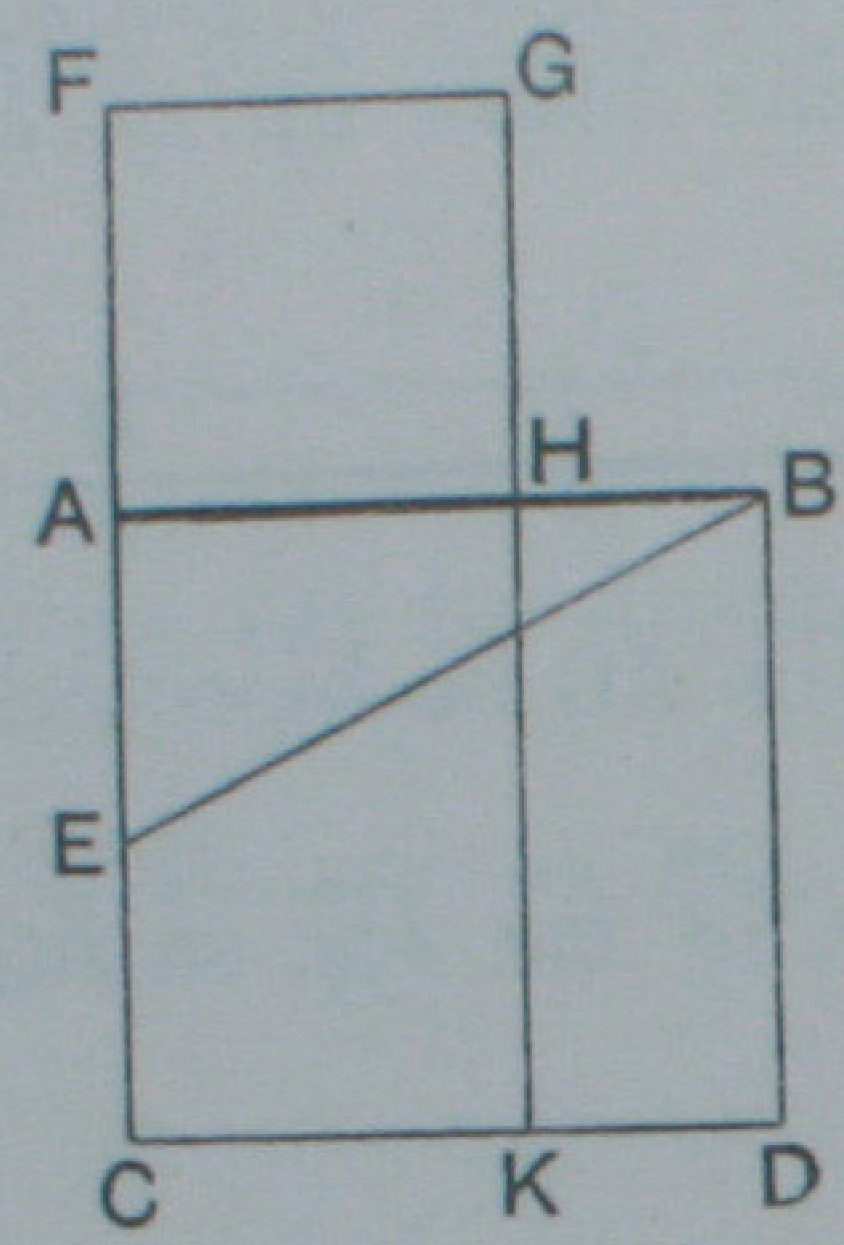
PROPOSITION 11. PROBLEM.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one part may be equal to the square on the other part.



Let AB be the given straight line.

It is required to divide AB *into two parts, so that the rectangle contained by the whole and one part may be equal to the square on the other part.*

Construction. On AB describe the square ACDB. I. 46.

Bisect AC at E. I. 10.

Join EB.

Produce CA to F, making EF equal to EB. I. 3.

On AF describe the square AFGH. I. 46.

Then shall AB *be divided at* H, *so that the rect.* AB, BH *is equal to the sq. on* AH.

Produce GH to meet CD in K.

Proof. Because CA is bisected at E, and produced to F,

∴ the rect. CF, FA with the sq. on EA = the sq. on EF II. 6.

= the sq. on EB. *Constr.*

But the sq. on EB = the sum of the sqq. on EA, AB,

for the angle EAB is a rt. angle. I. 47.

∴ the rect. CF, FA with the sq. on EA = the sum of the sqq. on EA, AB.

From these equals take the sq. on EA:

then the rect. CF, FA = the sq. on AB.

