3. Describe a circle that shall pass through two given points and have its centre in a given straight line. When is this impossible?

4. Describe a circle of given radius to pass through two given points. When is this impossible?

5. ABC is an isosceles triangle; and from the vertex A, as centre, a circle is described cutting the base, or the base produced, at X and Y. Shew that BX = CY.

6. If two circles which intersect are cut by a straight line parallel to the common chord, shew that the parts of it intercepted between the circumferences are equal.

7. If two circles cut one another, any two straight lines drawn through a point of section, making equal angles with the common chord, and terminated by the circumferences, are equal.

[Ex. 12, p. 171.]

8. If two circles cut one another, of all straight lines drawn through a point of section and terminated by the circumferences, the greatest is that which is parallel to the line joining the centres.

9. Two circles, whose centres are C and D, intersect at A, B; and through A a straight line PAQ is drawn terminated by the circumferences: if PC, QD intersect at X, shew that the angle PXQ is equal to the angle CAD.

10. Through a point of section of two circles which cut one another draw a straight line terminated by the circumferences and bisected at the point of section.

11. AB is a fixed diameter of a circle, whose centre is C; and from P, any point on the circumference, PQ is drawn perpendicular to AB; shew that the bisector of the angle CPQ always intersects the circle in one or other of two fixed points.

12. Circles are described on the sides of a quadrilateral as diameters: shew that the common chord of any two consecutive circles is parallel to the common chord of the other two.

[Ex. 9, p. 105.]

13. Two equal circles touch one another externally, and through the point of contact two chords are drawn, one in each circle, at right angles to each other: shew that the straight line joining their other extremities is equal to the diameter of either circle.

14. Straight lines are drawn from a given external point to the circumference of a circle: find the locus of their middle points. [Ex. 3, p. 105.]

15. Two equal segments of circles are described on opposite sides of the same chord AB; and through O, the middle point of AB, any straight line POQ is drawn, intersecting the arcs of the segments at P and Q: shew that OP = OQ.

II. ON THE TANGENT AND THE CONTACT OF CIRCLES.

[See Propositions 11, 12, 16, 17, 18, 19.]

1. All equal chords placed in a given circle touch a fixed concentric circle.

2. If from an external point two tangents are drawn to a circle, the angle contained by them is double the angle contained by the chord of contact and the diameter drawn through one of the points of contact.

3. Two circles touch one another externally, and through the point of contact a straight line is drawn terminated by the circumferences: shew that the tangents at its extremities are parallel.

4. Two circles intersect, and through one point of section any straight line is drawn terminated by the circumferences: shew that the angle between the tangents at its extremities is equal to the angle between the tangents at the point of section.

5. Shew that two parallel tangents to a circle intercept on any third tangent a segment which subtends a right angle at the centre.

6. Two tangents are drawn to a given circle from a fixed external point A, and any third tangent cuts them produced at P and Q: shew that PQ subtends a constant angle at the centre of the circle.

7. *In any quadrilateral circumscribed about a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.*

8. *If the sum of one pair of opposite sides of a quadrilateral is equal to the sum of the other pair, shew that a circle may be inscribed in the figure.*

[Bisect two adjacent angles of the figure, and so describe a circle to touch three of its sides. Then prove indirectly by means of the last exercise that this circle must also touch the fourth side.]

9. Two circles touch one another internally, the centre of the outer being within the inner circle: shew that of all chords of the outer circle which touch the inner, the greatest is that which is perpendicular to the straight line joining the centres.

10. In any triangle, if a circle is described from the middle point of one side as centre and with a radius equal to half the sum of the other two sides, it will touch the circles described on these sides as diameters.

11. Through a given point, draw a straight line to cut a circle, so that the part intercepted by the circumference may be equal to a given straight line.

In order that the problem may be possible, between what limits must the given line lie, when the given point is (i) without the circle, (ii) within it?

12. A series of circles touch a given straight line at a given point: shew that the tangents to them at the points where they cut a given parallel straight line all touch a fixed circle, whose centre is the given point.

13. If two circles touch one another internally, and any third circle be described touching one internally and the other externally; then the sum of the distances of the centre of this third circle from the centres of the two given circles is constant.

14. Find the locus of points such that the pairs of tangents drawn from them to a given circle contain a constant angle.

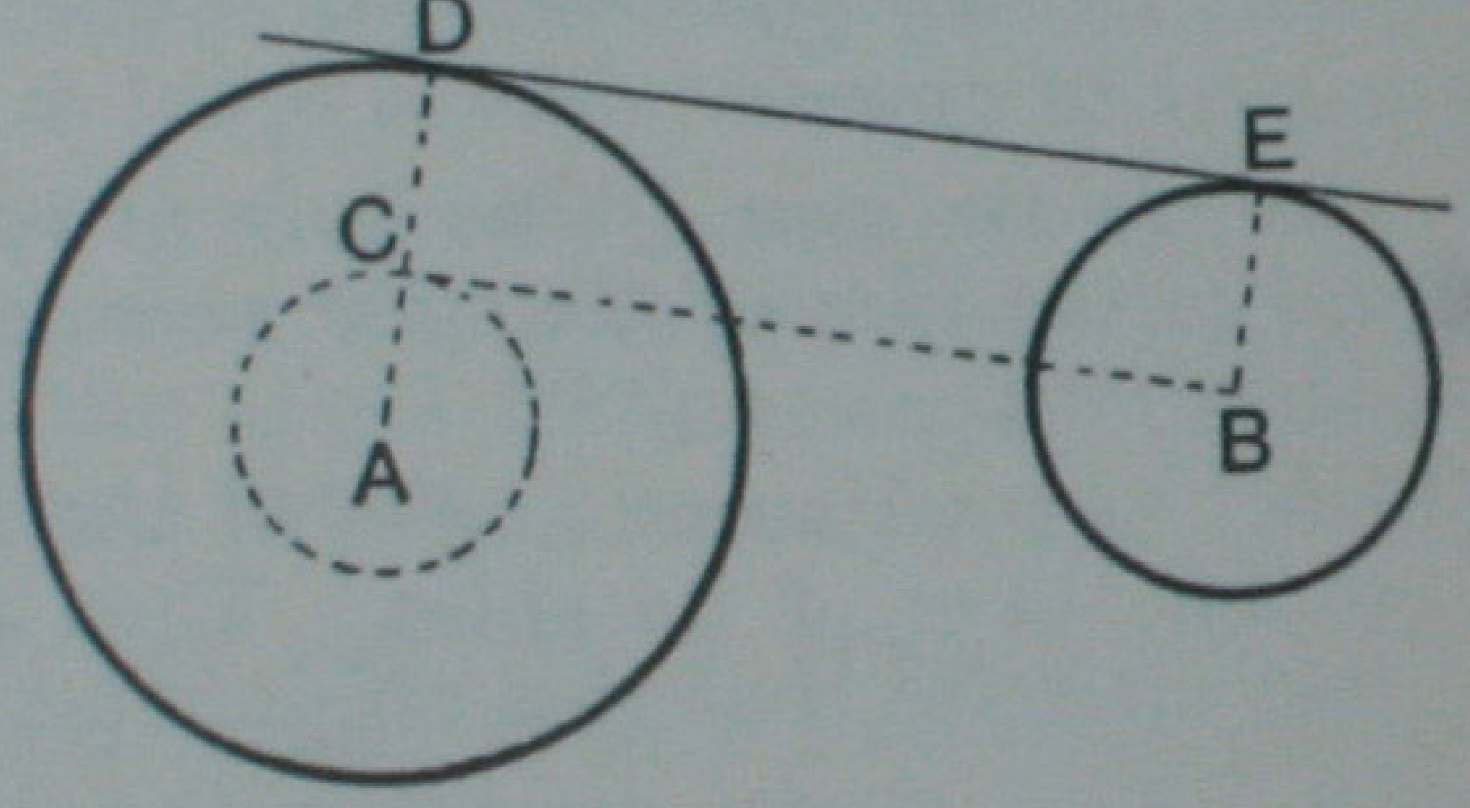
15. Find a point such that the tangents drawn from it to two given circles may be equal to two given straight lines. When is this impossible?

16. If three circles touch one another two and two; prove that the tangents drawn to them at the three points of contact are concurrent and equal.

The Common Tangents to Two Circles.

17. *To draw a common tangent to two circles.*

First. When the given circles are external to one another, or when they intersect.



Let A be the centre of the greater circle, and B the centre of the less.

From A, with radius equal to the diffce of the radii of the given circles, describe a circle: and from B draw BC to touch the last drawn circle. Join AC, and produce it to meet the greater of the given circles at D.

Through B draw the radius BE parl to AD, and in the same direction.

Join DE.

Then DE *shall be a common tangent to the two given circles.*

For since AC = the diffce between AD and BE, *Constr.*

∴ CD = BE :

and CD is parl to BE; *Constr.*

∴ DE is equal and parl to CB. I. 33.

But since BC is a tangent to the circle at C,

∴ the ∠ ACB is a rt. angle; III. 18.

hence each of the angles at D and E is a rt. angle: I. 29.

∴ DE is a tangent to both circles. Q.E.F.

It follows from hypothesis that the point B is outside the circle used in the construction :

∴ two tangents such as BC may always be drawn to it from B ; hence *two* common tangents may always be drawn to the given circles by the above method. These are called the **direct common tangents.**

Secondly. When the given circles are external to one another and do not intersect, two more common tangents may be drawn.

For, from centre A, with a radius equal to the *sum* of the radii of the given circles, describe a circle.

From B draw a tangent to this circle ;

and proceed as before, but draw BE in the direction *opposite* to AD.

It follows from hypothesis that B is external to the circle used in the construction ;

∴ two tangents may be drawn to it from B.

Hence *two* more common tangents may be drawn to the given circles : these will be found to pass between the given circles, and are called the **transverse common tangents.**

Thus, in general, *four* common tangents may be drawn to two given circles.

The student should investigate for himself the number of common tangents which may be drawn in the following special cases, noting in each case where the general construction fails, or is modified :—

(i) When the given circles intersect :

(ii) When the given circles have external contact :

(iii) When the given circles have internal contact :

(iv) When one of the given circles is wholly within the other.

18. *Draw the direct common tangents to two equal circles.*

19. If the two direct, or the two transverse, common tangents are drawn to two circles, the parts of the tangents intercepted between the points of contact are equal.

20. If four common tangents are drawn to two circles external to one another ; shew that the two direct, and also the two transverse, tangents intersect on the straight line which joins the centres of the circles.

21. Two given circles have external contact at A, and a direct common tangent is drawn to touch them at P and Q : shew that PQ subtends a right angle at the point A.

22. Two circles have external contact at A, and a direct common tangent is drawn to touch them at P and Q : shew that a circle described on PQ as diameter is touched at A by the straight line which joins the centres of the circles.

23. Two circles whose centres are C and C′ have external contact at A, and a direct common tangent is drawn to touch them at P and Q: shew that the bisectors of the angles PCA, QC′A meet at right angles in PQ. And if R is the point of intersection of the bisectors, shew that RA is also a common tangent to the circles.

24. Two circles have external contact at A, and a direct common tangent is drawn to touch them at P and Q: shew that the square on PQ is equal to the rectangle contained by the diameters of the circles.

25. Draw a tangent to a given circle, so that the part of it intercepted by another given circle may be equal to a given straight line. When is this impossible?

26. Draw a secant to two given circles, so that the parts of it intercepted by the circumferences may be equal to two given straight lines.

Problems on Tangency.

Obs. The following exercises are solved by the Method of Intersection of Loci, explained on page 125.

The student should begin by making himself familiar with the following loci.

(i) *The locus of the centres of circles which pass through two given points.*

(ii) *The locus of the centres of circles which touch a given straight line at a given point.*

(iii) *The locus of the centres of circles which touch a given circle at a given point.*

(iv) *The locus of the centres of circles which touch a given straight line, and have a given radius.*

(v) *The locus of the centres of circles which touch a given circle, and have a given radius.*

(vi) *The locus of the centres of circles which touch two given straight lines.*

In each exercise the student should investigate the limits and relations among the data, in order that the problem may be possible.

27. Describe a circle to touch three given straight lines.

28. Describe a circle to pass through a given point, and touch a given straight line at a given point.

29. Describe a circle to pass through a given point, and touch a given circle at a given point.

30. Describe a circle of given radius to pass through a given point, and touch a given straight line.

31. Describe a circle of given radius to touch two given circles.

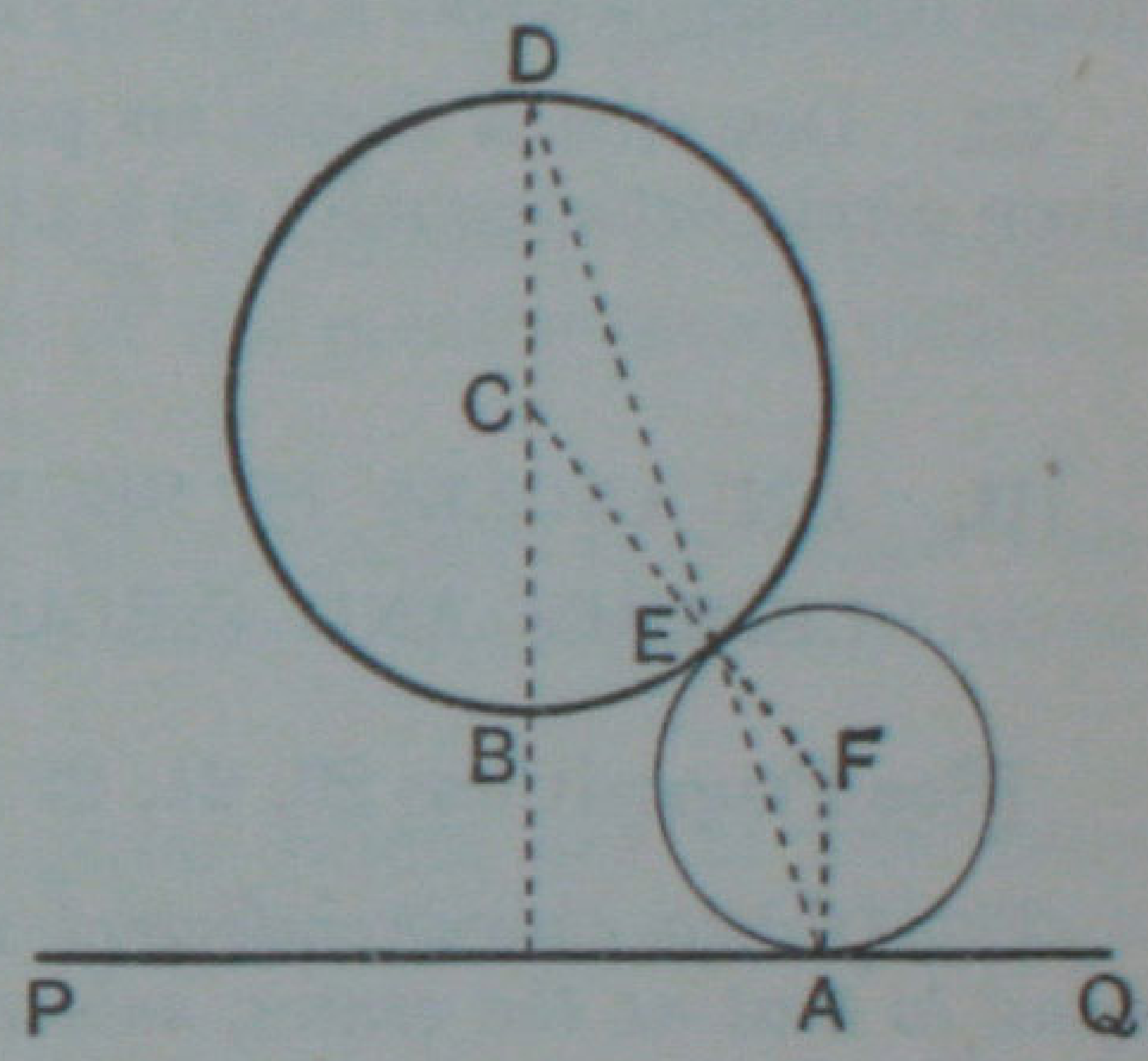
32. Describe a circle of given radius to touch two given straight lines.

33. Describe a circle of given radius to touch a given circle and a given straight line.

34. Describe two circles of given radii to touch one another and a given straight line, on the same side of it.

35. If a circle touches a given circle and a given straight line, shew that the points of contact and an extremity of the diameter of the given circle at right angles to the given line are collinear.

36. *To describe a circle to touch a given circle, and also to touch a given straight line at a given point.*



Let DEB be the given circle, PQ the given straight line, and A the given point in it.

It is required to describe a circle to touch the ⊙ DEB, and also to touch PQ at A.

At A draw AF perp. to PQ: I. 11.
then the centre of the required circle must lie in AF. III 19.

Find C, the centre of the ⊙ DEB, III. 1.
and draw a diameter BD perp. to PQ:
join A to one extremity D, cutting the ○ce at E.

Join CE, and produce it to cut AF at F.

Then F *shall be the centre, and* FA *the radius of the required circle.*

[Supply the proof: and shew that a second solution is obtained by joining AB, and producing it to meet the ○ce.
Also distinguish between the nature of the contact of the circles, when PQ cuts, touches, or is without the given circle.]

37. Describe a circle to touch a given straight line, and to touch a given circle at a given point.

38. Describe a circle to touch a given circle, have its centre in a given straight line, and pass through a given point in that straight line.

[For other problems of the same class see page 253.]

Orthogonal Circles.

Definition. Circles which intersect at a point, so that the two tangents at that point are at right angles to one another, are said to be **orthogonal**, or **to cut one another orthogonally**.

39. In two intersecting circles the angle between the tangents at one point of intersection is equal to the angle between the tangents at the other.

40. *If two circles cut one another orthogonally, the tangent to each circle at a point of intersection will pass through the centre of the other circle.*

41. *If two circles cut one another orthogonally, the square on the distance between their centres is equal to the sum of the squares on their radii.*

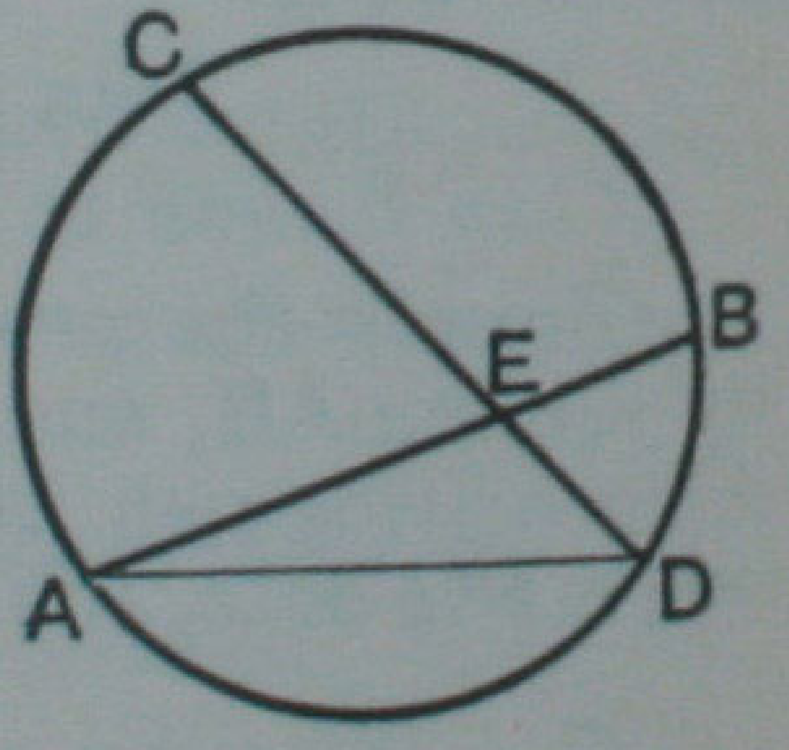
42. Find the locus of the centres of all circles which cut a given circle orthogonally at a given point.

43. Describe a circle to pass through a given point and cut a given circle orthogonally at a given point.

III. On Angles in Segments, and Angles at the Centres and Circumferences of Circles.

[See Propositions 20, 21, 22 ; 26, 27, 28, 29 ; 31, 32, 33, 34.]

1. *If two chords intersect within a circle, they form an angle equal to that at the centre, subtended by half the sum of the arcs they cut off.*



Let AB and CD be two chords, intersecting at E within the given ⊙ ADBC.

Then shall the ∠ AEC *be equal to the angle at the centre, subtended by half the sum of the arcs* AC, BD.

Join AD.

Then the ext. ∠ AEC = the sum of the int. opp. ∠s EDA, EAD ;

that is, the sum of the ∠s CDA, BAD.

But the ∠s CDA, BAD are the angles at the ○ce subtended by the arcs AC, BD ;

∴ their sum = half the sum of the angles at the centre subtended by the same arcs ;

or, the ∠ AEC = the angle at the centre subtended by half the sum of the arcs AC, BD.

Q.E.D.

2. *If two chords when produced intersect outside a circle, they form an angle equal to that at the centre subtended by half the difference of the arcs they cut off.*

3. The sum of the arcs cut off by two chords of a circle at right angles to one another is equal to the semi-circumference.

4. AB, AC are any two chords of a circle; and P, Q are the middle points of the minor arcs cut off by them: if PQ is joined, cutting AB and AC at X, Y, shew that AX=AY.

5. *If one side of a quadrilateral inscribed in a circle is produced, the exterior angle is equal to the opposite interior angle.*

6. If two circles intersect, and any straight lines are drawn, one through each point of section, terminated by the circumferences; shew that the chords which join their extremities towards the same parts are parallel.

7. ABCD is a quadrilateral inscribed in a circle; and the opposite sides AB, DC are produced to meet at P, and CB, DA to meet at Q: if the circles circumscribed about the triangles PBC, QAB intersect at R, shew that the points P, R, Q are collinear.

8. If a circle is described on one of the sides of a right-angled triangle, then the tangent drawn to it at the point where it cuts the hypotenuse bisects the other side.

9. Given three points not in the same straight line: shew how to find any number of points on the circle which passes through them, without finding the centre.

10. Through any one of three given points not in the same straight line, draw a tangent to the circle which passes through them, without finding the centre.

11. Of two circles which intersect at A and B, the circumference of one passes through the centre of the other: from A any straight line is drawn to cut the first at C, the second at D; shew that CB=CD.

12. Two tangents AP, AQ are drawn to a circle, and B is the middle point of the arc PQ, convex to A. Shew that PB bisects the angle APQ.

13. Two circles intersect at A and B; and at A tangents are drawn, one to each circle, to meet the circumferences at C and D; if CB, BD are joined, shew that the triangles ABC, DBA are equiangular to one another.

14. Two segments of circles are described on the same chord and on the same side of it; the extremities of the common chord are joined to any point on the arc of the exterior segment: shew that the arc intercepted on the interior segment is constant.

15. If a series of triangles are drawn standing on a fixed base, and having a given vertical angle, show that the bisectors of the vertical angles all pass through a fixed point.

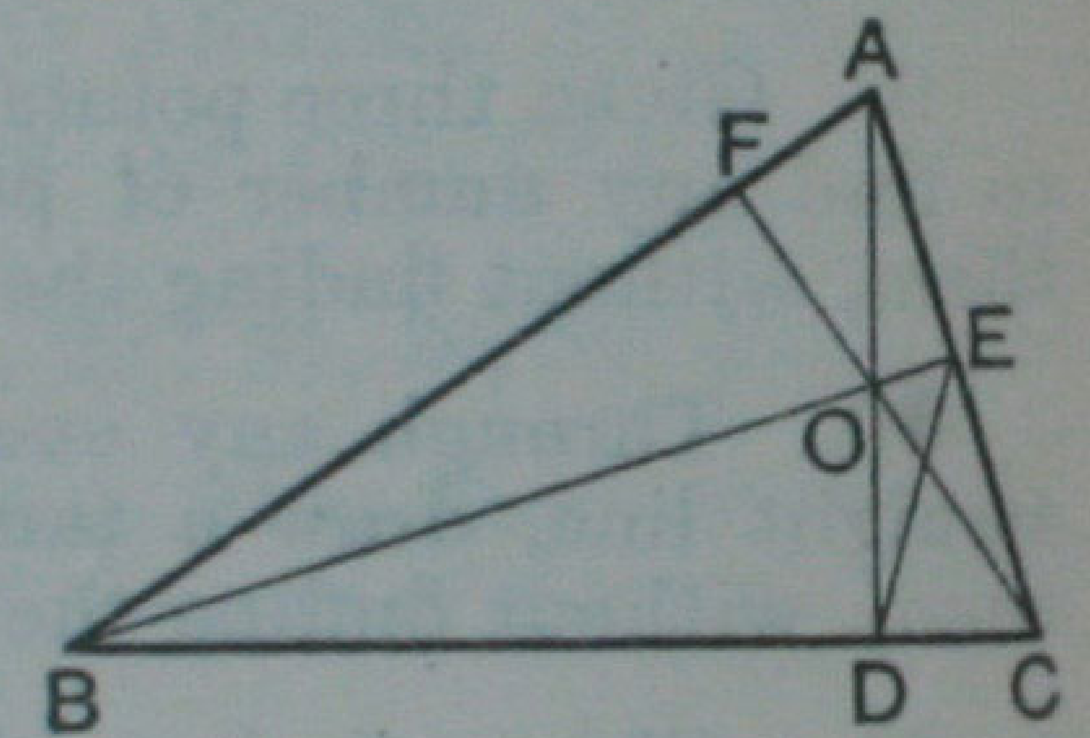
16. ABC is a triangle inscribed in a circle, and E the middle point of the arc subtended by BC on the side remote from A: if through E a diameter ED is drawn, shew that the angle DEA is half the difference of the angles at B and C. [See Ex. 7, p. 109.]

17. If two circles touch each other internally at a point A, any chord of the exterior circle which touches the interior is divided at its point of contact into segments which subtend equal angles at A.

18. If two circles touch one another internally, and a straight line is drawn to cut them, the segments of it intercepted between the circumferences subtend equal angles at the point of contact.

The Orthocentre of a Triangle.

19. *The perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*



In the △ ABC, let AD, BE be the perp^s drawn from A and B to the opposite sides; and let them intersect at O. *Join* CO; and produce it to meet AB at F.

It is required to shew that CF *is perp. to* AB.

Join DE.

Then, because the ∠^s OEC, ODC are rt. angles, *Hyp.*

∴ the points O, E, C, D are concyclic:
∴ the ∠ DEC = the ∠ DOC, in the same segment;
= the vert. opp. ∠ FOA.

Again, because the ∠^s AEB, ADB are rt. angles, *Hyp.*
∴ the points A, E, D, B are concyclic:
∴ the ∠ DEB = the ∠ DAB, in the same segment.

∴ the sum of the ∠^s FOA, FAO = the sum of the ∠^s DEC, DEB
= a rt. angle: *Hyp.*
∴ the remaining ∠ AFO = a rt. angle: I. 32.
that is, CF is perp. to AB.

Hence the three perp^s AD, BE, CF meet at the point O. Q.E.D.

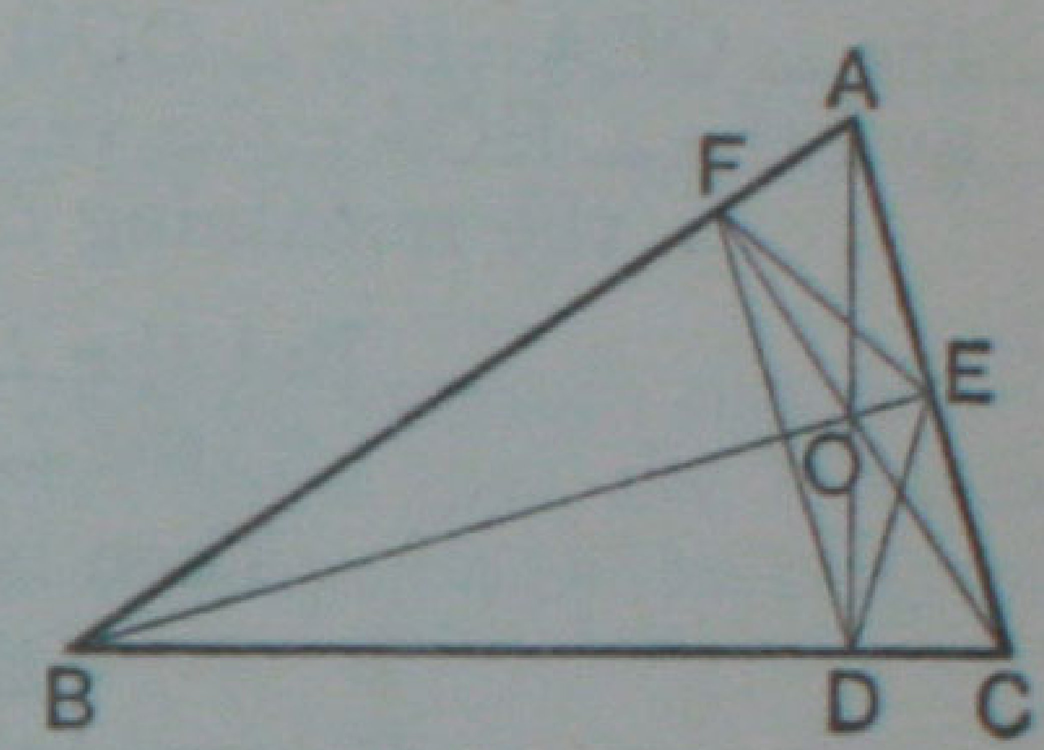
[For an Alternative Proof see p. 114.]

Definitions.

(i) The intersection of the perpendiculars drawn from the vertices of a triangle to the opposite sides is called its **orthocentre.**

(ii) The triangle formed by joining the feet of the perpendiculars is called the **pedal** or **orthocentric triangle.**

20. *In an acute-angled triangle the perpendiculars drawn from the vertices to the opposite sides bisect the angles of the pedal triangle through which they pass.*



In the acute-angled △ ABC, let AD, BE, CF be the perp^s drawn from the vertices to the opposite sides, meeting at the orthocentre O; and let DEF be the pedal triangle.

Then shall AD, BE, CF *bisect respectively the* ∠^{s} FDE, DEF, EFD.

For, as in the last theorem, it may be shewn that the points O, D, C, E are concyclic;

∴ the ∠ ODE = the ∠ OCE, in the same segment.

Similarly the points O, D, B, F are concyclic;

∴ the ∠ ODF = the ∠ OBF, in the same segment.

But the ∠ OCE = the ∠ OBF, each being the comp^t of the ∠ BAC.

∴ the ∠ ODE = the ∠ ODF.

Similarly it may be shewn that the ∠^{s} DEF, EFD are bisected by BE and CF. Q.E.D.

Corollary. (i) *Every two sides of the pedal triangle are equally inclined to that side of the original triangle in which they meet.*

For the ∠ EDC = the comp^t of the ∠ ODE
= the comp^t of the ∠ OCE
= the ∠ BAC.

Similarly it may be shewn that the ∠ FDB = the ∠ BAC,

∴ the ∠ EDC = the ∠ FDB = the ∠ A.

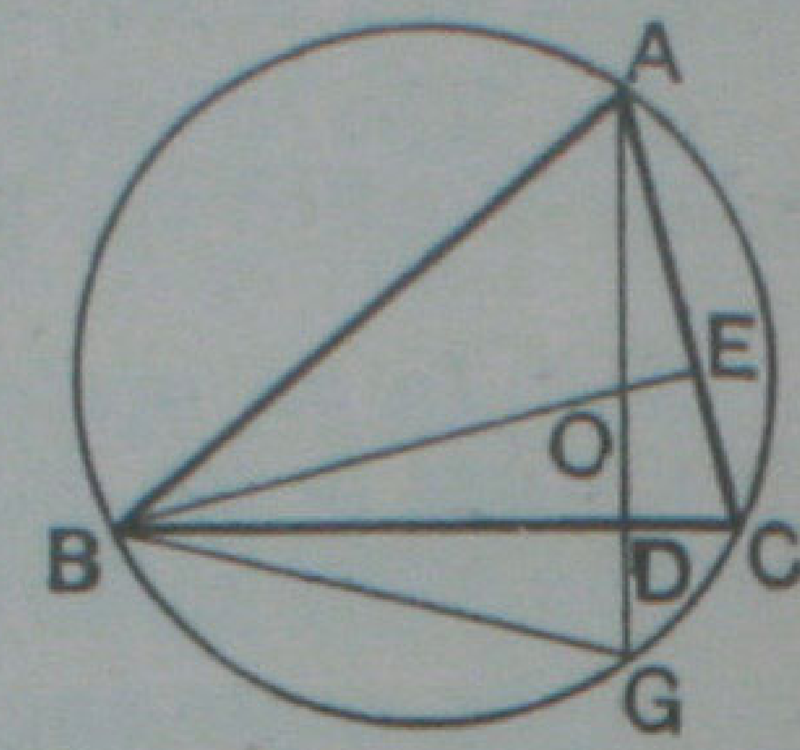
In like manner it may be proved that

the ∠ DEC = the ∠ FEA = the ∠ B,
and the ∠ DFB = the ∠ EFA = the ∠ C.

Corollary. (ii) *The triangles* DEC, AEF, DBF *are equiangular to one another and to the triangle* ABC.

Note. If the angle BAC is *obtuse*, then the perpendiculars BE, CF bisect *externally* the corresponding angles of the pedal triangle.

21. *In any triangle, if the perpendiculars drawn from the vertices on the opposite sides are produced to meet the circumscribed circle, then each side bisects that portion of the line perpendicular to it which lies between the orthocentre and the circumference.*



Let ABC be a triangle in which the perpendiculars AD, BE are drawn, intersecting at O the orthocentre, and let AD be produced to meet the $\bigcirc^{ce}$ of the circumscribing circle at G.

Then shall DO = DG.

Join BG.

