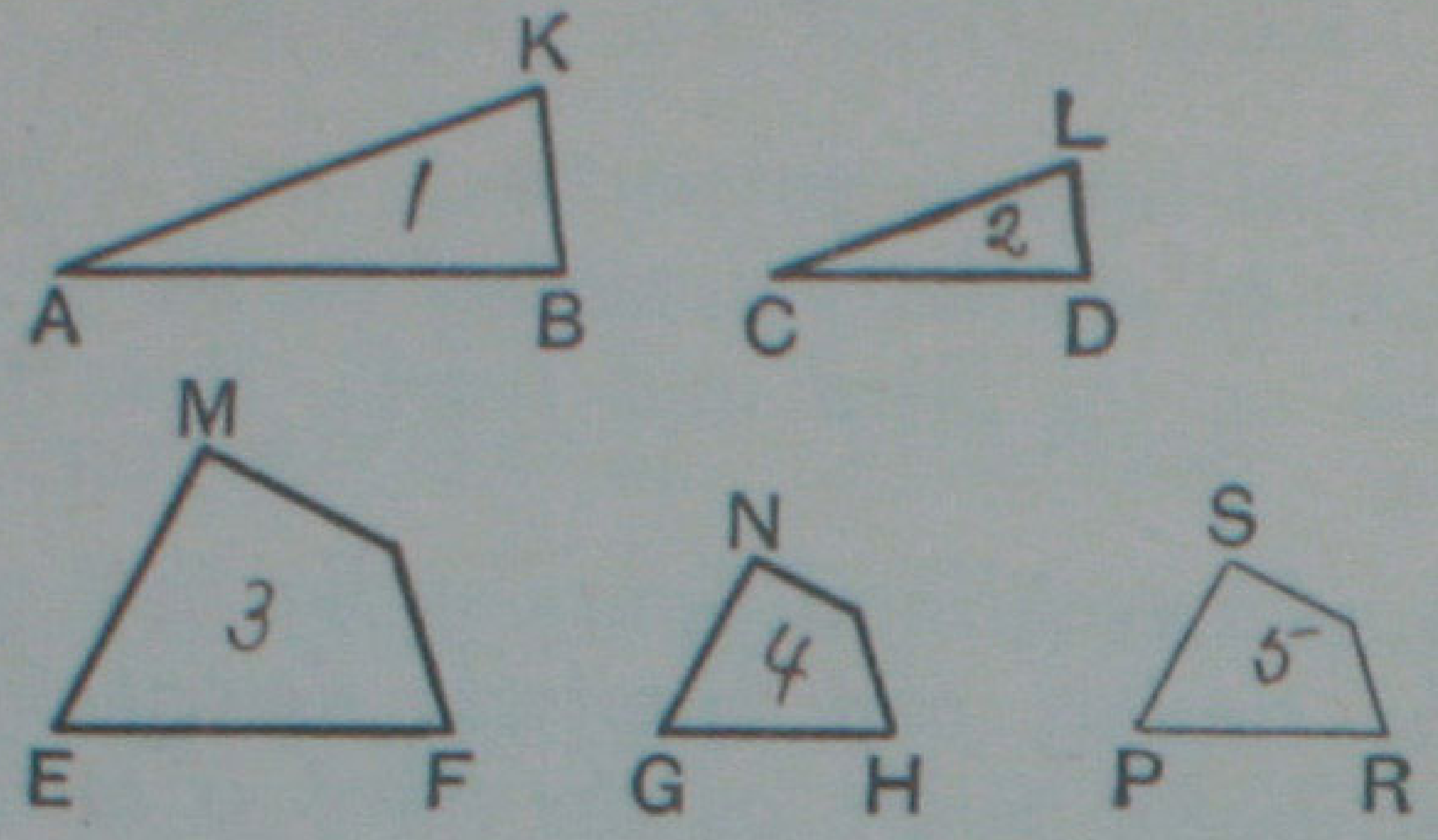
Conversely, if a rectilineal figure on the first of four straight lines be to the similar and similarly described figure on the second as a rectilineal figure on the third is to the similar and similarly described figure on the fourth, the four straight lines shall be proportional.



First. Let AB, CD, EF, GH be proportionals,
so that AB : CD :: EF : GH;
and let similar figures KAB, LCD be similarly described on AB, CD, and also let similar figures MF, NH be similarly described on EF, GH.

Then shall
the fig. KAB : *the fig.* LCD :: *the fig.* MF : *the fig.* NH.

To AB and CD take a *third* proportional X; VI. 11.
and to EF and GH take a *third* proportional O;
then AB : CD :: CD : X, *Constr.*
and EF : GH :: GH : O.
But AB : CD :: EF : GH; *Hyp.*
∴ CD : X :: GH : O, V. 1.
∴, *ex æquali*, AB : X :: EF : O. V. 14.
But AB : X :: the fig. KAB : the fig. LCD; VI. 20, *Cor.*
and EF : O :: the fig. MF : the fig. NH;
∴ the fig. KAB : the fig. LCD :: the fig. MF : the fig. NH.
V. 1.



Conversely.

Let the fig. KAB : the fig. LCD :: the fig. MF : the fig. NH.

Then shall AB : CD :: EF : GH.

To AB, CD, and EF take a *fourth* proportional PR : VI. 12.
and on PR describe the fig. SR similar and similarly situated to either of the figs. MF, NH. VI. 18.

Then because AB : CD :: EF : PR, *Constr.*

∴, by the former part of the proposition,

the fig. KAB : the fig. LCD :: the fig. MF : the fig. SR.

But, *by hypothesis*,

the fig. KAB : the fig. LCD :: the fig. MF : the fig. NH;

∴ the fig. MF : the fig. SR :: the fig. MF : the fig. NH, V. 1.

∴ the fig. SR = the fig. NH.

And since the figs. SR and NH are similar and similarly situated, *Constr.*

∴ PR = GH*.

Now AB : CD :: EF : PR; *Constr.*

∴ AB : CD :: EF : GH.

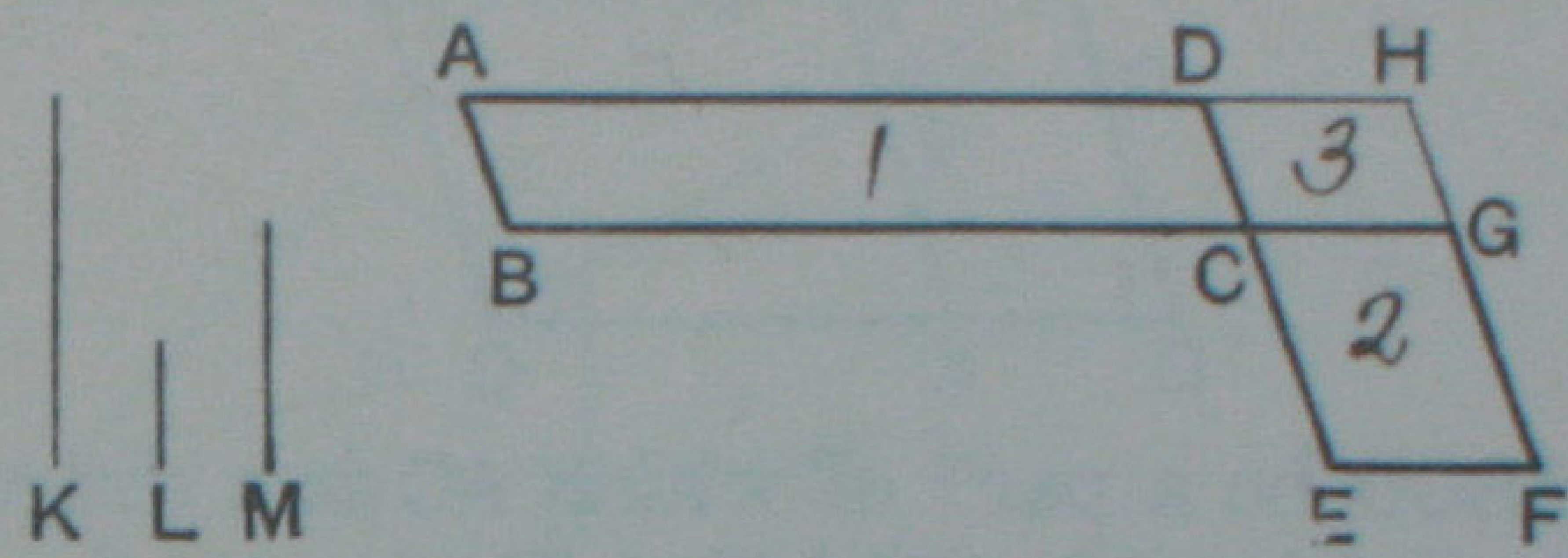
Q.E.D.

* Euclid here assumes that *if two similar and similarly situated figures are equal, their homologous sides are equal.* The proof is easy and may be left as an exercise for the student.

DEFINITION. When there are any number of magnitudes of the same kind, the *first* is said to have to the *last* the **ratio compounded** of the ratios of the *first* to the *second*, of the *second* to the *third*, and so on up to the ratio of the *last but one* to the *last* magnitude. [Book V. Def. 11.]

PROPOSITION 23. THEOREM.

Parallelograms which are equiangular to one another have to one another the ratio which is compounded of the ratios of their sides.



Let the par^{m} AC be equiangular to the par^{m} CF, having the ∠ BCD equal to the ∠ ECG.

Then shall the par^{m} AC *have to the par^{m}* CF *the ratio compounded of the ratios* BC : CG *and* DC : CE.

Let the par^{ms} be placed so that BC and CG are in a st. line; then DC and CE are also in a st. line. I. 14.

Complete the par^{m} DG.

Take any st. line K,

and to BC, CG, and K find a *fourth* proportional L; VI. 12.

and to DC, CE, and L take a *fourth* proportional M;

then BC : CG :: K : L,

and DC : CE :: L : M.

But K : M is the ratio compounded of the ratios K : L and L : M; V. *Def.* 11.

that is, K : M is the ratio compounded of the ratios BC : CG and DC : CE.

Now the par^{m} AC : the par^{m} DG :: BC : CG VI. 1.
:: K : L, *Constr.*

and the par^{m} DG : the par^{m} CF :: DC : CE VI. 1.
:: L : M; *Constr.*

∴, *ex æquali*, the par^{m} AC : the par^{m} CF :: K : M. V. 14.

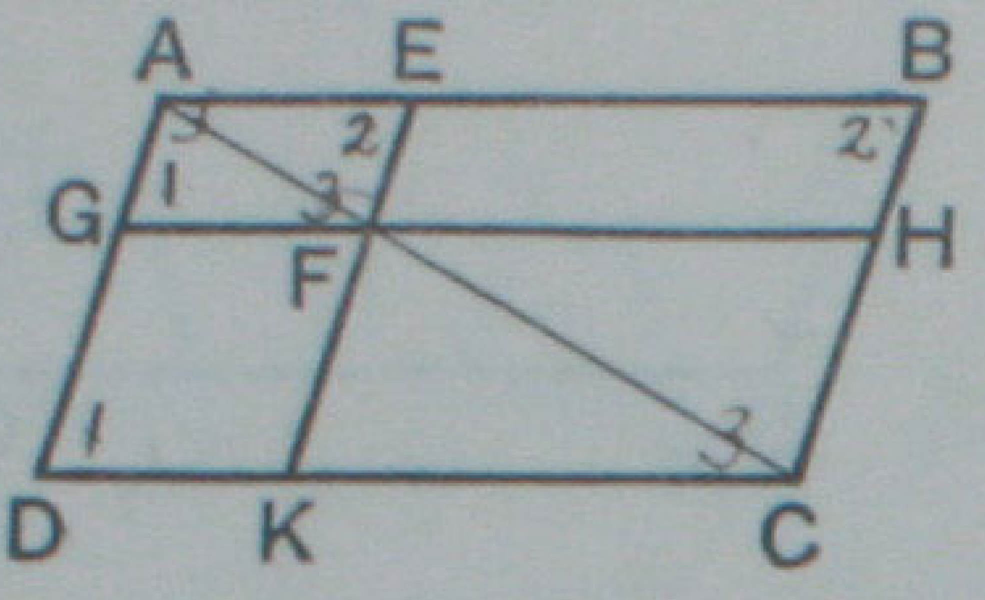
But K : M is the ratio compounded of the ratios of the sides;

∴ the par^{m} AC has to the par^{m} CF the ratio compounded of the ratios of the sides. Q.E.D.

EXERCISE. The areas of two triangles or parallelograms are to one another in the ratio compounded of the ratios of their bases and of their altitudes.

PROPOSITION 24. THEOREM.

Parallelograms about a diagonal of any parallelogram are similar to the whole parallelogram and to one another.



Let ABCD be a par^{m} of which AC is a diagonal;
and let EG, HK be par^{ms} about AC.

Then shall the par^{ms} EG, HK *be similar to the par^{m}* ABCD, *and to one another.*

For because DC is par^{l} to GF,
∴ the ∠ ADC = the ∠ AGF; I 29.
and because BC is par^{l} to EF,
∴ the ∠ ABC = the ∠ AEF; I. 29.
and each of the ∠^{s} BCD, EFG is equal to the opp. ∠ BAD,
∴ the ∠ BCD = the ∠ EFG; I. 34.
∴ the par^{m} ABCD is equiangular to the par^{m} AEFG.