But the rect. CF, FA = the fig. FK; for FA = FG;
and the sq. on AB = the fig. AD. *Constr.*
∴ the fig. FK = the fig. AD.
From these equals take the common fig. AK;
then the remaining fig. FH = the remaining fig. HD.
But the fig. HD = the rect. AB, BH; for BD = AB;
and the fig. FH is the sq. on AH.
∴ the rect. AB, BH = the sq. on AH. Q.E.F.

DEFINITION. A straight line is said to be divided in **Medial Section** when the rectangle contained by the given line and one of its segments is equal to the square on the other segment.

The student should observe that this division may be *internal* or *external*.

Thus if the straight line AB is divided internally at H, and externally at H′, so that

(i) $AB \, . \, BH = AH^2$,
(ii) $AB \, . \, BH' = AH'^2$,

we shall in either case consider that AB is divided in medial section.

The case of *internal* section is alone given in Euclid II. 11; but a straight line may be divided *externally* in medial section by a similar process. See Ex. 21, p. 160.

ALGEBRAICAL ILLUSTRATION.

It is required to find a point H in AB, or AB produced, such that

$$AB \, . \, BH = AH^2.$$

Let AB contain a units of length, and let AH contain x units;

$$\text{then } BH = a - x:$$

and x must be such that $a(a - x) = x^2$,

or
$$x^2 + ax - a^2 = 0.$$

Thus the construction for dividing a straight line in medial section corresponds to the solution of this quadratic equation, the two roots of which indicate the *internal* and *external* points of division.

EXERCISES.

In the figure of II. 11, shew that

(i) if CH is produced to meet BF at L, CL is at right angles to BF;

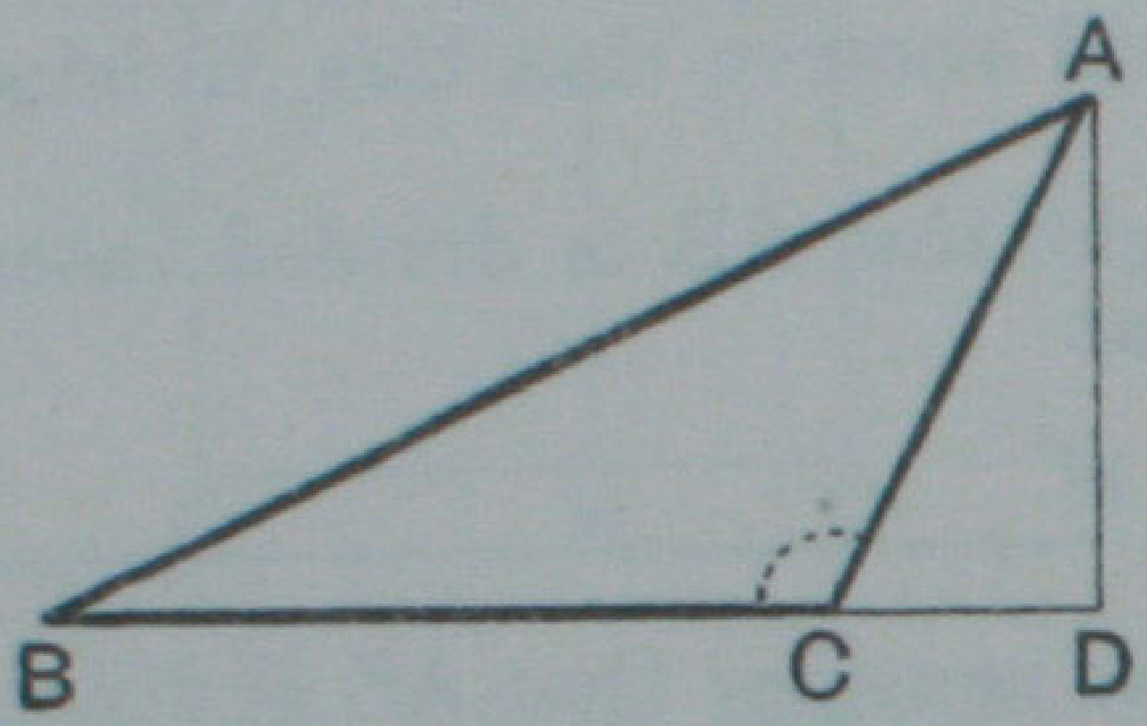
(ii) if BE and CH meet at O, AO is at right angles to CH.

(iii) the lines BG, DF, AK are parallel:

(iv) CF is divided in medial section at A.

PROPOSITION 12. THEOREM.