Then in the two $\triangle^s$ OEA, ODB,
the $\angle$ OEA = the $\angle$ ODB, being rt. angles;
and the $\angle$ EOA = the vert. opp. $\angle$ DOB;
$\therefore$ the remaining $\angle$ EAO = the remaining $\angle$ DBO. I. 32.

But the $\angle$ CAG = the $\angle$ CBG, in the same segment;
$\therefore$ the $\angle$ DBO = the $\angle$ DBG.

Then in the $\triangle^s$ DBO, DBG,

Because { the $\angle$ DBO = the $\angle$ DBG, *Proved.*
the $\angle$ BDO = the $\angle$ BDG,
and BD is common;

$\therefore$ DO = DG. I. 26.

Q.E.D.

22. *In an acute-angled triangle the three sides are the external bisectors of the angles of the pedal triangle: and in an obtuse-angled triangle the sides containing the obtuse angle are the internal bisectors of the corresponding angles of the pedal triangle.*

23. *If* O *is the orthocentre of the triangle* ABC, *shew that the angles* BOC, BAC *are supplementary.*

24. *If* O *is the orthocentre of the triangle* ABC, *then any one of the four points* O, A, B, C *is the orthocentre of the triangle whose vertices are the other three.*

25. *The three circles which pass through two vertices of a triangle and its orthocentre are each equal to the circle circumscribed about the triangle.*

26. D, E are taken on the circumference of a semicircle described on a given straight line AB: the chords AD, BE and AE, BD intersect (produced if necessary) at F and G: shew that FG is perpendicular to AB.

27. ABCD is a parallelogram; AE and CE are drawn at right angles to AB, and CB respectively: shew that ED, if produced, will be perpendicular to AC.

28. ABC is a triangle, O is its orthocentre, and AK a diameter of the circumscribed circle: shew that BOCK is a parallelogram.

29. The orthocentre of a triangle is joined to the middle point of the base, and the joining line is produced to meet the circumscribed circle: prove that it will meet it at the same point as the diameter which passes through the vertex.

30. The perpendicular from the vertex of a triangle on the base, and the straight line joining the orthocentre to the middle point of the base, are produced to meet the circumscribed circle at P and Q: shew that PQ is parallel to the base.

31. *The distance of each vertex of a triangle from the orthocentre is double of the perpendicular drawn from the centre of the circumscribed circle on the opposite side.*

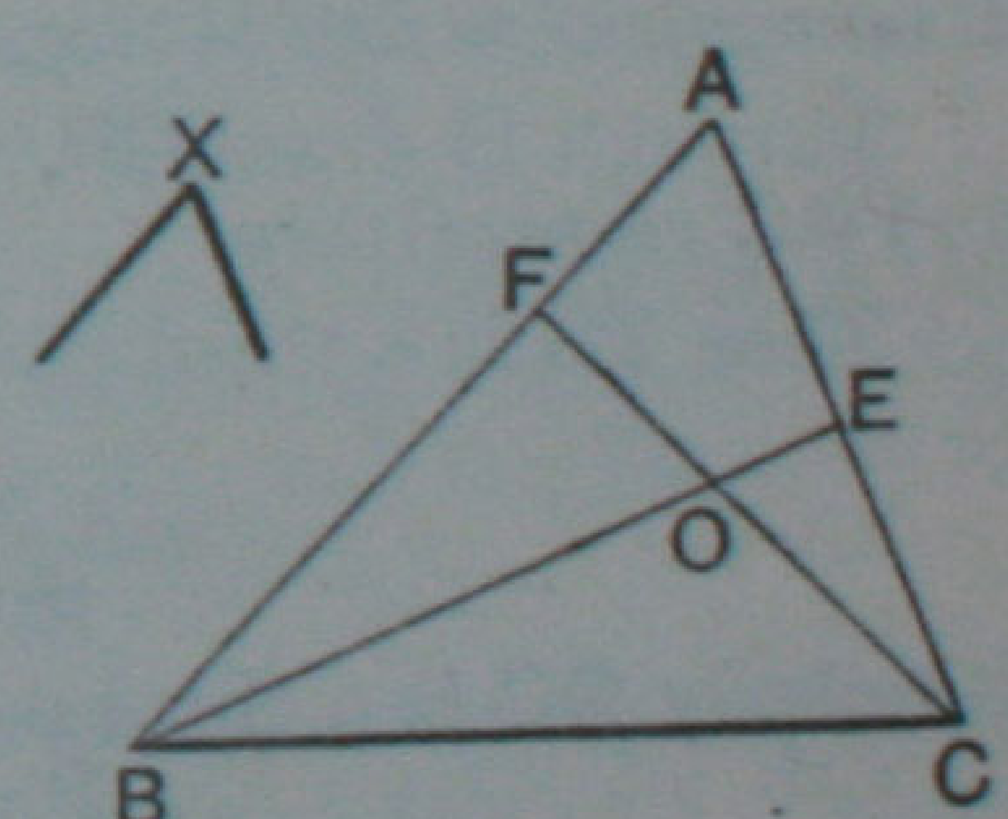
32. Three circles are described each passing through the orthocentre of a triangle and two of its vertices: shew that the triangle formed by joining their centres is equal in all respects to the original triangle.

33. ABC is a triangle inscribed in a circle, and the bisectors of its angles which intersect at O are produced to meet the circumference in PQR: shew that O is the orthocentre of the triangle PQR.

34. Construct a triangle, having given a vertex, the orthocentre, and the centre of the circumscribed circle.

Loci.

35. *Given the base and vertical angle of a triangle, find the locus of its orthocentre.*



Let BC be the given base, and X the given angle; and let BAC be any triangle on the base BC, having its vertical $\angle$ A equal to the $\angle$ X.

Draw the perp^s BE, CF, intersecting at the orthocentre O.

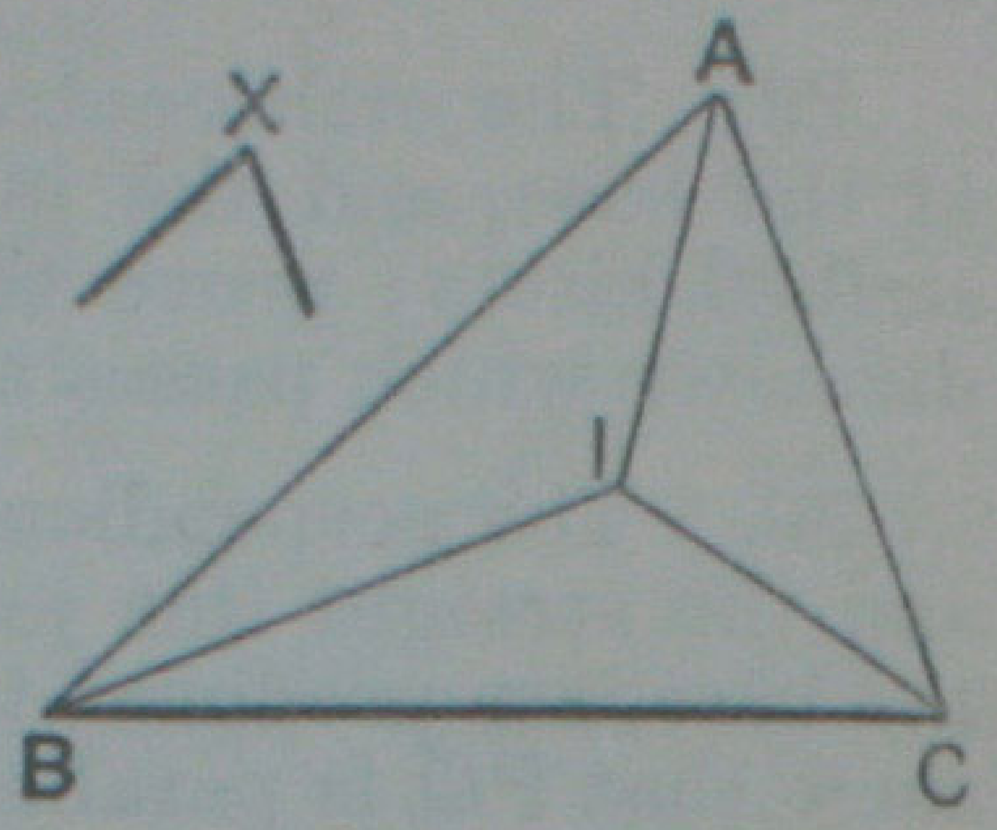
It is required to find the locus of O.

Since the $\angle^s$ OFA, OEA are rt. angles,
∴ the points O, F, A, E are concyclic;
∴ the $\angle$ FOE is the supplement of the $\angle$ A: III. 22.
∴ the vert. opp. $\angle$ BOC is the supplement of the $\angle$ A.

But the $\angle$ A is constant, being always equal to the $\angle$ X;
∴ its supplement is constant;

that is, the $\triangle$ BOC has a fixed base, and constant vertical angle; hence the locus of its vertex O is the arc of a segment of which BC is the chord. [See Corollary p. 201.]

36. *Given the base and vertical angle of a triangle, find the locus of the intersection of the bisectors of its angles.*



Let BAC be any triangle on the given base BC, having its vertical angle equal to the given ∠ X; and let AI, BI, CI be the bisectors of its angles. [See Ex. 2, p. 111.]

It is required to find the locus of the point I.

Denote the angles of the △ ABC by A, B, C; and let the ∠ BIC be denoted by I.

Then from the △ BIC,

(i) $I + \frac{1}{2}B + \frac{1}{2}C =$ two rt. angles, I. 32.

and from the △ ABC,

$A + B + C =$ two rt. angles; I. 32.

(ii) so that $\frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C =$ one rt. angle,

∴, taking the differences of the equals in (i) and (ii),

$I - \frac{1}{2}A =$ one rt. angle:

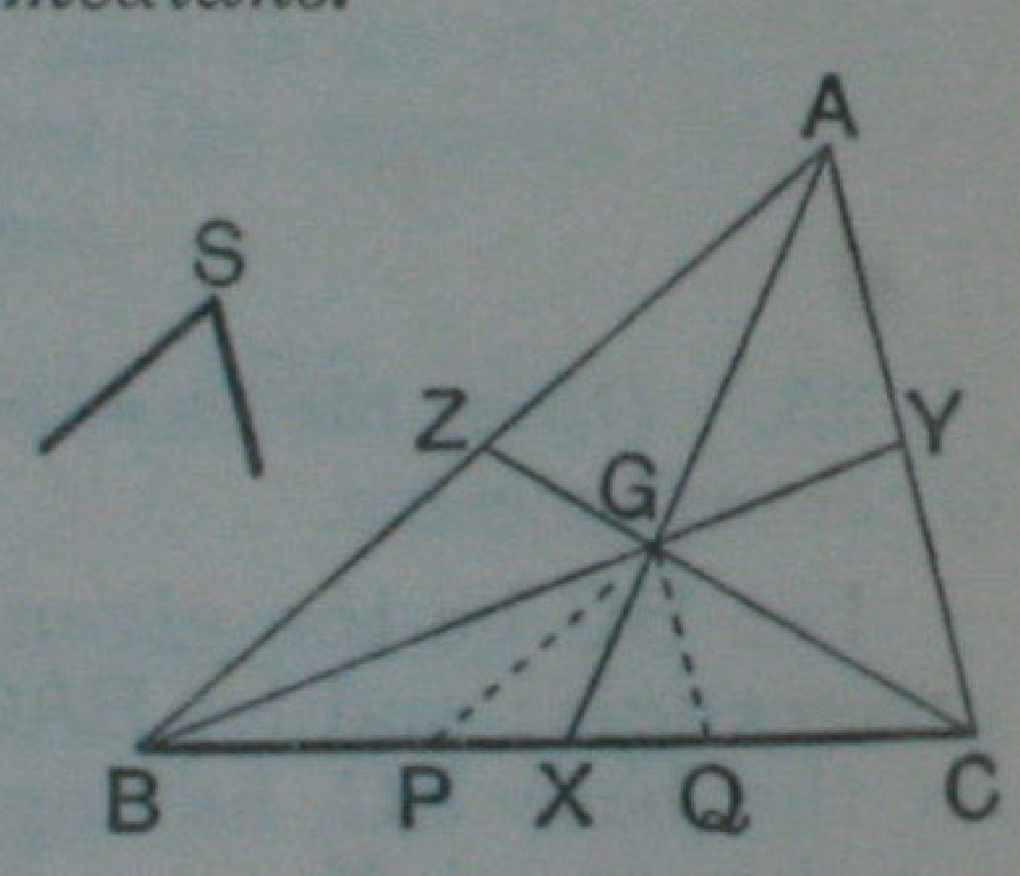
or, $I =$ one rt. angle $+ \frac{1}{2}A$.

But A is constant, being always equal to the ∠ X;

∴ I is constant:

∴ the locus of I is the arc of a segment on the fixed chord BC.

37. *Given the base and vertical angle of a triangle, find the locus of the centroid, that is, the intersection of the medians.*



Let BAC be any triangle on the given base BC, having its vertical angle equal to the given angle S; let the medians AX, BY, CZ intersect at the centroid G. [See Ex. 4, p. 113.]

It is required to find the locus of the point G.

Through G draw GP, GQ par^l to AB and AC respectively.

Then ZG is a third part of ZC;

Ex. 4, p. 113,

and since GP is par^l to ZB,

∴ BP is a third part of BC. Ex. 19, p. 107.

Similarly QC is a third part of BC;

∴ P and Q are fixed points.

Now since PG, GQ are par^l respectively to BA, AC, *Constr.*

∴ the ∠ PGQ = the ∠ BAC, I. 29.

= the ∠ S,

that is, the ∠ PGQ is constant;

∴ the locus of G is the arc of a segment on the fixed chord PQ.

Note. In this problem the points A and G move on the arcs of *similar segments.*

38. Given the base and the vertical angle of a triangle; find the locus of the intersection of the bisectors of the exterior base angles.

39. Through the extremities of a given straight line AB any two parallel straight lines AP, BQ are drawn; find the locus of the intersection of the bisectors of the angles PAB, QBA.

40. Find the locus of the middle points of chords of a circle drawn through a fixed point.

Distinguish between the cases when the given point is within, on, or without the circumference.

41. Find the locus of the points of contact of tangents drawn from a fixed point to a system of concentric circles.

42. Find the locus of the intersection of straight lines which pass through two fixed points on a circle and intercept on its circumference an arc of constant length.

43. A and B are two fixed points on the circumference of a circle, and PQ is any diameter: find the locus of the intersection of PA and QB.

44. BAC is any triangle described on the fixed base BC and having a constant vertical angle; and BA is produced to P, so that BP is equal to the sum of the sides containing the vertical angle: find the locus of P.

45. AB is a fixed chord of a circle, and AC is a moveable chord passing through A: if the parallelogram CB is completed, find the locus of the intersection of its diagonals.

46. A straight rod PQ slides between two rulers placed at right angles to one another, and from its extremities PX, QX are drawn perpendicular to the rulers: find the locus of X.

47. Two circles whose centres are C and D, intersect at A and B: through A, any straight line PAQ is drawn terminated by the circumferences; and PC, QD intersect at X: find the locus of X, and shew that it passes through B. [Ex. 9, p. 234.]

48. Two circles intersect at A and B, and through P, any point on the circumference of one of them, two straight lines PA, PB are drawn, and produced if necessary, to cut the other circle at X and Y: find the locus of the intersection of AY and BX.

49. Two circles intersect at A and B; HAK is a fixed straight line drawn through A and terminated by the circumferences, and PAQ is any other straight line similarly drawn: find the locus of the intersection of HP and QK.

50. Two segments of circles are on the same chord AB and on the same side of it; and P and Q are any points one on each arc: find the locus of the intersection of the bisectors of the angles PAQ, PBQ.

51. Two circles intersect at A and B; and through A any straight line PAQ is drawn terminated by the circumferences: find the locus of the middle point of PQ.

Miscellaneous Examples on Angles in a Circle.

52. ABC is a triangle, and circles are drawn through B, C, cutting the sides in P, Q, P′, Q′, ...: shew that PQ, P′Q′ ... are parallel to one another and to the tangent drawn at A to the circle circumscribed about the triangle.

53. Two circles intersect at B and C, and from any point A, on the circumference of one of them, AB, AC are drawn, and produced if necessary, to meet the other at D and E: shew that DE is parallel to the tangent at A.

54. A secant PAB and a tangent PT are drawn to a circle from an external point P; and the bisector of the angle ATB meets AB at C: shew that PC is equal to PT.

55. From a point A on the circumference of a circle two chords AB, AC are drawn, and also the diameter AF: if AB, AC are produced to meet the tangent at F in D and E, shew that the triangles ABC, AED are equiangular to one another.

56. O is any point within a triangle ABC, and OD, OE, OF are drawn perpendicular to BC, CA, AB respectively: shew that the angle BOC is equal to the sum of the angles BAC, EDF.

57. If two tangents are drawn to a circle from an external point, shew that they contain an angle equal to the difference of the angles in the segments cut off by the chord of contact.

58. Two circles intersect, and through a point of section a straight line is drawn bisecting the angle between the diameters through that point: shew that this straight line cuts off similar segments from the two circles.

59. Two equal circles intersect at A and B; and from centre A, with any radius less than AB a third circle is described cutting the given circles on the same side of AB at C and D: shew that the points B, C, D are collinear.

60. ABC and A′B′C′ are two triangles inscribed in a circle, so that AB, AC are respectively parallel to A′B′, A′C′: shew that BC′ is parallel to B′C.

61. Two circles intersect at A and B, and through A two straight lines HAK, PAQ are drawn terminated by the circumferences: if HP and KQ intersect at X, shew that the points H, B, K, X are concyclic.

62. Describe a circle touching a given straight line at a given point, so that tangents drawn to it from two fixed points in the given line may be parallel. [See Ex. 10, p. 197.]

63. C is the centre of a circle, and CA, CB two fixed radii: if from any point P on the arc AB perpendiculars PX, PY are drawn to CA and CB, shew that the distance XY is constant.

64. AB is a chord of a circle, and P any point in its circumference; PM is drawn perpendicular to AB, and AN is drawn perpendicular to the tangent at P: shew that MN is parallel to PB.

65. P is any point on the circumference of a circle of which AB is a fixed diameter, and PN is drawn perpendicular to AB; on AN and BN as diameters circles are described, which are cut by AP, BP at X and Y: shew that XY is a common tangent to these circles.

66. Upon the same chord and on the same side of it three segments of circles are described containing respectively a given angle, its supplement and a right angle: shew that the intercept made by the two former segments upon any straight line drawn through an extremity of the given chord is bisected by the latter segment.

67. Two straight lines of indefinite length touch a given circle, and any chord is drawn so as to be bisected by the chord of contact: if the former chord is produced, shew that the intercepts between the circumference and the tangents are equal.

68. Two circles intersect one another: through one of the points of section draw a straight line of given length terminated by the circumferences.

69. On the three sides of any triangle equilateral triangles are described remote from the given triangle: shew that the circles described about them intersect at a point.

70. On BC, CA, AB the sides of a triangle ABC, any points P, Q, R are taken; shew that the circles described about the triangles AQR, BRP, CPQ meet in a point.

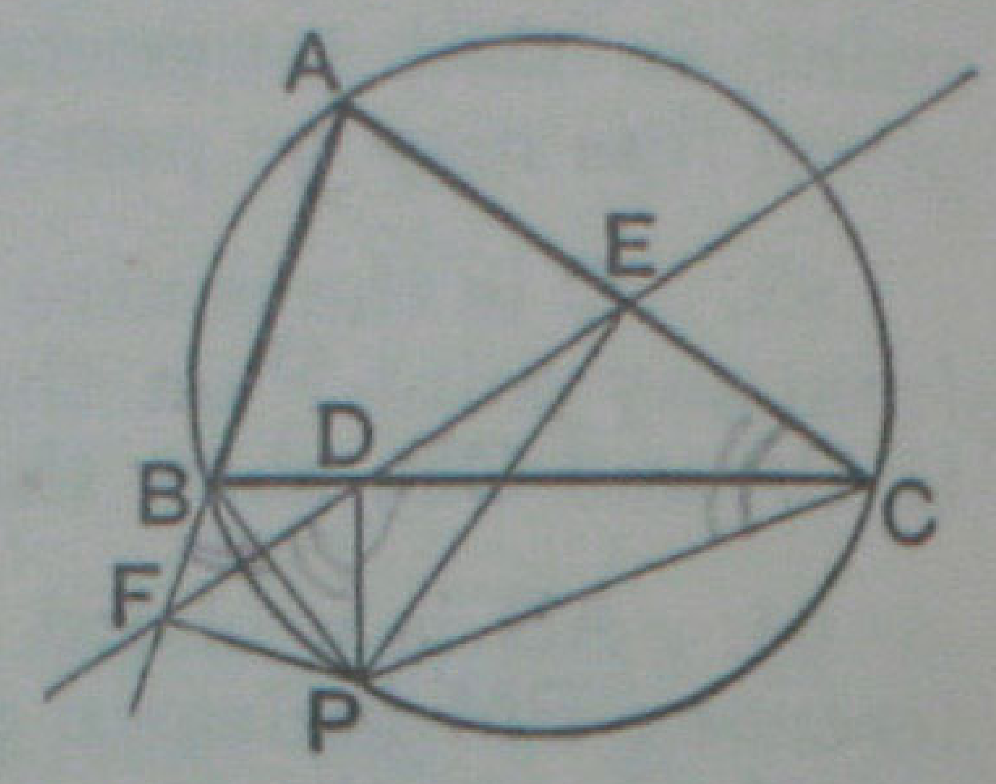
71. Find a point within a triangle at which the sides subtend equal angles.

72. Describe an equilateral triangle so that its sides may pass through three given points.

73. Describe a triangle equal in all respects to a given triangle, and having its sides passing through three given points.

SIMSON'S LINE.

74. *If from any point on the circumference of the circle circumscribed about a triangle, perpendiculars are drawn to the three sides, the feet of these perpendiculars are collinear.*



Let P be any point on the $\bigcirc^{ce}$ of the circle circumscribed about the △ ABC; and let PD, PE, PF be the perp[s] drawn from P to the three sides.

It is required to prove that the points D, E, F *are collinear.*

Join FD and DE:
then FD and DE shall be in the same st. line.

Join PB, PC.

Because the ∠[s] PDB, PFB are rt. angles, *Hyp.*
∴ the points P, D, B, F are concyclic:
∴ the ∠ PDF = the ∠ PBF, in the same segment. III. 21.

But since BACP is a quad[l] inscribed in a circle, having one of its sides AB produced to F,
∴ the ext. ∠ PBF = the opp. int. ∠ ACP. *Ex.* 3, *p.* 202.
∴ the ∠ PDF = the ∠ ACP.
To each add the ∠ PDE:
then the ∠[s] PDF, PDE = the ∠[s] ECP, PDE.
But since the ∠[s] PDC, PEC are rt. angles,
∴ the points P, D, E, C are concyclic;
∴ the ∠[s] ECP, PDE together = two rt. angles:
∴ the ∠[s] PDF, PDE together = two rt. angles;
∴ FD and DE are in the same st. line; I. 14.
that is, the points D, E, F are collinear. Q.E.D.

[The line FDE is called the **Pedal** or **Simson's Line** of the triangle ABC for the point P; though the tradition attributing the theorem to Robert Simson has been recently shaken by the researches of Dr. J. S. Mackay.]

75. ABC is a triangle inscribed in a circle; and from any point P on the circumference PD, PF are drawn perpendicular to BC and AB: if FD, or FD produced, cuts AC at E, shew that PE is perpendicular to AC.

76. Find the locus of a point which moves so that if perpendiculars are drawn from it to the sides of a given triangle, their feet are collinear.

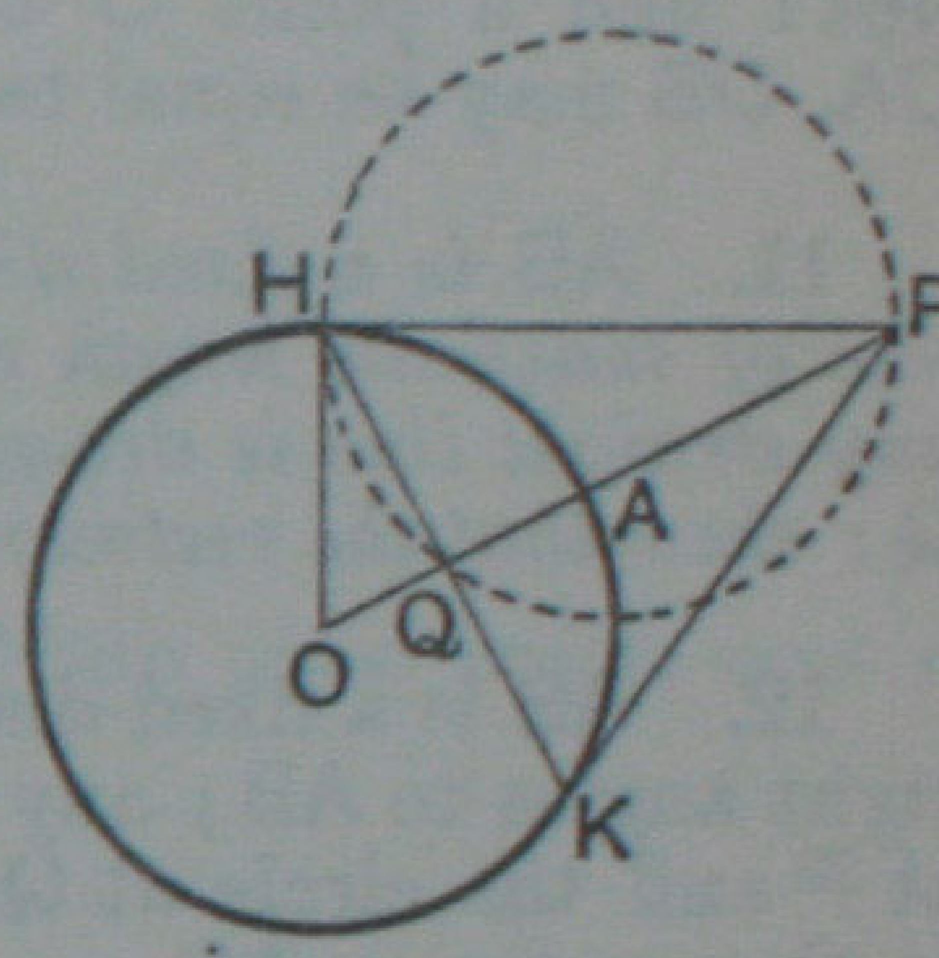
77. ABC and AB′C′ are two triangles having a common vertical angle, and the circles circumscribed about them meet again at P; shew that the feet of perpendiculars drawn from P to the four lines AB, AC, BC, B′C′ are collinear.

78. *A triangle is inscribed in a circle, and any point* P *on the circumference is joined to the orthocentre of the triangle: shew that this joining line is bisected by the pedal of the point* P.

IV. ON THE CIRCLE IN CONNECTION WITH RECTANGLES.

[See Propositions 35, 36, 37.]

1. *If from any external point* P *two tangents are drawn to a given circle whose centre is* O, *and if* OP *meets the chord of contact at* Q; *then the rectangle* OP, OQ *is equal to the square on the radius.*



Let PH, PK be tangents, drawn from the external point P to the ⊙ HAK, whose centre is O; and let OP meet HK the chord of contact at Q, and the ○ce at A.

Then shall the rect. OP, OQ=*the sq. on* OA.

On HP as diameter describe a circle: this circle must pass through Q, since the ∠ HQP is a rt. angle. III. 31.

Join OH.

Then since PH is a tangent to the ⊙ HAK,

∴ the ∠ OHP is a rt. angle.

And since HP is a diameter of the ⊙ HQP,

∴ OH touches the ⊙ HQP at H. III. 16.

∴ the rect. OP, OQ=the sq. on OH, III. 36.

=the sq. on OA. Q.E.D.

2. ABC is a triangle, and AD, BE, CF the perpendiculars drawn from the vertices to the opposite sides, meeting in the orthocentre O: shew that the rect. AO, OD=the rect. BO, OE=the rect. CO, OF.

3. ABC is a triangle, and AD, BE the perpendiculars drawn from A and B on the opposite sides: shew that the rectangle CA, CE is equal to the rectangle CB, CD.

4. ABC is a triangle right-angled at C, and from D, any point in the hypotenuse AB, a straight line DE is drawn perpendicular to AB and meeting BC at E: shew that the square on DE is equal to the difference of the rectangles AD, DB and CE, EB.

5. From an external point P two tangents are drawn to a given circle whose centre is O, and OP meets the chord of contact at Q: shew that any circle which passes through the points P, Q will cut the given circle orthogonally. [See Def. p. 240.]

6. *A series of circles pass through two given points, and from a fixed point in the common chord produced tangents are drawn to all the circles: shew that the points of contact lie on a circle which cuts all the given circles orthogonally.*

7. *All circles which pass through a fixed point, and cut a given circle orthogonally, pass also through a second fixed point.*

8. Find the locus of the centres of all circles which pass through a given point and cut a given circle orthogonally.

9. Describe a circle to pass through two given points and cut a given circle orthogonally.

10. A, B, C, D are four points taken in order on a given straight line: find a point O between B and C such that the rectangle OA, OB may be equal to the rectangle OC, OD.

11. AB *is a fixed diameter of a circle, and* CD *a fixed straight line of indefinite length cutting* AB *or* AB *produced at right angles; any straight line is drawn through* A *to cut* CD *at* P *and the circle at* Q: *shew that the rectangle* AP, AQ *is constant.*

12. AB is a fixed diameter of a circle, and CD a fixed chord at right angles to AB; any straight line is drawn through A to cut CD at P and the circle at Q: shew that the rectangle AP, AQ is equal to the square on AC.

13. A *is a fixed point, and* CD *a fixed straight line of indefinite length*; AP *is any straight line drawn through* A *to meet* CD *at* P; *and in* AP *a point* Q *is taken such that the rectangle* AP, AQ *is constant: find the locus of* Q.

14. Two circles intersect orthogonally, and tangents are drawn from any point on the circumference of one to touch the other: prove that the first circle passes through the middle point of the chord of contact of the tangents. [Ex. 1, p. 251.]

15. A semicircle is described on AB as diameter, and any two chords AC, BD are drawn intersecting at P: shew that

$$AB^2 = AC.AP + BD.BP.$$

16. Two circles intersect at B and C, and the two direct common tangents AE and DF are drawn: if the common chord is produced to meet the tangents at G and H, shew that $GH^2 = AE^2 + BC^2$.

17. If from a point P, without a circle, PM is drawn perpendicular to a diameter AB, and also a secant PCD, shew that

$$PM^2 = PC.PD + AM.MB.$$

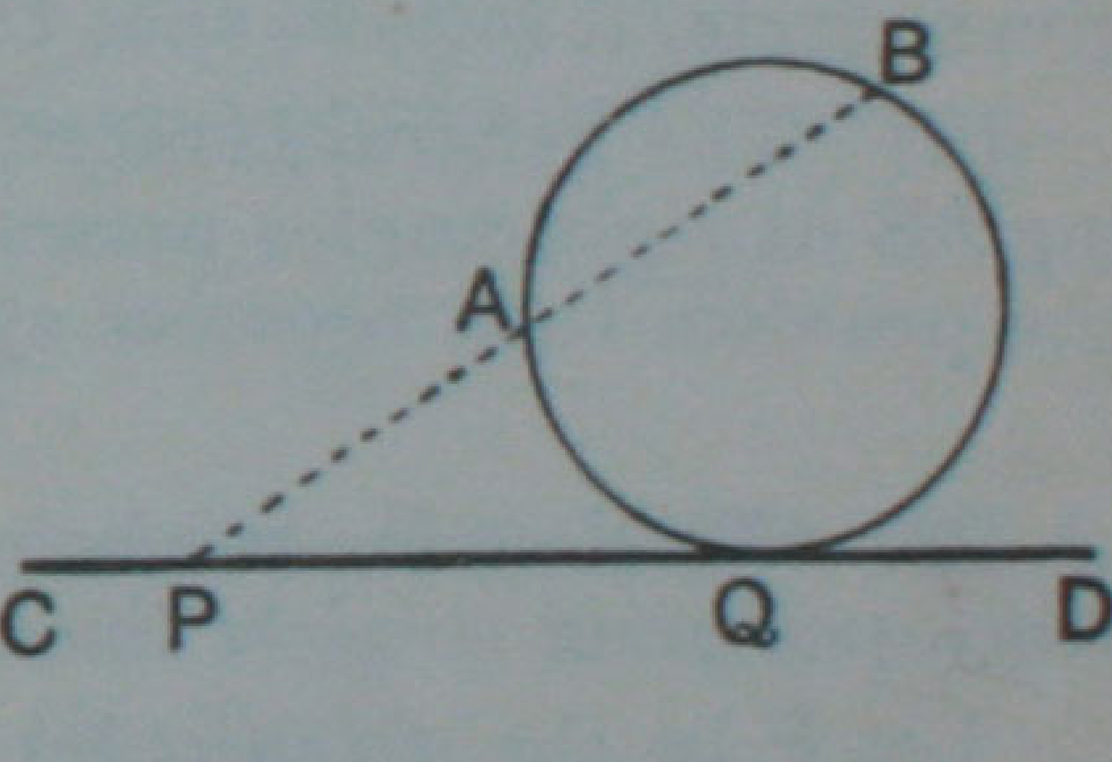
18. Three circles intersect at D, and their other points of intersection are A, B, C; AD cuts the circle BDC at E, and EB, EC cut the circles ADB, ADC respectively at F and G: show that the points F, A, G are collinear, and F, B, C, G concyclic.

19. A semicircle is described on a given diameter BC, and from B and C any two chords BE, CF are drawn intersecting within the semicircle at O; BF and CE are produced to meet at A: shew that the sum of the squares on AB, AC is equal to twice the square on the tangent from A together with the square on BC.

20. X and Y are two fixed points in the diameter of a circle equidistant from the centre C: through X any chord PXQ is drawn, and its extremities are joined to Y; shew that the sum of the squares on the sides of the triangle PYQ is constant. [See p. 161, Ex. 24.]