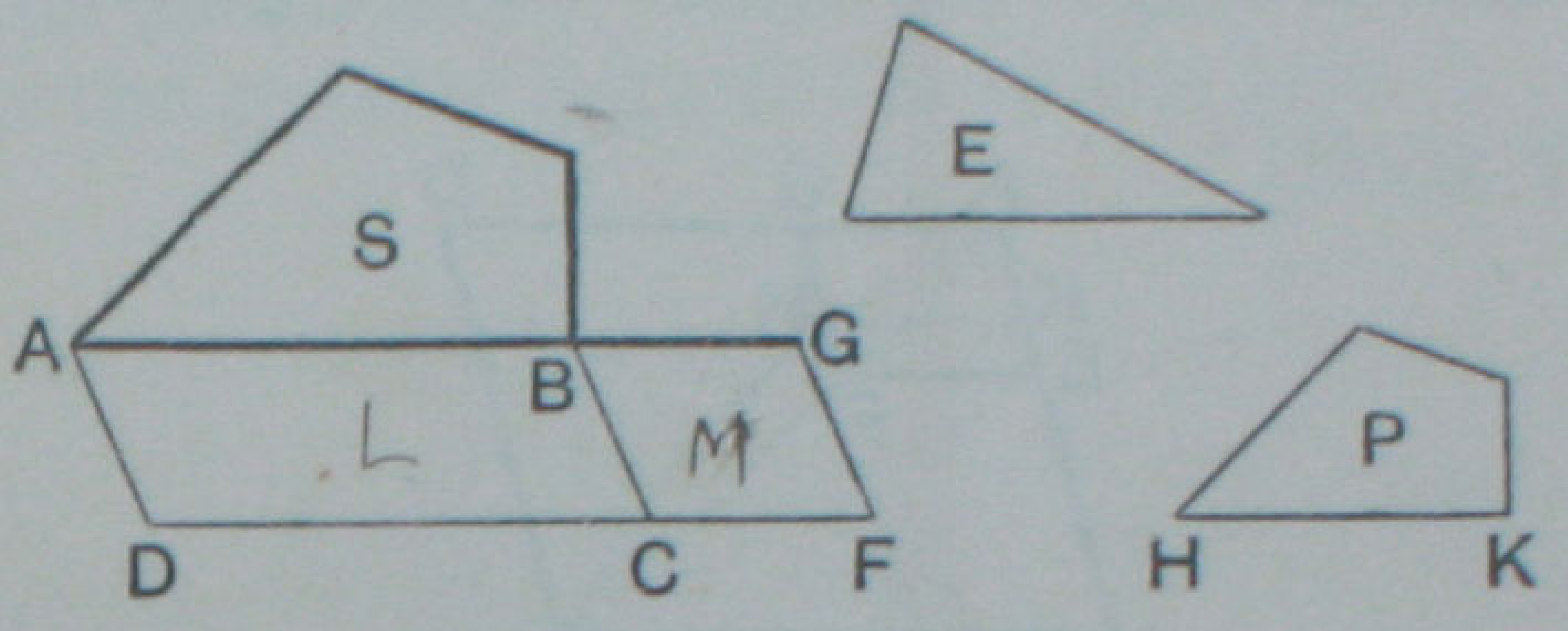
Again in the △^{s} BAC, EAF,
because the ∠ ABC = the ∠ AEF, I. 29.
and the ∠ BAC is common;
∴ the remaining ∠ BCA = the remaining ∠ EFA; I. 32.
∴ the △^{s} BAC, EAF are equiangular to one another;
∴ AB : BC :: AE : EF. VI. 4.
But BC = AD, and EF = AG; I. 34.
∴ AB : AD :: AE : AG.
Similarly DC : CB :: GF : FE,
and CD : DA :: FG : GA;
∴ the sides of the par^{ms} ABCD, AEFG about their equal angles are proportional;
∴ the par^{m} ABCD is similar to the par^{m} AEFG. VI. *Def.* 2.

In the same way the par^{m} ABCD may be proved similar to the par^{m} FHCK,
∴ each of the par^{ms} EG, HK is similar to the whole par^{m};
∴ the par^{m} EG is similar to the par^{m} HK. VI. 21.

Q.E.D.

PROPOSITION 25. PROBLEM.

To describe a rectilineal figure which shall be equal to one and similar to another rectilineal figure.



Let E and S be the two given rectilineal figures.

It is required to describe a figure equal to the fig. E *and similar to the fig.* S.

On AB a side of the fig. S describe a par^{m} ABCD equal to S; and on BC describe a par^{m} CBGF equal to the fig. E, and having the ∠ CBG equal to the ∠ DAB; I. 45.

then AB and BG are in one st. line, and also DC and CF in one st. line.

Between AB and BG find a *mean* proportional HK; VI. 13.
and on HK describe the fig. P, similar and similarly situated to the fig. S. VI. 18.

Then P *shall be the figure required.*

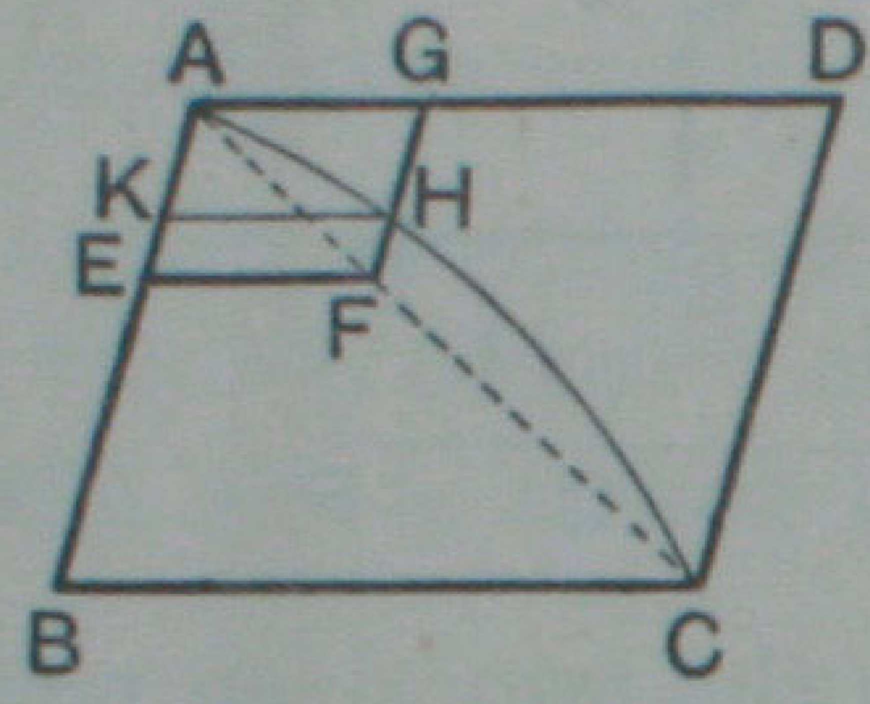
Because AB : HK :: HK : BG, *Constr.*
∴ AB : BG :: the fig. S : the fig. P. VI. 20, *Cor.*
But AB : BG :: the par^{m} AC : the par^{m} BF; VI. 1.
∴ the fig. S : the fig. P :: the par^{m} AC : the par^{m} BF; V. 1.
and the fig. S = the par^{m} AC; *Constr.*
∴ the fig. P = the par^{m} BF
= the fig. E. *Constr.*

And since, by construction, the fig. P is similar to the fig. S,
∴ P is the figure required.

Q.E.F.

PROPOSITION 26. THEOREM.

If two similar parallelograms have a common angle, and are similarly situated, they are about the same diagonal.



Let the par^{ms} ABCD, AEFG be similar and similarly situated, and have the common angle BAD.

Then shall the par^{ms} ABCD, AEFG *be about the same diagonal.*

Join AC.

Then if AC does not pass through F, if possible let it cut FG, or FG produced, at H.

Through H draw HK par^{l} to AD or BC. I. 31.

Then the par^{ms} BD and KG are similar, since they are about the same diagonal AHC; VI. 24.

∴ DA : AB :: GA : AK.

But because the par^{ms} BD and EG are similar; *Hyp.*

∴ DA : AB :: GA : AE; VI. *Def.* 2.

∴ GA : AK :: GA : AE;

∴ AK = AE, which is impossible;

∴ AC must pass through F;

that is, the par^{ms} BD, EG are about the same diagonal.

Q.E.D.

Obs. Propositions 27, 28, 29 being cumbrous in form and of little value as geometrical results are now very generally omitted.

DEFINITION. A straight line is said to be divided **in extreme and mean ratio,** when the whole is to the greater segment as the greater segment is to the less.
[Book VI. Def. 5.]

PROPOSITION 30. PROBLEM.

To divide a given straight line in extreme and mean ratio.

Let AB be the given st. line.
It is required to divide AB *in extreme and mean ratio.*

Divide AB in C so that the rect. AB, BC may be equal to the sq. on AC. II. 11.

Then because the rect. AB, BC = the sq. on AC,
∴ AB : AC :: AC : BC. VI. 17.
Q.E.F.

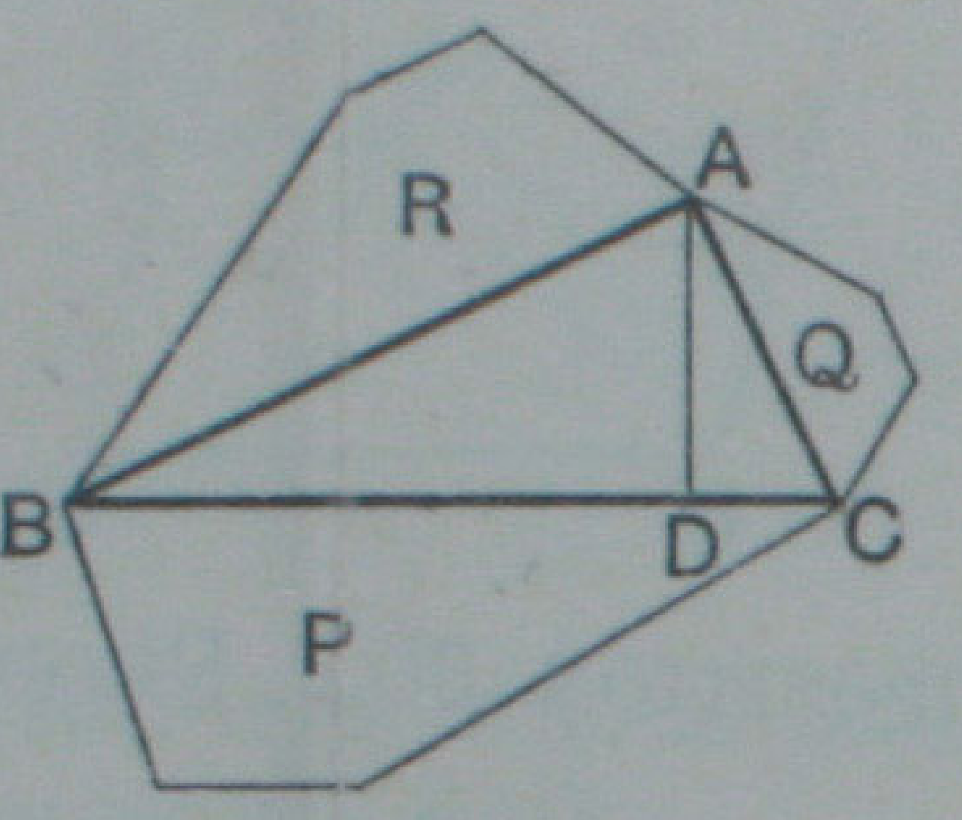
EXERCISES.

1. ABCDE is a regular pentagon; if the lines BE and AD intersect in O, shew that each of them is divided in extreme and mean ratio.

2. If the radius of a circle is cut in extreme and mean ratio, the greater segment is equal to the side of a regular decagon inscribed in the circle.

PROPOSITION 31. THEOREM.

In a right-angled triangle, any rectilineal figure described on the hypotenuse is equal to the sum of the two similar and similarly described figures on the sides containing the right angle.



Let ABC be a right-angled triangle of which BC is the hypotenuse; and let P, Q, R be similar and similarly described figures on BC, CA, AB respectively.

Then shall the fig. P *be equal to the sum of the figs.* Q *and* R.

Draw AD perp. to BC.

Then the $\triangle^s$ CBA, ABD are similar; VI. 8.

$\therefore$ CB : BA :: BA : BD;

$\therefore$ CB : BD :: the fig. P : the fig. R; VI. 20, *Cor.*

$\therefore$, *inversely*, BD : BC :: the fig. R : the fig. P. V. 2.

In like manner, DC : BC :: the fig. Q : the fig. P;

$\therefore$ the sum of BD, DC : BC :: the sum of figs. R, Q : fig. P; V. 15.

but BC = the sum of BD, DC;

$\therefore$ the fig. P = the sum of the figs. R and Q.

Q.E.D.

NOTE. This proposition is a generalization of Book I., Prop. 47. It will be a useful exercise for the student to deduce the general theorem (VI. 31) from the particular case (I. 47) with the aid of VI. 20, Cor. 2.

EXERCISES.

1. In a right-angled triangle if a perpendicular is drawn from the right angle to the opposite side, the segments of the hypotenuse are in the duplicate ratio of the sides containing the right angle.

2. If, in Proposition 31, the figure on the hypotenuse is equal to the given triangle, the figures on the other two sides are respectively equal to the parts into which the triangle is divided by the perpendicular from the right angle to the hypotenuse.

3. AX and BY are medians of the triangle ABC which meet in G: if XY is joined, compare the areas of the triangles AGB, XGY.

4. *Shew that similar triangles are to one another in the duplicate ratio of* (i) *corresponding medians,* (ii) *the radii of their inscribed circles,* (iii) *the radii of their circumscribed circles.*

5. DEF is the pedal triangle of the triangle ABC; prove that the triangle ABC is to the triangle DBF in the duplicate ratio of AB to BD. Hence shew that

$$\text{the fig. AFDC : the } \triangle \text{ BFD} :: AD^2 : BD^2.$$

6. The base BC of a triangle ABC is produced to a point D such that BD : DC in the duplicate ratio of BA : AC. Shew that AD is a mean proportional between BD and DC.