In an obtuse-angled triangle, if a perpendicular is drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the sum of the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the line intercepted without the triangle, between the perpendicular and the obtuse angle.



Let ABC be an obtuse-angled triangle, having the obtuse angle at C; and let AD be drawn from A perp. to the opp. side BC produced.

Then shall the sq. on AB *be greater than the sum of the sqq. on* BC, CA, *by twice the rect.* BC, CD.

Proof. Because BD is divided into two parts at C,

∴ the sq. on BD = the sum of the sqq. on BC, CD, with twice the rect. BC, CD. II. 4,

To each of these equals add the sq. on DA.

Then the sqq. on BD, DA = the sum of the sqq. on BC, CD, DA, with twice the rect. BC, CD.

But the sum of the sqq. on BD, DA = the sq. on AB,
for the angle at D is a rt. angle. I. 47.

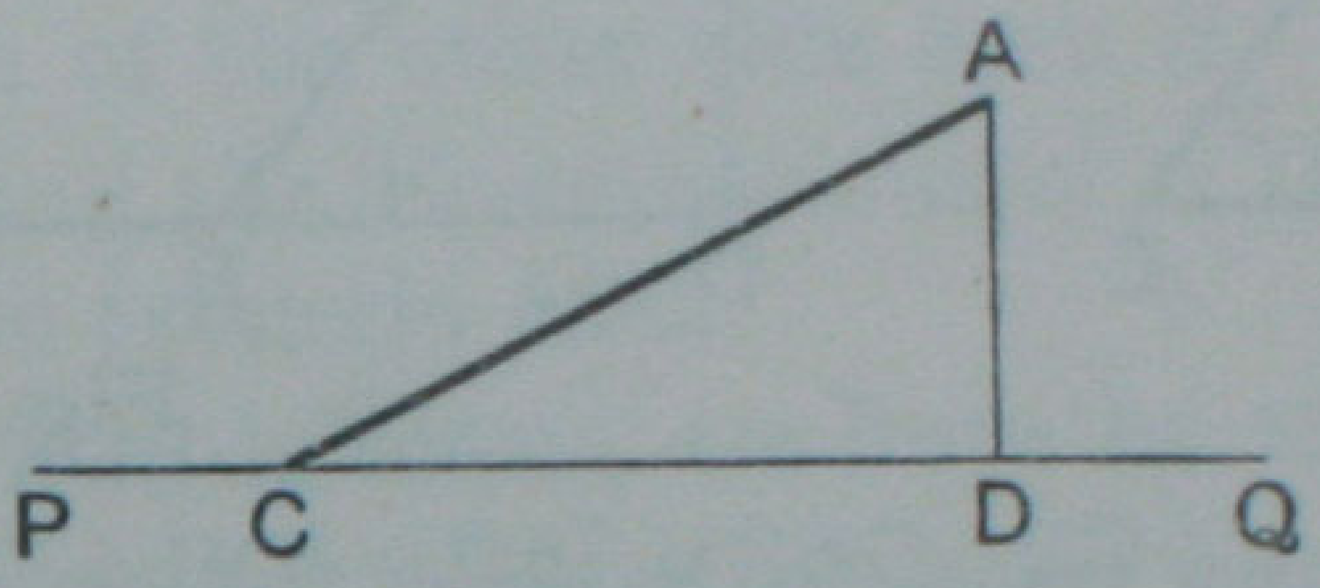
Similarly, the sum of the sqq. on CD, DA = the sq. on CA.

∴ the sq. on AB = the sum of the sqq. on BC, CA, with twice the rect. BC, CD.

That is, the sq. on AB is greater than the sum of the sqq. on BC, CA by twice the rect. BC, CD. Q.E.D.

NOTE ON PROP. 12.

A general definition of the **projection of one straight line on another** is given on page 105. The student's attention is here called to a special case of projection which will enable us to simplify the Enunciation of Proposition 12.



In the above diagram, CA is a given straight line drawn from a point C in PQ; and from A a perpendicular AD is drawn to PQ. In this case, CD is said to be the **projection** of CA on PQ.

By applying this definition to the figure of Prop. 12, we see that the statement

The sq. on AB *is greater than the sum of the sqq. on* BC, CA *by twice the rect.* BC, CD

is the particular form of the following general Enunciation:

In an obtuse-angled triangle the square on the side opposite the obtuse angle is greater than the sum of the squares on the sides containing the obtuse angle by twice the rectangle contained by one of those sides, and the **projection of the other side** *upon it.*

The Enunciation of Prop. 12 thus stated should be carefully compared with that of Prop. 13.