Problems on Tangency.

21. *To describe a circle to pass through two given points and to touch a given straight line.*



Let A and B be the given points, and CD the given st. line.

It is required to describe a circle to pass through A *and* B *and to touch* CD.

Join BA, and produce it to meet CD at P.

Describe a square equal to the rect. PA, PB; II. 14.
and from PD (or PC) cut off PQ equal to a side of this square.

Through A, B, and Q describe a circle. Ex. 4, p. 171.

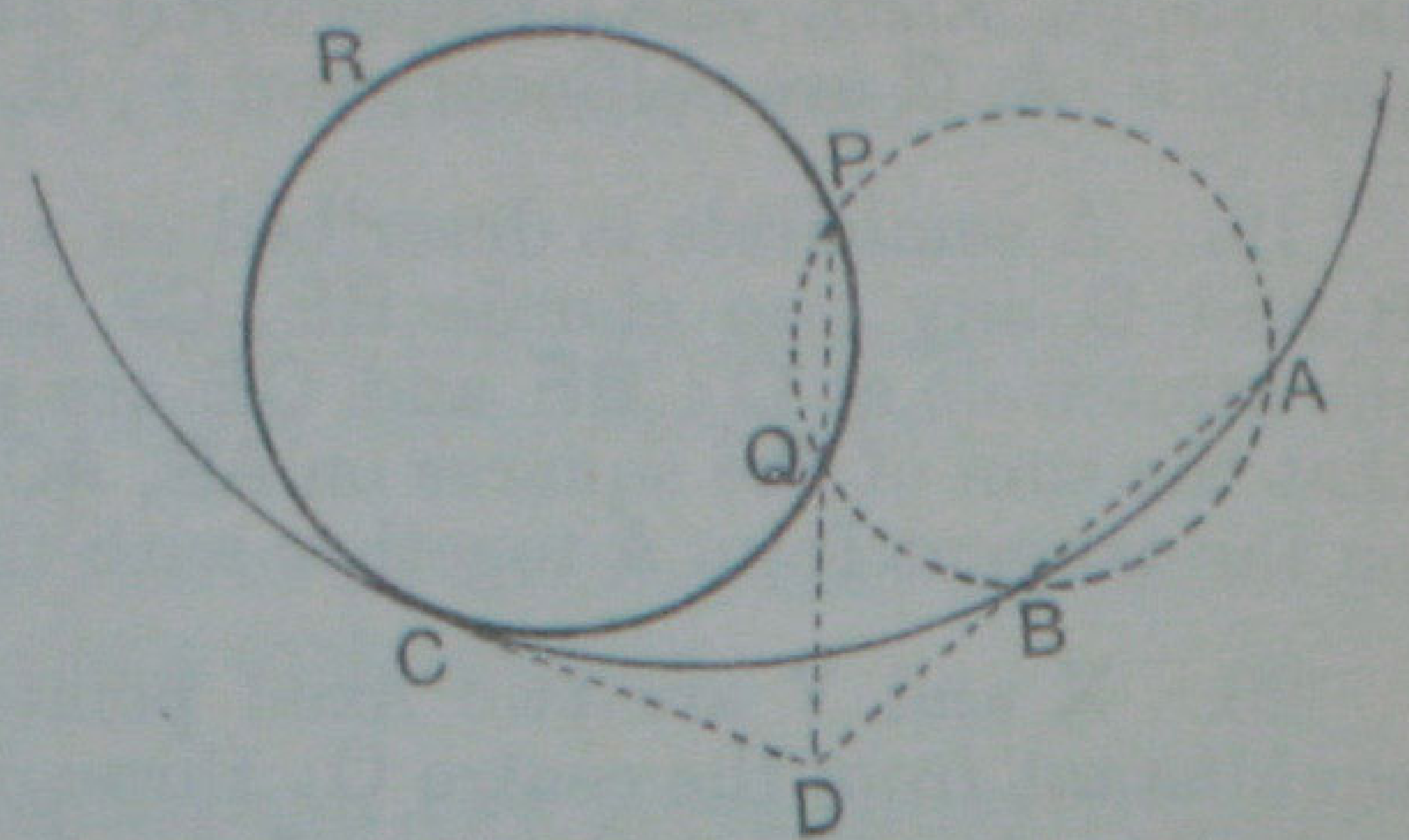
Then since the rect. PA, PB = the sq. on PQ,
∴ the ⊙ ABQ touches CD at Q. III. 37.

Q.E.F.

Notes. (i) Since PQ may be taken on either side of P, it is clear that there are in general two solutions of the problem.

(ii) When AB is parallel to the given line CD, the above method is not applicable. In this case a simple construction follows from III. 1, Cor. and III. 16, and it will be found that only one solution exists.

22. *To describe a circle to pass through two given points and to touch a given circle.*



Let A and B be the given points, and CRP the given circle.

It is required to describe a circle to pass through A *and* B, *and to touch the* ⊙ CRP.

Through A and B describe *any* circle to cut the given circle at P and Q.

Join AB, PQ, and produce them to meet at D.

From D draw DC to touch the given circle, and let C be the point of contact.

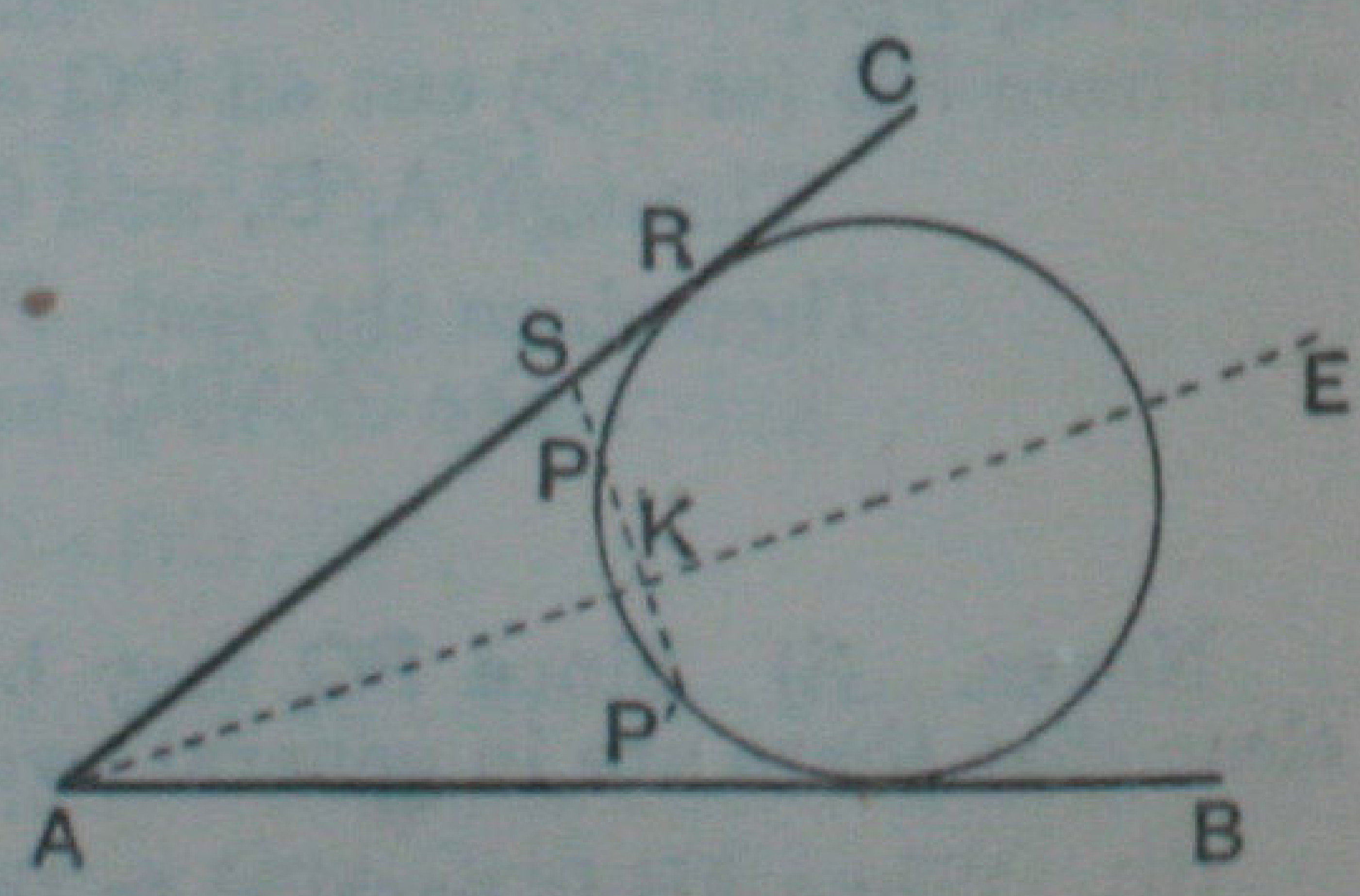
Then the circle described through A, B, C *will touch the given circle.*

For, from the ⊙ ABQP, the rect. DA, DB = the rect. DP, DQ:
and from the ⊙PQC, the rect. DP, DQ = the sq. on DC; III. 36.
∴ the rect. DA, DB = the sq. on DC:
∴ DC touches the ⊙ ABC at C. III. 37.
But DC touches the ⊙ PQC at C; *Constr.*
∴ the ⊙ ABC touches the given circle, and it passes through the given points A and B. Q.E.F.

NOTE. (i) Since two tangents may be drawn from D to the given circle, it follows that there will be two solutions of the problem.

(ii) The general construction fails when the straight line bisecting AB at right angles passes through the centre of the given circle: the problem then becomes symmetrical, and the solution is obvious.

23. *To describe a circle to pass through a given point and to touch two given straight lines.*



Let P be the given point, and AB, AC the given straight lines.

It is required to describe a circle to pass through P *and to touch* AB, AC.

Now the centre of every circle which touches AB and AC must lie on the bisector of the ∠ BAC. Ex. 7, p. 197.

Hence draw AE bisecting the ∠ BAC.

From P draw PK perp. to AE, and produce it to P′, making KP′ equal to PK.

Then every circle which has its centre in AE, and passes through P, must also pass through P′. Ex. 1, p. 233.

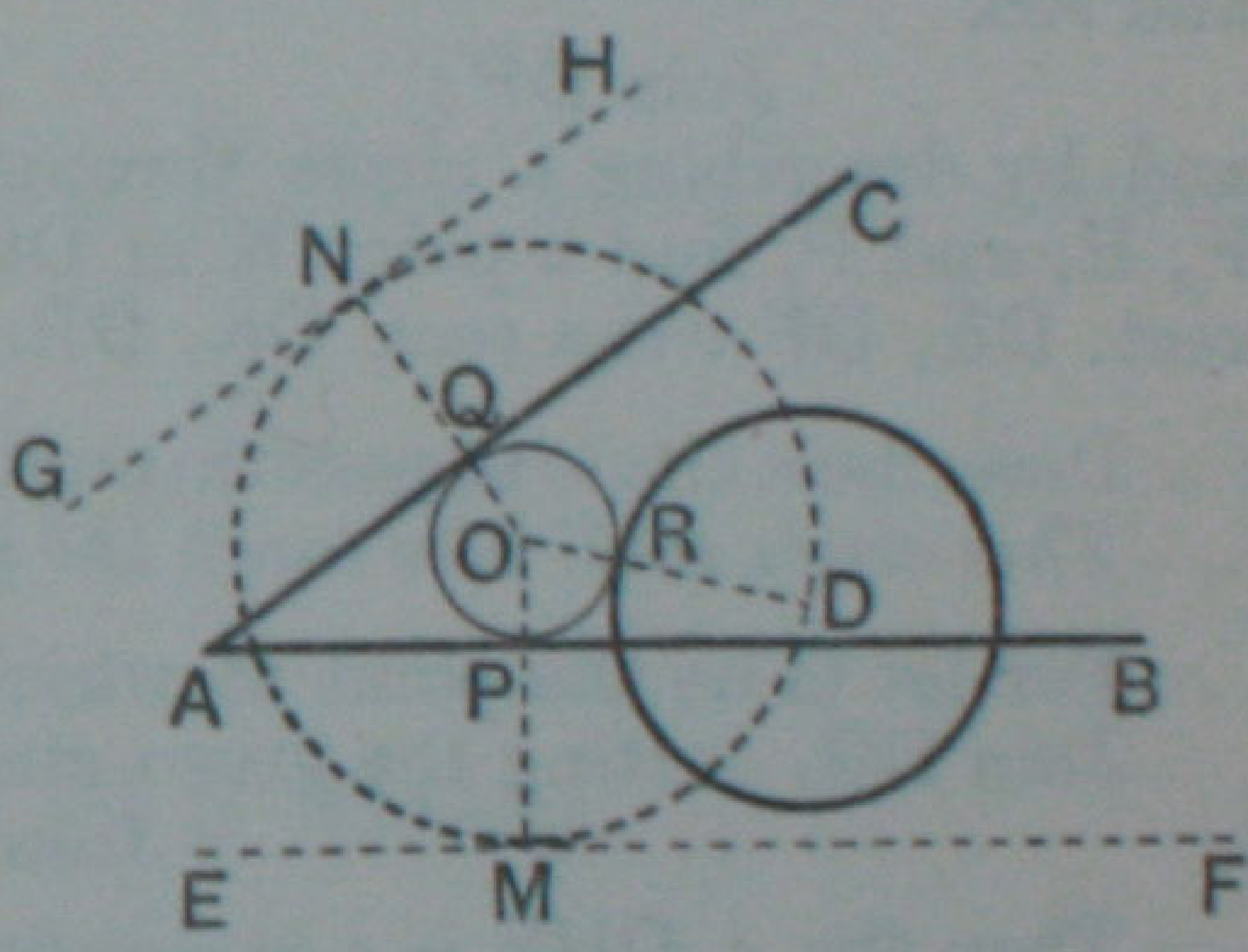
Hence the problem is now reduced to drawing a circle through P and P′ to touch *either* AC or AB. Ex. 21, p. 253.

Produce P′P to meet AC at S.
Describe a square equal to the rect. SP, SP′; II. 14.
and cut off SR equal to a side of the square.
Describe a circle through the points P′, P, R.
Then since the rect. SP, SP′ = the sq. on SR, *Constr.*
∴ this circle touches AC at R; III. 37,
and since its centre is in AE, the bisector of the ∠ BAC,
it may be shewn also to touch AB. Q.E.F.

NOTES. (i) Since SR may be taken on either side of S, it follows that there will be two solutions of the problem.

(ii) If the given straight lines are parallel, the centre lies on the parallel straight line mid-way between them, and the construction proceeds as before.

24. *To describe a circle to touch two given straight lines and a given circle.*



Let AB, AC be the two given st. lines, and D the centre of the given circle.

It is required to describe a circle to touch AB, AC *and the circle whose centre is* D.

Draw EF, GH par[l] to AB and AC respectively, on the sides remote from D, and at distances from them equal to the radius of the given circle.

Describe the ⊙ MND to touch EF and GH at M and N, and to pass through D. Ex. 23, p. 254.

Let O be the centre of this circle.

Join OM, ON, OD meeting AB, AC, and the given circle at P, Q, and R.

Then a circle described with centre O *and radius* OP *will touch* AB, AC *and the given circle.*

For since O is the centre of the ⊙ MND,
∴ OM = ON = OD.
But PM = QN = RD; *Constr.*
∴ OP = OQ = OR.

∴ a circle described with centre O, and radius OP, will pass through Q and R.

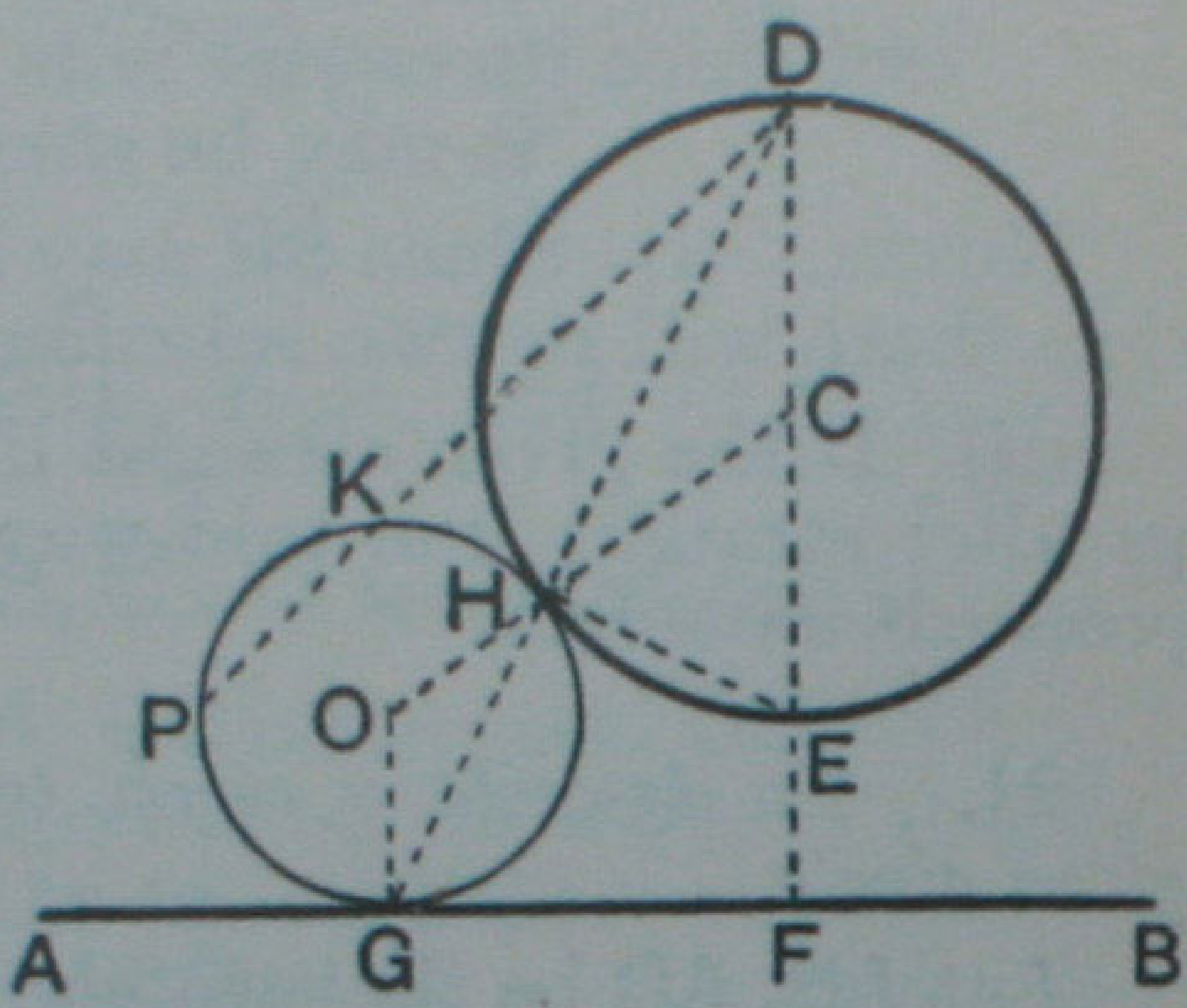
And since the ∠s at M and N are rt. angles, III. 18.
∴ the ∠s at P and Q are rt. angles; I. 29.
∴ the ⊙ PQR touches AB and AC.

And since R, the point in which the circles meet, is on the line of centres OD,

∴ the ⊙ PQR touches the given circle. Q.E.F.

NOTE. There will be two solutions of this problem, since two circles may be drawn to touch EF, GH and to pass through D.

25. *To describe a circle to pass through a given point and touch a given straight line and a given circle.*



Let P be the given point, AB the given st. line, and DHE the given circle, of which C is the centre.

It is required to describe a circle to pass through P, *and to touch* AB *and the* ⊙ DHE.

Through C draw DCEF perp. to AB, cutting the circle at the points D and E, of which E is between C and AB.

Join DP;

and by describing a circle through F, E, and P, find a point K in DP (or DP produced) such that the rect. DE, DF = the rect. DK, DP.

Describe a circle to pass through P, K, and touch AB: Ex. 21, p. 253.
This circle shall also touch the given ⊙ DHE.

For let G be the point at which this circle touches AB.
Join DG, cutting the given circle DHE at H.

Join HE.

Then the ∠ DHE is a rt. angle, being in a semicircle, III. 31.
also the angle at F is a rt. angle; *Constr.*
∴ the points E, F, G, H are concyclic:
∴ the rect. DE, DF = the rect. DH, DG: III. 36.
but the rect. DE, DF = the rect. DK, DP: *Constr.*
∴ the rect. DH, DG = the rect. DK, DP:
∴ the point H is on the ⊙ PKG.

Let O be the centre of the ⊙ PHG.

Join OG, OH, CH.

Then OG and DF are par[l], since they are both perp. to AB;
and DG meets them.
∴ the ∠ OGD = the ∠ GDC. I. 29.
But since OG = OH, and CD = CH,
∴ the ∠ OGH = the ∠ OHG; and the ∠ CDH = the ∠ CHD:
∴ the ∠ OHG = the ∠ CHD;
∴ OH and CH are in one st. line.
∴ the ⊙ PHG touches the given ⊙ DHE. Q.E.F.

NOTES. (i) Since two circles may be drawn to pass through P, K and to touch AB, it follows that there will be two solutions of the present problem.

(ii) Two more solutions may be obtained by joining PE, and proceeding as before.

The student should examine the nature of the contact between the circles in each case.

26. Describe a circle to pass through a given point, to touch a given straight line, and to have its centre on another given straight line.

27. Describe a circle to pass through a given point, to touch a given circle, and to have its centre on a given straight line.

28. Describe a circle to pass through two given points, and to intercept an arc of given length on a given circle.

29. Describe a circle to touch a given circle and a given straight line at a given point.

30. Describe a circle to touch two given circles and a given straight line.

V. ON MAXIMA AND MINIMA.

We gather from the Theory of Loci that the position of an angle, line or figure is capable under suitable conditions of gradual change; and it is usually found that change of *position* involves a corresponding and gradual change of *magnitude*.

Under these circumstances we may be required to note if any situations exist at which the magnitude in question, after increasing, begins to decrease; or after decreasing, to increase: in such situations the magnitude is said to have reached a **Maximum** or a **Minimum** value; for in the former case it is greater, and in the latter case less than in adjacent situations on either side. In the geometry of the circle and straight line we only meet with such cases of continuous change as admit of *one* transition from an increasing to a decreasing state—or vice versâ—so that in all the problems with which we have to deal (where a single circle is involved) there can be only one Maximum and one Minimum—the Maximum being the greatest, and the Minimum being the least value that the variable magnitude is capable of taking.

Thus a variable geometrical magnitude reaches its maximum or minimum value at a *turning point*, towards which the magnitude may mount or descend from either side: it is natural therefore to expect a maximum or minimum value to occur when, in the course of its change, the magnitude assumes a *symmetrical* form or position; and this is usually found to be the case.

This general connection between a symmetrical form or position and a maximum or minimum value is not exact enough to constitute a *proof* in any particular problem; but by means of it a situation is suggested, which on further examination may be shewn to give the maximum or minimum value sought for.

For example, suppose it is required

to determine the greatest straight line that may be drawn perpendicular to the chord of a segment of a circle and intercepted between the chord and the arc:

we immediately anticipate that the greatest perpendicular is that which occupies a *symmetrical* position in the figure, namely the perpendicular which passes through the middle point of the chord; and on further examination this may be proved to be the case by means of I. 19, and I. 34.

Again we are able to find at what point a geometrical magnitude, varying under certain conditions, assumes its Maximum or Minimum value, if we can discover a construction for drawing the magnitude so that it may have an *assigned* value: for we may then examine between what limits the assigned value must lie in order that the construction may be possible; and the higher or lower limit will give the Maximum or Minimum sought for.

It was pointed out in the chapter on the Intersection of Loci, [see page 125] that if under certain conditions existing among the data, *two* solutions of a problem are possible, and under other conditions, *no* solution exists, there will always be some intermediate condition under which *one* and *only one* distinct solution is possible.

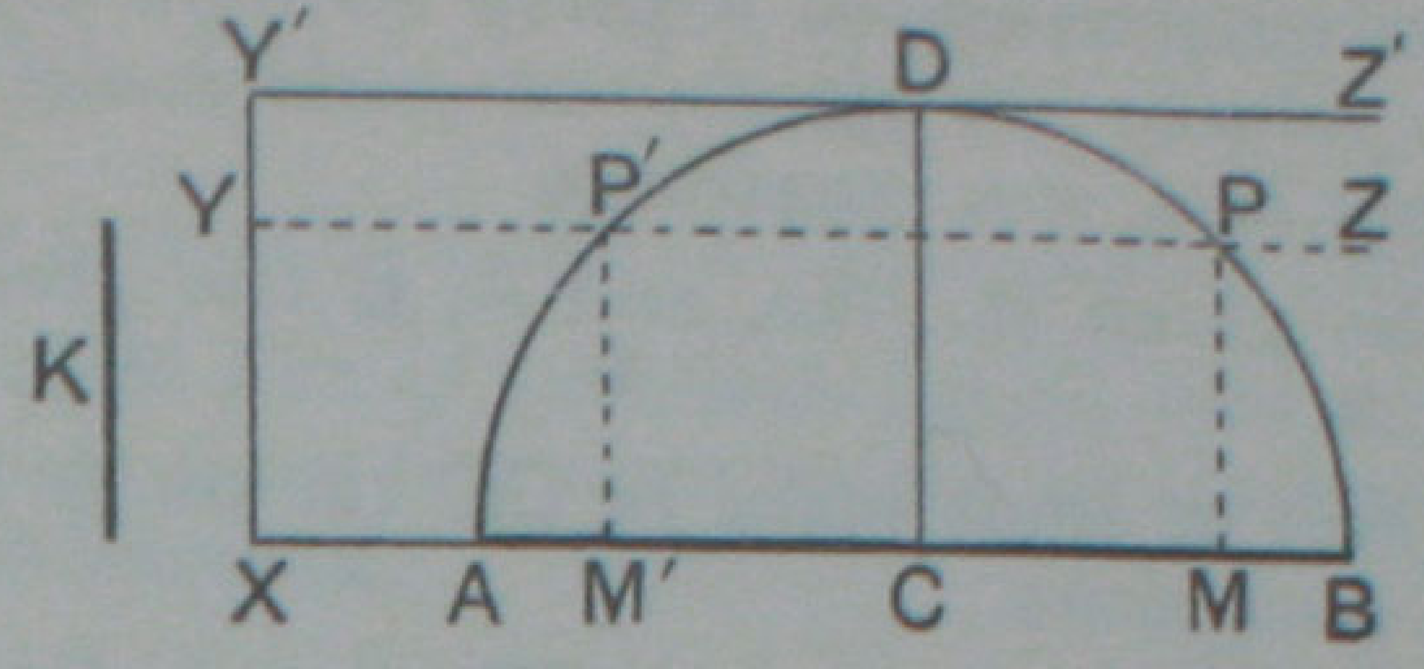
Under these circumstances this single or limiting solution will always be found to correspond to the maximum or minimum value of the magnitude to be constructed.

1. For example, suppose it is required

to divide a given straight line so that the rectangle contained by the two segments may be a maximum.

We may first attempt to divide the given straight line so that the rectangle contained by its segments may have a *given* area—that is, be equal to the square on a given straight line.

Let AB be the given straight line, and K the side of the given square.



It is required to divide the st. line AB *at a point* M, *so that the rect.* AM, MB *may be equal to the sq. on* K.

Adopting a construction suggested by II. 14,

describe a semicircle on AB; and at any point X in AB, or AB produced, draw XY perp. to AB, and equal to K.

Through Y draw YZ par[l] to AB, to meet the arc of the semicircle at P.

Then if the perp. PM is drawn to AB, it may be shewn after the manner of II. 14, or by III. 35 that

the rect. AM, MB = the sq. on PM
= the sq. on K.

So that the rectangle AM, MB increases as K increases.

Now if K is less than the radius CD, then YZ will meet the arc of the semicircle in two points P, P′; and it follows that AB may be divided at *two* points, so that the rectangle contained by its segments may be equal to the square on K. If K increases, the st. line YZ will recede from AB, and the points of intersection P, P′ will continually approach one another; until, when K is equal to the radius CD, the st. line YZ (now in the position Y′Z′) will meet the arc in *two coincident points*, that is, will touch the semicircle at D; and there will be only *one* solution of the problem.

If K is greater than CD, the straight line YZ will not meet the semicircle, and the problem is impossible.

Hence the greatest length that K may have, in order that the construction may be possible, is the radius CD.

∴ the rect. AM, MB is a maximum, when it is equal to the square on CD;

that is, when PM coincides with CD, and consequently when M is the middle point of AB.

NOTE. The special feature to be noticed in this problem is that the maximum is found at the transitional point between *two* solutions and *no* solution; that is, when the two solutions coincide and become identical.

The following example illustrates the same point.

2. *To find at what point in a given straight line the angle subtended by the line joining two given points, which are on the same side of the given straight line, is a maximum.*

Let CD be the given st. line, and A, B the given points on the same side of CD.

It is required to find at what point in CD *the an le subtended by the st. line* AB *is a maximum.*

First determine at what point in CD, the st. line AB subtends a *given* angle.

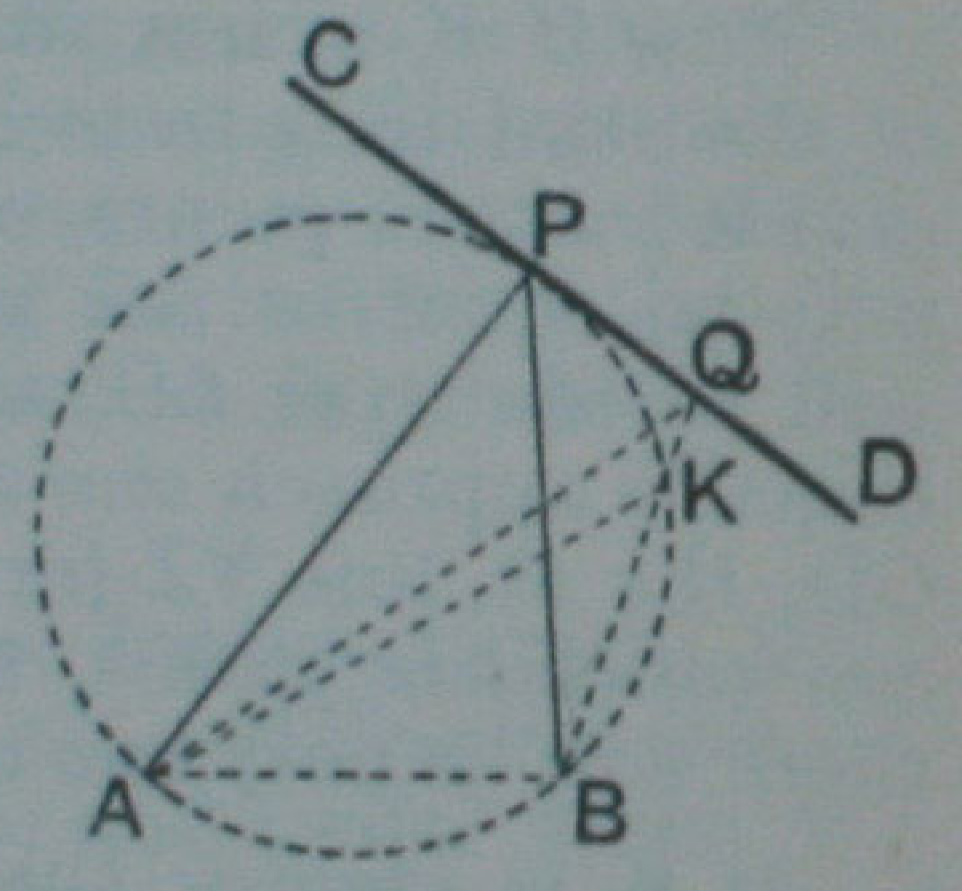
This is done as follows:—

On AB describe a segment of a circle containing an angle equal to the given angle. III. 33.

If the arc of this segment intersects CD, *two* points in CD are found at which AB subtends the given angle: but if the arc does not meet CD, *no* solution is given.

In accordance with the principles explained above, we expect that a maximum angle is determined at the limiting position; that is, when the arc touches CD, or meets it at two coincident points. [See page 231.]

This we may prove to be the case.



Describe a circle to pass through A and B, and to touch the st. line CD. [Ex. 21, p. 253.]

Let P be the point of contact.

Then shall the ∠ APB be greater than any other angle subtended by AB at a point in CD on the same side of AB as P.

For take Q, any other point in CD, on the same side of AB as P;

and join AQ, QB.

Since Q is a point in the tangent other than the point of contact, it must be without the circle;

∴ either BQ or AQ must meet the arc of the segment APB.

Let BQ meet the arc at K: join AK.

Then the ∠ APB = the ∠ AKB, in the same segment:
but the ext. ∠ AKB is greater than the int. opp. ∠ AQB.

∴ the ∠ APB is greater than AQB.