7. Bisect a triangle by a line drawn parallel to one of its sides.

8. Shew how to draw a line parallel to the base of a triangle so as to form with the other two sides produced a triangle double of the given triangle.

9. If through any point within a triangle lines are drawn from the angles to cut the opposite sides, the segments of any one side will have to each other the ratio compounded of the ratios of the segments of the other sides.

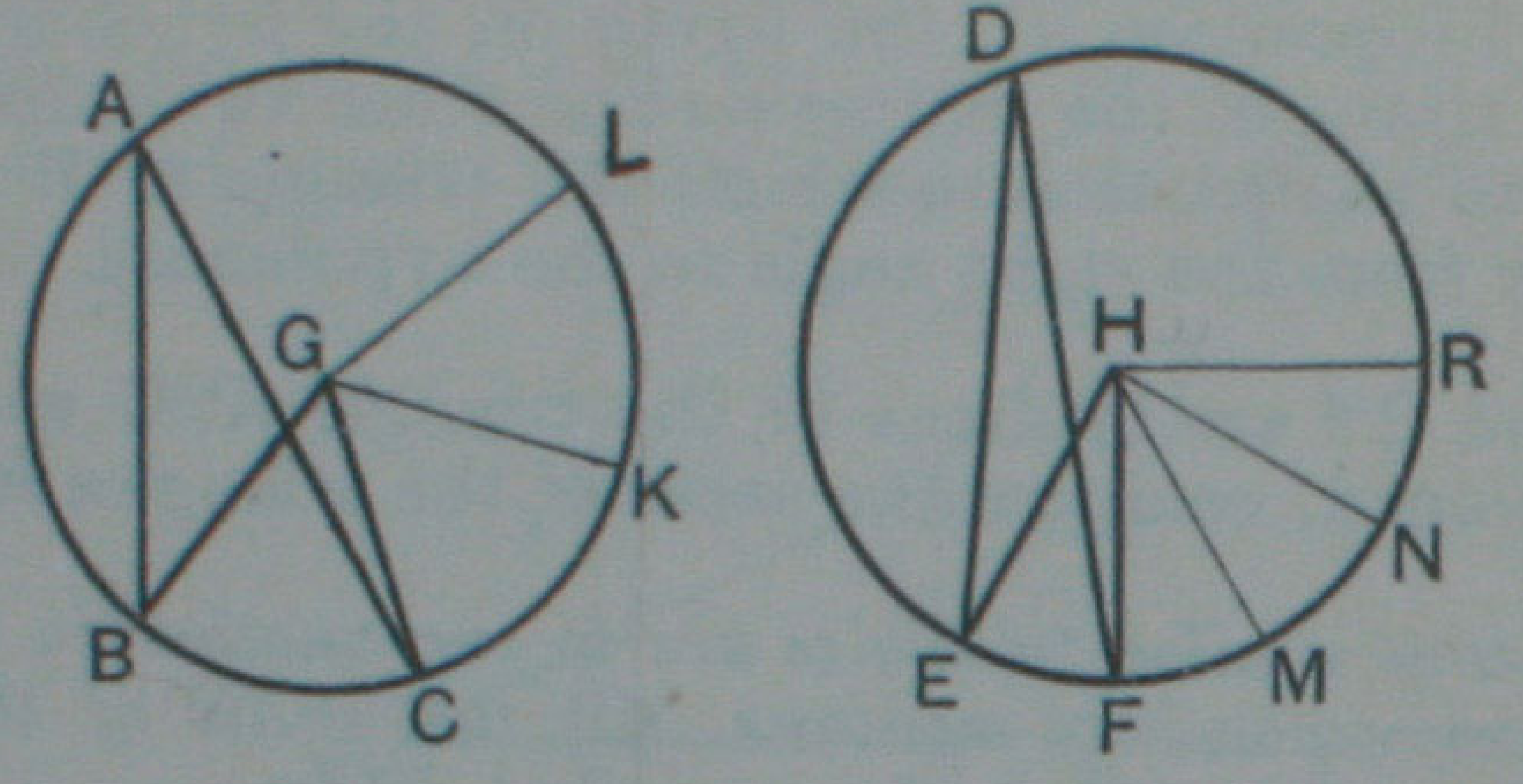
10. Draw a straight line parallel to the base of an isosceles triangle so as to cut off a triangle which has to the whole triangle the ratio of the base to a side.

11. Through a given point, between two straight lines containing a given angle, draw a line which shall cut off a triangle equal to a given rectilineal figure.

Obs. The 32nd Proposition as given by Euclid is defective, and as it is never applied, we have omitted it.

PROPOSITION 33. THEOREM.

In equal circles, angles, whether at the centres or the circumferences, have the same ratio as the arcs on which they stand: so also have the sectors.



Let ABC and DEF be equal circles, and let BGC, EHF be angles at the centres, and BAC and EDF angles at the ○ces.

Then shall

(i) *the* ∠ BGC : *the* ∠ EHF :: *the arc* BC : *the arc* EF;
(ii) *the* ∠ BAC : *the* ∠ EDF :: *the arc* BC : *the arc* EF;
(iii) *the sector* BGC : *the sector* EHF :: *the arc* BC : *the arc* EF.

Along the ○ce of the ⊙ ABC take *any* number of arcs CK, KL each equal to BC;
and along the ○ce of the ⊙ DEF take *any* number of arcs FM, MN, NR each equal to EF.

Join GK, GL, HM, HN, HR.

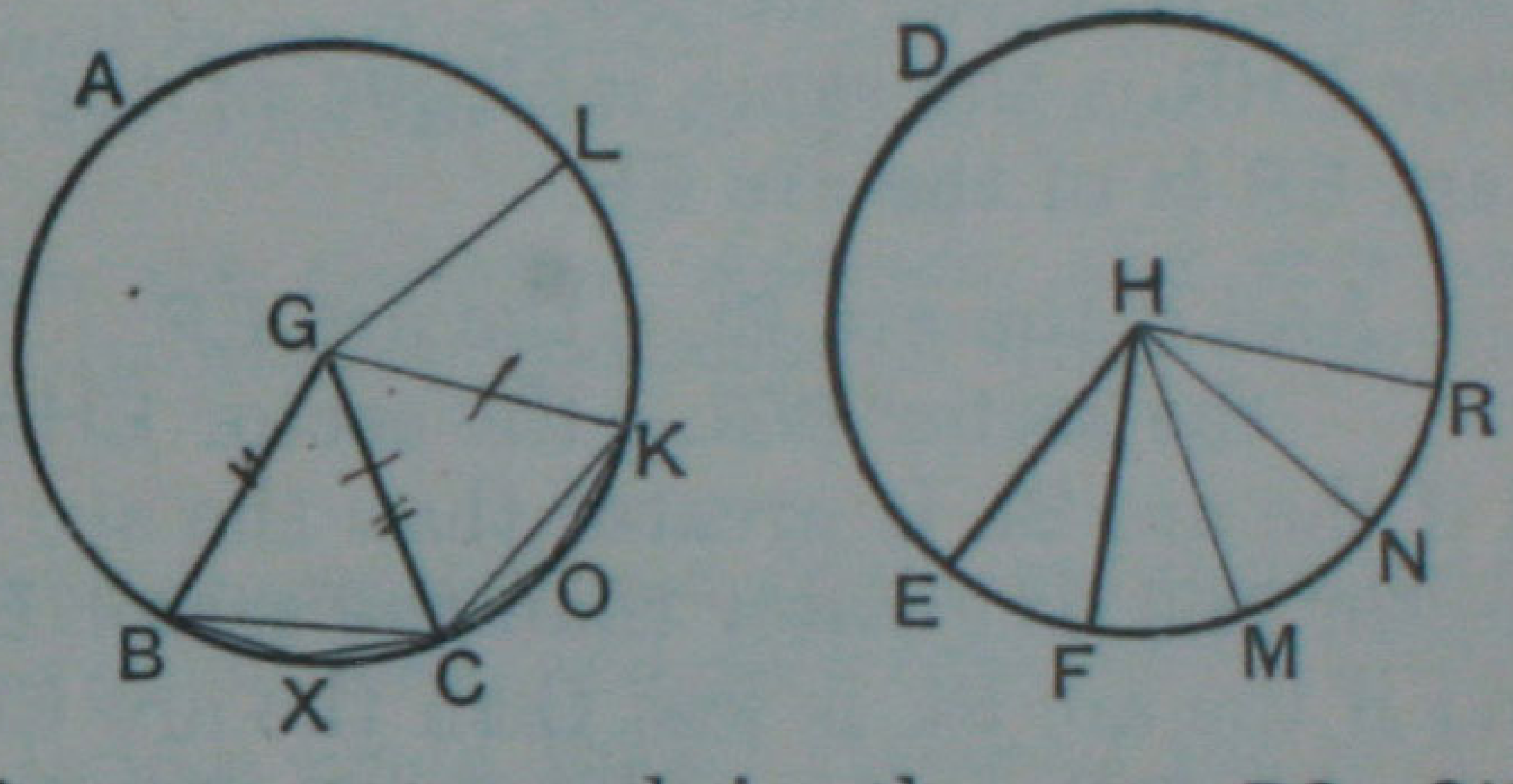
(i) Then the ∠s BGC, CGK, KGL are all equal,
for they stand on the equal arcs BC, CK, KL: III. 27.
∴ the ∠ BGL is the same multiple of the ∠ BGC that the arc BL is of the arc BC.

Similarly, the ∠ EHR is the same multiple of the ∠ EHF that the arc ER is of the arc EF.

And if the arc BL = the arc ER,
the ∠ BGL = the ∠ EHR; III. 27.
and if the arc BL is greater than the arc ER,
the ∠ BGL is greater than the ∠ EHR;
and if the arc BL is less than the arc ER,
the ∠ BGL is less than the ∠ EHR.

Now since there are four magnitudes, namely the ∠s BGC, EHF and the arcs BC, EF; and of the antecedents any equimultiples have been taken, namely the ∠ BGL and the arc BL; and of the consequents any equimultiples have been taken, namely the ∠ EHR and the arc ER:
and since it has been proved that the ∠ BGL is greater than, equal to, or less than the ∠ EHR, according as BL is greater than, equal to, or less than ER;
∴ the four original magnitudes are proportionals; V. *Def.* 5.
that is, the ∠ BGC : the ∠ EHF :: the arc BC : the arc EF.

(ii) And since the ∠ BGC = twice the ∠ BAC, III. 20.
and the ∠ EHF = twice the ∠ EDF;
∴ the ∠ BAC : the ∠ EDF :: the arc BC : the arc EF. V. 8.

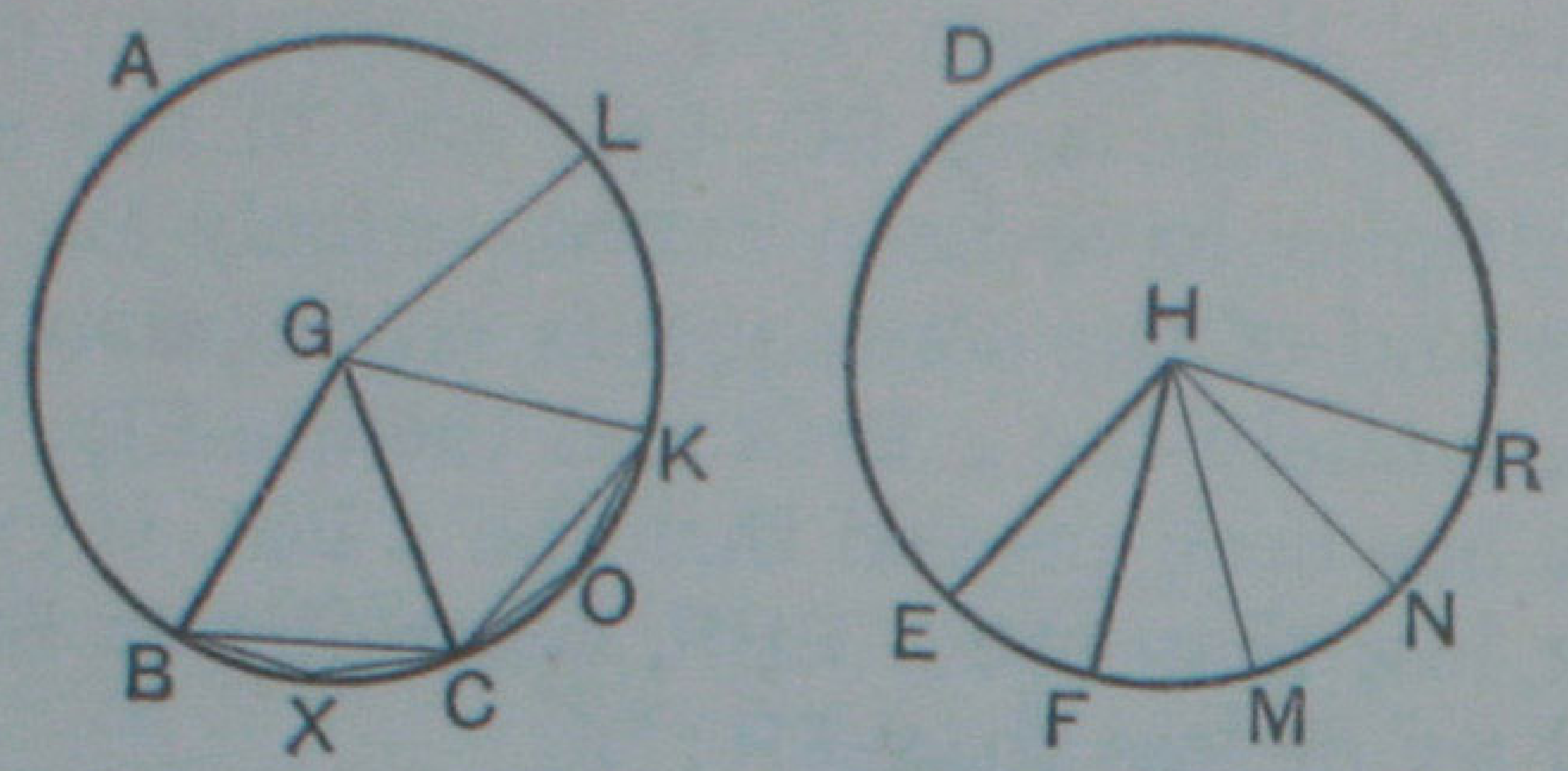


(iii) Join BC, CK; and in the arcs BC, CK take any points X, O.