Proposition 13. Theorem.

In every triangle, the square on the side subtending an acute angle is less than the sum of the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle, and the acute angle.

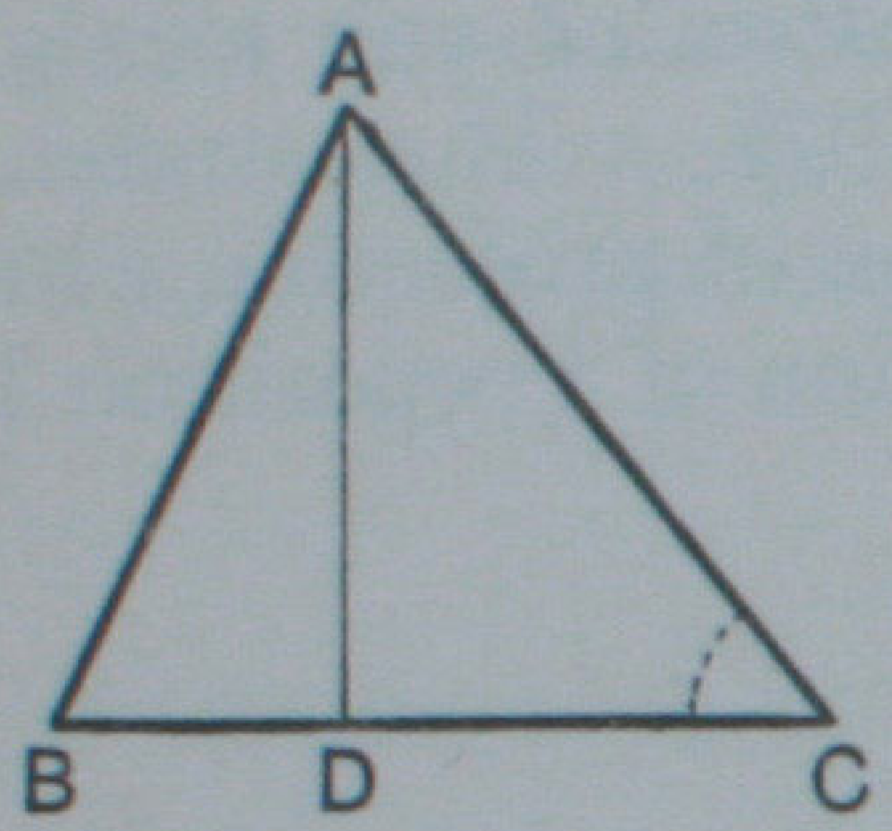


Fig. 1.