Similarly the ∠ APB may be shewn to be greater than any other angle subtended by AB at a point in CD on the same side of AB:

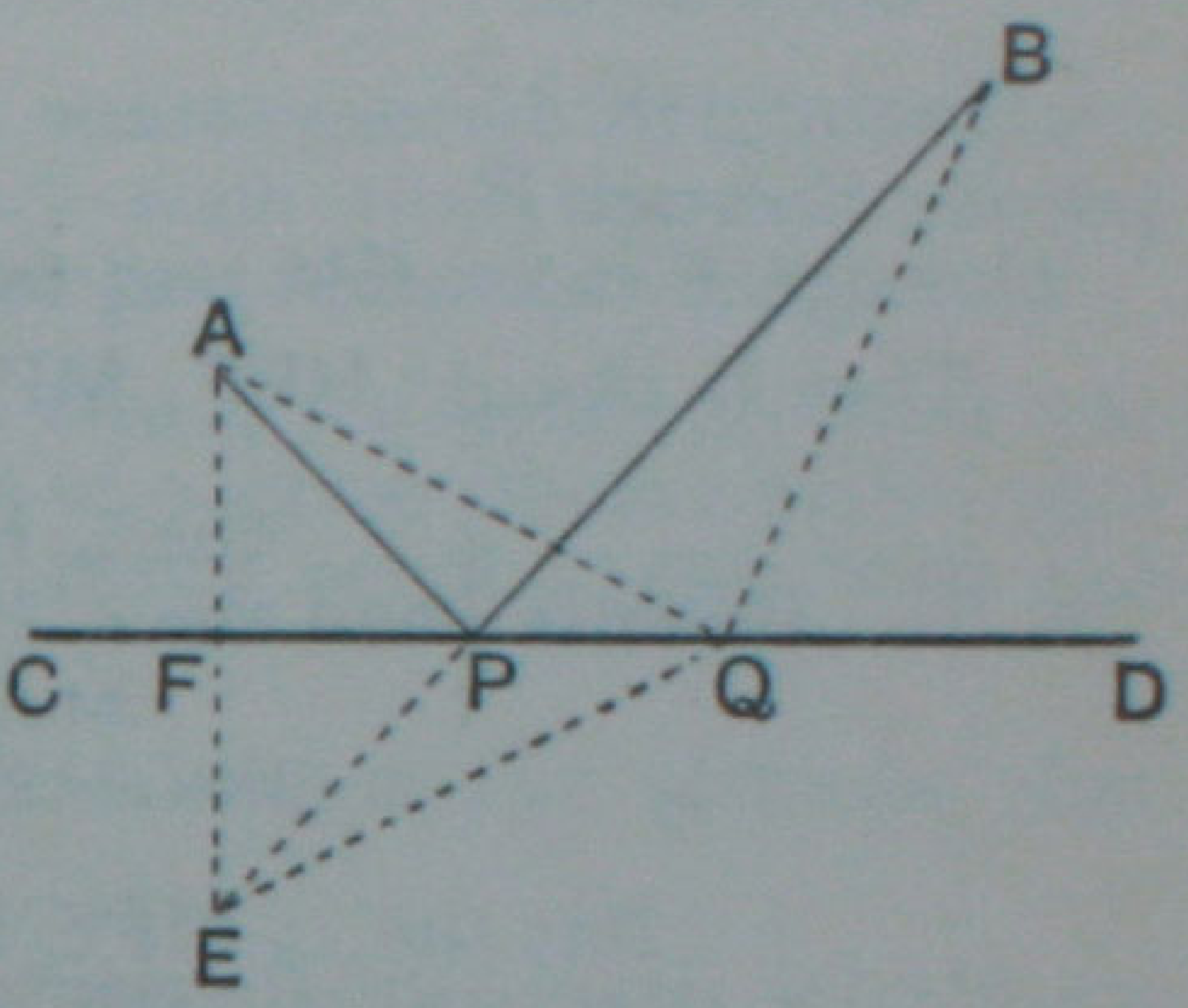
that is, the ∠ APB is the greatest of all such angles. Q.E.D.

NOTE. Two circles may be described to pass through A and B, and to touch CD, the points of contact being on opposite sides of AB;

hence two points in CD may be found such that the angle subtended by AB at each of them is greater than the angle subtended at any other point in CD *on the same side of* AB.

We add two more examples of considerable importance.

3. *In a straight line of indefinite length find a point such that the sum of its distances from two given points, on the same side of the given line, shall be a minimum.*



Let CD be the given st. line of indefinite length, and A, B the given points on the same side of CD.

It is required to find a point P *in* CD, *such that the sum of* AP, PB *is a minimum.*

Draw AF perp. to CD;
and produce AF to E, making FE equal to AF.

Join EB, cutting CD at P.

Join AP, PB.

Then of all lines drawn from A *and* B *to a point in* CD, *the sum of* AP, PB *shall be the least.*

For, let Q be any other point in CD.

Join AQ, BQ, EQ.

Now in the $\triangle^s$ AFP, EFP,

Because $\begin{cases} \text{AF} = \text{EF}, & \textit{Constr.} \\ \text{and FP is common;} \\ \text{and the } \angle \text{ AFP} = \text{the } \angle \text{ EFP}, & \text{being rt. angles.} \end{cases}$

$\therefore$ AP = EP. I. 4.

Similarly it may be shewn that

AQ = EQ.

Now in the $\triangle$ EQB, the two sides EQ, QB are together greater than EB;

hence, AQ, QB are together greater than EB,
that is, greater than AP, PB.

Similarly the sum of the st. lines drawn from A and B to any other point in CD may be shewn to be greater than AP, PB.

$\therefore$ the sum of AP, PB is a minimum.

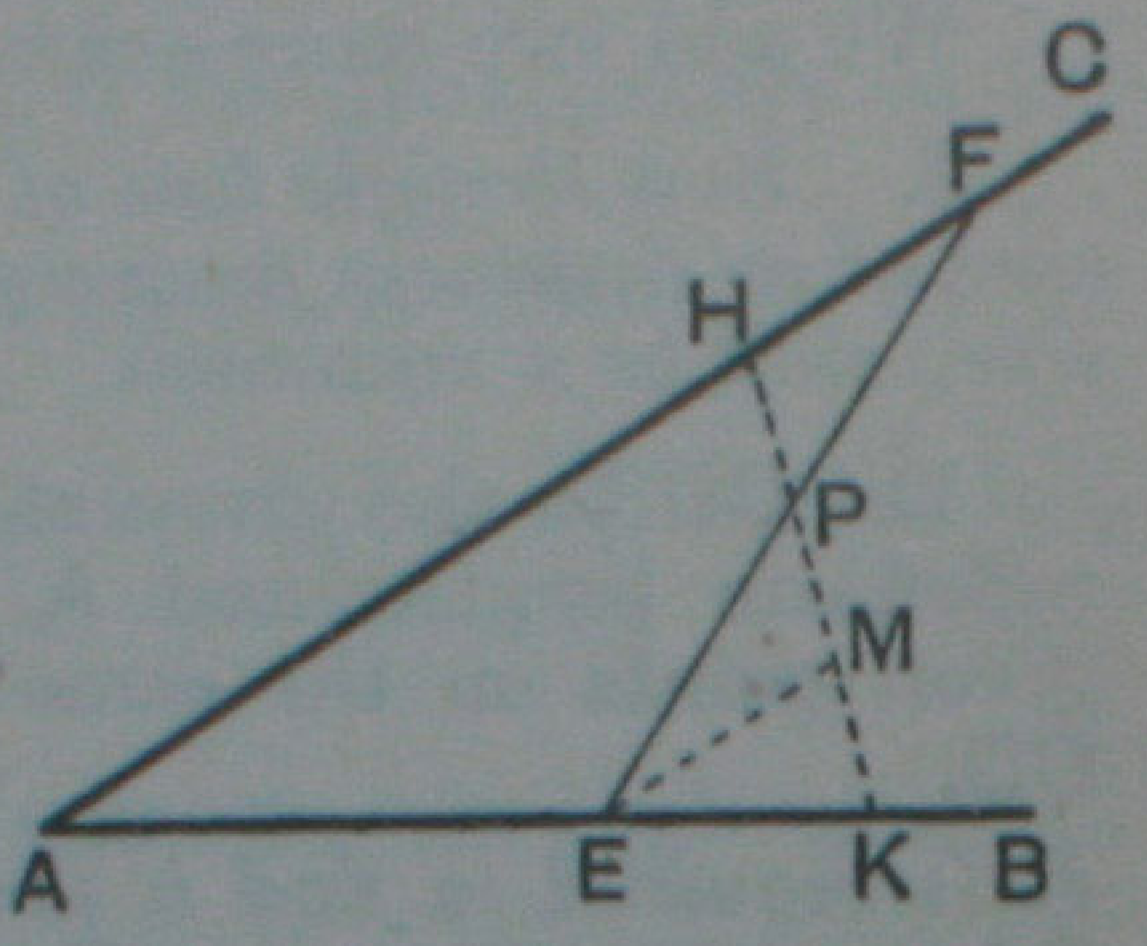
Q.E.D.

NOTE. It follows from the above proof that

the $\angle$ APF = the $\angle$ EPF I. 4.
= the $\angle$ BPD. I. 15.

Thus the sum of AP, PB is a minimum, when these lines are *equally inclined* to CD.

4. *Given two intersecting straight lines* AB, AC, *and a point* P *between them; shew that of all straight lines which pass through* P *and are terminated by* AB, AC, *that which is bisected at* P *cuts off the triangle of minimum area.*



Let EF be the st. line, terminated by AB, AC, which is bisected at P.

Then the △ FAE *shall be of minimum area.*

For let HK be any other st. line passing through P.

Through E draw EM par[l] to AC.

Then in the △[s] HPF, MPE,

Because { the ∠ HPF = the ∠ MPE, I. 15.
and the ∠ HFP = the ∠ MEP, I. 29.
and FP = EP; *Hyp.*

∴ the △ HPF = the △ MPE. I. 26, Cor.

But the △ MPE is less than the △ KPE;
∴ the △ HPF is less than the △ KPE:
to each add the fig. AHPE;
then the △ FAE is less than the △ HAK.

Similarly it may be shewn that the △ FAE is less than any other triangle formed by drawing a st. line through P:
that is, the △ FAE is a minimum.

Examples.

1. Two sides of a triangle are given in length; how must they be placed in order that the area of the triangle may be a maximum?

2. *Of all triangles of given base and area, the isosceles is that which has the least perimeter.*

3. Given the base and vertical angle of a triangle; construct it so that its area may be a maximum.

4. Find a point in a given straight line such that the tangents drawn from it to a given circle contain the greatest angle possible.

5. A straight rod slips between two straight rulers placed at right angles to one another; in what position is the triangle intercepted between the rulers and rod a maximum?

6. Divide a given straight line into two parts, so that the sum of the squares on the segments

(i) may be equal to a given square;

(ii) may be a minimum.

7. Through a point of intersection of two circles draw a straight line terminated by the circumferences,

(i) so that it may be of given length;

(ii) so that it may be a maximum.

8. Two tangents to a circle cut one another at right angles; find the point on the intercepted arc such that the sum of the perpendiculars drawn from it to the tangents may be a minimum.

9. Straight lines are drawn from two given points to meet one another on the convex circumference of a given circle: prove that their sum is a minimum when they make equal angles with the tangent at the point of intersection.

10. Of all triangles of given vertical angle and altitude, that which is isosceles has the least area.

11. Two straight lines CA, CB of indefinite length are drawn from the centre of a circle to meet the circumference at A and B; then of all tangents that may be drawn to the circle at points on the arc AB, that whose intercept is bisected at the point of contact cuts off the triangle of minimum area.

12. Given two intersecting tangents to a circle, draw a tangent to the convex arc so that the triangle formed by it and the given tangents may be of maximum area.

13. Of all triangles of given base and area, that which is isosceles has the greatest vertical angle.

14. Find a point on the circumference of a circle at which the straight line joining two given points (of which both are within, or both without the circle) subtends the greatest angle.

15. A bridge consists of three arches, whose spans are 49 ft., 32 ft. and 49 ft. respectively: shew that the point on either bank of the river at which the middle arch subtends the greatest angle is 63 feet distant from the bridge.

16. From a given point P without a circle whose centre is C, draw a straight line to cut the circumference at A and B, so that the triangle ACB may be of maximum area.

17. Shew that the greatest rectangle which can be inscribed in a circle is a square.

18. A and B are two fixed points without a circle: find a point P on the circumference, such that the sum of the squares on AP, PB may be a minimum. [See p. 161, Ex. 24.]

19. A segment of a circle is described on the chord AB: find a point C on its arc so that the sum of AC, BC may be a maximum.

20. *Of all triangles that can be inscribed in a circle that which has the greatest perimeter is equilateral.*

21. *Of all triangles that can be inscribed in a given circle that which has the greatest area is equilateral.*

22. *Of all triangles that can be inscribed in a given triangle that which has the least perimeter is the triangle formed by joining the feet of the perpendiculars drawn from the vertices on opposite sides.*

23. Of all rectangles of given area, the square has the least perimeter.

24. Describe the triangle of maximum area, having its angles equal to those of a given triangle, and its sides passing through three given points.

VI. HARDER MISCELLANEOUS EXAMPLES.

1. AB is a diameter of a given circle; and AC, BD, two chords on the same side of AB, intersect at E: shew that the circle which passes through D, E, C cuts the given circle orthogonally.

2. Two circles whose centres are C and D intersect at A and B; and a straight line PAQ is drawn through A and terminated by the circumferences: prove that

(i) the angle PBQ = the angle CAD
(ii) the angle BPC = the angle BQD.

3. Two chords AB, CD of a circle whose centre is O intersect at right angles at P: shew that

(i) $PA^2+PB^2+PC^2+PD^2=4\ (\text{radius})^2$.
(ii) $AB^2+CD^2+4OP^2 \qquad =8\ (\text{radius})^2$.

4. Two parallel tangents to a circle intercept on any third tangent a portion which is so divided at its point of contact that the rectangle contained by its two parts is equal to the square on the radius.

5. Two equal circles move between two straight lines placed at right angles, so that each straight line is touched by one circle, and the two circles touch one another: find the locus of the point of contact.

6. AB is a given diameter of a circle, and CD is any parallel chord: if any point X in AB is joined to the extremities of CD, shew that

$$XC^2+XD^2=XA^2+XB^2.$$

7. PQ is a fixed chord in a circle, and PX, QY any two parallel chords through P and Q; shew that XY touches a fixed concentric circle.

8. Two equal circles intersect at A and B; and from C, any point on the circumference of one of them, a perpendicular is drawn to AB, meeting the other circle at O and O′; shew that either O or O′ is the orthocentre of the triangle ABC. Distinguish between the two cases.

9. Three equal circles pass through the same point A, and their other points of intersection are B, C, D: shew that of the four points A, B, C, D, each is the orthocentre of the triangle formed by joining the other three.

10. From a given point without a circle draw a straight line to the concave circumference so as to be bisected by the convex circumference. When is this problem impossible?

11. Draw a straight line cutting two concentric circles so that the chord intercepted by the circumference of the greater circle may be double of the chord intercepted by the less.

12. ABC is a triangle inscribed in a circle, and A′, B′, C′ are the middle points of the arcs subtended by the sides (remote from the opposite vertices): find the relation between the angles of the two triangles ABC, A′B′C′; and prove that the pedal triangle of A′B′C′ is equiangular to the triangle ABC.

13. The opposite sides of a quadrilateral inscribed in a circle are produced to meet: shew that the bisectors of the two angles so formed are perpendicular to one another.

14. If a quadrilateral can have one circle inscribed in it, and another circumscribed about it; shew that the straight lines joining the opposite points of contact of the inscribed circle are perpendicular to one another.

15. Given the base of a triangle and the sum of the remaining sides; find the locus of the foot of the perpendicular from one extremity of the base on the bisector of the exterior vertical angle.

16. Two circles touch each other at C, and straight lines are drawn through C at right angles to one another, meeting the circles at P, P′ and Q, Q′ respectively: if the straight line which joins the centres is terminated by the circumferences at A and A′, shew that

$$P'P^2 + Q'Q^2 = A'A^2.$$

17. Two circles cut one another orthogonally at A and B; P is any point on the arc of one circle intercepted by the other, and PA, PB are produced to meet the circumference of the second circle at C and D: shew that CD is a diameter.

18. ABC is a triangle, and from any point P perpendiculars PD, PE, PF are drawn to the sides: if S_1, S_2, S_3 are the centres of the circles circumscribed about the triangles EPF, FPD, DPE, shew that the triangle $S_1S_2S_3$ is equiangular to the triangle ABC, and that the sides of the one are respectively half of the sides of the other.

19. Two tangents PA, PB are drawn from an external point P to a given circle, and C is the middle point of the chord of contact AB; if XY is any chord through P, shew that AB bisects the angle XCY.

20. Given the sum of two straight lines and the rectangle contained by them (equal to a given square): find the lines.

21. Given the sum of the squares on two straight lines and the rectangle contained by them: find the lines.

22. Given the sum of two straight lines and the sum of the squares on them: find the lines.

23. Given the difference between two straight lines, and the rectangle contained by them: find the lines.

24. Given the sum or difference of two straight lines and the difference of their squares: find the lines.

25. ABC is a triangle, and the internal and external bisectors of the angle A meet BC, and BC produced, at P and P′: if O is the middle point of PP′, shew that OA is a tangent to the circle circumscribed about the triangle ABC.

26. ABC is a triangle, and from P, any point on the circumference of the circle circumscribed about it, perpendiculars are drawn to the sides BC, CA, AB meeting the circle again in A′, B′, C′; prove that

(i) the triangle A′B′C′ is identically equal to the triangle ABC.

(ii) AA′, BB′, CC′ are parallel.

27. Two equal circles intersect at fixed points A and B, and from any point in AB a perpendicular is drawn to meet the circumferences on the same side of AB at P and Q: shew that PQ is of constant length.

28. The straight lines which join the vertices of a triangle to the centre of its circumscribed circle, are perpendicular respectively to the sides of the pedal triangle.

29. P is any point on the circumference of a circle circumscribed about a triangle ABC; and perpendiculars PD, PE are drawn from P to the sides BC, CA. Find the locus of the centre of the circle circumscribed about the triangle PDE.

30. P is any point on the circumference of a circle circumscribed about a triangle ABC : shew that the angle between Simson's Line for the point P and the side BC is equal to the angle between AP and the diameter of the circumscribed circle through A.

31. Shew that the circles circumscribed about the four triangles formed by two pairs of intersecting straight lines meet in a point.

32. Shew that the orthocentres of the four triangles formed by two pairs of intersecting straight lines are collinear.

On the Construction of Triangles.

33. Given the vertical angle, one of the sides containing it, and the length of the perpendicular from the vertex on the base : construct the triangle.

34. Given the feet of the perpendiculars drawn from the vertices on the opposite sides : construct the triangle.

35. Given the base, the altitude, and the radius of the circumscribed circle : construct the triangle.

36. Given the base, the vertical angle, and the sum of the squares on the sides containing the vertical angle : construct the triangle.

37. Given the base, the altitude and the sum of the squares on the sides containing the vertical angle : construct the triangle.

38. Given the base, the vertical angle, and the difference of the squares on the sides containing the vertical angle : construct the triangle.

39. Given the vertical angle, and the lengths of the two medians drawn from the extremities of the base : construct the triangle.

40. Given the base, the vertical angle, and the difference of the angles at the base : construct the triangle.

41. Given the base, and the position of the bisector of the vertical angle : construct the triangle.

42. Given the base, the vertical angle, and the length of the bisector of the vertical angle : construct the triangle.

43. Given the perpendicular from the vertex on the base, the bisector of the vertical angle, and the median which bisects the base : construct the triangle.

44. Given the bisector of the vertical angle, the median bisecting the base, and the difference of the angles at the base : construct the triangle.

BOOK IV.

Book IV. consists entirely of problems, dealing with various rectilineal figures in relation to the circles which pass through their angular points, or are touched by their sides.

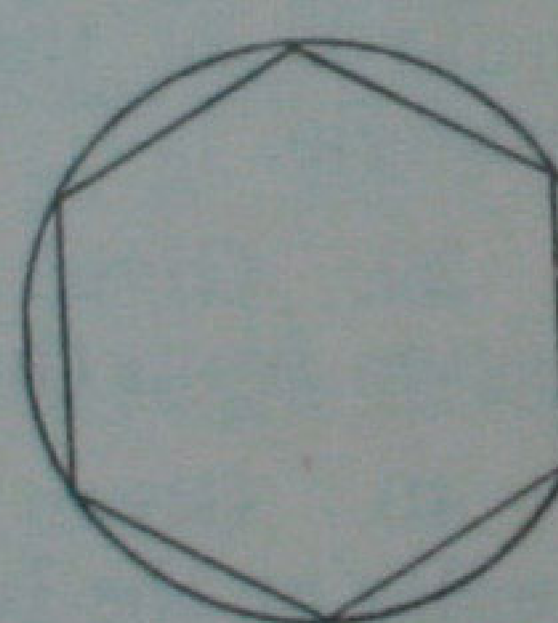
Definitions.

1. A **Polygon** is a rectilineal figure bounded by more than four sides.

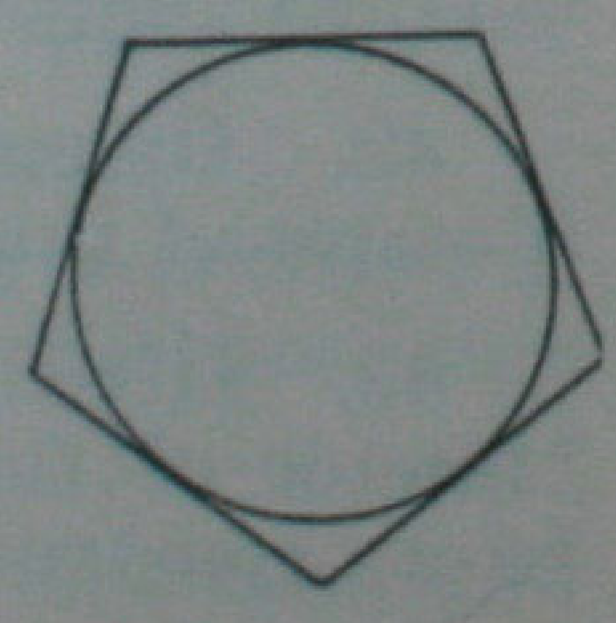
A Polygon of	*five* sides	is called a	**Pentagon,**
,,	*six* sides	,,	**Hexagon,**
,,	*seven* sides	,,	**Heptagon,**
,,	*eight* sides	,,	**Octagon,**
,,	*ten* sides	,,	**Decagon,**
,,	*twelve* sides	,,	**Dodecagon,**
,,	*fifteen* sides	,,	**Quindecagon.**

2. A Polygon is **Regular** when all its sides are equal, and all its angles are equal.

3. A rectilineal figure is said to be **inscribed** in a circle, when all its angular points are on the circumference of the circle; and a circle is said to be **circumscribed about** a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure.



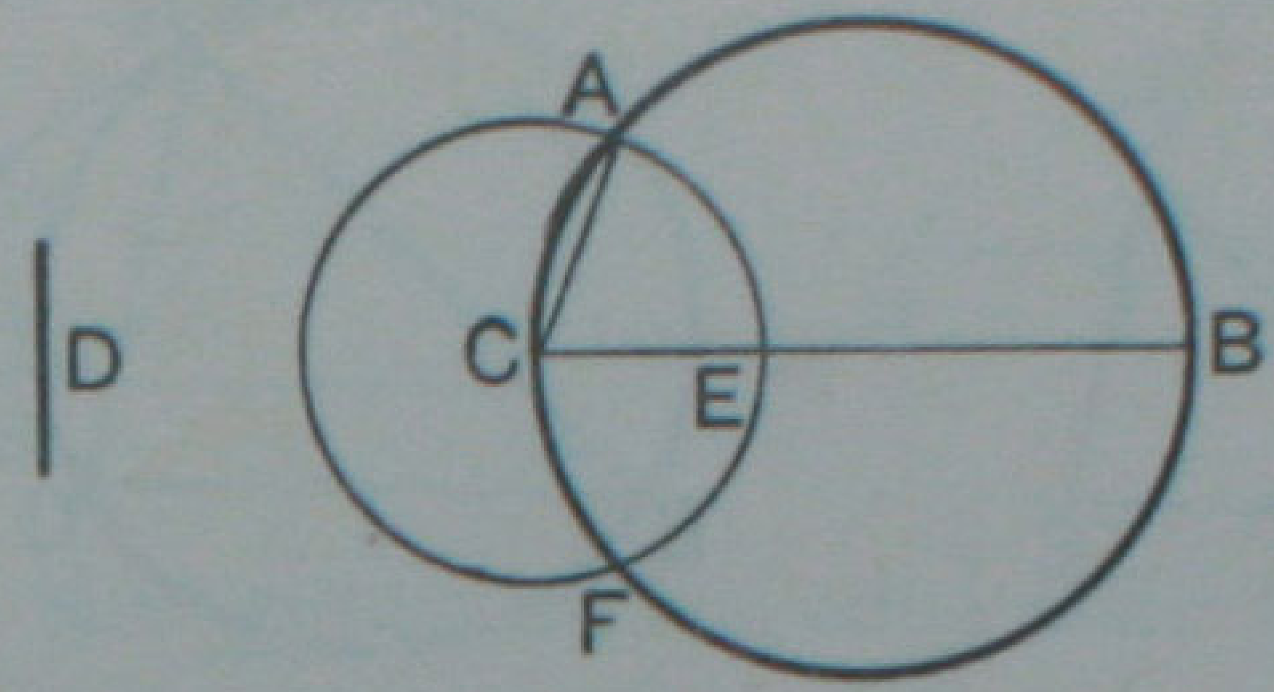
4. A circle is said to be **inscribed in** a rectilineal figure, when the circumference of the circle is touched by each side of the figure; and a rectilineal figure is said to be **circumscribed about** a circle, when each side of the figure is a tangent to the circle.



5. A straight line is said to be **placed in a circle**, when its extremities are on the circumference of the circle.

PROPOSITION 1. PROBLEM.

In a given circle to place a chord equal to a given straight line, which is not greater than the diameter of the circle.



Let ABC be the given circle, and D the given straight line not greater than the diameter of the circle.

It is required to place in the ⊙ ABC *a chord equal to* D.

Construction. Draw CB, a diameter of the ⊙ ABC.

Then if CB = D, the thing required is done.

But if not, CB must be greater than D. *Hyp.*

From CB cut off CE equal to D: I. 3.

and with centre C, and radius CE, describe the ⊙ AEF, cutting the given circle at A.

Join CA.

Then CA *shall be the chord required.*

Proof. For CA = CE, being radii of the ⊙ AEF;

and CE = D: *Constr.*

∴ CA = D.

Q.E.F.

EXERCISES.

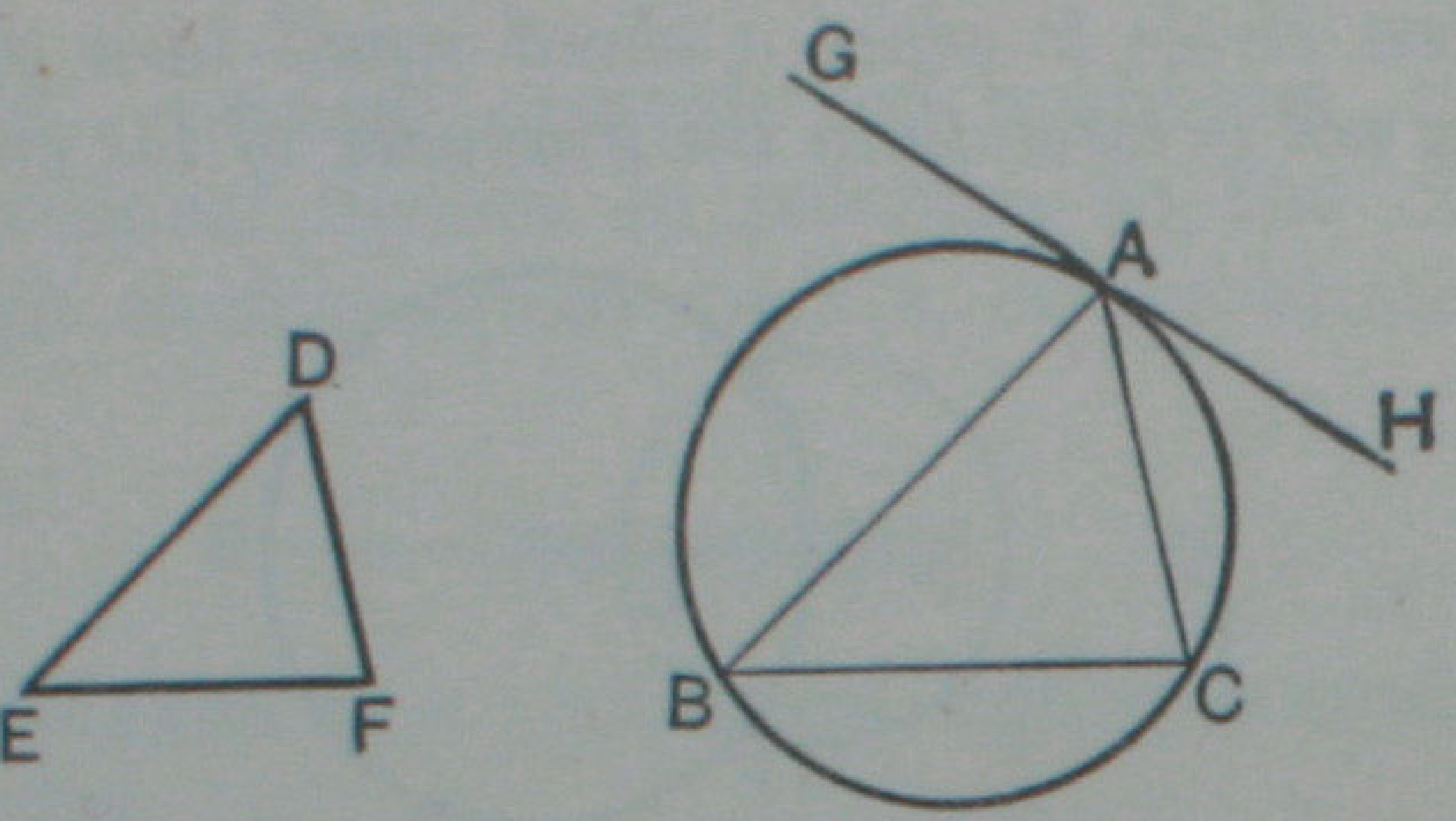
1. In a given circle place a chord of given length so as to pass through a given point (i) without, (ii) within the circle.

When is this problem impossible?

2. In a given circle place a chord of given length so that it may be parallel to a given straight line.

PROPOSITION 2. PROBLEM.

In a given circle to inscribe a triangle equiangular to a given triangle.



Let ABC be the given circle, and DEF the given triangle.

It is required to inscribe in the ⊙ ABC *a triangle equiangular to the* △ DEF.

Construction. At any point A, on the ○ce of the ⊙ ABC, draw the tangent GAH. III. 17.

At A make the ∠ GAB equal to the ∠ DFE; I. 23.

and make the ∠ HAC equal to the ∠ DEF. I. 23.

Join BC.

Then ABC *shall be the triangle required.*

Proof. Because GH is a tangent to the ⊙ ABC, and from A its point of contact the chord AB is drawn,

∴ the ∠ GAB = the ∠ ACB in the alt. segment : III. 32.

but the ∠ GAB = the ∠ DFE ; *Constr.*

∴ the ∠ ACB = the ∠ DFE.

Similarly the ∠ HAC = the ∠ ABC, in the alt. segment :

∴ the ∠ ABC = the ∠ DEF. *Constr.*

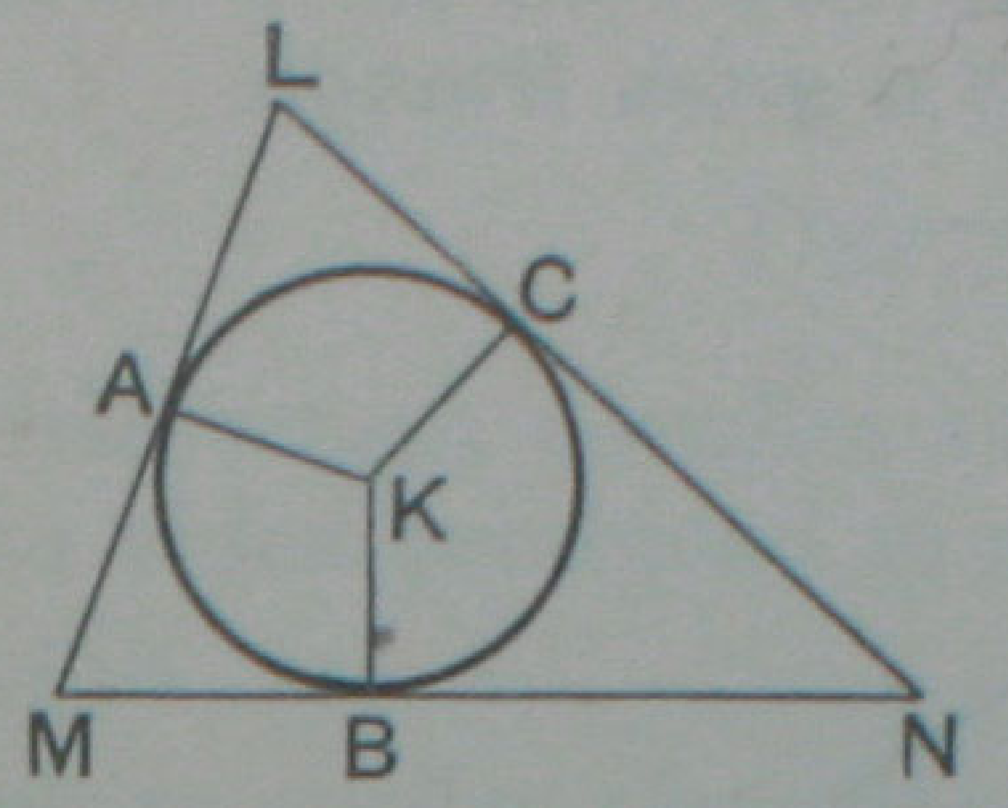
Hence the third ∠ BAC = the third ∠ EDF, for the three angles in each triangle are together equal to two rt. angles. I. 32.