Join BX, XC, CO, OK.
Then in the △s BGC, CGK,

Because { BG = CG,
GC = GK,
and the ∠ BGC = the ∠ CGK; III. 27.

∴ BC = CK; I. 4.
and the △ BGC = the △ CGK.
And because the arc BC = the arc CK, *Constr.*
∴ the remaining arc BAC = the remaining arc CAK:
∴ the ∠ BXC = the ∠ COK; III. 27.
∴ the segment BXC is similar to the segment COK;
III. *Def.* 10.
and these segments stand on equal chords BC, CK;
∴ the segment BXC = the segment COK. III. 24.
And the △ BGC = the △ CGK;
∴ the sector BGC = the sector CGK.



Similarly it may be shewn that the sectors BGC, CGK, KGL are all equal;
and likewise the sectors EHF, FHM, MHN, NHR are all equal.

∴ the sector BGL is the same multiple of the sector BGC that the arc BL is of the arc BC;
and the sector EHR is the same multiple of the sector EHF that the arc ER is of the arc EF.

And if the arc BL = the arc ER,
the sector BGL = the sector EHR: *Proved.*
and if the arc BL is greater than the arc ER,
the sector BGL is greater than the sector EHR:
and if the arc BL is less than the arc ER,
the sector BGL is less than the sector EHR.

Now since there are four magnitudes, namely, the sectors BGC, EHF and the arcs BC, EF; and of the antecedents any equimultiples have been taken, namely the sector BGL and the arc BL; and of the consequents any equimultiples have been taken, namely the sector EHR and the arc ER:
and since it has been shewn that the sector BGL is greater than, equal to, or less than the sector EHR, according as the arc BL is greater than, equal to, or less than the arc ER;
∴ the four original magnitudes are proportionals;
that is, V. *Def.* 5.
the sector BGC : the sector EHF :: the arc BC : the arc EF.

Q.E.D.

QUESTIONS FOR REVISION.

1. Explain why the operation known as *Alternately* requires that the four terms of a proportion should be *of the same kind.* Shew that this is unnecessary in the case of *Inversely.*

2. State and prove algebraically the theorem known as *Componendo.* In what proposition is this principle applied?

3. Enunciate and prove algebraically the operation used in Book VI. under the name *Ex Æquali.*

Also prove the same theorem in the following more general form:

If there are two sets of magnitudes, such that the first *is to the* second *of the first set as the* first *to the* second *of the other set, and the* second *to the* third *of the first set as the* second *to the* third *of the other, and so on throughout: then the* first *shall be to the* last *of the first set as the* first *to the* last *of the other.*

4. Explain the operation *Addendo*, and give an algebraical proof of it. In what proposition of Book VI. is this operation employed?

5. Give the geometrical and algebraical definitions of the *ratio compounded of given ratios*, and shew that the two definitions agree.

By what artifice would Euclid represent the ratio compounded of the ratios $A : B$ and $C : D$?

6. Two parallelograms ABCD, EFGH are equiangular to one another: if AB, BC are respectively 21 and 18 inches in length, and if EF, FG are 27 and 35 inches; shew that the areas of the parallelograms are in the ratio 2 : 5.

7. If $A : B = X : Y$, and $C : B = Z : Y$; shew that $A + C : B = X + Z : Y$.

In what proposition of Book VI. is this principle used?

Explain and illustrate the necessity of the step *invertendo* in this proposition.

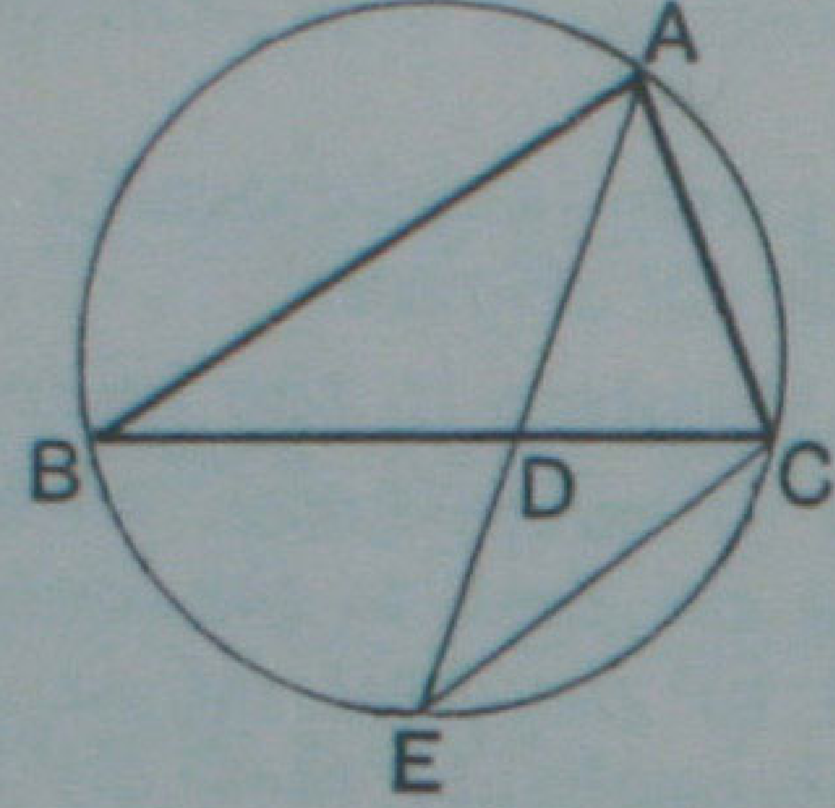
8. When is a straight line said to be divided *in extreme and mean ratio*?

If a line 10 inches in length is so divided, shew that the lengths of the segments are approximately 6·2 inches and 3·8 inches.

Shew also that the segments of *any* line divided in *extreme and mean ratio* are incommensurable.

Proposition B. Theorem.

If the vertical angle of a triangle be bisected by a straight line which cuts the base, the rectangle contained by the sides of the triangle shall be equal to the rectangle contained by the segments of the base, together with the square on the straight line which bisects the angle.



Let ABC be a triangle, having the ∠ BAC bisected by AD.

Then shall

the rect. BA, AC = *the rect.* BD, DC, *with the sq. on* AD.

Describe a circle about the △ ABC, IV. 5.
and produce AD to meet the ○ᶜᵉ in E.
Join EC.

Then in the △ˢ BAD, EAC,
because the ∠ BAD = the ∠ EAC, *Hyp.*
and the ∠ ABD = the ∠ AEC in the same segment; III. 21.
∴ the remaining ∠ BDA = the remaining ∠ ECA; I. 32.
that is, the △ BAD is equiangular to the △ EAC.
∴ BA : AD :: EA : AC; VI. 4.
∴ the rect. BA, AC = the rect. EA, AD, VI. 16.
= the rect. ED, DA, with the sq. on AD. II. 3.
But the rect. ED, DA = the rect. BD, DC; III. 35.
∴ the rect. BA, AC = the rect. BD, DC, with the sq. on AD.

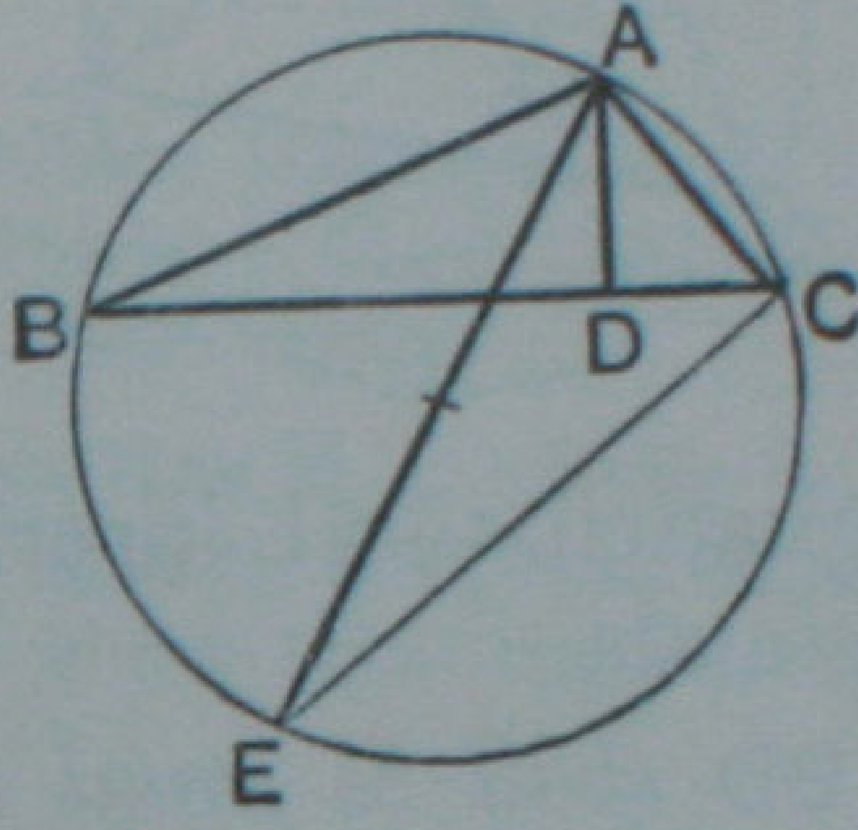
Q.E.D.

EXERCISE.

If the vertical angle BAC is *externally* bisected by a straight line which meets the base in D, shew that the rectangle contained by BA, AC together with the square on AD is equal to the rectangle contained by the segments of the base.

Proposition C. Theorem.

If from the vertical angle of a triangle a straight line be drawn perpendicular to the base, the rectangle contained by the sides of the triangle shall be equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.



Let ABC be a triangle, and let AD be the perp. from A to the base BC.

Then the rect. BA, AC *shall be equal to the rectangle contained by* AD *and the diameter of the circle circumscribed about the* △ ABC.

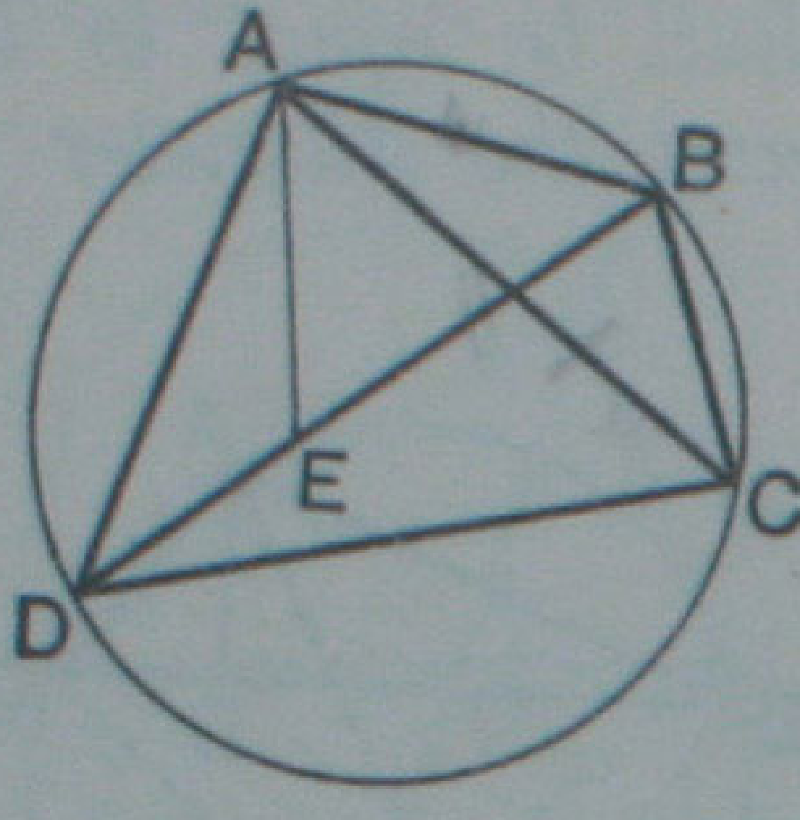
Describe a circle about the △ ABC; IV. 5.
draw the diameter AE, and join EC.

Then in the △s BAD, EAC,
the rt. angle BDA = the rt. angle ECA, in the semicircle ECA,
and the ∠ ABD = the ∠ AEC, in the same segment; III. 21.
∴ the remaining ∠ BAD = the remaining ∠ EAC; I. 32.
that is, the △ BAD is equiangular to the △ EAC;
∴ BA : AD :: EA : AC; VI. 4.
∴ the rect. BA, AC = the rect. EA, AD. VI. 16.

Q.E.D.

PROPOSITION D. THEOREM.

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the two rectangles contained by its opposite sides.



Let ABCD be a quadrilateral inscribed in a circle, and let AC, BD be its diagonals.

Then the rect. AC, BD *shall be equal to the sum of the rectangles* AB, CD *and* BC, AD.

Make the ∠ DAE equal to the ∠ BAC; I. 23.
to each add the ∠ EAC,
then the ∠ DAC = the ∠ EAB.