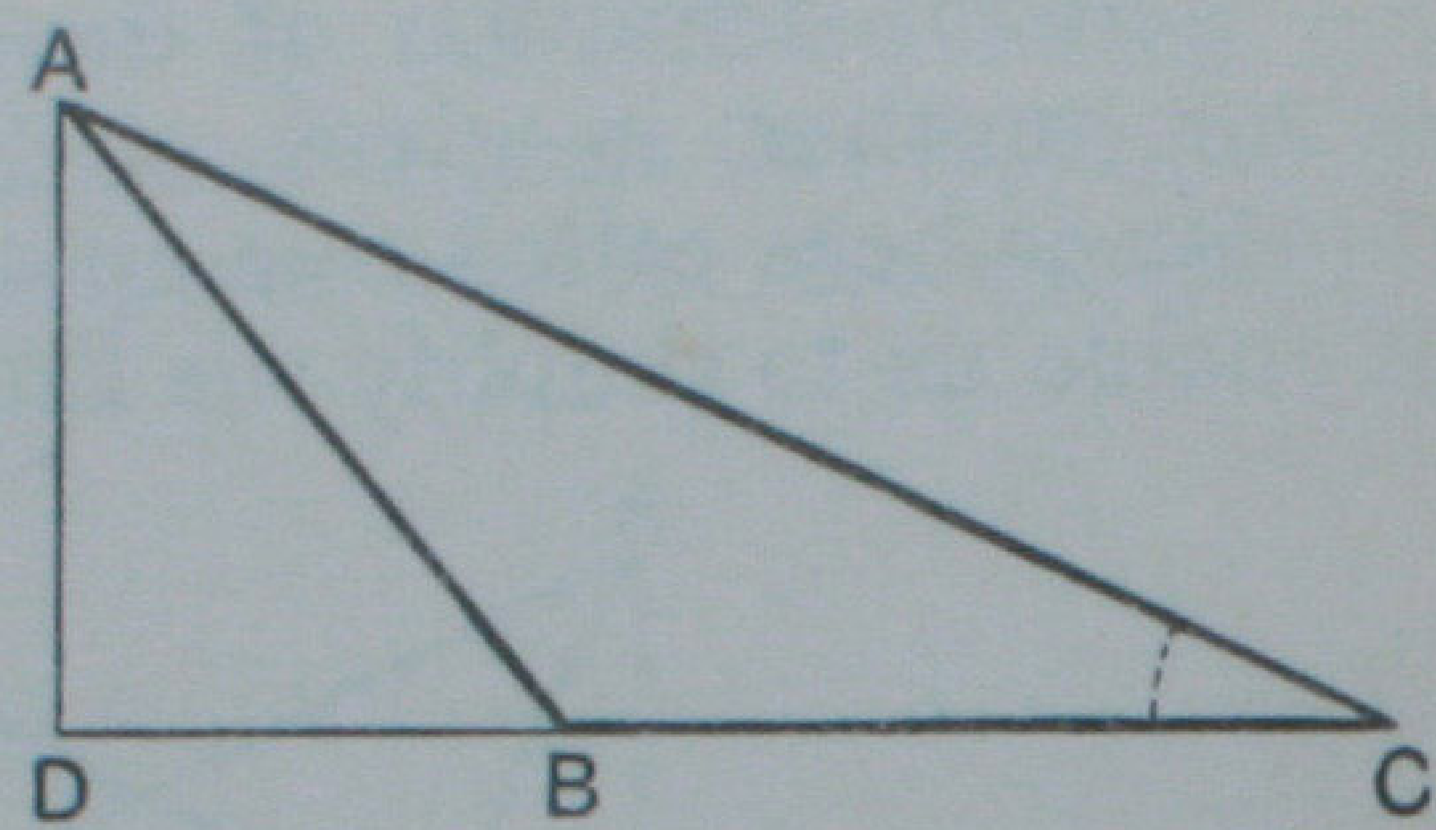


Fig. 2.

Let ABC be any triangle having the angle at C an acute angle; and let AD be the perp. drawn from A to the opp. side BC.

Then shall the sq. on AB *be less than the sum of the sqq. on* BC, CA, *by twice the rect.* BC, CD.

Proof. Now AD may fall within the triangle ABC, as in fig. 1, or without it, as in fig. 2.

Because {in fig. 1, BC is divided into two parts at D,
in fig. 2, DC is divided into two parts at B,

∴ *in both cases*

the sum of the sqq. on BC, CD = twice the rect. BC, CD with the sq. on BD. II. 7.

To each of these equals add the sq. on DA.

Then the sum of the sqq. on BC, CD, DA = twice the rect. BC, CD with the sum of the sqq. on BD, DA.

But the sum of the sqq. on CD, DA = the sq. on CA, I. 47.
for the angle ADC is a rt. angle.

Similarly, the sum of the sqq. on BD, DA = the sq. on AB.

∴ the sum of the sqq. on BC, CA = twice the rect. BC, CD with the sq. on AB.

That is, the sq. on AB is less than the sum of the sqq. on BC, CA by twice the rect. BC, CD. Q.E.D.

Obs. If the perpendicular AD *coincides* with AB, that is, if ABC is a right angle, then *twice the rect.* BC, CD becomes *twice the sq. on* BC; and it may be shewn that the proposition merely repeats the result of I. 47.

NOTES ON PROP. 13.

(i) Remembering the definition of the **projection** of a straight line given on p. 153, we may enunciate Prop. 13 as follows;

In every triangle, the square on the side subtending an acute angle is less than the sum of the squares on the sides containing that angle, by twice the rectangle contained by one of these sides and the **projection of the other side** *upon it.*

(ii) Comparing the Enunciations of II. 12, I. 47, II. 13, we see that in the triangle ABC,

if the angle ACB is *obtuse*, we have by II. 12,

$$AB^2 = BC^2 + CA^2 + 2\,BC\,.\,CD;$$

if the angle ACB is *a right angle*, we have by I. 47,

$$AB^2 = BC^2 + CA^2;$$

if the angle ACB is *acute*, we have by II. 13,

$$AB^2 = BC^2 + CA^2 - 2\,BC\,.\,CD.$$

These results may be collected as follows:

The square on a side of a triangle is greater than, equal to, or less than the sum of the squares on the other sides, according as the angle opposite to the first is obtuse, a right angle, or acute.

EXERCISES ON II. 12 AND 13.