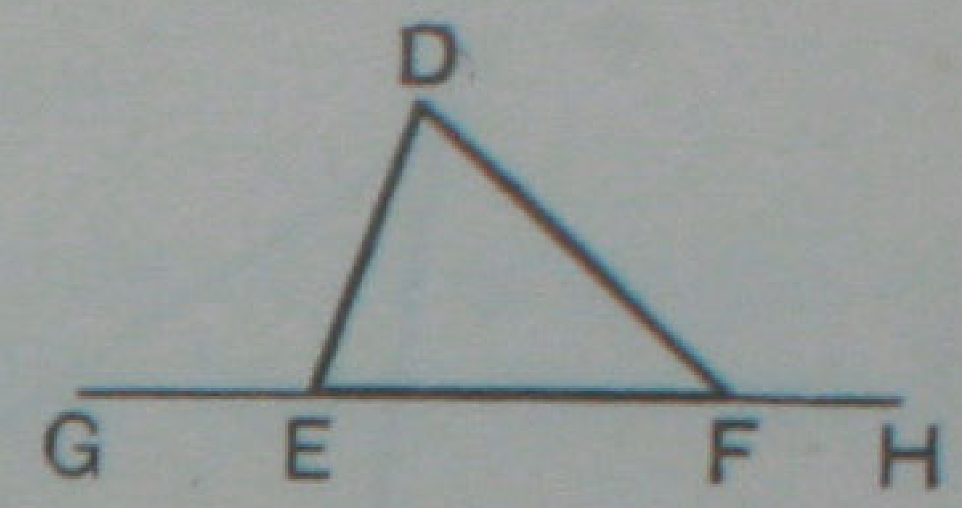
∴ the △ ABC is equiangular to the △ DEF, and it is inscribed in the ⊙ ABC.

Q.E.F.

PROPOSITION 3. PROBLEM.

About a given circle to circumscribe a triangle equiangular to a given triangle.





Let ABC be the given circle, and DEF the given triangle.

It is required to circumscribe about the ⊙ ABC *a triangle equiangular to the* △ DEF.

Construction. Produce EF both ways to G and H.

Find K the centre of the ⊙ ABC, III. 1.
and draw any radius KB.

At K make the ∠ BKA equal to the ∠ DEG; I. 23.
and make the ∠ BKC equal to the ∠ DFH.

Through A, B, C draw LM, MN, NL perp. to KA, KB, KC.

Then LMN *shall be the triangle required.*

Proof. Because LM, MN, NL are drawn perp. to radii at their extremities.

∴ LM, MN, NL are tangents to the circle. III. 16.

And because the four angles of the quadrilateral AKBM together = four rt. angles; I. 32. Cor.
and of these, the ∠s KAM, KBM are rt. angles; *Constr.*
∴ the ∠s AKB, AMB together = two rt. angles.

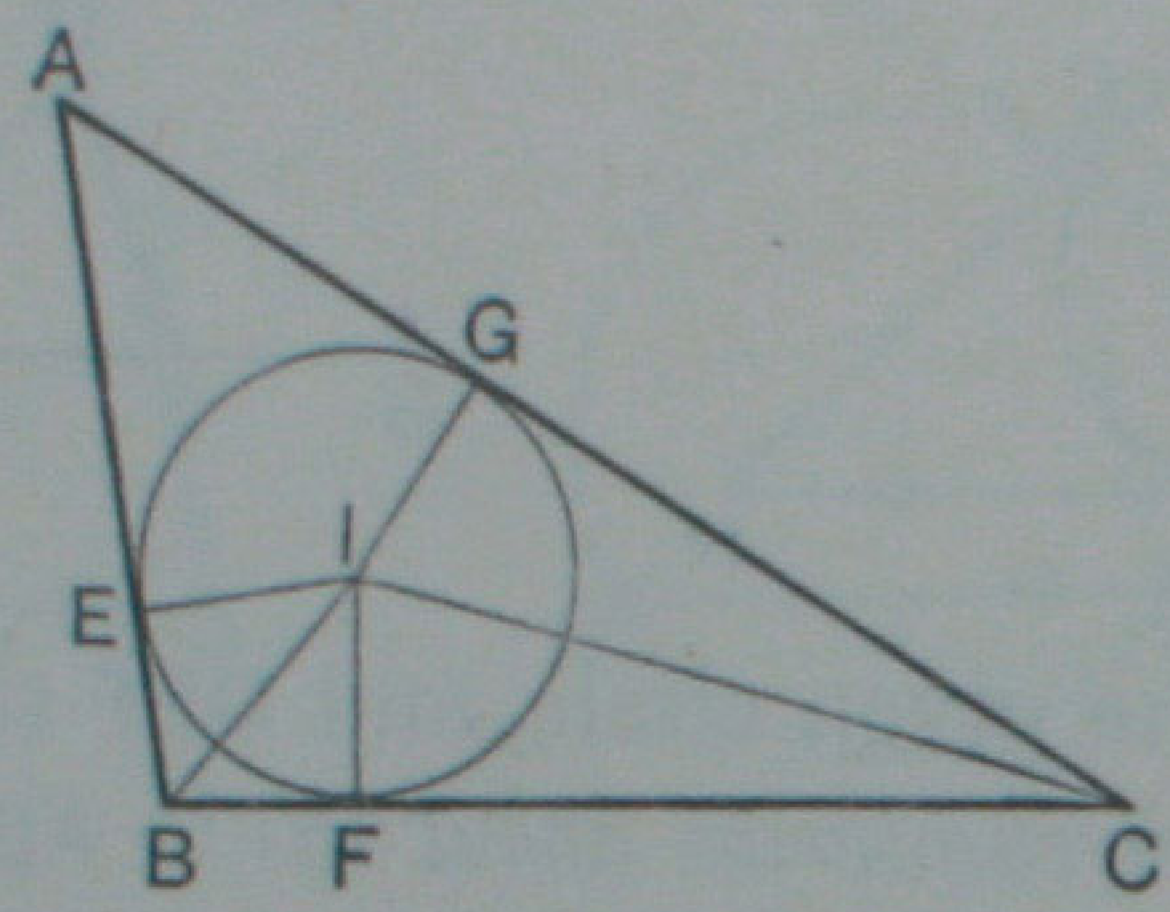
But the ∠s DEG, DEF together = two rt. angles; I. 13.
∴ the ∠s AKB, AMB = the ∠s DEG, DEF;
and of these, the ∠ AKB = the ∠ DEG; *Constr.*
∴ the ∠ AMB = the ∠ DEF.

Similarly it may be shewn that the ∠ LNM = the ∠ DFE.
∴ the third ∠ MLN = the third ∠ EDF. I. 32.

∴ the △ LMN is equiangular to the △ DEF, and it is circumscribed about the ⊙ ABC. Q.E.F.

PROPOSITION 4. PROBLEM.

To inscribe a circle in a given triangle.



Let ABC be the given triangle.

It is required to inscribe a circle in the △ ABC.

Construction. Bisect the ∠ˢ ABC, ACB by the st. lines BI, CI, which intersect at I. I. 9.

From I draw IE, IF, IG perp. to AB, BC, CA. I. 12.

Proof. Then in the △ˢ EIB, FIB,

Because { the ∠ EBI = the ∠ FBI; *Constr.*
and the ∠ BEI = the ∠ BFI, being rt. angles;
and BI is common; }

∴ IE = IF. I. 26.

Similarly it may be shewn that IF = IG.

∴ IE, IF, IG are all equal.

With centre I, and radius IE, describe a circle.

This circle must pass through the points E, F, G;
and it will be inscribed in the △ ABC.

For since IE, IF, IG, being equal, are radii of the ⊙ EFG;
and since the ∠ˢ at E, F, G are rt. angles; *Constr.*
∴ the ⊙ EFG is touched at these points by AB, BC, CA: III. 16.

∴ the ⊙ EFG is inscribed in the △ ABC.

Q.E.F.

NOTE. From page 111 it is seen that if AI is joined, then AI bisects the angle BAC : hence it follows that

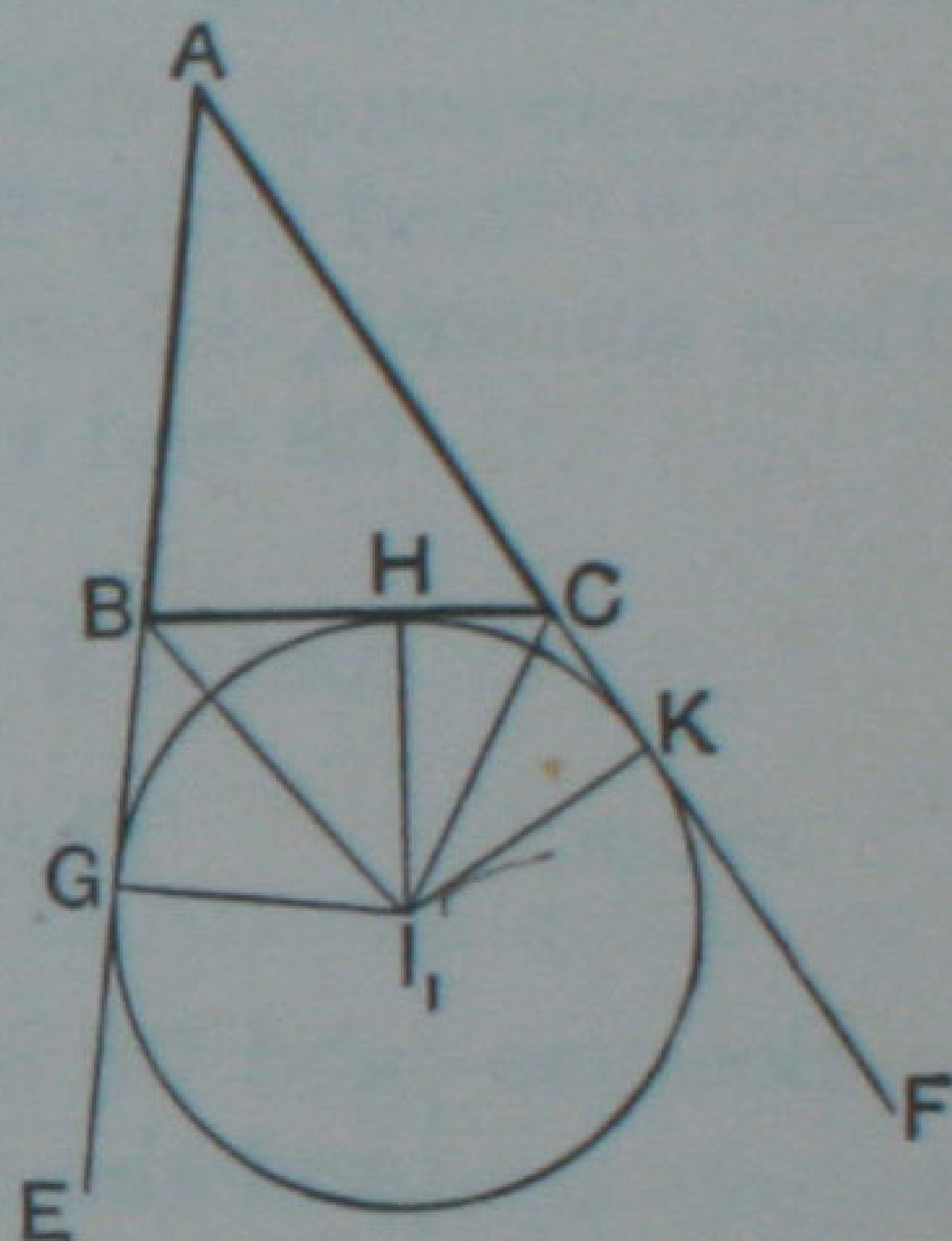
The bisectors of the angles of a triangle are concurrent, the point of intersection being the centre of the inscribed circle.

The centre of the circle inscribed in a triangle is usually called its **in-centre**.

DEFINITION.

A circle which touches one side of a triangle and the other two sides produced is said to be an **escribed** circle of the triangle.

To draw an escribed circle of a given triangle.



Let ABC be the given triangle, of which the two sides AB, AC are produced to E and F.

It is required to describe a circle touching BC, *and* AB, AC *produced.*

Bisect the $\angle^s$ CBE, BCF by the st. lines BI_1, CI_1, which intersect at I_1. I. 9.

From I_1 draw I_1G, I_1H, I_1K perp. to AE, BC, AF. I. 12.

Then in the $\triangle^s$ I_1BG, I_1BH,

Because $\begin{cases} \text{the } \angle I_1BG = \text{the } \angle I_1BH, & \textit{Constr.} \\ \text{and the } \angle I_1GB = \text{the } \angle I_1HB, \\ \quad \text{being rt. angles;} \\ \quad \text{also } I_1B \text{ is common;} \end{cases}$

$\therefore$ $I_1G = I_1H$.

Similarly it may be shewn that $I_1H = I_1K$;

$\therefore$ I_1G, I_1H, I_1K are all equal.

With centre I_1 and radius I_1G, describe a circle.

This circle must pass through the points G, H, K;
and it will be an escribed circle of the $\triangle$ ABC.

For since I_1H, I_1G, I_1K, being equal, are radii of the $\odot$ HGK,
and since the angles at H, G, K are rt. angles,

$\therefore$ the $\odot$ GHK is touched at these points by BC, and by AB, AC produced:

$\therefore$ the $\odot$ GHK is an escribed circle of the $\triangle$ ABC. Q.E.F.

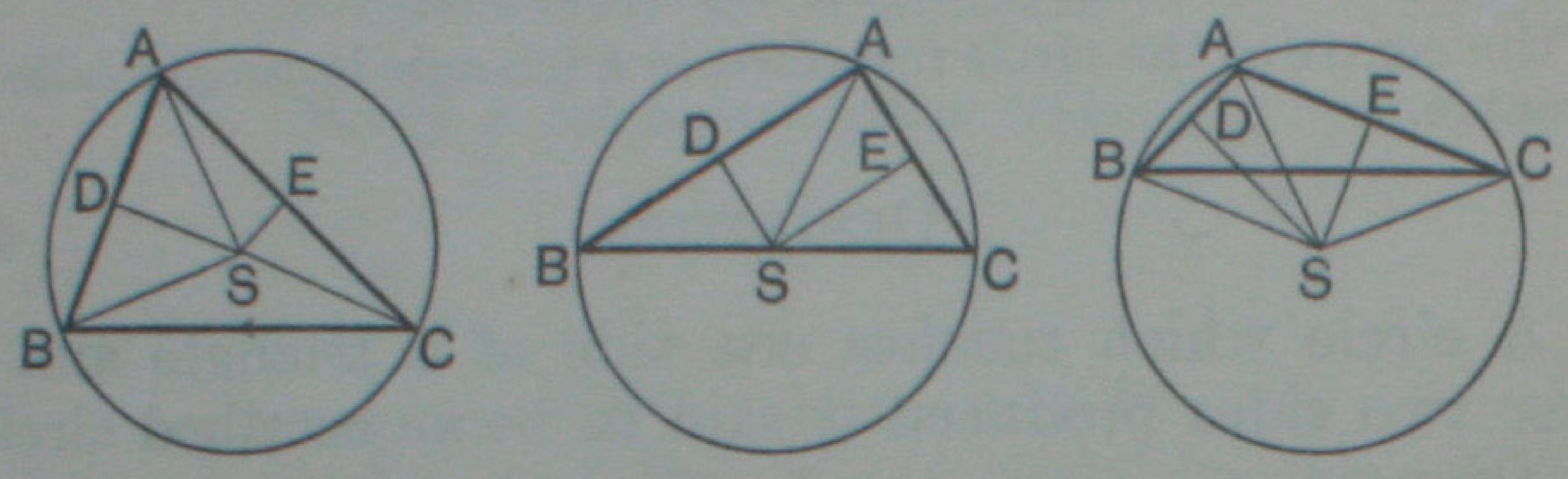
It is clear that every triangle has *three* escribed circles.

NOTE. From page 112 it is seen that if AI_1 is joined, then AI_1 bisects the angle BAC : hence it follows that

The bisectors of two exterior angles of a triangle and the bisector of the third angle are concurrent, the point of intersection being the centre of an escribed circle.

PROPOSITION 5. PROBLEM.

To circumscribe a circle about a given triangle.



Let ABC be the given triangle.

It is required to circumscribe a circle about the △ ABC.

Construction. Draw DS bisecting AB at rt. angles; I. 11.
and draw ES bisecting AC at rt. angles.

Then since AB, AC are neither par[l], nor in the same st. line,
∴ DS and ES must meet at some point S.
Join SA;
and if S be not in BC, join SB, SC.

Proof. Then in the △[s] ADS, BDS,

Because { AD = BD,
and DS is common to both;
and the ∠ ADS = the ∠ BDS, being rt. angles;
∴ SA = SB. I. 4.

Similarly it may be shewn that SC = SA.
∴ SA, SB, SC are all equal.

With centre S, and radius SA, describe a circle: this circle must pass through the points A, B, C, and is therefore circumscribed about the △ ABC. Q.E.F.

It follows that

(i) when the centre of the circumscribed circle falls *within* the triangle, each of its angles must be acute, for each angle is then in a segment greater than a semicircle:

(ii) when the centre falls *on one of the sides* of the triangle, the angle opposite to this side must be a right angle, for it is the angle in a semicircle:

(iii) when the centre falls *without* the triangle, the angle opposite to the side beyond which the centre falls, must be obtuse, for it is the angle in a segment less than a semicircle.

Therefore, conversely, if the given triangle be acute-angled, the centre of the circumscribed circle falls within it: if it be a right-angled triangle, the centre falls on the hypotenuse: if it be an obtuse-angled triangle, the centre falls without the triangle.

NOTE. From page 111 it is seen that if S is joined to the middle point of BC, then the joining line is perpendicular to BC.

Hence the perpendiculars drawn to the sides of a triangle from their middle points are concurrent, the point of intersection being the centre of the circle circumscribed about the triangle.

The centre of the circle circumscribed about a triangle is usually called its **circum-centre**.

EXERCISES.

ON THE INSCRIBED, CIRCUMSCRIBED, AND ESCRIBED CIRCLES OF A TRIANGLE.

1. An equilateral triangle is inscribed in a circle, and tangents are drawn at its vertices, prove that
   (i) the resulting figure is an equilateral triangle:
   (ii) its area is four times that of the given triangle.

2. Describe a circle to touch two parallel straight lines and a third straight line which meets them. Shew that two such circles can be drawn, and that they are equal.

3. *Triangles which have equal bases and equal vertical angles have equal circumscribed circles.*

4. I *is the centre of the circle inscribed in the triangle* ABC, *and* I_1 *is the centre of the circle which touches* BC *and* AB, AC *produced: shew that* A, I, I_1 *are collinear.*

5. *If the inscribed and circumscribed circles of a triangle are concentric, shew that the triangle is equilateral; and that the diameter of the circumscribed circle is double that of the inscribed circle.*

6. ABC is a triangle, and I, S are the centres of the inscribed and circumscribed circles; if A, I, S are collinear, shew that AB=AC

7. The sum of the diameters of the inscribed and circumscribed circles of a right-angled triangle is equal to the sum of the sides containing the right angle.

8. If the circle inscribed in a triangle ABC touches the sides at D, E, F, shew that the triangle DEF is acute-angled; and express its angles in terms of the angles at A, B, C.

9. If I is the centre of the circle inscribed in the triangle ABC, and I_1 the centre of the escribed circle which touches BC; shew that I, B, I_1, C are concyclic.

10. In any triangle the difference of two sides is equal to the difference of the segments into which the third side is divided at the point of contact of the inscribed circle.

11. In the triangle ABC the bisector of the angle BAC meets the base at D, and from I the centre of the inscribed circle a perpendicular IE is drawn to BC: shew that the angle BID is equal to the angle CIE.

12. In the triangle ABC, I and S are the centres of the inscribed and circumscribed circles: shew that IS subtends at A an angle equal to half the difference of the angles at the base of the triangle.

13. In a triangle ABC, I and S are the centres of the inscribed and circumscribed circles, and AD is drawn perpendicular to BC: shew that AI is the bisector of the angle DAS.

14. Shew that the area of a triangle is equal to the rectangle contained by its semi-perimeter and the radius of the inscribed circle.

15. The diagonals of a quadrilateral ABCD intersect at O: shew that the centres of the circles circumscribed about the four triangles AOB, BOC, COD, DOA are at the angular points of a parallelogram.

16. In any triangle ABC, if I is the centre of the inscribed circle, and if AI is produced to meet the circumscribed circle at O; shew that O is the centre of the circle circumscribed about the triangle BIC.

17. Given the base, altitude, and the radius of the circumscribed circle; construct the triangle.

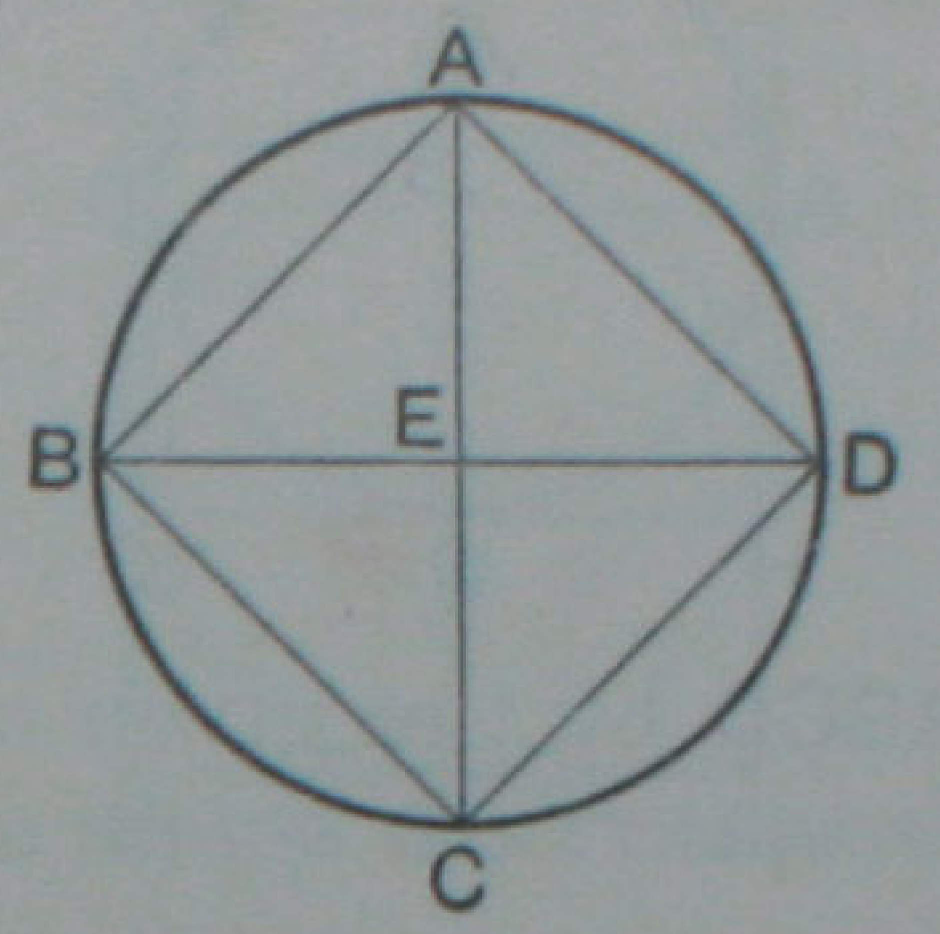
18. Describe a circle to intercept equal chords of given length on three given straight lines.

19. In an equilateral triangle the radii of the circumscribed and escribed circles are respectively double and treble of the radius of the inscribed circle.

20. Three circles whose centres are A, B, C touch one another externally two by two at D, E, F: shew that the inscribed circle of the triangle ABC is the circumscribed circle of the triangle DEF.

PROPOSITION 6. PROBLEM.

To inscribe a square in a given circle.



Let ABCD be the given circle.

It is required to inscribe a square in the ⊙ ABCD.

Construction. Find E the centre of the circle: III. 1.
and draw two diameters AC, BD perp. to one another. I. 11.
Join AB, BC, CD, DA.

Then the fig. ABCD *shall be the square required.*

Proof. For in the △s BEA, DEA,

Because { BE = DE, I. *Def.* 15.
and EA is common;
and the ∠ BEA = the ∠ DEA, being rt. angles; }

∴ BA = DA. I. 4.

Similarly it may be shewn that CD = DA, and that BC = CD.
∴ the fig. ABCD is equilateral.

And since BD is a diameter of the ⊙ ABCD,
∴ BAD is a semicircle;
∴ the ∠ BAD is a rt. angle. III. 31.

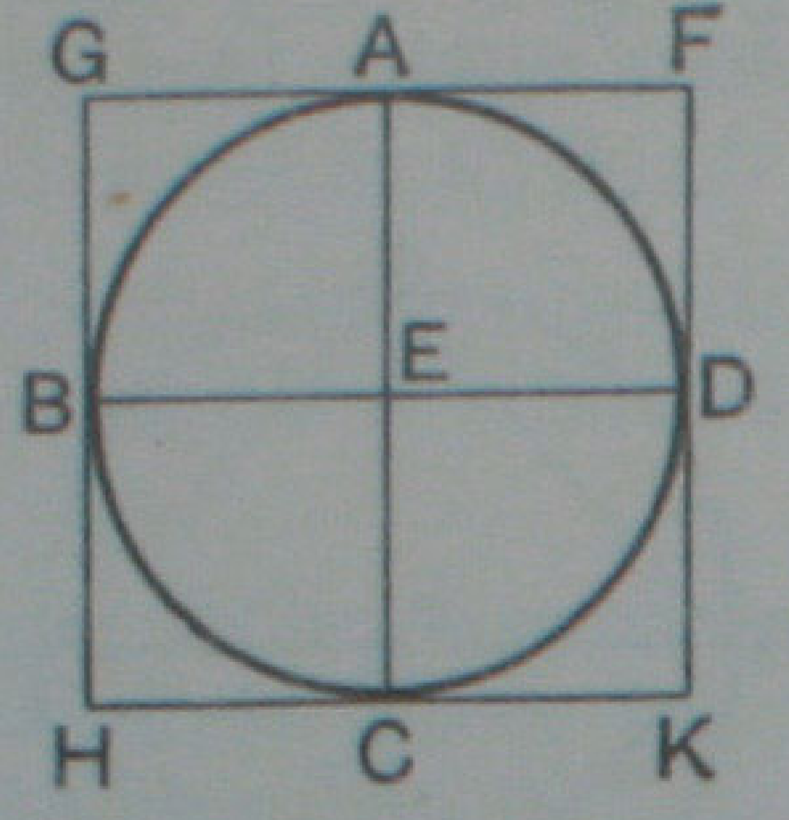
Similarly the other angles of the fig. ABCD are rt. angles.
∴ the fig. ABCD is a square;
and it is inscribed in the given circle.

Q.E.F.

[For Exercises see page 281.]

PROPOSITION 7. PROBLEM.

To circumscribe a square about a given circle.



Let ABCD be the given circle.

It is required to circumscribe a square about the ⊙ ABCD.

Construction. Find E the centre of the ⊙ ABCD: III. 1.
and draw two diameters AC, BD perp. to one another. I. 11.
Through A, B, C, D draw FG, GH, HK, KF perp. to EA, EB, EC, ED.

Then the fig. GK *shall be the square required.*

Proof. Because FG, GH, HK, KF are drawn perp. to radii at their extremities,

∴ FG, GH, HK, KF are tangents to the circle. III. 16.

And because the ∠ˢ AEB, EBG are both rt. angles, *Constr.*

∴ GH is par' to AC. I. 28.

Similarly FK is par' to AC:

and in like manner GF, BD, HK are par'.

Hence the figs. GK, GC, AK, GD, BK, GE are parᵐˢ.

∴ GF and HK each = BD;

also GH and FK each = AC:

but AC = BD;

∴ GF, FK, KH, HG are all equal:

that is, the fig. GK is equilateral.

And since the fig. GE is a parᵐ,

∴ the ∠ BGA = the ∠ BEA; I. 34.

but the ∠ BEA is a rt. angle; *Constr.*

∴ the ∠ at G is a rt. angle.

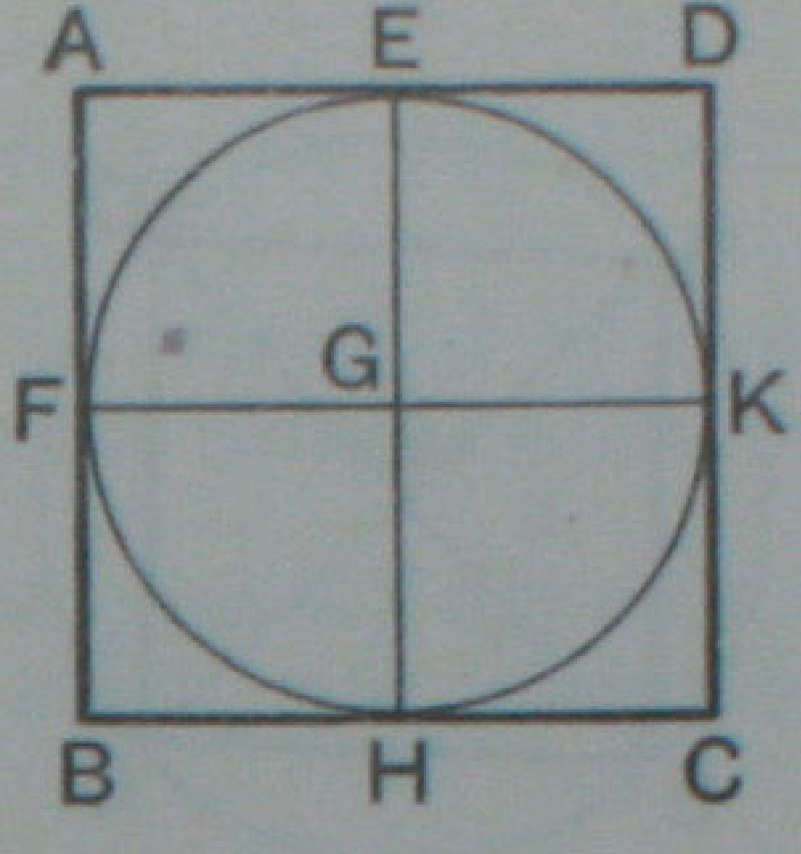
Similarly the ∠ˢ at F, K, H are rt. angles.

∴ the fig. GK is a square, and it has been circumscribed about the ⊙ ABCD.

Q.E.F.

PROPOSITION 8. PROBLEM.

To inscribe a circle in a given square.



Let ABCD be the given square.

It is required to inscribe a circle in the square ABCD.

Construction. Bisect the sides AB, AD at F and E. I. 10.
Through E draw EH par^l to AB or DC: I. 31.
and through F draw FK par^l to AD or BC, meeting EH at G.

Proof. Now AB = AD, being the sides of a square;
and their halves are equal *Ax.* 7.
∴ AF = AE.
But the fig. AG is a par^m; *Constr.*
∴ AF = GE, and AE = GF;
∴ GE = GF.

Similarly it may be shewn that GE = GK, and GK = GH:
∴ GF, GE, GK, GH are all equal.

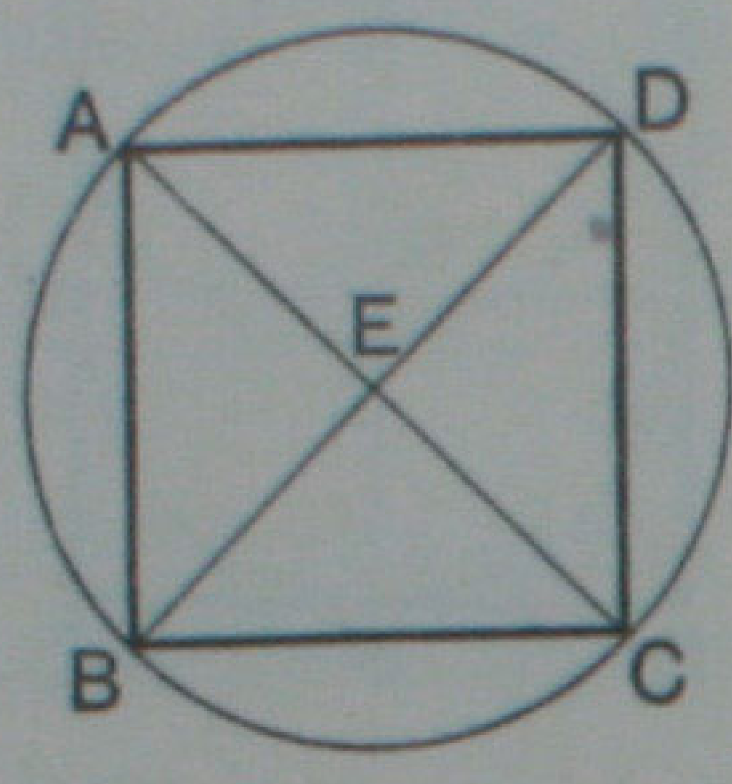
With centre G, and radius GE, describe a circle.
This circle must pass through the points F, E, K, H;
and it will be touched by BA, AD, DC, CB; III. 16.
for GF, GE, GK, GH, being equal, are radii;
and the angles at F, E, K, H are rt. angles. I. 29.
Hence the ⊙ FEKH is inscribed in the sq. ABCD.

Q.E.F.

[For Exercises see p. 281.]

PROPOSITION 9. PROBLEM

To circumscribe a circle about a given square.



Let ABCD be the given square.

It is required to circumscribe a circle about the square ABCD.

Construction. Join AC, BD, intersecting at E.

Proof. Then in the $\triangle^s$ BAC, DAC,

Because { BA = DA, — I. *Def.* 30.
and AC is common;
and BC = DC; — I. *Def.* 30.

$\therefore$ the $\angle$ BAC = the $\angle$ DAC; — I. 8.

that is, the diagonal AC bisects the $\angle$ BAD.

Similarly the remaining angles of the square are bisected by the diagonals AC or BD.

Hence each of the $\angle^s$ EAD, EDA is half a rt. angle;

$\therefore$ the $\angle$ EAD = the $\angle$ EDA:

$\therefore$ EA = ED. — I. 6.

Similarly it may be shewn that ED = EC, and EC = EB.

$\therefore$ EA, EB, EC, ED are all equal.