Then in the △^{s} EAB, DAC,
the ∠ EAB = the ∠ DAC,
and the ∠ ABE = the ∠ ACD in the same segment; III. 21.
∴ the △^{s} EAB, DAC are equiangular to one another; I. 32.
∴ AB : BE :: AC : CD; VI. 4.
∴ the rect. AB, CD = the rect. AC, EB. VI. 16.

Again in the △^{s} DAE, CAB,
the ∠ DAE = the ∠ CAB, *Constr.*
and the ∠ ADE = the ∠ ACB, in the same segment, III. 21.
∴ the △^{s} DAE, CAB are equiangular to one another; I. 32.
∴ AD : DE :: AC : CB; VI. 4.
∴ the rect. BC, AD = the rect. AC, DE. VI. 16.
But the rect. AB, CD = the rect. AC, EB. *Proved.*
∴ the sum of the rects. BC, AD and AB, CD = the sum of the rects. AC, DE and AC, EB;
that is, the sum of the rects. BC, AD and AB, CD
= the rect. AC, BD. II. 1.

Q.E.D.

NOTE. Propositions B, C, and D do not occur in Euclid, but were added by Robert Simson, who edited Euclid's text in 1756.

Prop. D is usually known as Ptolemy's theorem, and it is the particular case of the following more general theorem:

The rectangle contained by the diagonals of a quadrilateral is less than the sum of the rectangles contained by its opposite sides, unless a circle can be circumscribed about the quadrilateral, in which case it is equal to that sum.

EXERCISES.

1. ABC is an isosceles triangle, and on the base, or base produced, any point X is taken: shew that the circumscribed circles of the triangles ABX, ACX are equal.

2. From the extremities B, C of the base of an isosceles triangle ABC, straight lines are drawn perpendicular to AB, AC respectively, and intersecting at D: shew that the rectangle BC, AD is double of the rectangle AB, DB.

3. If the diagonals of a quadrilateral inscribed in a circle are at right angles, the sum of the rectangles contained by the opposite sides is double the area of the figure.

4. ABCD is a quadrilateral inscribed in a circle, and the diagonal BD bisects AC: shew that the rectangle AD, AB is equal to the rectangle DC, CB.

5. If the vertex A of a triangle ABC is joined to any point in the base, it will divide the triangle into two triangles such that their circumscribed circles have radii in the ratio of AB to AC.

6. Construct a triangle, having given the base, the vertical angle, and the rectangle contained by the sides.

7. Two triangles of equal area are inscribed in the same circle: shew that the rectangle contained by any two sides of the one is to the rectangle contained by any two sides of the other as the base of the second is to the base of the first.

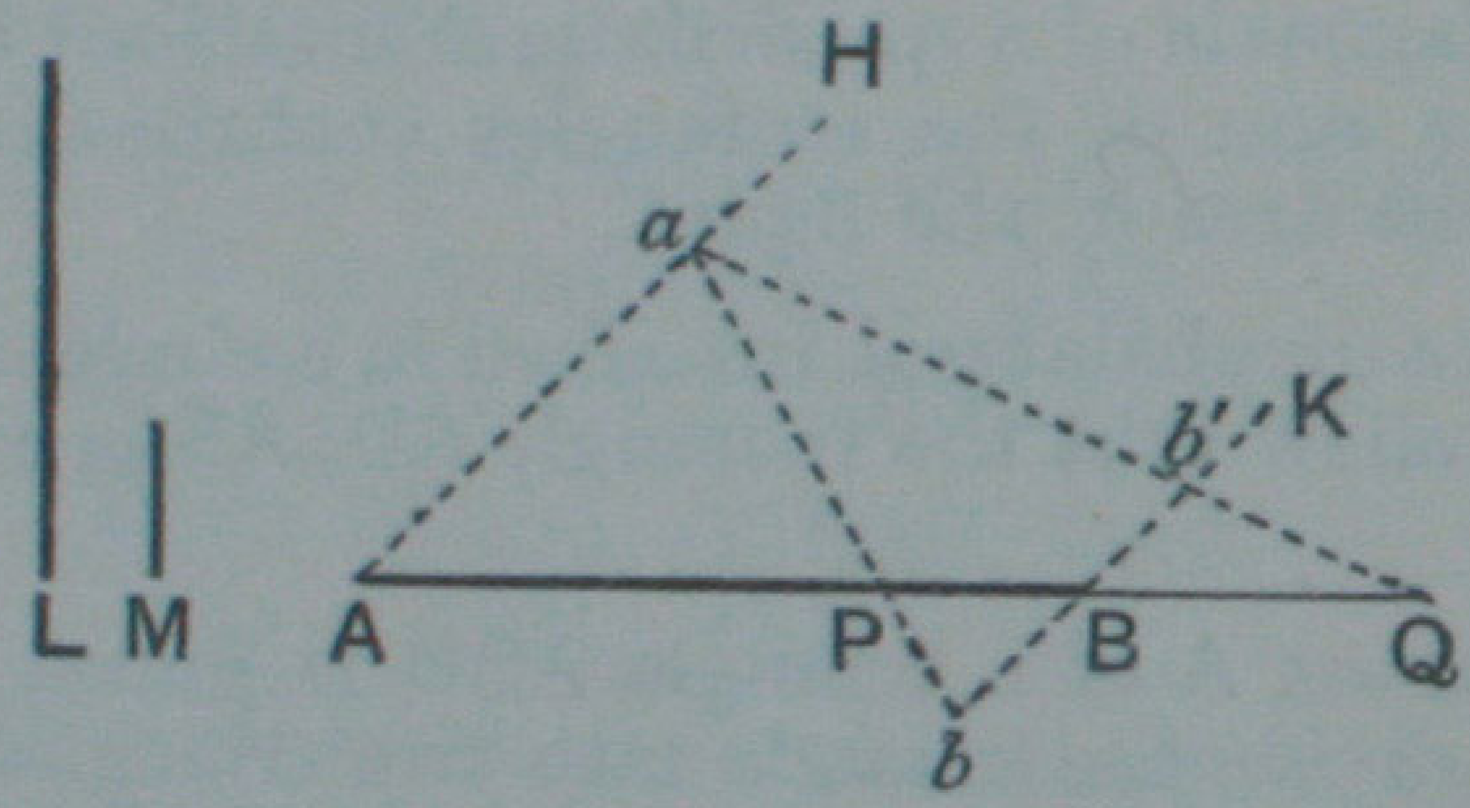
8. A circle is described round an equilateral triangle, and from any point in the circumference straight lines are drawn to the angular points of the triangle: shew that one of these straight lines is equal to the sum of the other two.

9. ABCD is a quadrilateral inscribed in a circle, and BD bisects the angle ABC: if the points A and C are fixed on the circumference of the circle and B is variable in position, shew that the sum of AB and BC has a constant ratio to BD.

THEOREMS AND EXAMPLES ON BOOK VI.

I. ON HARMONIC SECTION.

1. *To divide a given straight line internally and externally so that its segments may be in a given ratio.*



Let AB be the given st. line, and L, M two other st. lines which determine the given ratio.

It is required to divide AB *internally and externally in the ratio* L : M.

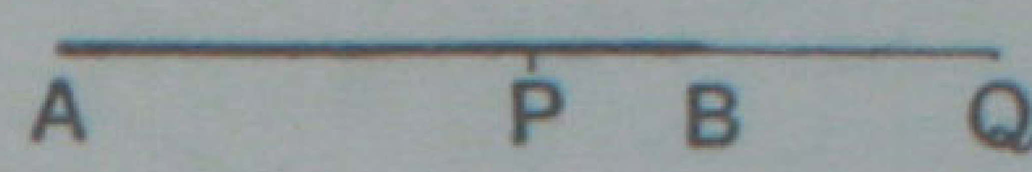
Through A and B draw any two par[l] st. lines AH, BK.
From AH cut off A*a* equal to L,
and from BK cut off B*b* and B*b′* each equal to M, B*b′* being taken in the *same* direction as A*a*, and B*b* in the opposite direction.
Join *ab*, cutting AB in P;
join *ab′*, and produce it to cut AB externally at Q.

Then AB *shall be divided internally at* P *and externally at* Q,
so that AP : PB = L : M.
and AQ : QB = L : M.

The proof follows at once from Euclid VI. 4.

Note. The solution is *singular*; that is, only *one* internal and *one* external point can be found that will divide the given straight line into segments which have the given ratio.

DEFINITION. A finite straight line is said to be **cut harmonically** when it is divided internally and externally into segments which have the same ratio.



Thus AB is divided harmonically at P and Q, if

$$AP : PB = AQ : QB.$$

P and Q are said to be **harmonic conjugates** of A and B.

Now by taking the above proportion *alternately*, we have

$$PA : AQ = PB : BQ;$$

from which it is seen that if P and Q divide AB internally and externally in the same ratio, then A and B divide PQ internally and externally in the same ratio; hence A and B are harmonic conjugates of P and Q.

Example. The base of a triangle is divided harmonically by the internal and external bisectors of the vertical angle: for in each case the segments of the base are in the ratio of the other sides of the triangle. [Euclid VI. 3 and A.]

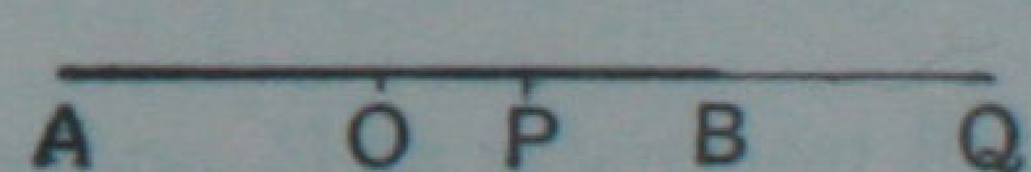
Obs. We shall use the terms *Arithmetic*, *Geometric*, and *Harmonic Means* in their ordinary Algebraical sense.

1. *If* AB *is divided internally at* P *and externally at* Q *in the same ratio, then* AB *is the harmonic mean between* AQ *and* AP.

For, *by hypothesis*, $AQ : QB = AP : PB$;
$\therefore$, *alternately*, $AQ : AP = QB : PB$,
that is, $AQ : AP = AQ - AB : AB - AP$;
$\therefore$ AP, AB, AQ are in Harmonic Progression.

2. *If* AB *is divided harmonically at* P *and* Q, *and* O *is the middle point of* AB;

then $OP \cdot OQ = OA^2$.



For since AB is divided harmonically at P and Q,

$$\therefore\ AP : PB = AQ : QB;$$

$$\therefore\ AP - PB : AP + PB = AQ - QB : AQ + QB,$$

or,

$$2OP : 2OA = 2OA : 2OQ;$$

$$\therefore\ OP \cdot OQ = OA^2.$$

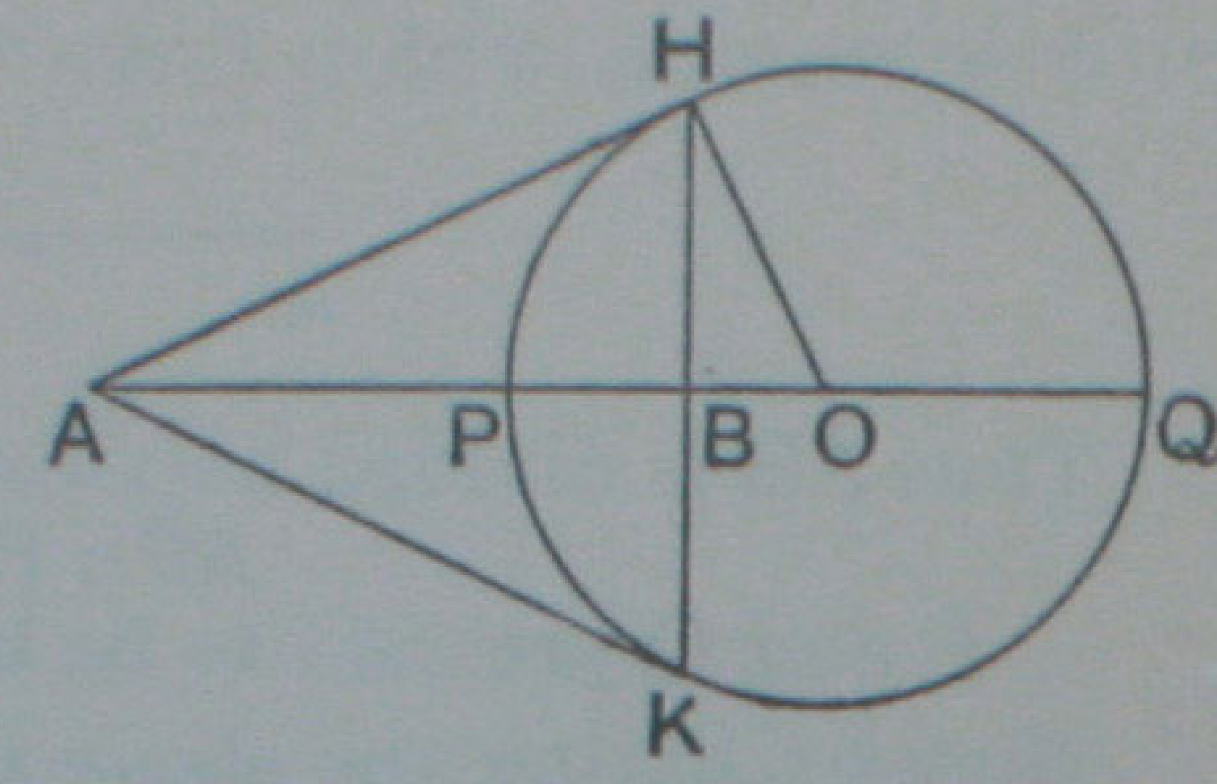
Conversely, if $OP \cdot OQ = OA^2$,
it may be shewn that

$$AP : PB = AQ : QB;$$

that is, that AB is divided harmonically at P and Q.