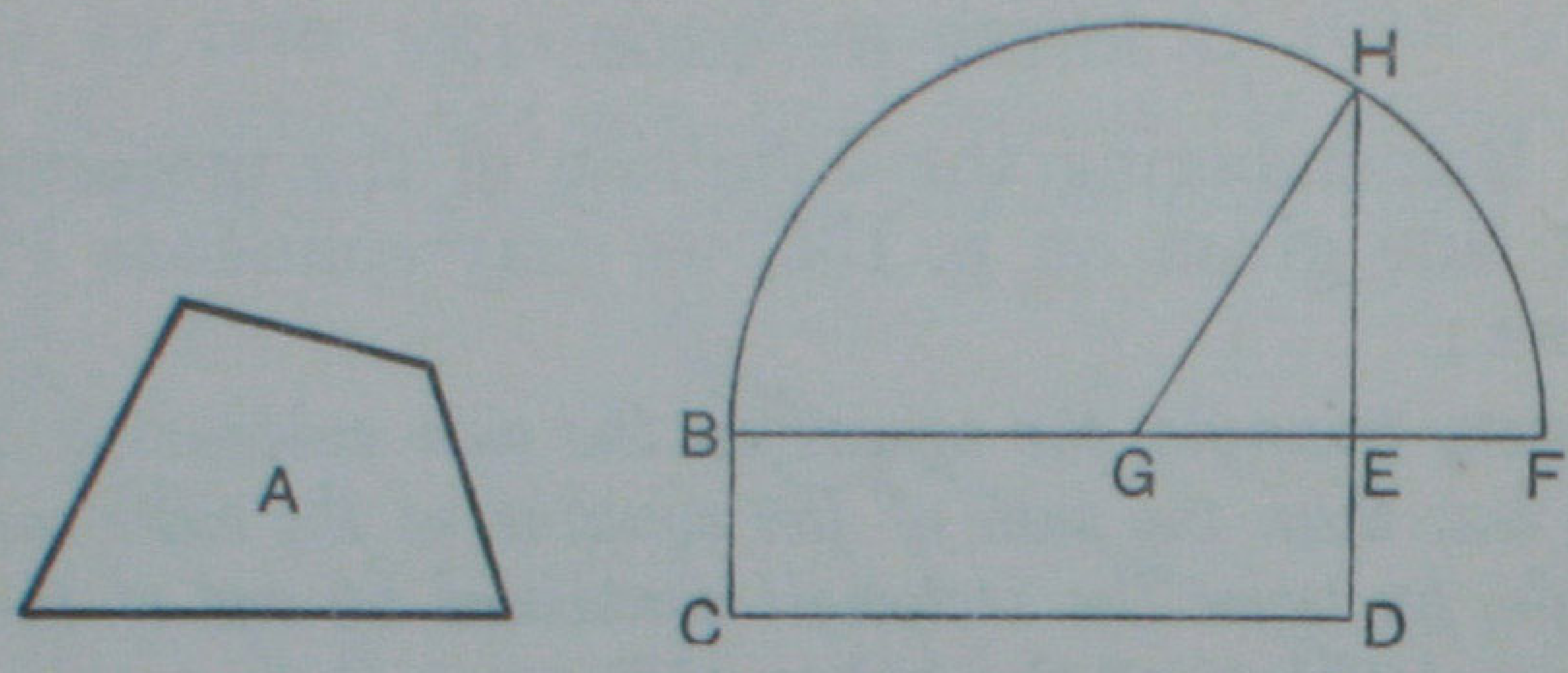
1. If from one of the base angles of an isosceles triangle a perpendicular is drawn to the opposite side, then twice the rectangle contained by that side and the segment adjacent to the base is equal to the square on the base.

2. If one angle of a triangle is one-third of two right angles, shew that the square on the opposite side is less than the sum of the squares on the sides forming that angle, by the rectangle contained by these two sides. [See Ex. 10, p. 109.]

3. If one angle of a triangle is two-thirds of two right angles, shew that the square on the opposite side is greater than the sum of the squares on the sides forming that angle, by the rectangle contained by these sides. [See Ex. 10, p. 109.]

PROPOSITION 14. PROBLEM.

To describe a square that shall be equal to a given rectilineal figure.



Let A be the given rectilineal figure.
It is required to describe a square equal to A.

Construction. Describe a par^{m} BCDE equal to the fig. A, and having the angle CBE a right angle. I. 45.

Then if BC = BE, the fig. BD is a square; and what was required is done.

But if not, produce BE to F, making EF equal to ED; I. 3.
and bisect BF at G. I. 10.

With centre G, and radius GF, describe the semicircle BHF;
produce DE to meet the semicircle at H.