With centre E, and radius EA, describe a circle: this circle must pass through the points A, B, C, D, and is therefore circumscribed about the sq. ABCD. Q.E.F.

DEFINITION. A rectilineal figure about which a circle may be described is said to be **Cyclic**.

EXERCISES ON PROPOSITIONS 6–9.

1. *If a circle can be inscribed in a quadrilateral, shew that the sum of one pair of opposite sides is equal to the sum of the other pair.*

2. *If the sum of one pair of opposite sides of a quadrilateral is equal to the sum of the other pair, shew that a circle may be inscribed in the figure.*

[Bisect two adjacent angles of the figure, and so describe a circle to touch three of its sides. Then prove indirectly by means of the last exercise that this circle must also touch the fourth side.]

3. *Prove that a rhombus and a square are the only parallelograms in which a circle can be inscribed.*

4. *All cyclic parallelograms are rectangular.*

5. *The greatest rectangle which can be inscribed in a given circle is a square.*

6. Circumscribe a rhombus about a given circle.

7. All squares circumscribed about a given circle are equal.

8. The area of a square circumscribed about a circle is double of the area of the inscribed square.

9. ABCD is a square inscribed in a circle, and P is any point on the arc AD: shew that the side AD subtends at P an angle three times as great as that subtended at P by any one of the other sides.

10. Inscribe a square in a given square ABCD, so that one of its angular points shall be at a given point X in AB.

11. In a given square inscribe the square of minimum area.

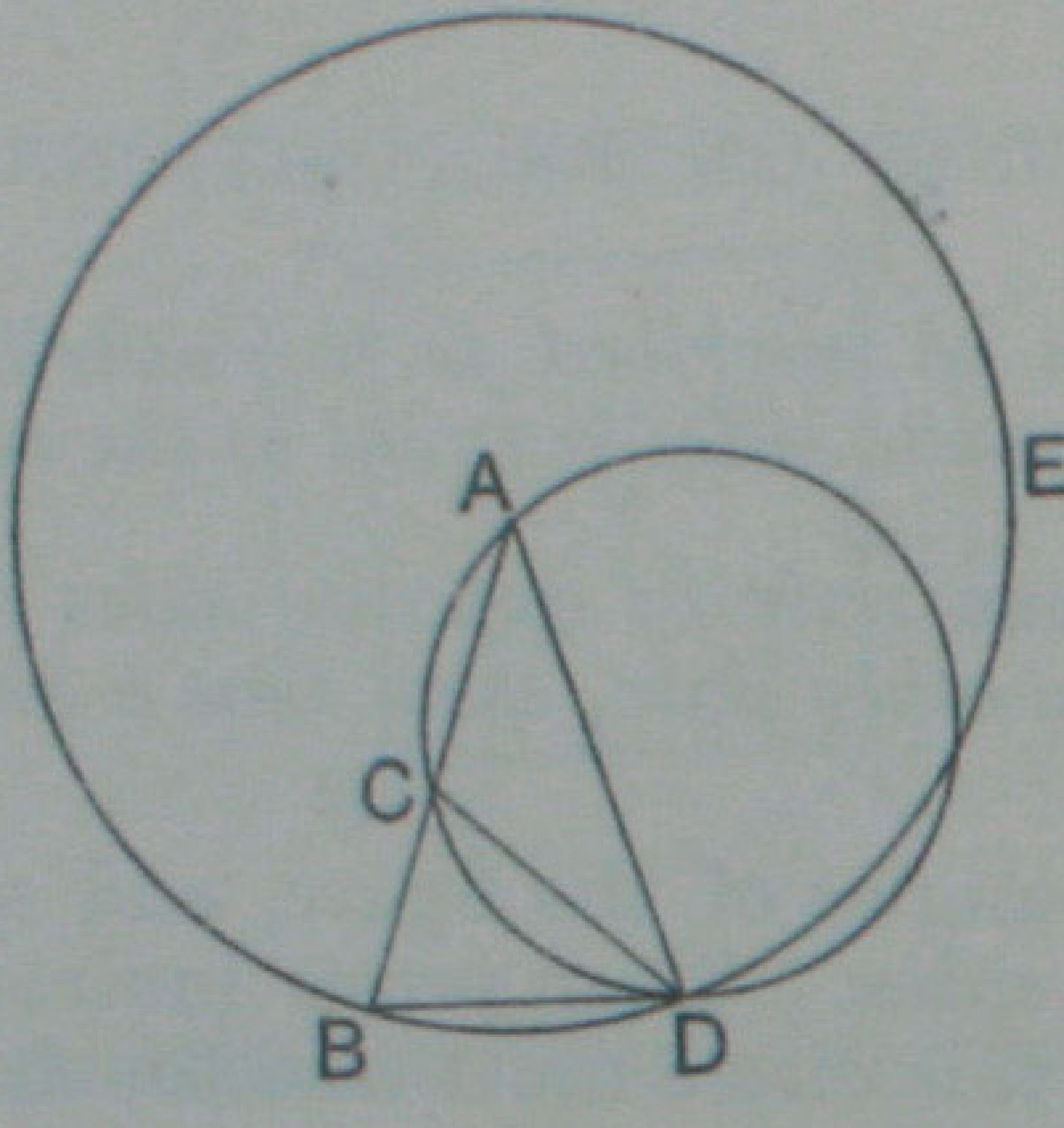
12. Describe (i) a circle, (ii) a square about a given rectangle.

13. Inscribe (i) a circle, (ii) a square in a given quadrant.

14. ABCD is a square inscribed in a circle, and P is any point on the circumference; shew that the sum of the squares on PA, PB, PC, PD is double the square on the diameter. [See Ex. 24, p. 161.]

PROPOSITION 10. PROBLEM.

To describe an isosceles triangle having each of the angles at the base double of the third angle.



Construction. Take any straight line AB.
Divide AB at C, so that the rect. BA, BC = the sq. on AC. II. 11.

With centre A, and radius AB, describe the ⊙ BDE;
and in it place the chord BD equal to AC. IV. 1.
Join DA.

Then ABD *shall be the triangle required.*

C. Join CD;
and about the △ ACD circumscribe a circle. IV. 5.

Proof. Now the rect. BA, BC = the sq. on AC *Constr.*
= the sq. on BD. *Constr.*
Hence BD is a tangent to the ⊙ ACD: III. 37.
and from the point of contact D a chord DC is drawn;
∴ the ∠ BDC = the ∠ CAD in the alt. segment. III. 32.

To each of these equals add the ∠ CDA:
then the whole ∠ BDA = the sum of the ∠s CAD, CDA.

But the ext. ∠ DCB = the sum of the ∠s CAD, CDA; I. 32.
∴ the ∠ DCB = the ∠ BDA.
And since AB = AD, being radii of the ⊙ BDE,
∴ the ∠ DBA = the ∠ BDA; I. 5.
∴ the ∠ DBC = the ∠ DCB;

∴ DC = DB; I. 6.

that is, DC = CA: *Constr.*

∴ the ∠ CAD = the ∠ CDA; I. 5.

∴ the sum of the ∠s CAD, CDA = twice the angle at A.

But the ∠ ADB = the sum of the ∠s CAD, CDA; *Proved.*

∴ each of the ∠s ABD, ADB = twice the angle at A.

Q.E.F.

EXERCISES ON PROPOSITION 10.

1. In an isosceles triangle in which each of the angles at the base is double of the vertical angle, shew that the vertical angle is one-fifth of two right angles.

2. *Divide a right angle into five equal parts.*

3. Describe an isosceles triangle whose vertical angle shall be three times either angle at the base. Point out a triangle of this kind in the figure of Proposition 10.

4. *In the figure of Proposition* 10, *if the two circles intersect at* F, *shew that* BD = DF.

5. *In the figure of Proposition* 10, *shew that the circle* ACD *is equal to the circle circumscribed about the triangle* ABD.

6. In the figure of Proposition 10, if the two circles intersect at F, shew that

(i) BD, DF are sides of a regular decagon inscribed in the circle EBD.

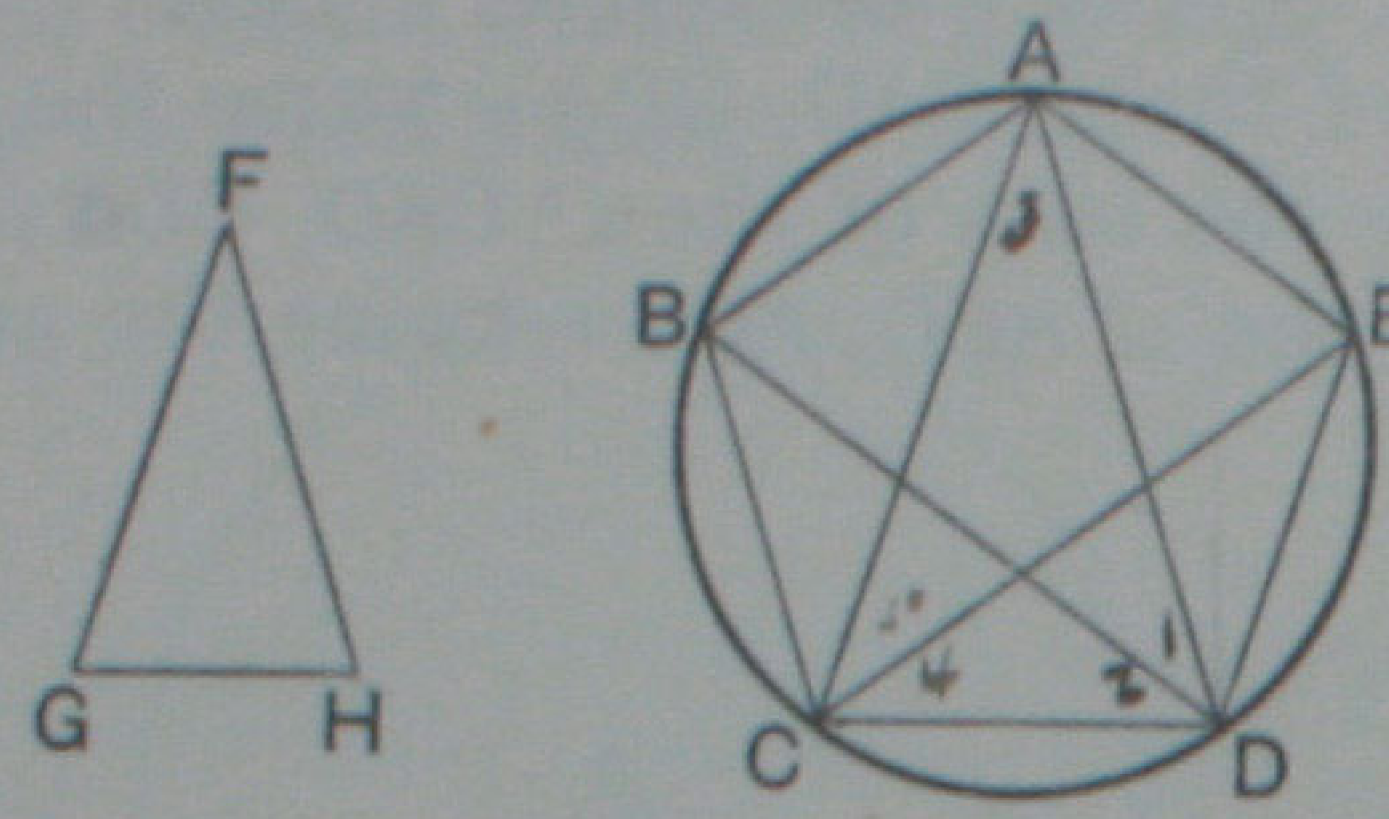
(ii) AC, CD, DF are sides of a regular pentagon inscribed in the circle ACD.

7. In the figure of Proposition 10, shew that the centre of the circle circumscribed about the triangle DBC is the middle point of the arc CD.

8. In the figure of Proposition 10, if I is the centre of the circle inscribed in the triangle ABD, and I′, S′ the centres of the inscribed and circumscribed circles of the triangle DBC, shew that S′I = S′I′.

PROPOSITION 11. PROBLEM.

To inscribe a regular pentagon in a given circle.



Let ABC be a given circle.

It is required to inscribe a regular pentagon in the ⊙ ABC.

Construction. Describe an isosceles △ FGH, having each of the angles at G and H double of the angle at F. IV. 10.

In the ⊙ ABC inscribe the △ ACD equiangular to the △ FGH, IV. 2.
so that each of the ∠ˢ ACD, ADC is double of the ∠ CAD.

Bisect the ∠ˢ ACD, ADC by CE and DB, which meet the ○ᶜᵉ at E and B. I. 9.

Join AB, BC, AE, ED.

Then ABCDE *shall be the required regular pentagon.*

Proof. Since each of the ∠ˢ ACD, ADC = twice the ∠ CAD;
and since the ∠ˢ ACD, ADC are bisected by CE, DB,
∴ the five ∠ˢ ADB, BDC, CAD, DCE, ECA are all equal.
∴ the five arcs AB, BC, CD, DE, EA are all equal. III. 26.
∴ the five chords AB, BC, CD, DE, EA are all equal. III. 29.
∴ the pentagon ABCDE is equilateral.

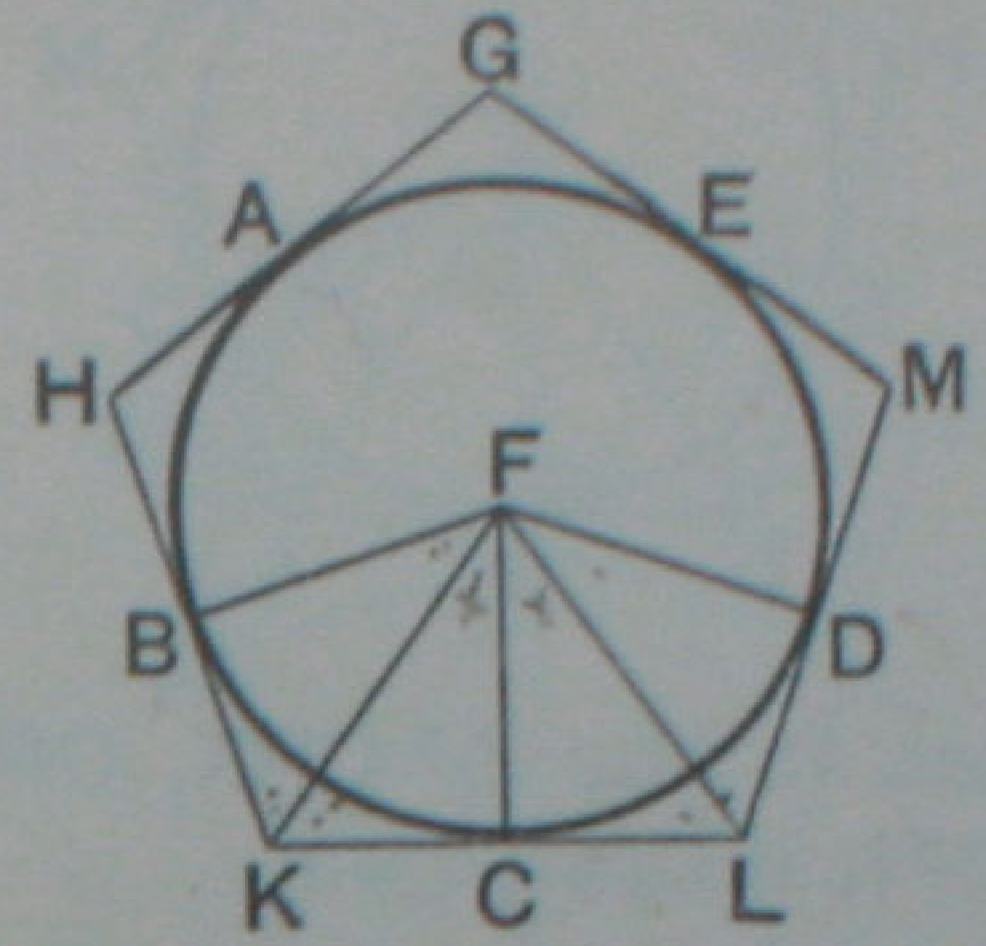
Again the arc AB = the arc DE; *Proved.*
to each of these equals add the arc BCD;
∴ the arc ABCD = the arc BCDE:
hence the angles at the ○ᶜᵉ which stand upon these equal arcs are equal; III. 27.
that is, the ∠ AED = the ∠ BAE.

In like manner the remaining angles of the pentagon may be shewn to be equal;
∴ the pentagon ABCDE is equiangular.

Hence the pentagon, being both equilateral and equiangular, is regular; and it is inscribed in the ⊙ ABC. Q.E.F.

PROPOSITION 12. PROBLEM.

To circumscribe a regular pentagon about a given circle.



Let ABCD be the given circle.

It is required to circumscribe a regular pentagon about the ⊙ ABCD.

Construction.

Inscribe a regular pentagon in the ⊙ ABCD, IV. 11.
and let A, B, C, D, E be its angular points.

At the points A, B, C, D, E draw GH, HK, KL, LM, MG, tangents to the circle. III. 17.

Then shall GHKLM *be the required regular pentagon.*

Find F the centre of the ⊙ ABCD; III. 1.
and join FB, FK, FC, FL, FD.

Proof. In the △ˢ BFK, CFK,

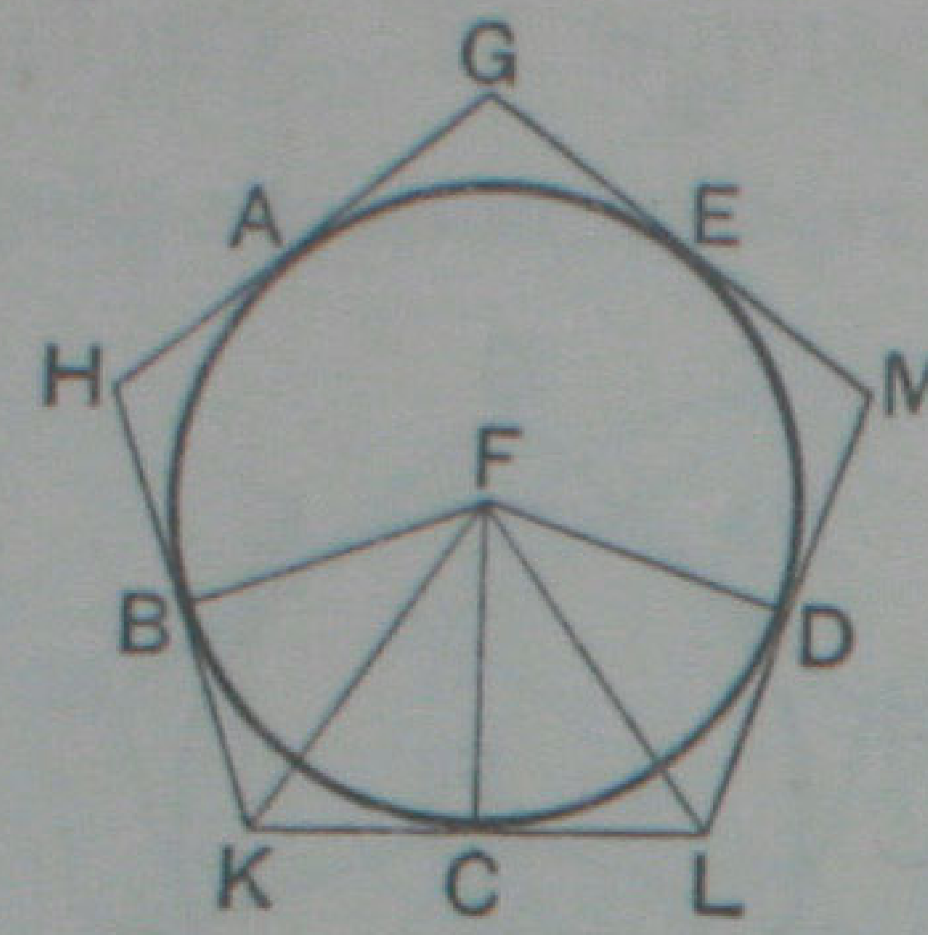
Because { BF = CF, being radii of the circle,
and FK is common;
and KB = KC, being tangents to the circle from the same point K; III. 17, *Cor.*

∴ the ∠ BFK = the ∠ CFK, I. 8.
also the ∠ BKF = the ∠ CKF. I. 8, *Cor.*

Hence the ∠ BFC = twice the ∠ CFK,
and the ∠ BKC = twice the ∠ CKF.

Similarly it may be shewn
that the ∠ CFD = twice the ∠ CFL,
and that the ∠ CLD = twice the ∠ CLF.

But since the arc BC = the arc CD, IV. 11.
∴ the ∠ BFC = the ∠ CFD; III. 27.
and the halves of these angles are equal,
that is, the ∠ CFK = the ∠ CFL.



Then in the $\triangle^s$ CFK, CFL,

Because { the $\angle$ CFK = the $\angle$ CFL, *Proved.*
and the $\angle$ FCK = the $\angle$ FCL, being rt. angles, III. 18.
and FC is common ;

$\therefore$ CK = CL, I. 26.

and the $\angle$ FKC = the $\angle$ FLC.

Hence KL is double of KC; similarly HK is double of KB.

And since KC = KB, III. 17, *Cor.*

$\therefore$ KL = HK.

In the same way it may be shewn that every two consecutive sides are equal ;

$\therefore$ the pentagon GHKLM is equilateral.

Again, it has been proved that the $\angle$ FKC = the $\angle$ FLC, and that the $\angle^s$ HKL, KLM are respectively double of these angles :

$\therefore$ the $\angle$ HKL = the $\angle$ KLM.

In the same way it may be shewn that every two consecutive angles of the figure are equal ;

$\therefore$ the pentagon GHKLM is equiangular.

$\therefore$ the pentagon is regular, and it is circumscribed about the $\odot$ ABCD.

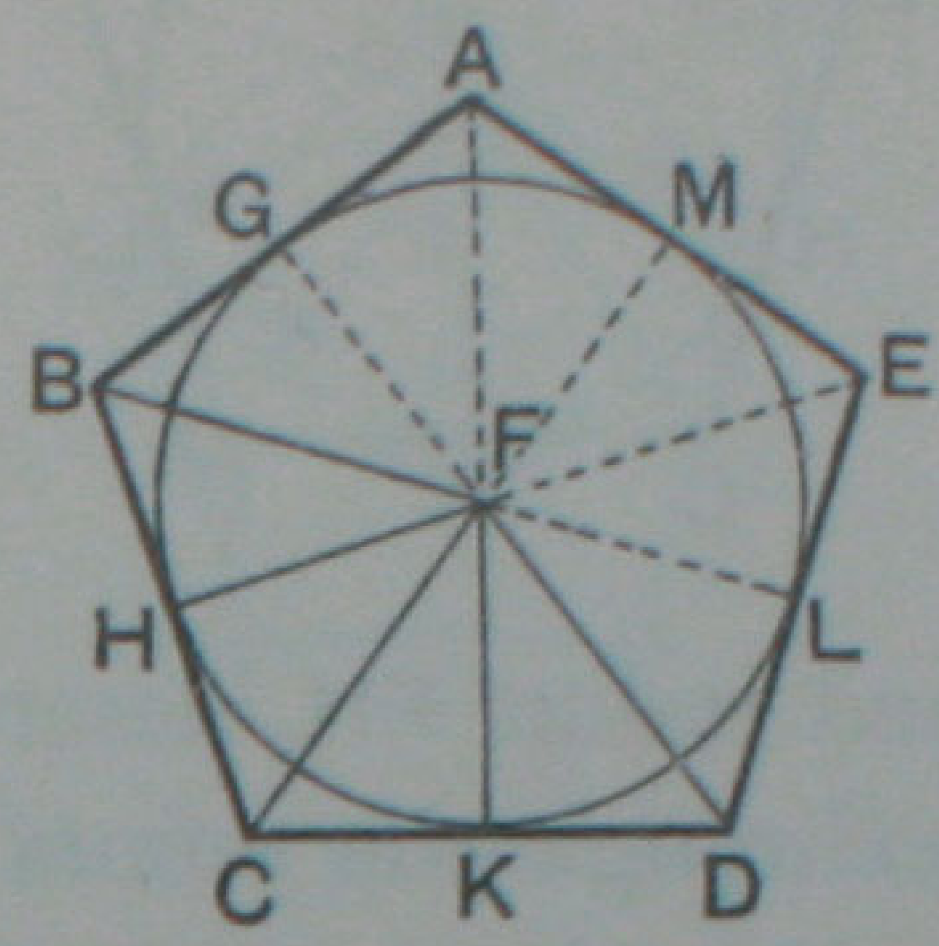
Q.E.F.

COROLLARY. *Similarly it may be proved that if tangents are drawn at the vertices of any regular polygon inscribed in a circle, they will form another regular polygon of the same species circumscribed about the circle.*

[For Exercises see p. 293.]

PROPOSITION 13. PROBLEM.

To inscribe a circle in a given regular pentagon.



Let ABCDE be the given regular pentagon.

It is required to inscribe a circle within the figure ABCDE.

Construction. Bisect two consecutive ∠^s BCD, CDE by CF and DF which intersect at F. I. 9.

Join FB;

and draw FH, FK perp. to BC, CD. I. 12.

Proof. In the △^s BCF, DCF,

Because { BC = DC, *Hyp.*
and CF is common to both;
and the ∠ BCF = the ∠ DCF; *Constr.*

∴ the ∠ CBF = the ∠ CDF. I. 4.

But the ∠ CDF is half an angle of the regular pentagon:
∴ also the ∠ CBF is half an angle of the regular pentagon:
that is, FB bisects the ∠ ABC.

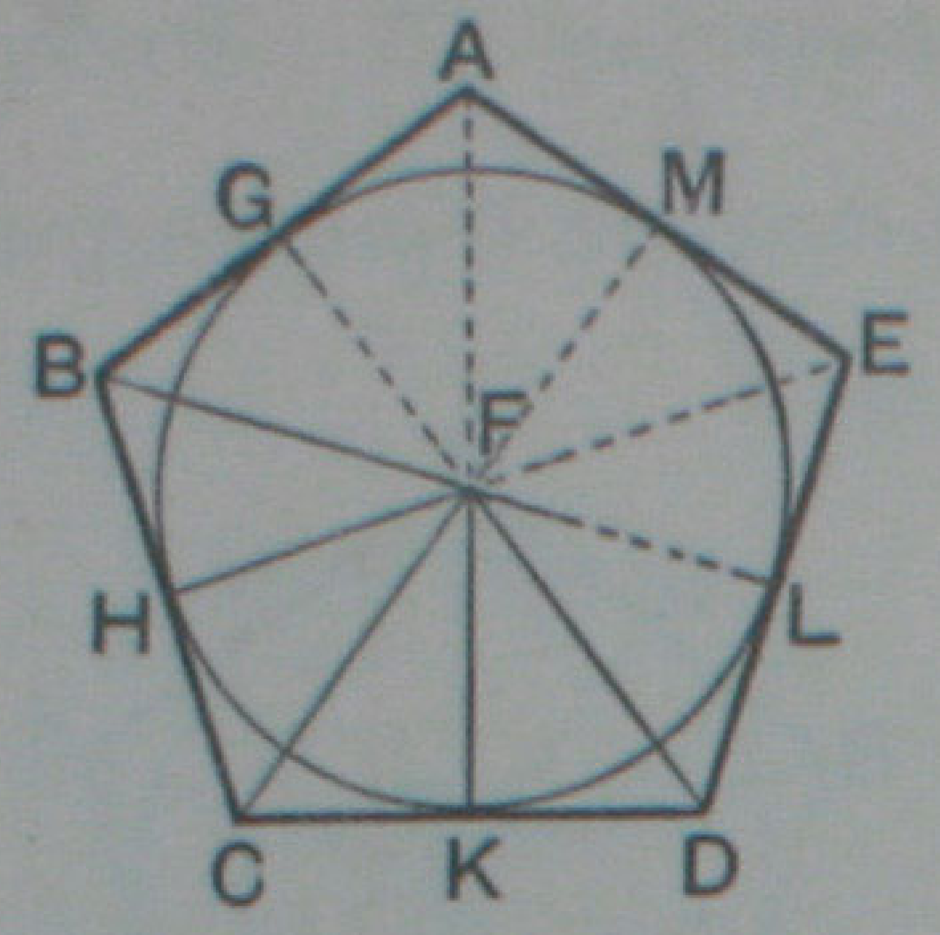
So it may be shewn that if FA, FE were joined, these lines would bisect the ∠^s at A and E.

Again, in the △^s FCH, FCK,

Because { the ∠ FCH = the ∠ FCK, *Constr.*
and the ∠ FHC = the ∠ FKC, being rt. angles;
also FC is common;

∴ FH = FK. I. 26.

Similarly if FG, FM, FL be drawn perp. to BA, AE, ED, it may be shewn that the five perpendiculars drawn from F to the sides of the pentagon are all equal.



With centre F, and radius FH, describe a circle;
this circle must pass through the points H, K, L, M, G;
and it will be touched at these points by the sides of the pentagon, for the ∠s at H, K, L, M, G are rt. ∠s. *Constr.*
∴ the ⊙ HKLMG is inscribed in the given pentagon. Q.E.F.

COROLLARY. *The bisectors of the angles of a regular pentagon meet at a point.*

NOTE. In the same way it may be shewn that the bisectors of the angles of any regular polygon meet at a point. [See Ex. 1, p. 294.]

[For Exercises on Regular Polygons see p. 293.]

MISCELLANEOUS EXERCISES.

1. Two tangents AB, AC are drawn from an external point A to a given circle: describe a circle to touch AB, AC and the convex arc intercepted by them on the given circle.

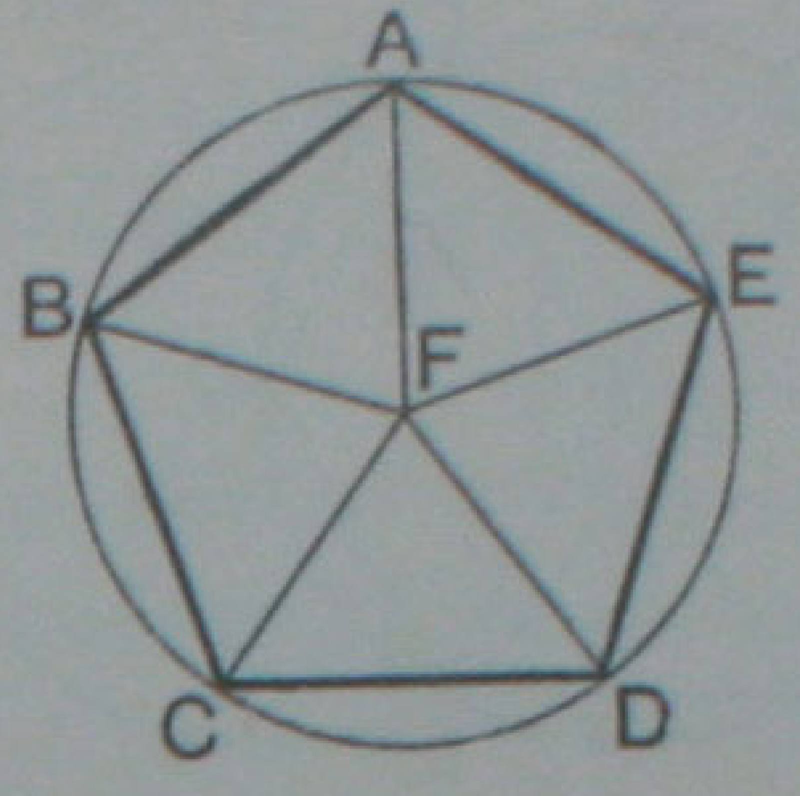
2. ABC is an isosceles triangle, and from the vertex A a straight line is drawn to meet the base at D and the circumference of the circumscribed circle at E: shew that AB is a tangent to the circle circumscribed about the triangle BDE.

3. An equilateral triangle is inscribed in a given circle: shew that twice the square on one of its sides is equal to three times the area of the square inscribed in the same circle.

4. ABC is an isosceles triangle in which each of the angles at B and C is double of the angle at A; shew that the square on AB is equal to the rectangle AB, BC with the square on BC.

PROPOSITION 14. PROBLEM.

To circumscribe a circle about a given regular pentagon.



Let ABCDE be the given regular pentagon.

It is required to circumscribe a circle about the figure ABCDE.

Construction. Bisect the $\angle^s$ BCD, CDE by CF, DF, intersecting at F. I. 9.

Join FB, FA, FE.

Proof. In the $\triangle^s$ BCF, DCF,

Because { BC = DC, *Hyp.*
and CF is common to both;
and the $\angle$ BCF = the $\angle$ DCF; *Constr.*

$\therefore$ the $\angle$ CBF = the $\angle$ CDF. I. 4.

But the $\angle$ CDF is half an angle of the regular pentagon:
$\therefore$ also the $\angle$ CBF is half an angle of the regular pentagon:
that is, FB bisects the $\angle$ ABC.

So it may be shewn that FA, FE bisect the $\angle^s$ at A and E.

Now the $\angle^s$ FCD, FDC are each half an angle of the given regular pentagon;

$\therefore$ the $\angle$ FCD = the $\angle$ FDC, IV. *Def.* 2.

$\therefore$ FC = FD. I. 6.