3. *The Arithmetic, Geometric and Harmonic means of two straight lines may be thus represented graphically.*

In the adjoining figure, two tangents AH, AK are drawn from any external point A to the circle PHQK; HK is the chord of contact, and the st. line joining A to the centre O cuts the ⊙ce at P and Q.



Then (i) AO is the Arithmetic mean between AP and AQ: for clearly

$$AO = \tfrac{1}{2}(AP + AQ).$$

(ii) AH is the Geometric mean between AP and AQ:

for $AH^2 = AP \,.\, AQ$. III. 36.

(iii) AB is the Harmonic mean between AP and AQ:

for $OA \,.\, OB = OP^2$; Ex. 1, p. 251.

∴ AB is cut harmonically at P and Q. Ex. 2, p. 385.

That is, AB is the Harmonic mean between AP and AQ.

And from the similar triangles OAH, HAB,

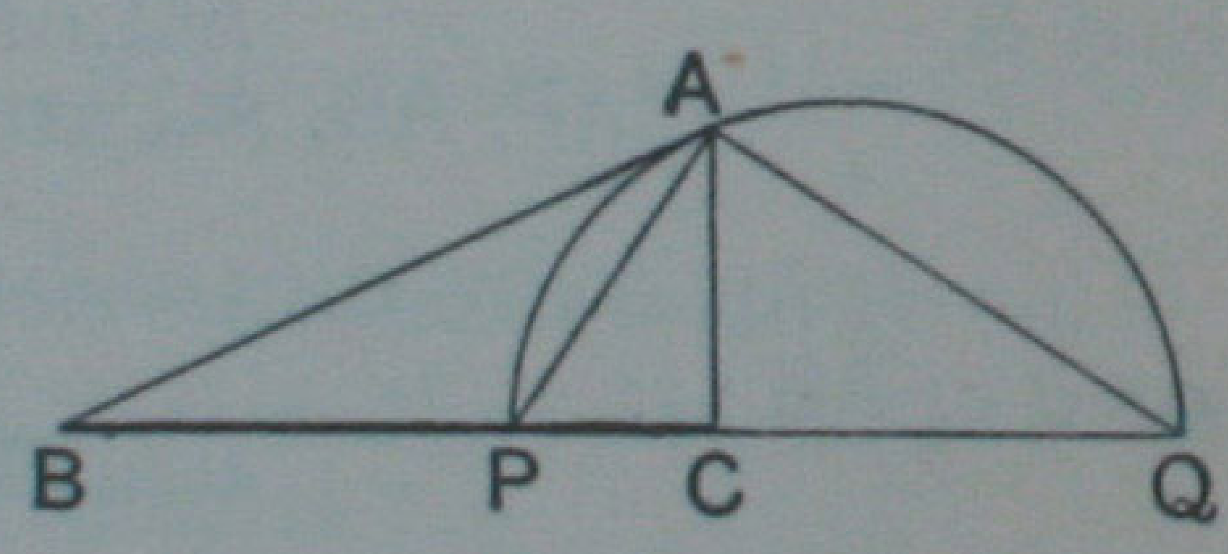
$$OA : AH = AH : AB,$$

$$\therefore\ AO \,.\, AB = AH^2; \qquad \text{VI. 17.}$$

∴ *the Geometric mean between two straight lines is the mean proportional between their Arithmetic and Harmonic means.*

4. *Given the base of a triangle and the ratio of the other sides, to find the locus of the vertex.*

Let BC be the given base, and let BAC be any triangle standing upon it, such that BA : AC = the given ratio.



It is required to find the locus of A.

Bisect the ∠ BAC internally and externally by AP, AQ.

Then BC is divided internally at P, and externally at Q, so that BP : PC = BQ : QC = the given ratio;

∴ P and Q are fixed points.

And since AP, AQ are the internal and external bisectors of the ∠ BAC,

∴ the ∠ PAQ is a rt. angle;

∴ the locus of A is a circle described on PQ as diameter.

Exercise. *Given three points* B, P, C *in a straight line: find the locus of points at which* BP *and* PC *subtend equal angles.*

DEFINITIONS.

1. A series of points in a straight line is called a **range**. If the range consists of four points, of which one pair are harmonic conjugates with respect to the other pair, it is said to be a **harmonic range**.

2. A series of straight lines drawn through a point is called a **pencil**.

The point of concurrence is called the **vertex** of the pencil, and each of the straight lines is called a **ray**.

A pencil of four rays drawn from any point to a harmonic range is said to be a **harmonic pencil**.

3. A straight line drawn to cut a system of lines is called a **transversal**.

4. A system of four straight lines, no three of which are concurrent, is called a **complete quadrilateral**.

These straight lines will intersect two and two in *six* points, called the **vertices** of the quadrilateral; the *three* straight lines which join the opposite vertices are **diagonals**.

THEOREMS ON HARMONIC SECTION.

1. *If a transversal is drawn parallel to one ray of a harmonic pencil, the other three rays intercept equal parts upon it: and conversely.*

2. *Any transversal is cut harmonically by the rays of a harmonic pencil.*

3. *In a harmonic pencil, if one ray bisect the angle between the other pair of rays, it is perpendicular to its conjugate ray. Conversely, if one pair of rays form a right angle, then they bisect internally and externally the angle between the other pair.*

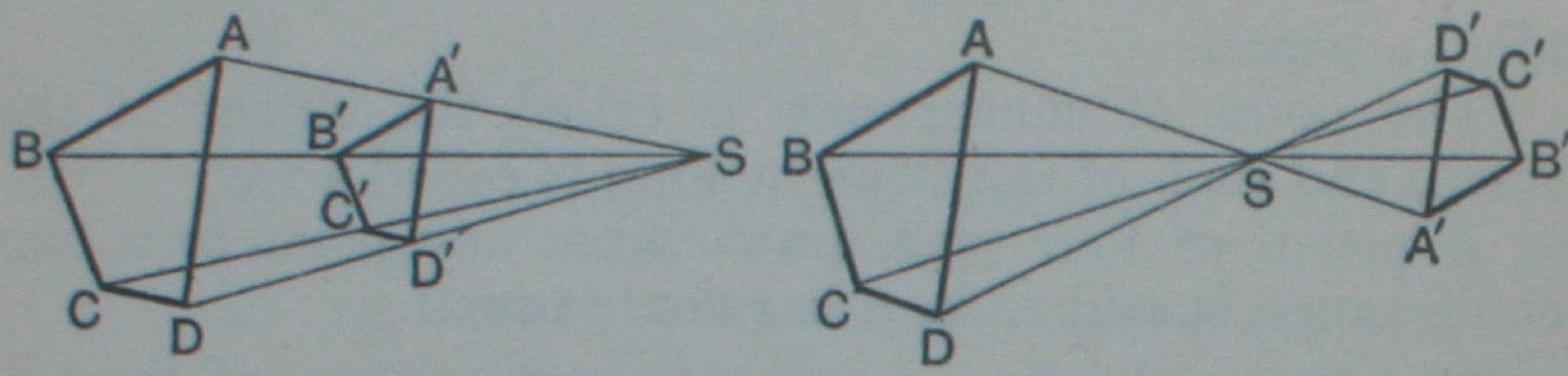
4. *If* A, P, B, Q *and* a, p, b, q *are harmonic ranges, one on each of two given straight lines, and if* Aa, Pp, Bb, *the straight lines which join three pairs of corresponding points, meet at* S; *then will* Qq *also pass through* S.

5. *If two straight lines intersect at* A, *and if* A, P, B, Q *and* A, p, b, q *are two harmonic ranges one on each straight line (the points corresponding as indicated by the letters), then* Pp, Bb, Qq *will be concurrent: also* Pq, Bb, Qp *will be concurrent.*

6. *Use Theorem* 5 *to prove that in a complete quadrilateral in which the three diagonals are drawn, the straight line joining any pair of opposite vertices is cut harmonically by the other two diagonals.*

II. ON CENTRES OF SIMILARITY AND SIMILITUDE.

1. *If any two unequal similar figures are placed so that their homologous sides are parallel, the lines joining corresponding points in the two figures meet in a point, whose distances from any two corresponding points are in the ratio of any pair of homologous sides.*



Let ABCD, A′B′C′D′ be two similar figures, and let them be placed so that their homologous sides are parallel; namely, AB, BC, CD, DA parallel to A′B′, B′C′, C′D′, D′A′ respectively.

Then shall AA′, BB′, CC′, DD′ *meet in a point, whose distances from any two corresponding points shall be in the ratio of any pair of homologous sides.*

Let AA′ meet BB′, produced if necessary, in S.

Then because AB is par[l] to A′B′; *Hyp.*

∴ the △[s] SAB, SA′B′ are equiangular;

∴ SA : SA′ = AB : A′B′; VI. 4.

∴ AA′ divides BB′, externally or internally, in the ratio of AB to A′B′.

Similarly it may be shewn that CC′ divides BB′ in the ratio of BC to B′C′.

But since the figures are similar,

BC : B′C′ = AB : A′B′;

∴ AA′ and CC′ divide BB′ in the same ratio;

that is, AA′, BB′, CC′ meet in the same point S.

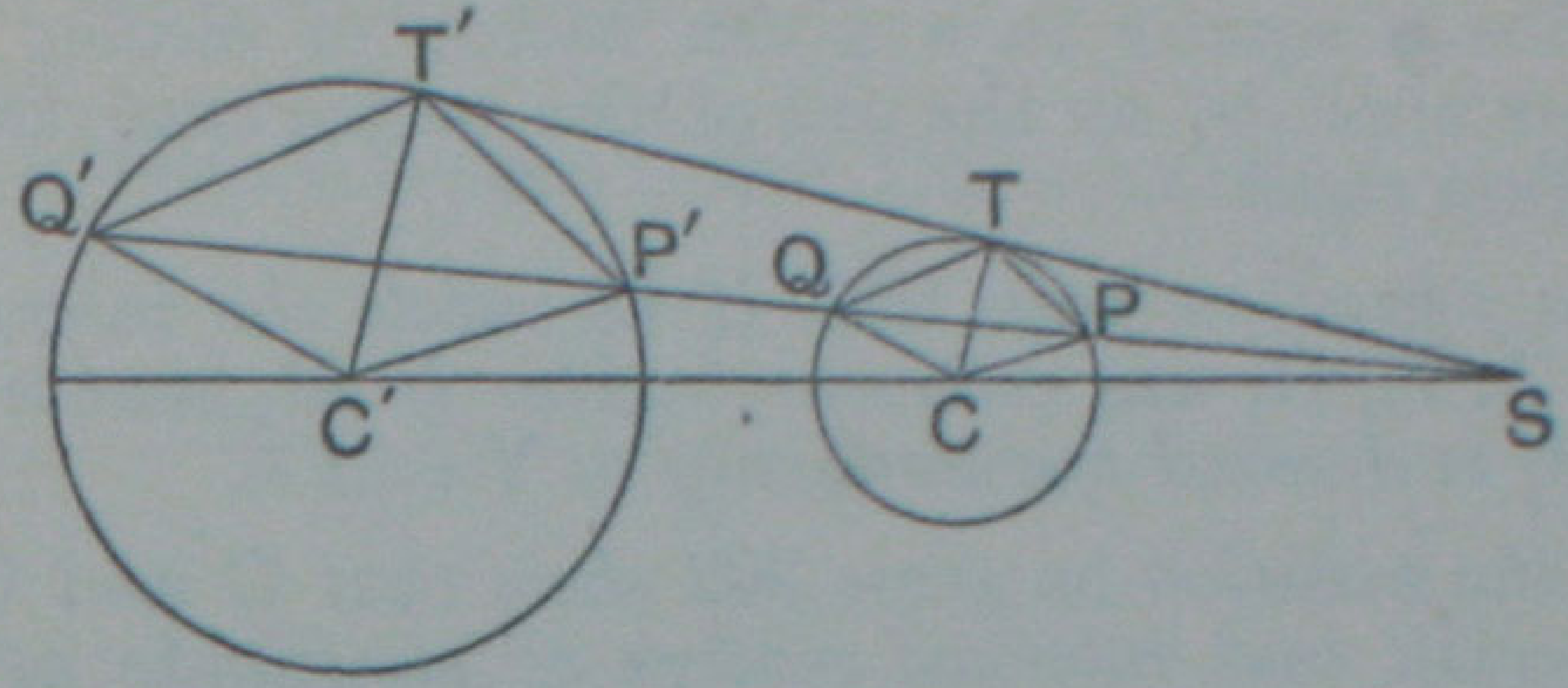
In like manner it may be proved that DD′ meets CC′ in the point S.

∴ AA′, BB′, CC′, DD′ are concurrent, and each of these lines is divided at S, externally or internally, in the ratio of a pair of homologous sides of the two figures. Q.E.D.

COR. *If any line is drawn through* S *meeting any pair of homologous sides in* K *and* K′, *the ratio* SK : SK′ *is constant, and equal to the ratio of any pair of homologous sides.*

NOTE. It will be seen that the lines joining corresponding points are divided externally or internally at S according as the corresponding sides are drawn in the same or in opposite directions. In either case the point of concurrence S is called a **centre of similarity** of the two figures.

2. *A common tangent* STT′ *to two circles whose centres are* C, C′, *meets the line of centres in* S. *If through* S *any straight line is drawn meeting these two circles in* P, Q, *and* P′, Q′, *respectively, then the radii* CP, CQ *shall be respectively parallel to* C′P′, C′Q′. *Also the rectangles* SQ . SP′, SP . SQ′ *shall each be equal to the rectangle* ST . ST′.



Join CT, CP, CQ and C′T′, C′P′, C′Q′.

Then since each of the $\angle^s$ CTS, C′T′S is a right angle, III. 18.

∴ CT is par[l] to C′T′;

∴ the $\triangle^s$ SCT, SC′T′ are equiangular;

∴ SC : SC′ = CT : C′T′

= CP : C′P′;

∴ the $\triangle^s$ SCP, SC′P′ are similar; VI. 7.