Then shall the sq. on EH *be equal to the given fig.* A.

Join GH.

Proof. Because BF is divided equally at G and unequally at E,

∴ the rect. BE, EF with the sq. on GE = the sq. on GF II. 5.
= the sq. on GH.

But the sq. on GH = the sum of the sqq. on GE, EH;
for the angle HEG is a rt. angle. I. 47.

∴ the rect. BE, EF with the sq. on GE = the sum of the sqq. on GE, EH.

From these equals take the sq. on GE:
then the rect. BE, EF = the sq. on HE.

But the rect. BE, EF = the fig. BD; for EF = ED; *Constr.*
and the fig. BD = the given fig. A. *Constr.*
∴ the sq. on EH = the given fig. A. Q.E.F.

QUESTIONS FOR REVISION ON BOOK II.

1. Explain the phrase, *the rectangle contained by* AB, CD; and shew by superposition that if AB=PQ, and CD=RS, then the rectangle contained by AB, CD=the rectangle contained by PQ, RS.

2. Shew that Prop. 2 is a *special case* of Prop. 1, explaining under what conditions Prop. 1 becomes identical with Prop. 2.

3. What must be the relation between the divided and undivided lines in the enunciation of Prop. 1 in order to give the result proved in Prop. 3?

4. Define the *segments into which a straight line is divided at a point* in such a way as to be applicable to the case when the dividing point is in the given line produced.

Hence frame a statement which includes the enunciations of both II. 5 and II. 6, and find the algebraical formulae corresponding to these enunciations.

Also combine in a single enunciation the results of I. 9 and II. 10.

5. Compare the results proved in Propositions 4 and 7 by finding the algebraical formulae corresponding to their enunciations.

6. *The difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference.* Deduce this theorem from Prop. 5.

7. Define *the projection of one straight line on another.*

How may the enunciations of II. 12 and II. 13 be simplified by means of this definition?

8. In the figure of Proposition 14,

(i) If BE=8 inches, and ED=2 inches, find the length of EH.

(ii) If BE=12·5 inches, and EH=2·5 inches, find the length of ED.

(iii) If BE=9 inches, and EH=3 inches, find the length of GH.

9. When is a straight line said to be divided in *medial section*?

If a straight line 8 inches in length is divided internally in medial section, shew that the lengths of the segments are approximately 4·9 inches and 3·1 inches.

[Frame a quadratic equation as explained on page 151, and solve.]

THEOREMS AND EXAMPLES ON BOOK II.

ON II. 4 AND 7.

1. *Shew by* II. 4 *that the square on a straight line is four times the square on half the line.*

[This result is constantly used in solving examples on Book II., especially those which follow from II. 12 and 13.]

2. If a straight line is divided into any three parts, the square on the whole line is equal to the sum of the squares on the three parts together with twice the rectangles contained by each pair of these parts.

Shew that the algebraical formula corresponding to this theorem is

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2bc + 2ca + 2ab.$$

3. *In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the square on this perpendicular is equal to the rectangle contained by the segments of the hypotenuse.*

4. In an isosceles triangle, if a perpendicular is drawn from one of the angles at the base to the opposite side, shew that the square on the perpendicular is equal to twice the rectangle contained by the segments of that side together with the square on the segment adjacent to the base.

5. Any rectangle is half the rectangle contained by the diagonals of the squares described upon its two sides.

6. In any triangle if a perpendicular is drawn from the vertical angle to the base, the sum of the squares on the sides forming that angle, together with twice the rectangle contained by the segments of the base, is equal to the square on the base together with twice the square on the perpendicular.