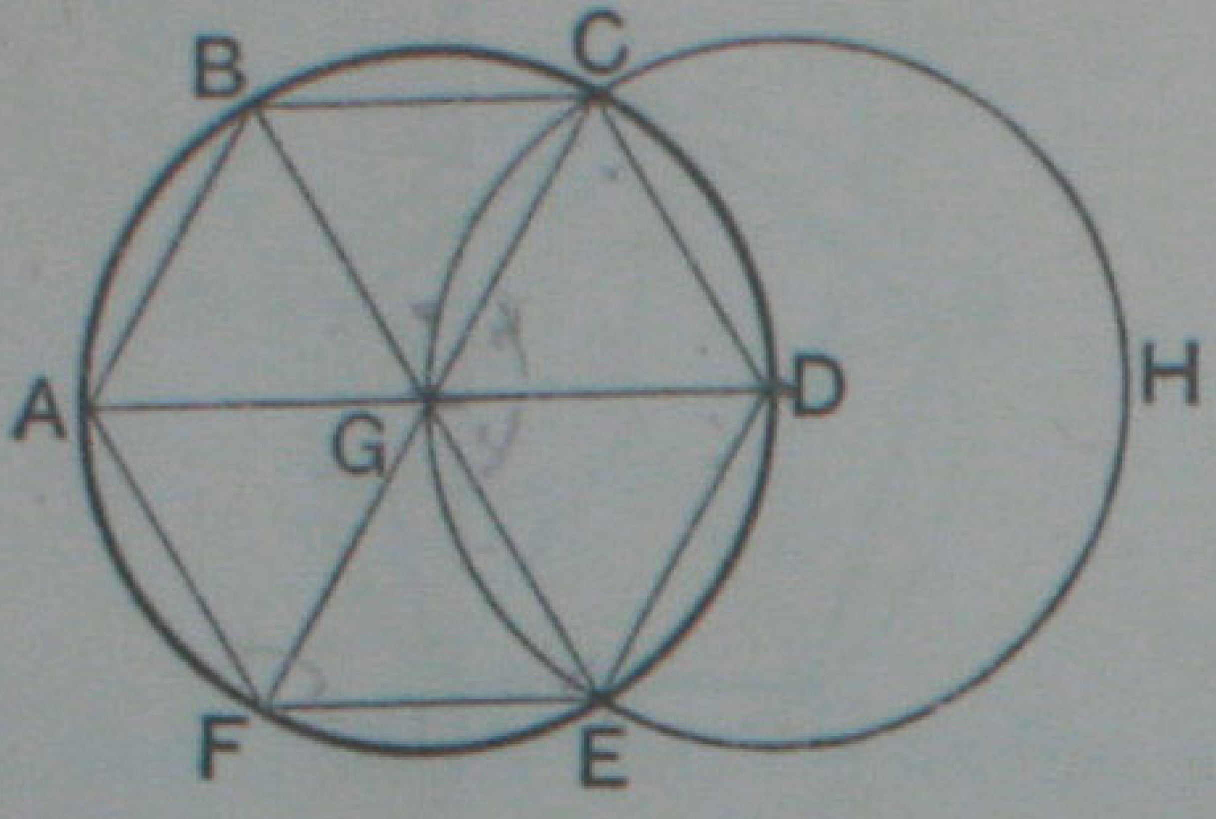
Similarly it may be shewn that FA, FB, FC, FD, FE are all equal.

With centre F, and radius FA, describe a circle:
this circle must pass through the points A, B, C, D, E,
and therefore is circumscribed about the pentagon. Q.E.F.

NOTE. In the same way a circle may be circumscribed about any regular polygon.

PROPOSITION 15. PROBLEM.

To inscribe a regular hexagon in a given circle.



Let ABDF be the given circle.

It is required to inscribe a regular hexagon in the ⊙ ABDF.

Construction. Find G the centre of the ⊙ ABDF; III. 1.
and draw a diameter AGD.

With centre D, and radius DG, describe the ⊙ EGCH.

Join CG, EG, and produce them to cut the $\bigcirc^{ce}$ of the given circle at F and B.

Join AB, BC, CD, DE, EF, FA.

Then ABCDEF *shall be the required regular hexagon.*

Proof. Now GE = GD, being radii of the ⊙ ACE;
and DG = DE, being radii of the ⊙ EHC:

∴ GE, ED, DG are all equal, and the △ EGD is equilateral.

Hence the ∠ EGD = one-third of two rt. angles. I. 32.

Similarly the ∠ DGC = one-third of two rt. angles.

But the $\angle^s$ EGD, DGC, CGB together = two rt. angles; I. 13.

∴ the remaining ∠ CGB = one-third of two rt. angles.

∴ the three $\angle^s$ EGD, DGC, CGB are equal to one another.

And to these angles the vert. opp. $\angle^s$ BGA, AGF, FGE are respectively equal:

∴ the $\angle^s$ EGD, DGC, CGB, BGA, AGF, FGE are all equal;

∴ the arcs ED, DC, CB, BA, AF, FE are all equal: III. 26.

∴ the chords ED, DC, CB, BA, AF, FE are all equal: III. 29.

∴ the hexagon is equilateral.

Again the arc FA = the arc DE: *Proved.*

to each of these equals add the arc ABCD;

then the arc FABCD = the arc ABCDE:

hence the angles at the $\bigcirc^{ce}$ which stand on these equal arcs are equal.

that is, the ∠ FED = the ∠ AFE. III. 27.

In like manner the remaining angles of the hexagon may be shewn to be equal.

∴ the hexagon is equiangular;

∴ the hexagon ABCDEF is regular, and it is inscribed in the ⊙ ABDF. Q.E.F.

COROLLARY. *The side of a regular hexagon inscribed in a circle is equal to the radius of the circle.*

SUMMARY OF THE PROPOSITIONS OF BOOK IV.

The following summary will assist the student in remembering the sequence of the Propositions of Book IV.

(i) Of the sixteen Propositions of this Book, Props. 1, 10, 15, 16 deal with isolated constructions.

(ii) The remaining twelve Propositions may be divided into *three* groups of *four* each, as follows:

(*a*) Group 1. Props. 2, 3, 4, 5 deal with *triangles* and *circles.*

(*b*) Group 2. Props. 6, 7, 8, 9 deal with *squares* and *circles.*

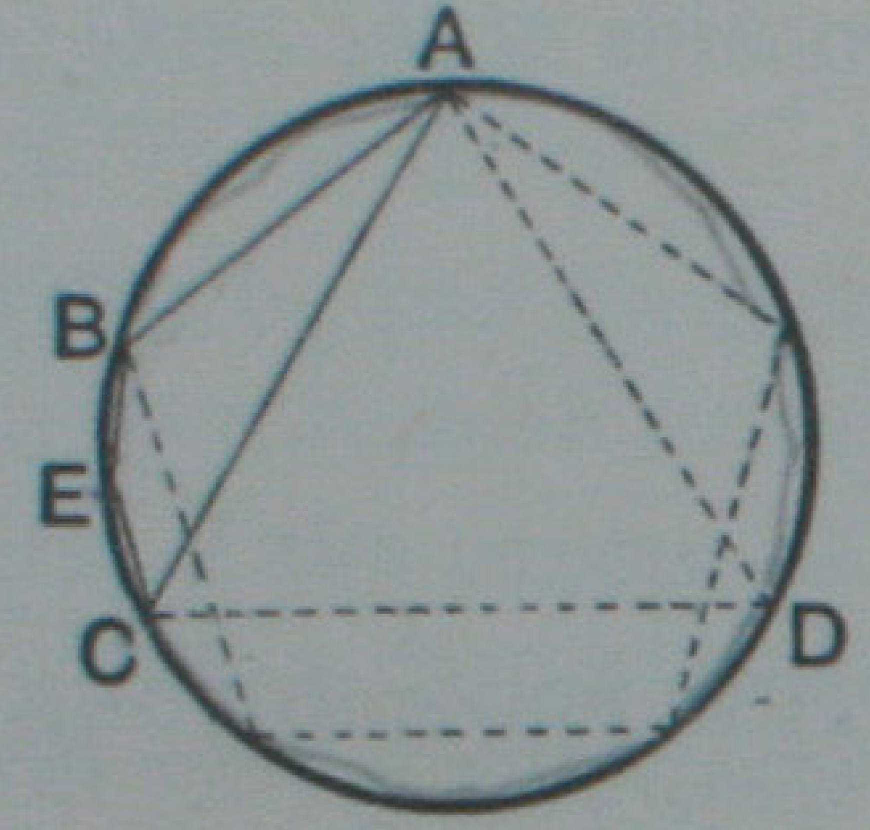
(*c*) Group 3. Props. 11, 12, 13, 14 deal with *pentagons* and *circles.*

(iii) In each group the problem of *inscription* precedes the corresponding problem of *circumscription.*

Further, each group deals with the inscription and circumscription of *rectilineal* figures first and of *circles* afterwards.

PROPOSITION 16. PROBLEM.

To inscribe a regular quindecagon in a given circle.



Let ABCD be the given circle.

It is required to inscribe a regular quindecagon in the ⊙ ABCD.

Construction.

In the ⊙ ABCD inscribe an equilateral triangle, IV. 2.
and let AC be one of its sides.

In the same circle inscribe a regular pentagon, IV. 11.
and let AB be one of its sides.

Proof.

Now of such equal parts as the whole $\bigcirc^{ce}$ contains fifteen,
the arc AC, which is one-third of the $\bigcirc^{ce}$, contains five,
and the arc AB, which is one-fifth of the $\bigcirc^{ce}$, contains three;
∴ their difference, the arc BC, contains two.

Bisect the arc BC at E: III. 30.
then each of the arcs BE, EC is one-fifteenth of the $\bigcirc^{ce}$.

∴ if BE, EC be joined, and st. lines equal to them be placed successively round the circle, a regular quindecagon will be inscribed in it.

Q.E.F.

EXERCISES ON PROPOSITIONS 11—16.

1. Express in terms of a right angle the magnitude of an angle of the following *regular* polygons:

(i) a pentagon, (ii) a hexagon, (iii) an octagon,
(iv) a decagon, (v) a quindecagon.

2. Any angle of a regular pentagon is trisected by the straight lines which join it to the opposite vertices.

3. In a polygon of n sides the straight lines which join any angular point to the vertices not adjacent to it, divide the angle into $n-2$ equal parts.

4. Shew how to construct on a given straight line
(i) a regular pentagon, (ii) a regular hexagon, (iii) a regular octagon.

5. An equilateral triangle and a regular hexagon are inscribed in a given circle; shew that

(i) the area of the triangle is half that of the hexagon;

(ii) the square on the side of the triangle is three times the square on the side of the hexagon.

6. ABCDE is a regular pentagon, and AC, BE intersect at H: shew that

(i) AB = CH = EH.

(ii) AB is a tangent to the circle circumscribed about the triangle BHC.

(iii) AC and BE cut one another in medial section.

7. The straight lines which join alternate vertices of a regular pentagon intersect so as to form another regular pentagon.

8. The straight lines which join alternate vertices of a regular polygon of n sides, intersect so as to form another regular polygon of n sides.

If $n=6$, shew that the area of the resulting hexagon is one-third of the given hexagon.

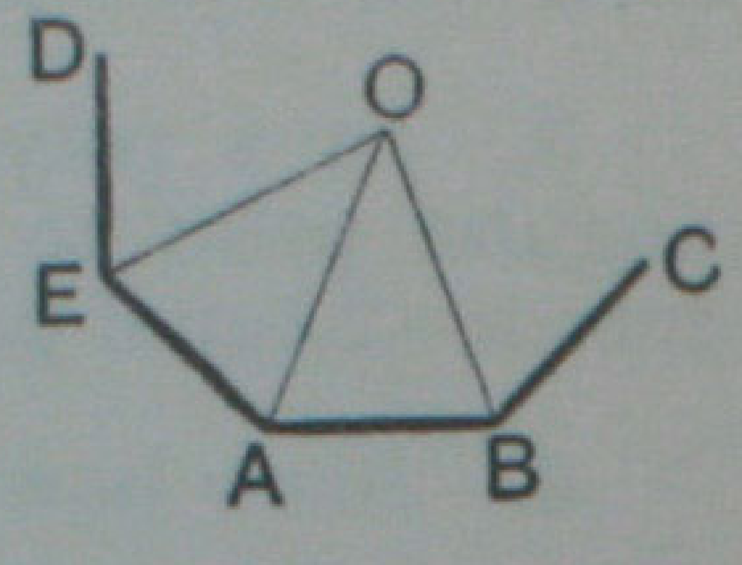
9. By means of IV. 16, inscribe in a circle a triangle whose angles are as the numbers 2, 5, 8.

10. Shew that the area of a regular hexagon inscribed in a circle is three-fourths of that of the corresponding circumscribed hexagon.

NOTE ON REGULAR POLYGONS.

The following propositions, proved by Euclid for a regular pentagon, hold good for all regular polygons.

1. *The bisectors of the angles of any regular polygon are concurrent.*



Let D, E, A, B, C be consecutive angular points of a regular polygon of any number of sides.

Bisect the $\angle$s EAB, ABC by AO, BO, which intersect at O.

Join EO.

It is required to prove that EO *bisects the* $\angle$ DEA.

For in the $\triangle$s EAO, BAO,

Because { EA = BA, being sides of a regular polygon ; and AO is common ; and the $\angle$ EAO = the $\angle$ BAO ; *Constr.*

∴ the $\angle$ OEA = the $\angle$ OBA. I. 4.

But the $\angle$ OBA is half the $\angle$ ABC ; *Constr.*
also the $\angle$ ABC = the $\angle$ DEA, since the polygon is regular ;
∴ the $\angle$ OEA is half the $\angle$ DEA :
that is, EO bisects the $\angle$ DEA.

Similarly if O be joined to the remaining angular points of the polygon, it may be proved that each joining line bisects the angle to whose vertex it is drawn.

That is to say, the bisectors of the angles of the polygon meet at the point O.

Q.E.D.

COROLLARIES. Since the $\angle$ EAB = the $\angle$ ABC ; *Hyp.*
and since the $\angle$s OAB, OBA are respectively half of the $\angle$s EAB, ABC ;
∴ the $\angle$ OAB = the $\angle$ OBA ;
∴ OA = OB. I. 6.
Similarly OE = OA.

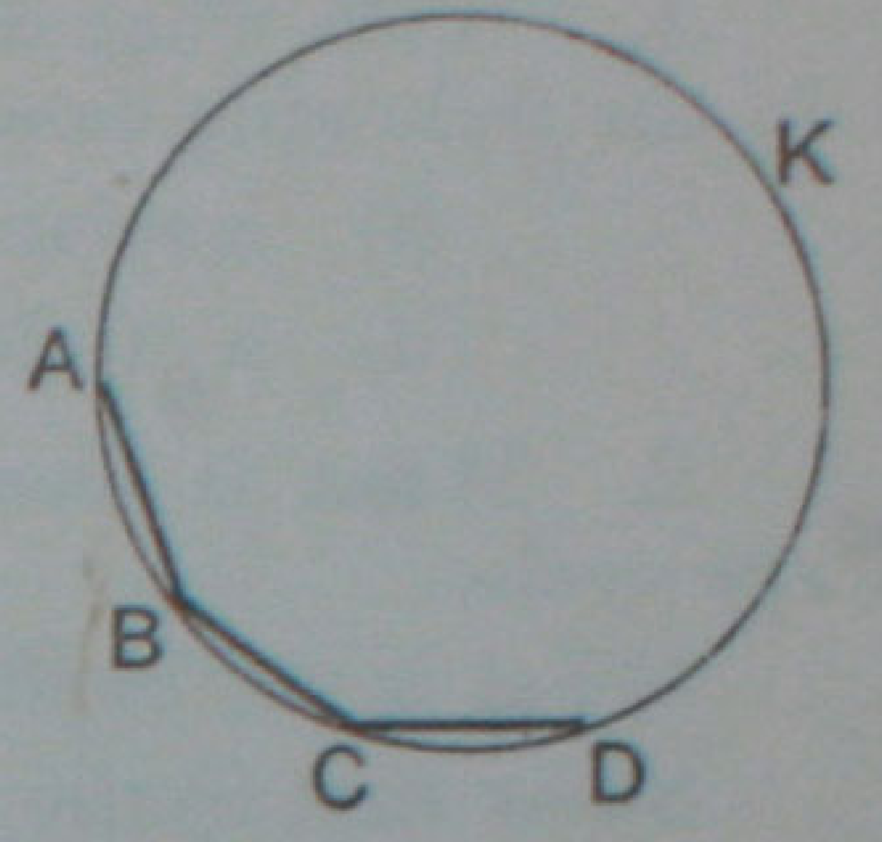
Hence the bisectors of the angles of a regular polygon are all equal.

Therefore a circle described with centre O, *and radius* OA, *will be circumscribed about the polygon.*

Also it may be shewn, as in Proposition 13, that perpendiculars drawn from O to the sides of the polygon are all equal.

Therefore a circle described with centre O, *and any one of these perpendiculars as radius, will be inscribed in the polygon.*

2. *If a polygon inscribed in a circle is equilateral, it is also equiangular.*



Let AB, BC, CD be consecutive sides of an equilateral polygon inscribed in the ⊙ ADK.

Then shall this polygon be equiangular.

Because the chord AB = the chord DC. *Hyp.*

∴ the minor arc AB = the minor arc DC. III. 28.

To each of these equals add the arc AKD :

then the arc BAKD = the arc AKDC ;

∴ the angles at the ○ce, which stand on these equal arcs, are equal ;

that is, the ∠ BCD = the ∠ ABC. III. 27.

Similarly the remaining angles of the polygon may be shewn to be equal :

the polygon is equiangular. Q.E.D.

3. *If a polygon inscribed in a circle is equiangular, it is also equilateral, provided that the number of its sides is odd.*

[Observe that Theorems 2 and 3 are only true of polygons *inscribed in a circle.*

Fig. 1.

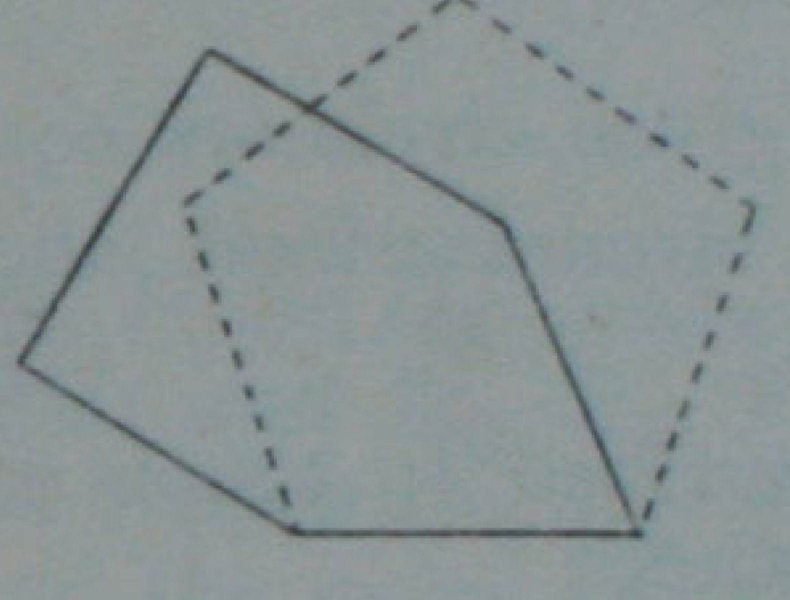
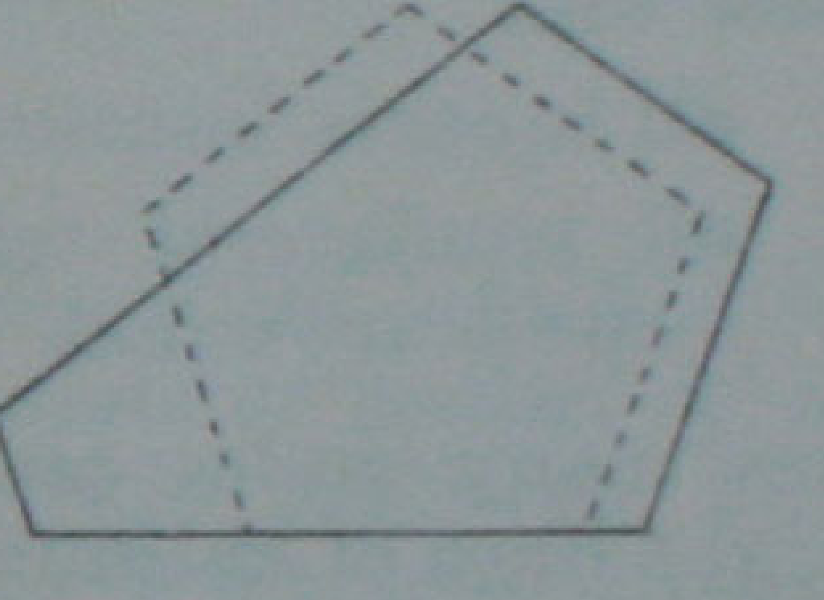


Fig. 2.



The above figures are sufficient to shew that otherwise a polygon may be equilateral without being equiangular, Fig. 1 ; or equiangular without being equilateral, Fig. 2.]

NOTE. The following extensions of Euclid's constructions for Regular Polygons should be noticed.

By continual bisection of arcs, we are enabled to divide the circumference of a circle,

by means of Proposition 6, into 4, 8, 16,..., $2 . 2^n$,... equal parts ;
by means of Proposition 15, into 3, 6, 12,..., $3 . 2^n$,... equal parts ;
by means of Proposition 11, into 5, 10, 20,..., $5 . 2^n$,... equal parts ;
by means of Proposition 16, into 15, 30, 60,..., $15 . 2^n$,... equal parts.

Hence we can inscribe in a circle a regular polygon the number of whose sides is included in any one of the formulæ $2 . 2^n$, $3 . 2^n$, $5 . 2^n$, $15 . 2^n$, n being any positive integer. It has also been shewn (by Gauss, 1800) that a regular polygon of 2^n+1 sides may be inscribed in a circle, provided 2^n+1 is a prime number.

QUESTIONS FOR REVISION ON BOOK IV.

1. With what difference of meaning is the word *inscribed* used in the following cases?

    (i) a triangle *inscribed* in a circle;

    (ii) a circle *inscribed* in a triangle.

2. What is meant by a *cyclic figure*? Shew that *all triangles* are cyclic.

    What is the condition that a quadrilateral may be *cyclic*?

    Shew that cyclic parallelograms must be rectangular.

3. Shew that the only *regular* figures which may be fitted together so as to form a plane surface are (i) *equilateral triangles*, (ii) *squares*, (iii) *regular hexagons*.

4. Employ the first Corollary of I. 32 to shew that in any regular polygon of n sides each interior angle contains $\frac{2(n-2)}{n}$ right angles?

5. *The bisectors of the angles of a regular polygon are concurrent.* State the *method of proof* employed in this and similar theorems.

6. Shew that

    (i) all squares inscribed in a given circle are equal; and

    (ii) all equilateral triangles circumscribed about a given circle are equal.

7. How many circles can be described to touch each of three given straight lines of unlimited length?

    (i) when no two of the lines are parallel;

    (ii) when two only are parallel;

    (iii) when all three are parallel.

8. Prove that the greatest triangle which can be inscribed in a circle on a diameter as base, is one-fourth of the circumscribed square.

9. The radius of a given circle is 10 inches: find the length of a side of

    (i) the circumscribed square; [20 inches.]

    (ii) the inscribed square; $\sqrt{2}$ inches.]

    (iii) the inscribed equilateral triangle; [$10\sqrt{3}$ inches.]

    (iv) the circumscribed equilateral triangle; [$20\sqrt{3}$ inches.]

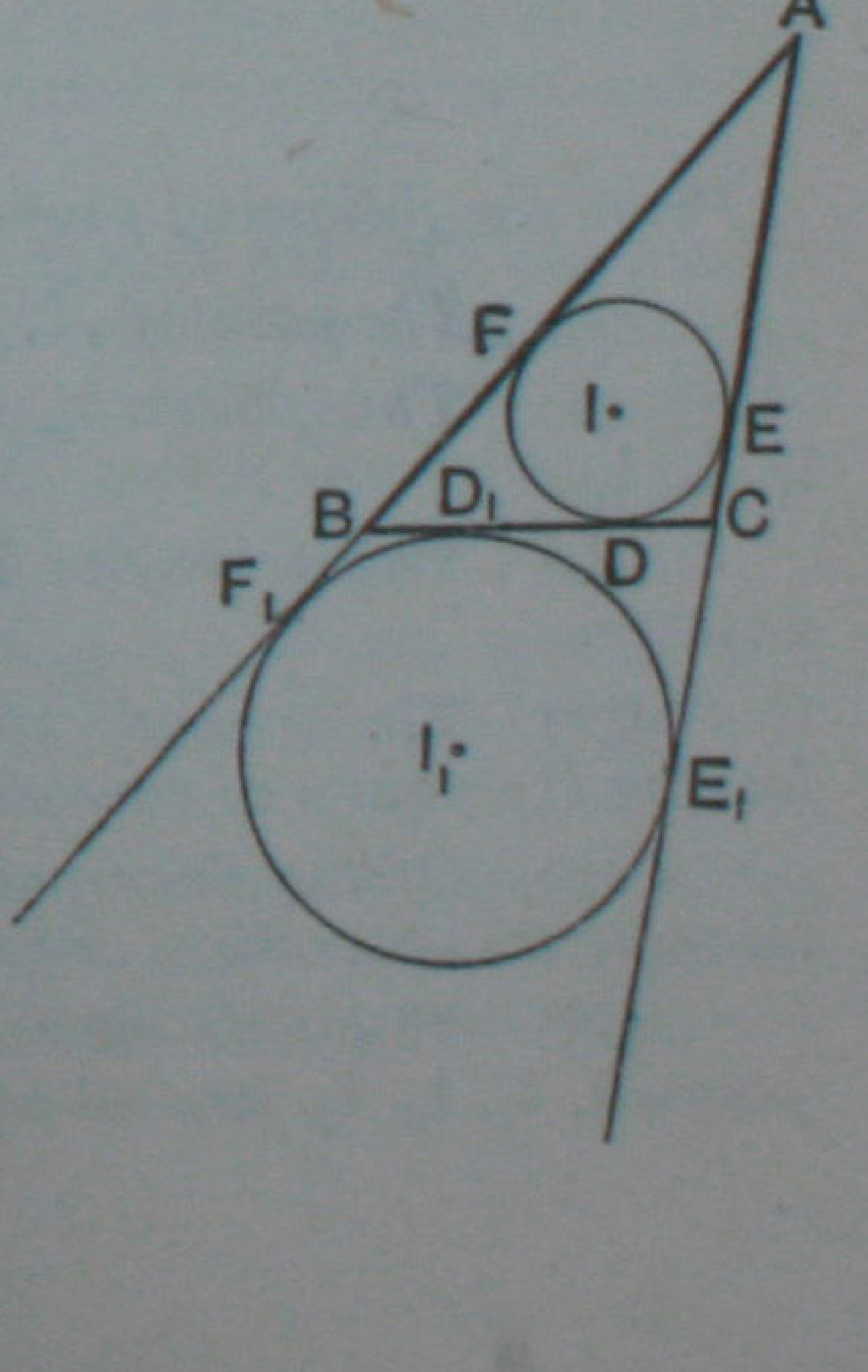
    (v) the inscribed regular hexagon. [10 inches.]

    Shew also that the areas of these figures are respectively 400, 200, $75\sqrt{3}$, $300\sqrt{3}$, and $150\sqrt{3}$ square inches.

THEOREMS AND EXAMPLES ON BOOK IV.

I. ON THE TRIANGLE AND ITS CIRCLES.

1. D, E, F *are the points of contact of the inscribed circle of the triangle* ABC, *and* D_1, E_1, F_1 *the points of contact of the escribed circle, which touches* BC *and the other sides produced:* a, b, c *denote the length of the sides* BC, CA, AB; s *the semi-perimeter of the triangle, and* r, r_1 *the radii of the inscribed and escribed circles.*



Prove the following equalities:

(i) $AE = AF = s - a$,
$BD = BF = s - b$,
$CD = CE = s - c$,

(ii) $AE_1 = AF_1 = s$.

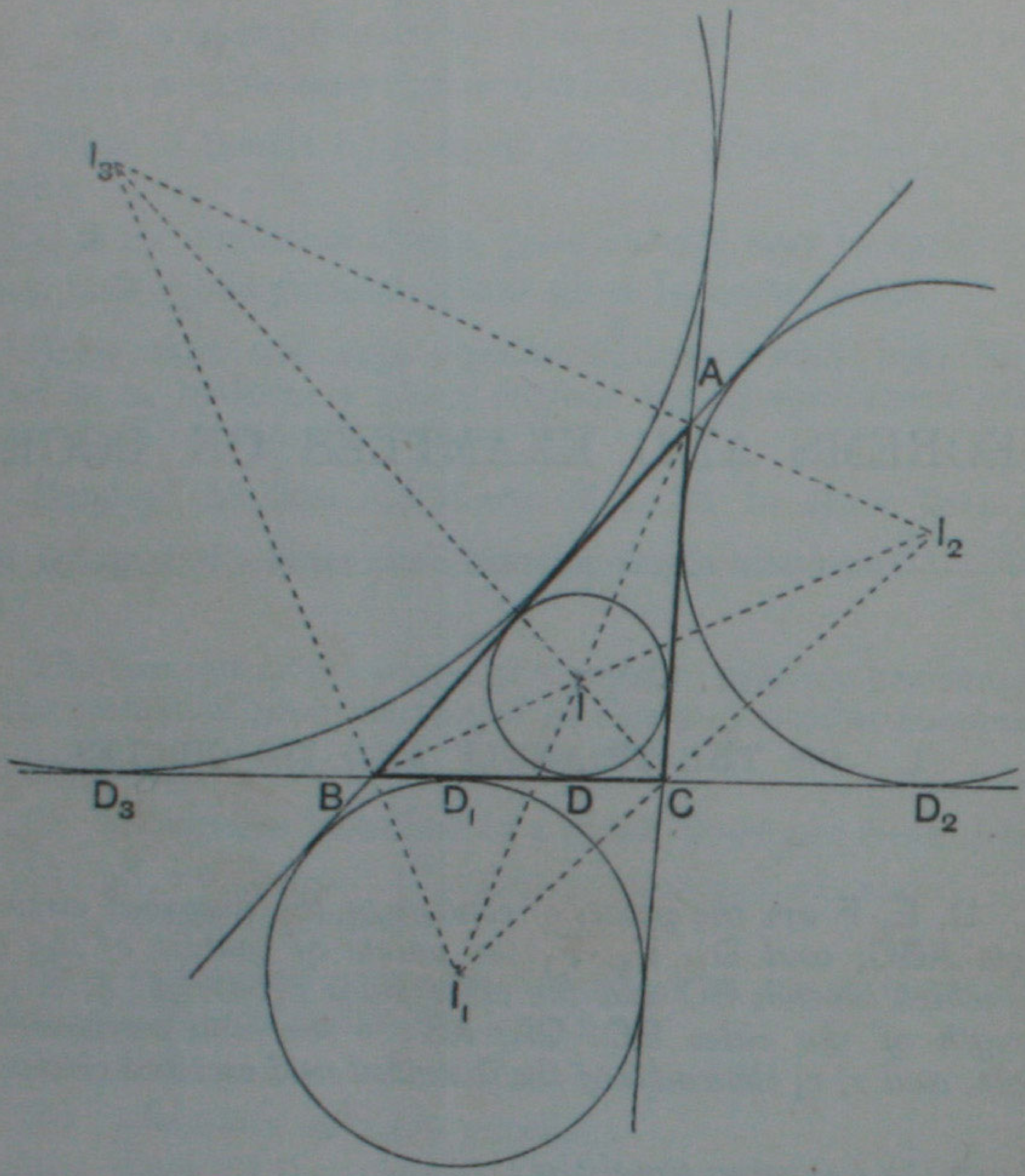
(iii) $CD_1 = CE_1 = s - b$,
$BD_1 = BF_1 = s - c$.

(iv) $CD = BD_1$ and $BD = CD_1$.

(v) $EE_1 = FF_1 = a$.

(vi) The area of the $\triangle$ ABC
$= rs = r_1(s - a)$.

2. *In the triangle* ABC, I *is the centre of the inscribed circle, and* I_1, I_2, I_3 *the centres of the escribed circles touching respectively the sides* BC, CA, AB *and the other sides produced.*



Prove the following properties:

(i) *The points* A, I, I_1 *are collinear: so are* B, I, I_2; *and* C, I, I_3.

(ii) *The points* I_2, A, I_3 *are collinear; so are* I_3, B, I_1; *and* I_1, C, I_2.

(iii) *The triangles* BI_1C, CI_2A, AI_3B *are equiangular to one another.*

(iv) *The triangle* $I_1I_2I_3$ *is equiangular to the triangle formed by joining the points of contact of the inscribed circle.*

(v) *Of the four points* I, I_1, I_2, I_3 *each is the orthocentre of the triangle whose vertices are the other three.*

(vi) *The four circles, each of which passes through three of the points* I, I_1, I_2, I_3, *are all equal.*

3. With the notation of page 297, shew that in a triangle ABC, if the angle at C is a right angle,

$$r = s - c;\quad r_1 = s - b;\quad r_2 = s - a;\quad r_3 = s.$$

4. With the figure given on page 298, shew that if the circles whose centres are I, I_1, I_2, I_3 touch BC at D, D_1, D_2, D_3, then

(i) $DD_2 = D_1D_3 = b$. (ii) $DD_3 = D_1D_2 = c$.

(iii) $D_2D_3 = b + c$. (iv) $DD_1 = b \sim c$.

5. *Shew that the orthocentre and vertices of a triangle are the centres of the inscribed and escribed circles of the pedal triangle.*
[See Ex. 20, p. 243.]

6. *Given the base and vertical angle of a triangle, find the locus of the centre of the inscribed circle.* [See Ex. 36, p. 246.]

7. *Given the base and vertical angle of a triangle, find the locus of the centre of the escribed circle which touches the base.*

8. *Given the base and vertical angle of a triangle, shew that the centre of the circumscribed circle is fixed.*

9. Given the base BC, and the vertical angle A of a triangle, find the locus of the centre of the escribed circle which touches AC.

10. Given the base, the vertical angle, and the radius of the inscribed circle; construct the triangle.

11. Given the base, the vertical angle, and the radius of the escribed circle, (i) which touches the base, (ii) which touches one of the sides containing the vertical angle; construct the triangle.

12. Given the base, the vertical angle, and the point of contact with the base of the inscribed circle; construct the triangle.

13. Given the base, the vertical angle, and the point of contact with the base, or base produced, of an escribed circle; construct the triangle.