∴ the ∠ SCP = the SC′P′;

∴ CP is par[l] to C′P′.

Similarly CQ is par[l] to C′Q′.

Again, it easily follows that TP, TQ are par[l] to T′P′, T′Q′ respectively;

∴ the $\triangle^s$ STP, ST′P′ are similar.

Now the rect. SP . SQ = the sq. on ST; III. 36.

∴ SP : ST = ST : SQ, VI. 16.

and SP : ST = SP′ : ST′;

∴ ST : SQ = SP′ : ST′;

∴ the rect. ST . ST′ = SQ . SP′.

In the same way it may be proved that

the rect. SP . SQ′ = the rect. ST . ST′.

Q.E.D.

Cor. 1. It has been proved that

SC : SC′ = CP : C′P′;

thus the external common tangents to the two circles meet at a point S which divides the line of centres externally in the ratio of the radii.

Similarly it may be shewn that the transverse common tangents meet at a point S′ which divides the line of centres internally in the ratio of the radii.

Cor. 2. CC′ is divided harmonically at S and S′.

Definition. The points S and S′ which divide externally and internally the line of centres of two circles in the ratio of their radii are called the **external and internal centres of similitude** respectively.

EXAMPLES ON CENTRES OF SIMILITUDE.

1. Inscribe a square in a given triangle.

2. In a given triangle inscribe a triangle similar and similarly situated to a given triangle.

3. Inscribe a square in a given sector of circle, so that two angular points shall be on the arc of the sector and the other two on the bounding radii.

4. *In the figure on page* 298, *if* DI *meets the inscribed circle in* X, *shew that* A, X, D_1 *are collinear.* *Also if* AI_1 meets the base in Y *shew that* II_1 *is divided harmonically at* Y *and* A.

5. *With the notation on page* 302 *shew that* O *and* G *are respectively the external and internal centres of similitude of the circumscribed and nine-points circle.*

6. *If a variable circle touches two fixed circles, the line joining their points of contact passes through a centre of similitude. Distinguish between the different cases.*

7. *Describe a circle which shall touch two given circles and pass through a given point.*

8. *Describe a circle which shall touch three given circles.*

9. C_1, C_2, C_3 *are the centres of three given circles;* S'_1, S_1, *are the internal and external centres of similitude of the pair of circles whose centres are* C_2, C_3, *and* S'_2, S_2, S'_3, S_3, *have similar meanings with regard to the other two pairs of circles: shew that*

(i) S'_1C_1, S'_2C_2, S'_3C_3 *are concurrent;*

(ii) *the six points* S_1, S_2, S_3, S'_1, S'_2, S'_3, *lie three and three on four straight lines.* [See Ex. 1 and 2, pp. 400, 401.]

III. ON POLE AND POLAR.

DEFINITIONS.

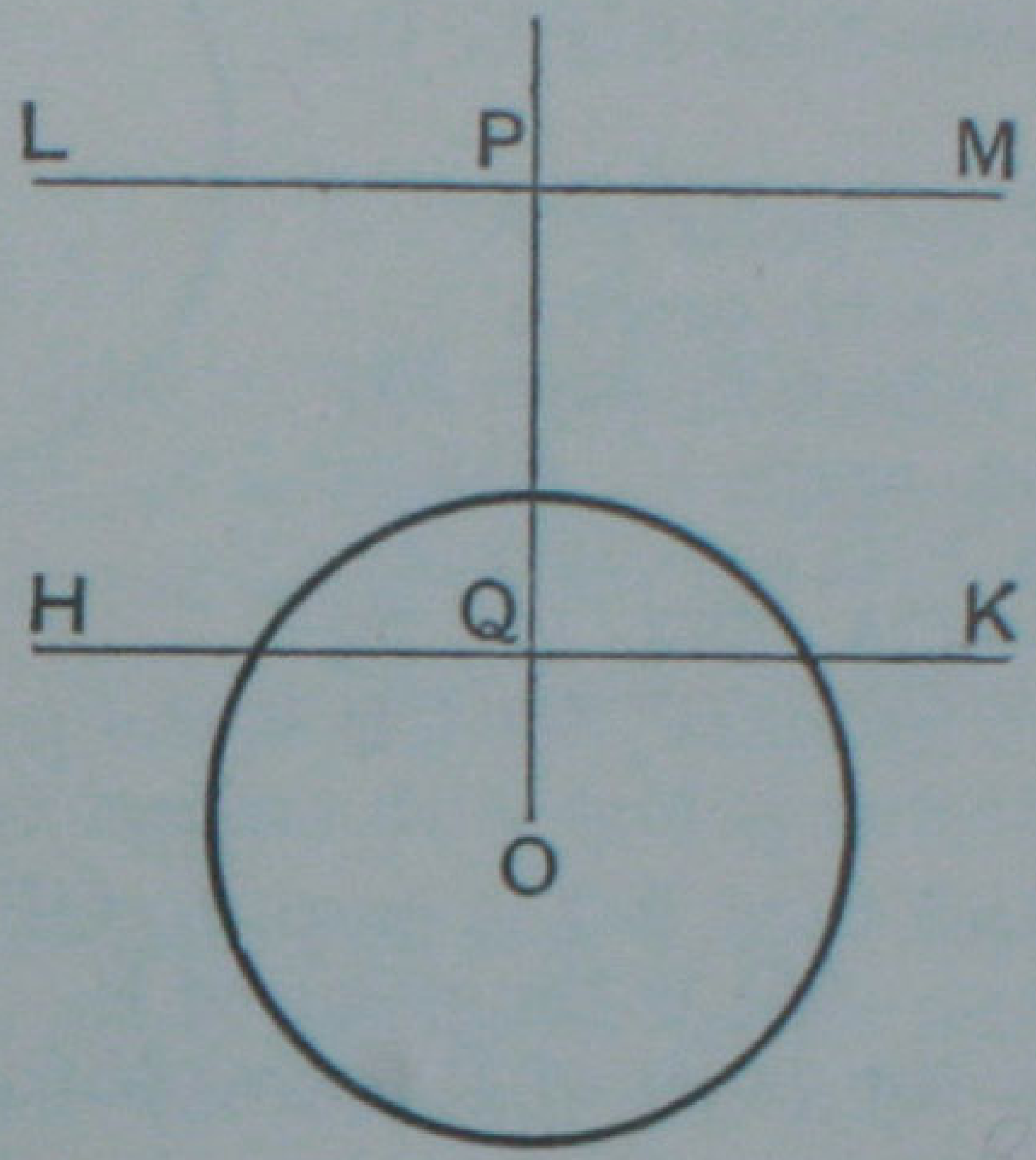
1. If in any straight line drawn from the centre of a circle two points are taken such that the rectangle contained by their distances from the centre is equal to the square on the radius, each point is said to be the **inverse** of the other.

Thus in the figure given on the following page, if O is the centre of the circle, and if OP . OQ = (radius)^2, then each of the points P and Q is the inverse of the other.

It is clear that if one of these points is within the circle the other must be without it.

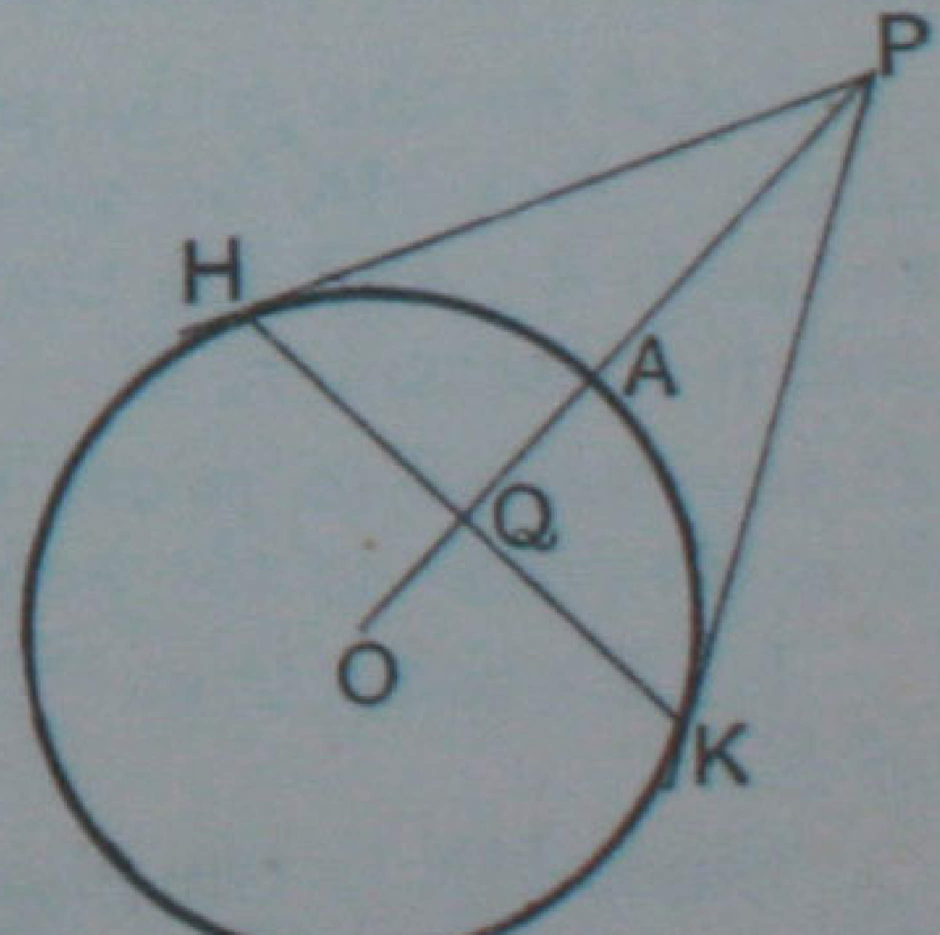
2. The **polar** of a given point with respect to a given circle is the straight line drawn through the inverse of the given point at right angles to the line which joins the given point to the centre: and with reference to the polar the given point is called the **pole**.

Thus in the adjoining figure, if OP . OQ = (radius)^2, and if through



P and Q, LM and HK are drawn perp. to OP; then HK is the polar of the point P, and P is the pole of the st. line HK with respect to the given circle: also LM is the polar of the point Q, and Q the pole of LM.

It is clear that the polar of an *external* point must intersect the circle, and that the polar of an *internal* point must fall without it: also that the polar of a point *on the circumference* is the tangent at that point.



1. Now it has been proved [see Ex. 1, page 251] that if from an external point P two tangents PH, PK are drawn to a circle, of which O is the centre, then OP cuts the chord of contact HK at right angles at Q, so that

$$OP \,.\, OQ = (\text{radius})^2;$$

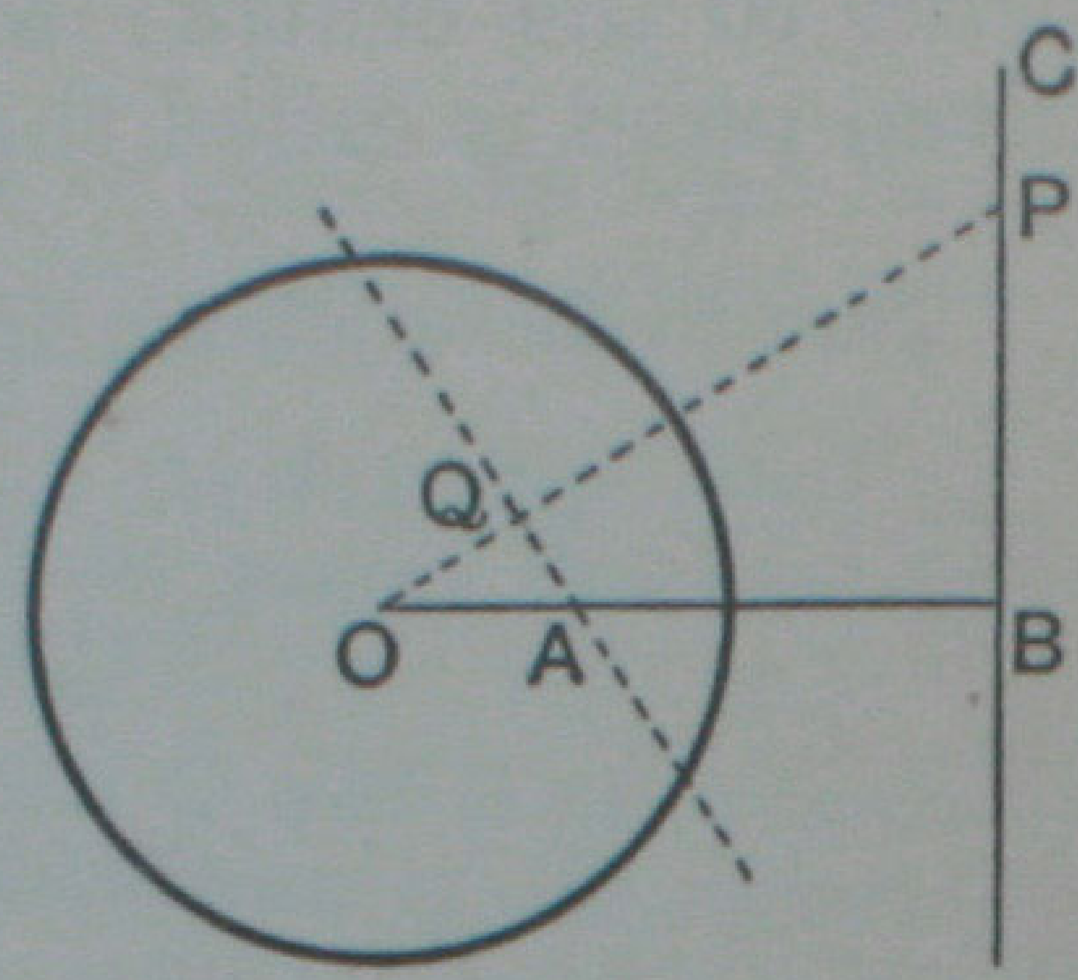
∴ HK is the polar of P with respect to the circle. *Def.* 2.

Hence we conclude that

The polar of an external point with reference to a circle is the chord of contact of tangents drawn from the given point to the circle.

2. *If* A *and* P *are any two points, and if the polar of* A *with respect to any circle passes through* P, *then the polar of* P *must pass through* A.

Let BC be the polar of the point A with respect to a circle whose centre is O, and let BC pass through P.



Then shall the polar of P *pass through* A.

Join OP; and from A draw AQ perp. to OP. We shall shew that AQ is the polar of P.

Now since BC is the polar of A,

∴ the ∠ ABP is a rt. angle; *Def.* 2, page 391.

and the ∠ AQP is a rt. angle: *Constr.*

∴ the four points A, B, P, Q are concyclic;

∴ OQ . OP = OA . OB III. 36.

= $(\text{radius})^2$, for CB is the polar of A:

∴ P and Q are inverse points with respect to the given circle.

And since AQ is perp. to OP,

∴ AQ is the polar of P.

That is, the polar of P passes through A.

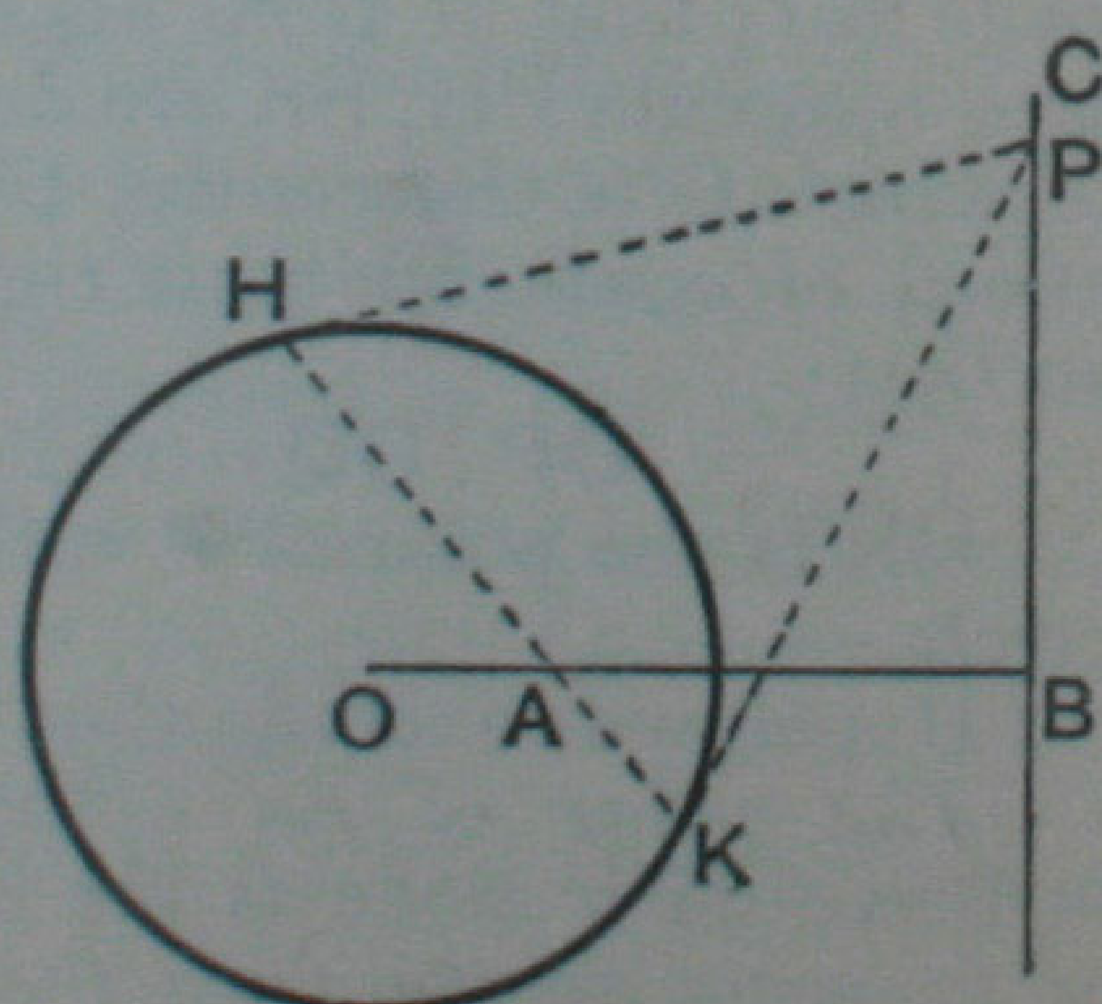
Q. E. D.

NOTE. A similar proof applies to the case when the given point A is without the circle, and the polar BC cuts it.

The above Theorem is known as the **Reciprocal Property** of Pole and Polar.

3. *To prove that the locus of the intersection of tangents drawn to a circle at the extremities of all chords which pass through a given point within the circle is the polar of that point.*

Let A be the given point within the circle. Let HK be *any* chord passing through A; and let the tangents at H and K intersect at P.



It is required to prove that the locus of P *is the polar of the point* A.

I. To shew that P lies on the polar of A.

Since HK is the chord of contact of tangents drawn from P,

∴ HK is the polar of P. Ex. 1, p. 391.

But HK, the polar of P, passes through A;

∴ the polar of A passes through P: Ex. 2, p. 392.

that is, the point P lies on the polar of A.

II. To shew that *any* point on the polar of A satisfies the given conditions.

Let BC be the polar of A, and let P be any point on it.

Draw tangents PH, PK, and let HK be the chord of contact.

Now from Ex. 1, p. 391, we know that the chord of contact HK is the polar of P,

and we also know that the polar of P must pass through A; for P is on BC, the polar of A: Ex. 2, p. 392.

that is, HK passes through A.

∴ P is the point of intersection of tangents drawn at the extremities of a chord passing through A.

From I. and II. we conclude that the required locus is the polar of A.

NOTE. If A is *without* the circle, the theorem demonstrated in Part I. of the above proof still holds good; but the converse theorem in Part II. is not true for *all* points in BC. For if A is without the circle, the polar BC will intersect it; and no point on that part of the polar which is within the circle can be the point of intersection of tangents.

We now see that

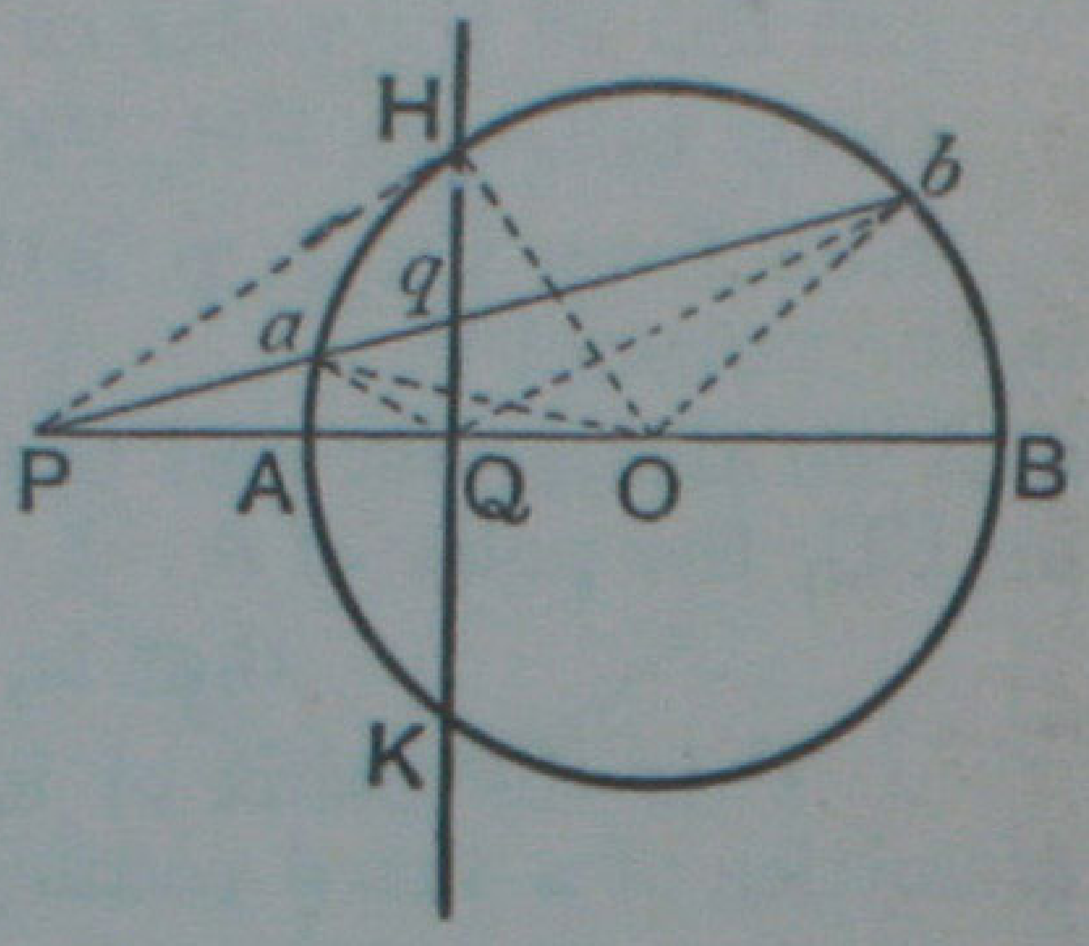
(i) *The Polar of an* external *point with respect to a circle is the chord of contact of tangents drawn from it.*

(ii) *The Polar of an* internal *point is the locus of the intersections of tangents drawn at the extremities of all chords which pass through it.*

(iii) *The Polar of a* point on the circumference *is the tangent at that point.*

The following theorem is known as the Harmonic Property of Pole and Polar.

4. *Any straight line drawn through a point is cut harmonically by the point, its polar, and the circumference of the circle.*



Let AHB be a circle, P the given point and HK its polar; let Paqb be any straight line drawn through P meeting the polar at q and the ○ᶜᵉ of the circle at a and b.

Then shall P, a, q, b *be a harmonic range.*

In the case here considered, P is an external point.

Join P to the centre O, and let PO cut the ○ᶜᵉ at A and B: let the polar of P cut the ○ᶜᵉ at H and K, and PO at Q.

Join Qa, Qb, Oa, OH, Ob, PH.

Then PH is a tangent to the ⊙ AHB. Ex. 1, p. 391.

From the similar triangles OPH, HPQ,

$$OP : PH = PH : PQ.$$

$$\therefore\ PQ \,.\, PO = PH^2$$

$$= Pa \,.\, Pb.$$

∴ the points O, Q, a, b are concyclic :

∴ the ∠ aQA = the ∠ abO Ex. 5, p. 241.

= the ∠ Oab I. 5.

= the ∠ OQb, in the same segment.

And since QH is perp. to AB,

∴ the ∠ aQH = the ∠ bQH.

∴ Qq and QP are the internal and external bisectors of the ∠ aQb:

∴ P, a, q, b is a harmonic range. Ex. 1, p. 385.

The student should investigate for himself the case when P is an internal point.

Conversely, it may be shewn that if through a fixed point P *any secant is drawn cutting the circumference of a given circle at* a *and* b, *and if* q *is the harmonic conjugate of* P *with respect to* a, b; *then the locus of* q *is the polar of* P *with respect to the given circle.*

DEFINITION.

A triangle so related to a circle that each side is the polar of the opposite vertex is said to be **self-conjugate** with respect to the circle.

EXAMPLES ON POLE AND POLAR.

1. *The straight line which joins any two points is the polar with respect to a given circle of the point of intersection of their polars.*

2. *The point of intersection of any two straight lines is the pole of the straight line which joins their poles.*

3. *Find the locus of the poles of all straight lines which pass through a given point.*

4. *Find the locus of the poles, with respect to a given circle, of tangents drawn to a concentric circle.*

5. *If two circles cut one another orthogonally and* PQ *be any diameter of one of them; shew that the polar of* P *with regard to the other circle passes through* Q.

√ 6. *If two circles cut one another orthogonally, the centre of each circle is the pole of their common chord with respect to the other circle.*

7. *Any two points subtend at the centre of a circle an angle equal to one of the angles formed by the polars of the given points.*

√ 8. O *is the centre of a given circle, and* AB *a fixed straight line.* P *is* any *point in* AB; *find the locus of the point inverse to* P *with respect to the circle.*

9. *Given a circle, and a fixed point* O *on its circumference:* P *is* any *point on the circle. find the locus of the point inverse to* P *with respect to any circle whose centre is* O.

10. *Given two points* A *and* B, *and a circle whose centre is* O; *shew that the rectangle contained by* OA *and the perpendicular from* B *on the polar of* A *is equal to the rectangle contained by* OB *and the perpendicular from* A *on the polar of* B.

11. *Four points* A, B, C, D *are taken in order on the circumference of a circle;* DA, CB *intersect at* P, AC, BD *at* Q, *and* BA, CD *in* R: *shew that the triangle* PQR *is self-conjugate with respect to the circle.*

12. *Give a* linear *construction for finding the polar of a given point with respect to a given circle. Hence find a linear construction for drawing a tangent to a circle from an external point.*

13. *If a triangle is self-conjugate with respect to a circle, the centre of the circle is at the orthocentre of the triangle.*

14. *The polars, with respect to a given circle, of the four points of a harmonic range form a harmonic pencil: and conversely.*