14. From an external point A two tangents AB, AC are drawn to a given circle; and the angle BAC is bisected by a straight line which meets the circumference in I and I_1: shew that I is the centre of the circle inscribed in the triangle ABC, and I_1 the centre of one of the escribed circles.

15. *I is the centre of the circle inscribed in a triangle, and I_1, I_2, I_3 the centres of the escribed circles; shew that II_1, II_2, II_3 are bisected by the circumference of the circumscribed circle.*

16. ABC is a triangle, and I_2, I_3 the centres of the escribed circles which touch AC, and AB respectively: shew that the points B, C, I_2, I_3 lie upon a circle whose centre is on the circumference of the circle circumscribed about ABC.

17. With three given points as centres describe three circles touching one another two by two. How many solutions will there be?

18. Two tangents AB, AC are drawn to a given circle from an external point A; and in AB, AC two points D and E are taken so that DE is equal to the sum of DB and EC: shew that DE touches the circle.

19. Given the perimeter of a triangle, and one angle in magnitude and position: shew that the opposite side always touches a fixed circle.

20. Given the centres of the three escribed circles; construct the triangle.

21. Given the centre of the inscribed circle, and the centres of two escribed circles; construct the triangle.

22. Given the vertical angle, perimeter, and the length of the bisector of the vertical angle; construct the triangle.

23. Given the vertical angle, perimeter, and altitude; construct the triangle.

24. Given the vertical angle, perimeter, and radius of the inscribed circle; construct the triangle.

25. Given the vertical angle, the radius of the inscribed circle, and the length of the perpendicular from the vertex to the base; construct the triangle.

26. Given the base, the difference of the sides containing the vertical angle, and the radius of the inscribed circle; construct the triangle. [See Ex. 10, p. 276.]

27. Given a vertex, the centre of the circumscribed circle, and the centre of the inscribed circle, construct the triangle.

28. In a triangle ABC, I is the centre of the inscribed circle; shew that the centres of the circles circumscribed about the triangles BIC, CIA, AIB lie on the circumference of the circle circumscribed about the given triangle.

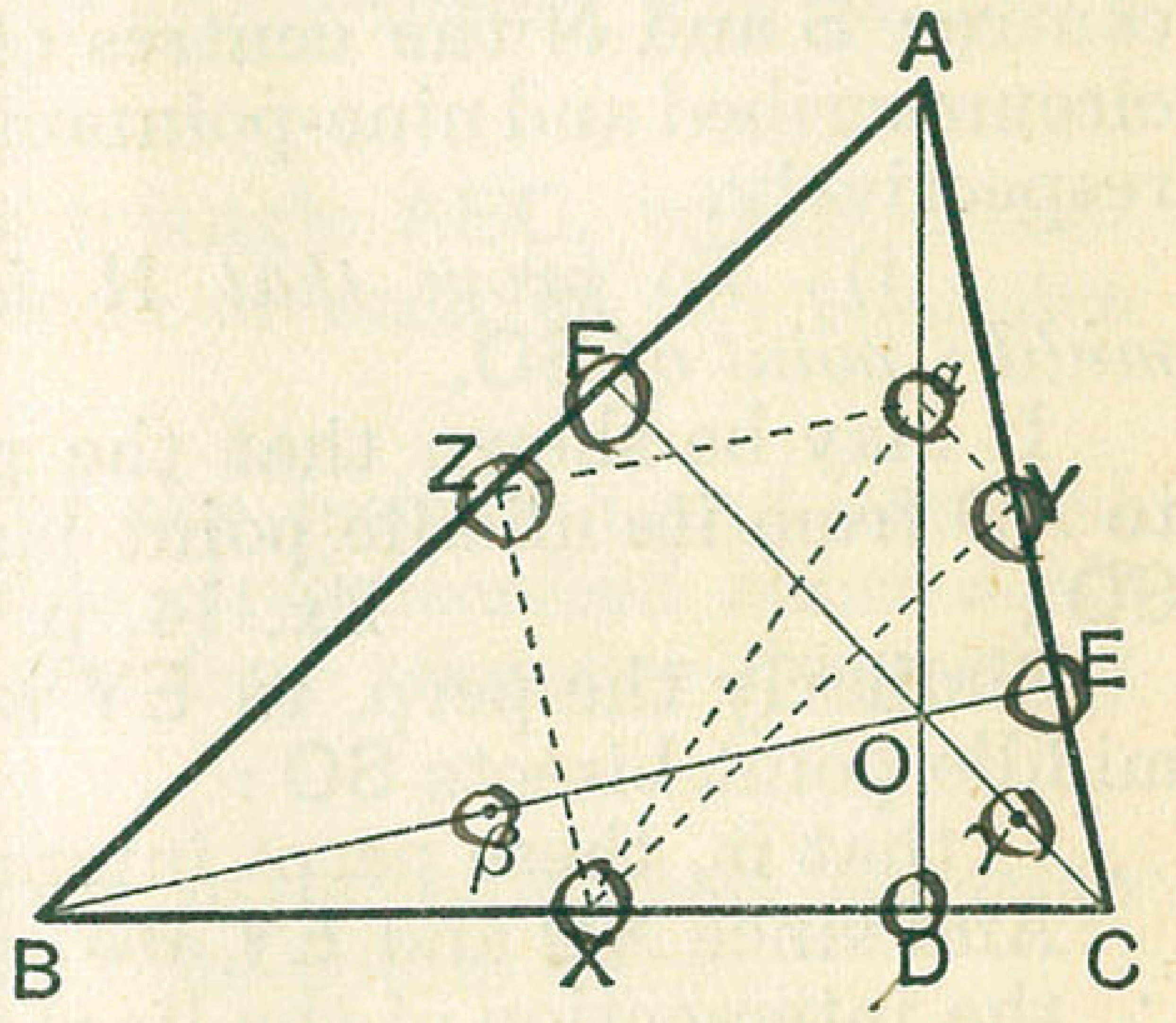
29. In a triangle ABC, the inscribed circle touches the base BC at D; and r, r_1 are the radii of the inscribed circle and of the escribed circle which touches BC: shew that $r \cdot r_1 = BD \cdot DC$.

30. ABC is a triangle, D, E, F the points of contact of its inscribed circle; and D′E′F′ is the pedal triangle of the triangle DEF: shew that the sides of the triangle D′E′F′ are parallel to those of ABC.

31. In a triangle ABC the inscribed circle touches BC at D. Shew that the circles inscribed in the triangles ABD, ACD touch one another.

On the Nine-Points Circle.

32. *In any triangle the middle points of the sides, the feet of the perpendiculars drawn from the vertices to the opposite sides, and the middle points of the lines joining the orthocentre to the vertices are concyclic.*



In the △ ABC, let X, Y, Z be the middle points of the sides BC, CA, AB; let D, E, F be the feet of the perp^s drawn to these sides from A, B, C; let O be the orthocentre, and α, β, γ the middle points of OA, OB, OC.

Then shall the nine points X, Y, Z, D, E, F, α, β, γ *be concyclic.*

Join XY, XZ, Xα, Yα, Zα.

Now from the △ ABO, since AZ = ZB, and Aα = αO, *Hyp.*
∴ Zα is par^l to BO. Ex. 2, p. 104.

And from the △ ABC, since BZ = ZA, and BX = XC, *Hyp.*
∴ ZX is par^l to AC.

But BO produced makes a rt. angle with AC; *Hyp.*
∴ the ∠ XZα is a rt. angle.

Similarly, the ∠ XYα is a rt. angle. I. 29.
∴ the points X, Z, α, Y are concyclic:

that is, α lies on the O^{ce} of the circle, which passes through X, Y, Z; and Xα is a diameter of this circle.

Similarly it may be shewn that β and γ lie on the O^{ce} of the circle which passes through X, Y, Z.

Again, since αDX is a rt. angle, *Hyp.*
∴ the circle on Xα as diameter passes through D.

Similarly it may be shewn that E and F lie on the circumference of the same circle.

∴ the points X, Y, Z, D, E, F, α, β, γ are concyclic. Q.E.D.

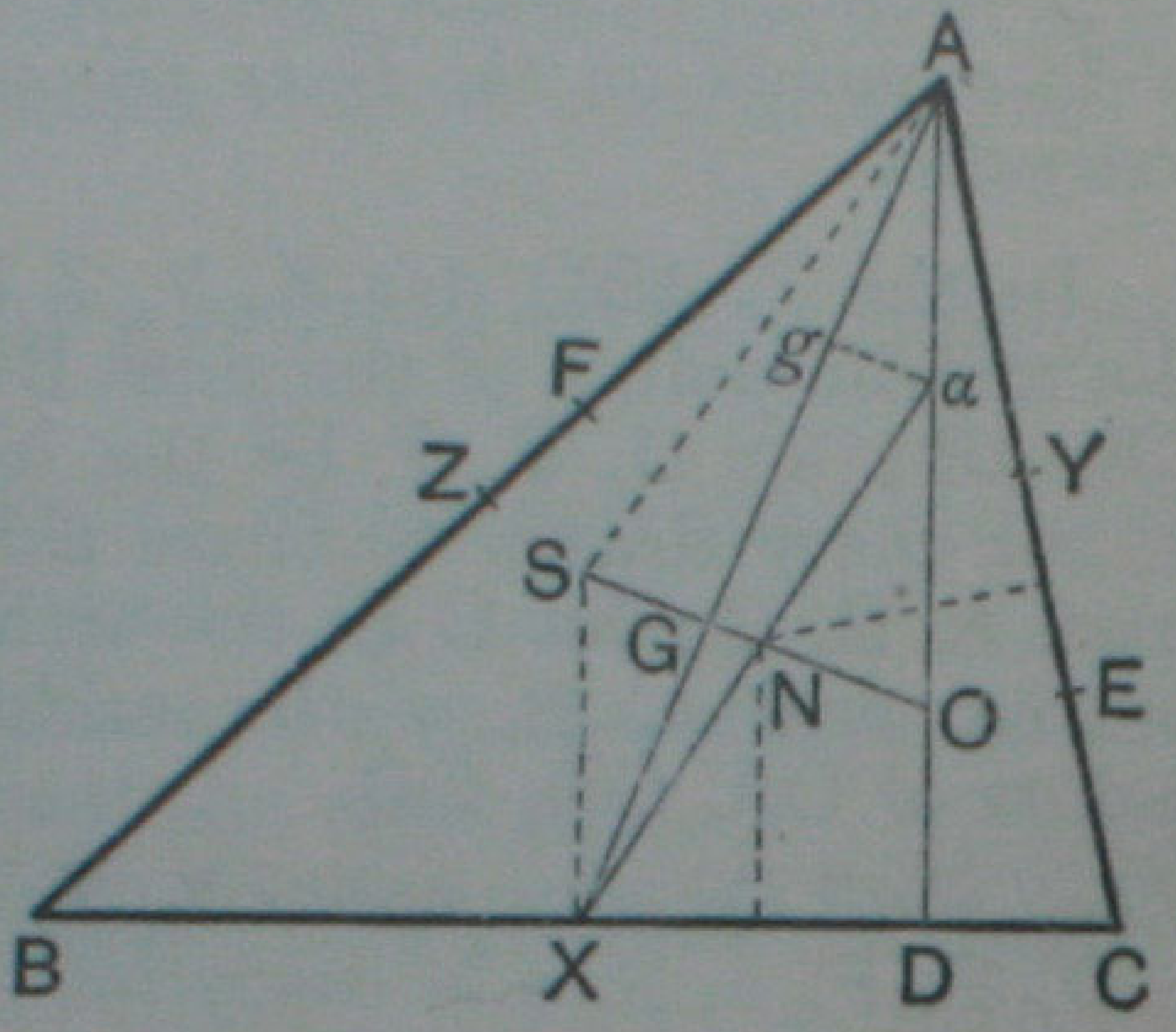
From this property the circle which passes through the middle points of the sides of a triangle is called the **Nine-Points Circle**; many of its properties may be derived from the fact of its being the circle circumscribed about the pedal triangle.

33. *To prove that*

(i) *the centre of the nine-points circle is the middle point of the straight line which joins the orthocentre to the circumscribed centre.*

(ii) *the radius of the nine-points circle is half the radius of the circumscribed circle.*

(iii) *the centroid is collinear with the circumscribed centre, the nine-points centre, and the orthocentre.*



In the △ ABC, let X, Y, Z be the middle points of the sides; D, E, F the feet of the perp^s; O the orthocentre; S and N the centres of the circumscribed and nine-points circles respectively.

(i) *To prove that* N *is the middle point of* SO.

It may be shewn that the perp. to XD from its middle point bisects SO; Ex. 14, p. 106.

Similarly the perp. to EY at its middle point bisects SO:

that is, these perp^s intersect at the middle point of SO:

And since XD and EY are chords of the nine-points circle,

∴ the intersection of the lines which bisect XD and EY at rt. angles is its centre: III. 1.

∴ the centre N is the middle point of SO

(ii) *To prove that the radius of the nine-points circle is half the radius of the circumscribed circle.*

By the last Proposition, Xα is a diameter of the nine-points circle.

∴ the middle point of Xα is its centre:

but the middle point of SO is also the centre of the nine-points circle. (*Proved.*)

Hence Xα and SO bisect one another at N.

Then from the △^s SNX, ONα,

Because { SN = ON,
and NX = Nα,
and the ∠ SNX = the ∠ ONα; I. 15.

∴ SX = Oα I. 4.
= Aα.

And SX is also par^l to Aα,

∴ SA = Xα. I. 33.

But SA is a *radius* of the circumscribed circle;
and Xα is a *diameter* of the nine-points circle;

∴ the radius of the nine-points circle is half the radius of the circumscribed circle.

(iii) *To prove that the centroid is collinear with points* S, N, O.

Join AX and draw ag par[1] to SO.

Let AX meet SO at G.

Then from the △ AGO, since Aa = aO, and ag is par[1] to OG,

∴ Ag = gG. Ex. 1, p. 104.

And from the △ Xag, since aN = NX, and NG is par[1] to ag,

∴ gG = GX.

∴ AG = $\frac{2}{3}$ of AX ;

∴ G is the centroid of the triangle ABC. Ex. 4, p. 113.

That is, the centroid is collinear with the points S, N, O. Q.E.D.

34. *Given the base and vertical angle of a triangle, find the locus of the centre of the nine-points circle.*

35. The nine-points circle of any triangle ABC, whose orthocentre is O, is also the nine-points circle of each of the triangles AOB, BOC, COA.

36. If I, I_1, I_2, I_3 are the centres of the inscribed and escribed circles of a triangle ABC, then the circle circumscribed about ABC is the nine-points circle of each of the four triangles formed by joining three of the points I, I_1, I_2, I_3.

37. All triangles which have the same orthocentre and the same circumscribed circle, have also the same nine-points circle.

38. Given the base and vertical angle of a triangle, shew that one angle and one side of the pedal triangle are constant.

39. Given the base and vertical angle of a triangle, find the locus of the centre of the circle which passes through the three escribed centres.

NOTE. For another important property of the Nine-points Circle see Miscellaneous Examples on Book VI., Ex. 60.

II. MISCELLANEOUS EXAMPLES.

1. If four circles are described to touch every three sides of a quadrilateral, shew that their centres are concyclic.

2. If the straight lines which bisect the angles of a rectilineal figure are concurrent, a circle may be inscribed in the figure.

3. Within a given circle describe three equal circles touching one another and the given circle.

4. The perpendiculars drawn from the centres of the three escribed circles of a triangle to the sides which they touch, are concurrent.

5. Given an angle and the radii of the inscribed and circumscribed circles; construct the triangle.

6. Given the base, an angle at the base, and the distance between the centre of the inscribed circle and the centre of the escribed circle which touches the base; construct the triangle.

7. In a given circle inscribe a triangle such that two of its sides may pass through two given points, and the third side be of given length.

8. In any triangle ABC, I, I_1, I_2, I_3 are the centres of the inscribed and escribed circles, and S_1, S_2, S_3 are the centres of the circles circumscribed about the triangles BIC, CIA, AIB: shew that the triangle $S_1S_2S_3$ has its sides parallel to those of the triangle $I_1I_2I_3$, and is one-fourth of it in area: also that the triangles ABC and $S_1S_2S_3$ have the same circumscribed circle.

9. O is the orthocentre of a triangle ABC: shew that

$$AO^2 + BC^2 = BO^2 + CA^2 = CO^2 + AB^2 = d^2,$$

where d is the diameter of the circumscribed circle.

10. If from any point within a regular polygon of n sides perpendiculars are drawn to the sides, the sum of the perpendiculars is equal to n times the radius of the inscribed circle.

11. The sum of the perpendiculars drawn from the vertices of a regular polygon of n sides on any straight line is equal to n times the perpendicular drawn from the centre of the inscribed circle.

12. The area of a cyclic quadrilateral is independent of the order in which the sides are placed in the circle.

13. Given the orthocentre, the centre of the nine-points circle, and the middle point of the base; construct the triangle.

14. Of all polygons of a given number of sides, which may be inscribed in a given circle, that which is regular has the maximum area and the maximum perimeter.

15. Of all polygons of a given number of sides circumscribed about a given circle, that which is regular has the minimum area and the minimum perimeter.

16. Given the vertical angle of a triangle in position and magnitude, and the sum of the sides containing it: find the locus of the centre of the circumscribed circle.

17. P is any point on the circumference of a circle circumscribed about an equilateral triangle ABC: shew that $PA^2 + PB^2 + PC^2$ is constant.

*** Book V. is now very rarely read. The subject-matter, so far as it is introductory to Book VI., is dealt with in a simpler manner at page 317, in the chapter called 'Elementary Principles of Proportion.' The student is advised to proceed at once to that chapter, leaving Book V. in its stricter form to be studied at a later stage, if it is thought desirable.*

BOOK V.

Book V. treats of Ratio and Proportion, and the method adopted is such as to place these subjects on a basis independent of arithmetical principles.

The following notation will be employed throughout this section.

Capital letters, A, B, C, ... will be used to denote the magnitudes themselves, *not any numerical or algebraical measures of them*, and small letters, m, n, p, ... will be used to denote whole numbers. Also it will be assumed that multiplication, in the sense of repeated addition, can be applied to any magnitude, so that $m \,.\, A$ or mA will denote the magnitude A taken m times.

The symbol $>$ will be used for the words *greater than*, and $<$ for *less than.*

Definitions.

Definition 1. One magnitude is said to be a **multiple** of another, when the first contains the second an *exact* number of times.

Definition 2. One magnitude is said to be a **submultiple** of another, when the first is contained an *exact* number of times in the second.

The following properties of multiples will be assumed as self-evident.

(1) $mA >, =,$ or $< mB$ according as $A >, =,$ or $< B$; and conversely.

(2) $mA + mB + \ldots = m(A + B + \ldots)$.

(3) If $A > B$, then $mA - mB = m(A - B)$.

(4) $mA + nA + \ldots = (m + n + \ldots)A$.

(5) If $m > n$, then $mA - nA = (m - n)A$.

(6) $m \,.\, nA = mn \,.\, A = nm \,.\, A = n \,.\, mA$.

Definition 3. The **Ratio** of one magnitude to another of the same kind is the relation which the first bears to the second in respect of *quantuplicity*.

The ratio of A to B is denoted thus, $A : B$; and A is called the **antecedent,** B the **consequent** of the ratio.

The term *quantuplicity* denotes the capacity of the first magnitude to contain the second with or without remainder.

If the magnitudes are commensurable, their quantuplicity may be expressed *numerically* by observing what multiples of the two magnitudes are equal to one another.

Thus if $A=ma$, and $B=na$, it follows that $nA=mB$. In this case $A=\frac{m}{n}B$, and the quantuplicity of A with respect to B is the arithmetical fraction $\frac{m}{n}$.

But if the magnitudes are incommensurable, no multiple of the first can be equal to any multiple of the second, and therefore the quantuplicity of one with respect to the other cannot exactly be expressed numerically: in this case it is determined by examining how the multiples of one magnitude are distributed among the multiples of the other.

Thus, let all the multiples of A be formed, the scale extending *ad infinitum;* also let all the multiples of B be formed and placed in their proper order of magnitude among the multiples of A. This forms the relative scale of the two magnitudes, and the quantuplicity of A with respect to B is estimated by examining how the multiples of A are distributed among those of B in their relative scale.

In other words, the ratio of A to B is known, if for all integral values of m we know the multiples nB and $(n+1)B$ between which mA lies.

In the case of two given magnitudes A and B, the relative scale of multiples is definite, and is different from that of A to C, if C differs from B by any magnitude however small.

For let D be the difference between B and C; then however small D may be, it will be possible to find a number m such that $mD>A$. In this case, mB and mC would differ by a magnitude greater than A, and therefore could not lie between the same two multiples of A; so that after a certain point the relative scale of A and B would differ from that of A and C.

Definition 4. **Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.**

Definition 5. The ratio of one magnitude to another is equal to that of a third magnitude to a fourth, when if any equimultiples whatever of the antecedents of the ratios are taken, and also any equimultiples whatever of the consequents, the multiple of one antecedent is greater than, equal to, or less than that of its consequent, according as the multiple of the other antecedent is greater than, equal to, or less than that of its consequent.

Thus the ratio A to B is equal to that of C to D when $mC >, =$, or $< nD$ according as $mA >, =$, or $< nB$, whatever whole numbers m and n may be.

Again, let m be any whole number whatever, and n another whole number determined in such a way that either mA is equal to nB, or mA lies between nB and $(n+1)B$; then the definition asserts that the ratio of A to B is equal to that of C to D if $mC=nD$ when $mA=nB$; or if mC lies between nD and $(n+1)D$ when mA lies between nB and $(n+1)B$.

In other words, the ratio of A to B is equal to that of C to D when the multiples of A are distributed among those of B in the same manner as the multiples of C are distributed among those of D.

When the ratio of A to B is equal to that of C to D the four magnitudes are called **proportionals**. This is expressed by saying "*A is to B as C is to D*," and the **proportion** is written

$$A : B :: C : D, \text{ or } A : B = C : D.$$

A and D are called the **extremes**, B and C the **means**; also D is said to be a **fourth proportional** to A, B, and C.

Definition 6. Two terms in a proportion are said to be **homologous** when they are both antecedents, or both consequents of the ratios.

Definition 7. The ratio of one magnitude to another is greater than that of a third magnitude to a fourth, when it is possible to find equimultiples of the antecedents and equimultiples of the consequents such that while the multiple of the antecedent of the first ratio is greater than, or equal to, that of its consequent, the multiple of the antecedent of the second is not greater, or is less, than that of its consequent.

This definition asserts that if whole numbers m and n can be found such that while mA is greater than nB, mC is not greater than nD, or while $mA=nB$, mC is less than nD, then the ratio of A to B is greater than that of C to D.

If A is equal to B, the ratio of A to B is called a **ratio of equality**.

If A is greater than B, the ratio of A to B is called a **ratio of greater inequality**.

If A is less than B, the ratio of A to B is called a **ratio of less inequality**.

Definition 8. Two ratios are said to be **reciprocal** when the antecedent and consequent of one are the consequent and antecedent of the other respectively; thus $B : A$ is the reciprocal of $A : B$.

Definition 9. Three magnitudes of the same kind are said to be proportionals, when the ratio of the first to the second is equal to that of the second to the third.

Thus A, B, C are proportionals if

$$A : B :: B : C.$$

B is called a **mean proportional** to A and C, and C is called a **third proportional** to A and B.

Definition 10. Three or more magnitudes are said to be **in continued proportion** when the ratio of the first to the second is equal to that of the second to the third, and the ratio of the second to the third is equal to that of the third to the fourth, and so on.

Definition 11. When there are any number of magnitudes of the same kind, the first is said to have to the last the **ratio compounded** of the ratios of the first to the second, of the second to the third, and so on up to the ratio of the last but one to the last magnitude.

For example, if A, B, C, D, E be magnitudes of the same kind, $A : E$ is the ratio compounded of the ratios $A : B$, $B : C$, $C : D$, and $D : E$.

This is sometimes expressed by the following notation:

$$A : E = \begin{cases} A : B \\ B : C \\ C : D \\ D : E. \end{cases}$$

Definition 12. If there are any number of ratios, and a set of magnitudes is taken such that the ratio of the first to the second is equal to the first ratio, and the ratio of the second to the third is equal to the second ratio, and so on, then the first of the set of magnitudes is said to have to the last the **ratio compounded** of the given ratios.

Thus, if $A : B$, $C : D$, $E : F$ be given ratios, and if P, Q, R, S be magnitudes taken so that

$$P : Q :: A : B,$$
$$Q : R :: C : D,$$
$$R : S :: E : F;$$

then

$$P : S = \begin{cases} A : B \\ C : D \\ E : F. \end{cases}$$

Definition 13. When three magnitudes are proportionals, the first is said to have to the third the **duplicate ratio** of that which it has to the second.

Thus if $$A : B :: B : C,$$
then A is said to have to C the duplicate ratio of that which it has to B.

Since $$A : C = \begin{cases} A : B \\ B : C, \end{cases}$$
it is clear that the ratio compounded of two equal ratios is the duplicate ratio of either of them.

Definition 14. When four magnitudes are in *continued proportion*, the first is said to have to the fourth the **triplicate ratio** of that which it has to the second.

It may be shewn as above that the ratio compounded of three equal ratios is the triplicate ratio of any one of them.

PROPOSITIONS.

Obs. Of the propositions of Book V., which, it may be noticed are all theorems, we here give only the more important.

Proposition 1.

Ratios which are equal to the same ratio are equal to one another.

Let $A : B :: P : Q$, and also $C : D :: P : Q$; then shall $A : B :: C : D$.

For it is evident that two scales or arrangements of multiples which agree in every respect with a third scale, will agree with one another.

Proposition 2.

If two ratios are equal, the antecedent of the second is greater than, equal to, or less than its consequent according as the antecedent of the first is greater than, equal to, or less than its consequent.

Let $$A : B :: C : D,$$
then $$C >, =, \text{ or } < D,$$
according as $$A >, =, \text{ or } < B.$$

This follows at once from Def. 5, by taking m and n each equal to unity.

PROPOSITION 3.

Invertendo or Inversely. *If two ratios are equal, their reciprocal ratios are equal.*

Let $A : B :: C : D,$

then shall $B : A :: D : C.$

For, by hypothesis, the multiples of A are distributed among those of B in the same manner as the multiples of C are among those of D.

therefore also, the multiples of B are distributed among those of A in the same manner as the multiples of D are among those of C.

That is, $B : A :: D : C.$

NOTE. This proposition is sometimes enunciated thus:

If four magnitudes are proportionals, they are also proportionals when taken inversely.

PROPOSITION 4.

Equal magnitudes have the same ratio to the same magnitude; and the same magnitude has the same ratio to equal magnitudes.

Let A, B, C be three magnitudes of the same kind, and let A be equal to B;

then shall $A : C :: B : C$

and $C : A :: C : B.$

Since $A = B$, their multiples are identical and therefore are distributed in the same way among the multiples of C.

$\therefore$ $A : C :: B : C,$ *Def.* 5.

$\therefore$ also, *invertendo*, $C : A :: C : B.$ v. 3.

PROPOSITION 5.

Of two unequal magnitudes, the greater has a greater ratio to a third magnitude than the less has; and the same magnitude has a greater ratio to the less of two magnitudes than it has to the greater.

First, let A be $> B$;

then shall $A : C$ be $> B : C$.

Since $A > B$, it will be possible to find m such that mA exceeds mB by a magnitude greater than C;

hence if mA lies between nC and $(n+1)C$, $mB < nC$:

and if $mA = nC$, then $mB < nC$;

$\therefore$ $A : C > B : C.$ *Def.* 7.

Secondly, let B be $< A$;
then shall $C : B$ be $> C : A$.
For taking m and n as before,

$nC > mB$, while nC is not $> mA$;

$\therefore\ C : B > C : A$. *Def.* 7.

Proposition 6.

Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.

First, let $A : C :: B : C$;
then shall $A = B$.

For if $A > B$, then $A : C > B : C$,
and if $B > A$, then $B : C > A : C$, v. 5.

which contradict the hypothesis;

$\therefore\ A = B$.

Secondly, let $C : A :: C : B$;
then shall $A = B$.

Because $C : A :: C : B$,
$\therefore$ *invertendo,* $A : C :: B : C$, v. 3.

$\therefore\ A = B$,

by the first part of the proof.

Proposition 7.

That magnitude which has a greater ratio than another has to the same magnitude is the greater of the two; and that magnitude to which the same has a greater ratio than it has to another magnitude is the less of the two.

First, let $A : C$ be $> B : C$;
then shall A be $> B$.

For if $A = B$, then $A : C :: B : C$, v. 4.
which is contrary to the hypothesis.

And if $A < B$, then $A : C < B : C$; v. 5.
which is contrary to the hypothesis;

$\therefore\ A > B$.

Secondly, let $C : A$ be $> C : B$;
then shall A be $< B$.

For if $A = B$, then $C : A :: C : B$, v. 4.
which is contrary to the hypothesis.

And if $A > B$, then $C : A < C : B$; v. 5.
which is contrary to the hypothesis;

$\therefore\ A < B$.

Proposition 8.

Magnitudes have the same ratio to one another which their equimultiples have.

Let A, B be two magnitudes;
then shall $A : B :: mA : mB$.

If p, q be any two whole numbers,
then $m \,.\, pA >, =,$ or $< m \,.\, qB$
according as $pA >, =,$ or $< qB$.
But $m \,.\, pA = p \,.\, mA$, and $m \,.\, qB = q \,.\, mB$;
$\therefore\ p \,.\, mA >, =,$ or $< q \,.\, mB$
according as $pA >, =,$ or $< qB$;
$\therefore\ A : B :: mA : mB$. *Def.* 5.

Cor. Let $A : B :: C : D$.
Then since $A : B :: mA : mB$,
and $C : D :: nC : nD$;
$\therefore\ mA : mB :: nC : nD$. v. 1.

Proposition 9.

If two ratios are equal, and any equimultiples of the antecedents and also of the consequents are taken, the multiple of the first antecedent has to that of its consequent the same ratio as the multiple of the other antecedent has to that of its consequent.

Let $A : B :: C : D$;
then shall $mA : nB :: mC : nD$.

Let p, q be any two whole numbers;
then because $A : B :: C : D$,
$pm \,.\, C >, =,$ or $< qn \,.\, D$
according as $pm \,.\, A >, =,$ or $< qn \,.\, B$, *Def.* 5.
that is, $p \,.\, mC >, =,$ or $< q \,.\, nD$,
according as $p \,.\, mA >, =,$ or $< q \,.\, nB$;
$\therefore\ mA : nB :: mC : nD$. *Def.* 5.

Proposition 10.

If four magnitudes of the same kind are proportionals, the first is greater than, equal to, or less than the third, according as the second is greater than, equal to, or less than the fourth.

Let A, B, C, D be four magnitudes of the same kind such that

$$A : B :: C : D;$$

then $A >, =, \text{ or } < C$

according as $B >, =, \text{ or } < D$.

If $B > D$, then $A : B < A : D$; v. 5

but $A : B :: C : D$;

$\therefore \; C : D < A : D$;

$\therefore \; A : D > C : D$;

$\therefore \; A > C$. v. 7.

Similarly it may be shewn that

if $B < D$, then $A < C$,

and if $B = D$, then $A = C$.

Proposition 11.

Alternando or Alternately. *If four magnitudes of the same kind are proportionals, they are also proportionals when taken alternately.*

Let A, B, C, D be four magnitudes of the same kind such that

$$A : B :: C : D;$$

then shall $A : C :: B : D$.

Because $A : B :: mA : mB$, v. 8.

and $C : D :: nC : nD$;

$\therefore \; mA : mB :: nC : nD$. v. 1.

$\therefore \; mA >, =, \text{ or } < nC$

according as $mB >, =, \text{ or } < nD$. v. 10.

And m and n are any whole numbers;

$\therefore \; A : C :: B : D$. *Def.* 5.

Proposition 12.

Addendo. *If any number of magnitudes of the same kind are proportionals, as one of the antecedents is to its consequent, so is the sum of the antecedents to the sum of the consequents.*

Let A, B, C, D, E, F, ... be magnitudes of the same kind such that

$$A : B :: C : D :: E : F :: \ldots\ldots;$$

then shall $A : B :: A+C+E+\ldots : B+D+F+\ldots$.

Because $A : B :: C : D :: E : F :: \ldots$,

$\therefore$ according as $mA >, =,$ or $< nB$,

so is $mC >, =,$ or $< nD$,

and $mE >, =,$ or $< nF$,

....

$\therefore$ so is $mA+mC+mE+\ldots >, =,$ or $< nB+nD+nF+\ldots$

or $m(A+C+E+\ldots) >, =,$ or $< n(B+D+F+\ldots)$;

and m and n are any whole numbers;

$\therefore$ $A : B :: A+C+E+\ldots : B+D+F+\ldots$. *Def.* 5.

Proposition 13.

Componendo. *If four magnitudes are proportionals, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.*

Let $A : B :: C : D$;

then shall $A+B : B :: C+D : D$.

If m be any whole number, it is possible to find another number n such that $mA=nB$, or lies between nB and $(n+1)B$,

$\therefore$ $mA+mB=mB+nB$, or lies between $mB+nB$ and $mB+(n+1)B$.

But $mA+mB=m(A+B)$, and $mB+nB=(m+n)B$;

$\therefore$ $m(A+B)=(m+n)B$, or lies between $(m+n)B$ and $(m+n+1)B$.

Also because $A : B :: C : D$,

$\therefore$ $mC=nD$, or lies between nD and $(n+1)D$; *Def.* 5.

$\therefore$ $m(C+D)=(m+n)D$ or lies between $(m+n)D$ and $(m+n+1)D$;

that is, the multiples of $C+D$ are distributed among those of D in the same way as the multiples of $A+B$ among those of B;

$\therefore$ $A+B : B :: C+D : D$.

Dividendo. In the same way it may be proved that

$A-B : B :: C-D : D$,

or $B-A : B :: D-C : D$,

according as A is $>$ or $< B$.