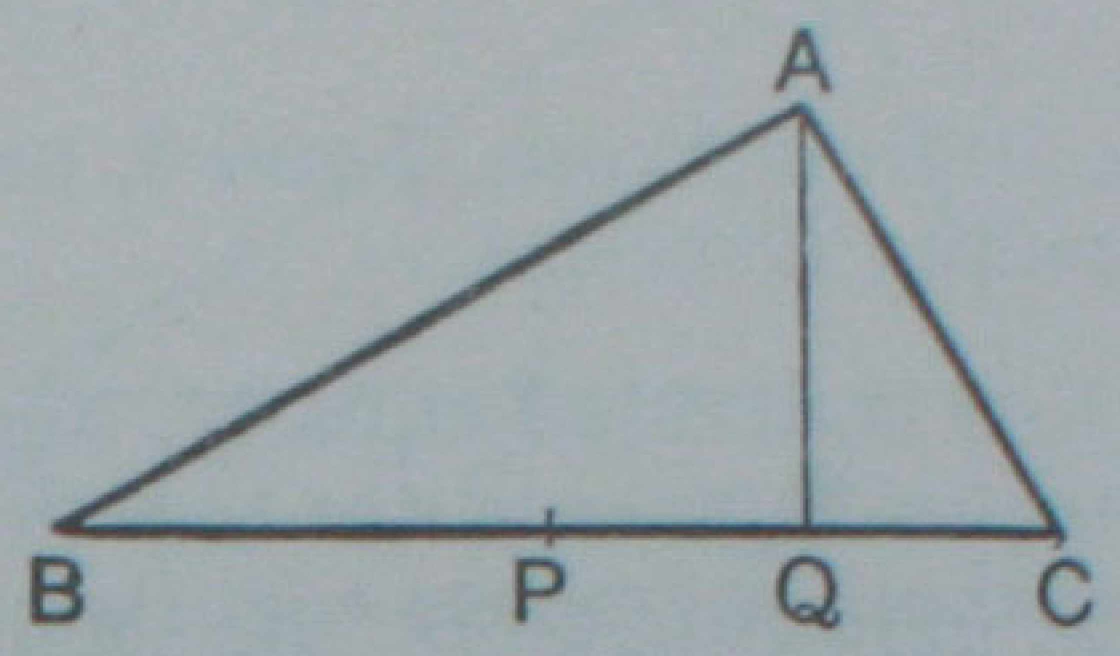
ON II. 5 AND 6.

Obs. The student is reminded that these important propositions are both included in the following enunciation:

The difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference. [See Cor., p. 137].

7. In a right-angled triangle the square on one of the sides forming the right angle is equal to the rectangle contained by the sum and difference of the hypotenuse and the other side. [I. 47 and II. 5, Cor.]

8. *The difference of the squares on two sides of a triangle is equal to twice the rectangle contained by the base and the intercept between the middle point of the base and the foot of the perpendicular drawn from the vertical angle to the base.*



Let ABC be a triangle, and let P be the middle point of the base BC: let AQ be drawn perp. to BC.

Then shall $AB^2 - AC^2 = 2BC \cdot PQ$.

First, let AQ fall within the triangle.

Now $AB^2 = BQ^2 + QA^2$, I. 47.

also $AC^2 = QC^2 + QA^2$,

$\therefore AB^2 - AC^2 = BQ^2 - QC^2$ Ax. 3.

$= (BQ + QC)(BQ - QC)$ II. 5, Cor.

$= BC \cdot 2PQ$ Ex., p. 137.

$= 2BC \cdot PQ$ Q.E.D.

The case in which AQ falls outside the triangle presents no difficulty.

9. *The square on any straight line drawn from the vertex of an isosceles triangle to the base is less than the square on one of the equal sides by the rectangle contained by the segments of the base.*

10. The square on any straight line drawn from the vertex of an isosceles triangle to the base produced, is greater than the square on one of the equal sides by the rectangle contained by the *segments into which the base is divided externally.*

11. If a straight line is drawn through one of the angles of an equilateral triangle to meet the opposite side produced, so that the rectangle contained *by the segments of the base* is equal to the square on the side of the triangle; shew that the square on the line so drawn is double of the square on a side of the triangle.

12. If XY is drawn parallel to the base BC of an isosceles triangle ABC, then the difference of the squares on BY and CY is equal to the rectangle contained by BC, XY. [See above, Ex. 8.]

13. In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the square on either side forming the right angle is equal to the rectangle contained by the hypotenuse and the segment of it adjacent to that side.

ON II. 9 AND 10.

14. Deduce Prop. 9 from Props. 4 and 5, using also the theorem that the square on a straight line is four times the square on half the line.

15. Deduce Prop. 10 from Props. 7 and 6, using also the theorem mentioned in the preceding Exercise.

16. If a straight line is divided equally, and also unequally, the squares on the two unequal segments are together equal to twice the rectangle contained by these segments together with four times the square on the line between the points of section.

ON II. 11.

17. *If a straight line is divided internally in medial section, and from the greater segment a part be taken equal to the less, shew that the greater segment is also divided in medial section.*

18. If a straight line is divided in medial section, the rectangle contained by the sum and difference of the segments is equal to the rectangle contained by the segments.

19. If AB is divided at H in medial section, and if X is the middle point of the greater segment AH, shew that a triangle whose sides are equal to AH, XH, BX respectively must be right-angled.

20. If a straight line AB is divided internally in medial section at H, prove that the sum of the squares on AB, BH is three times the square on AH.

21. *Divide a straight line externally in medial section.*

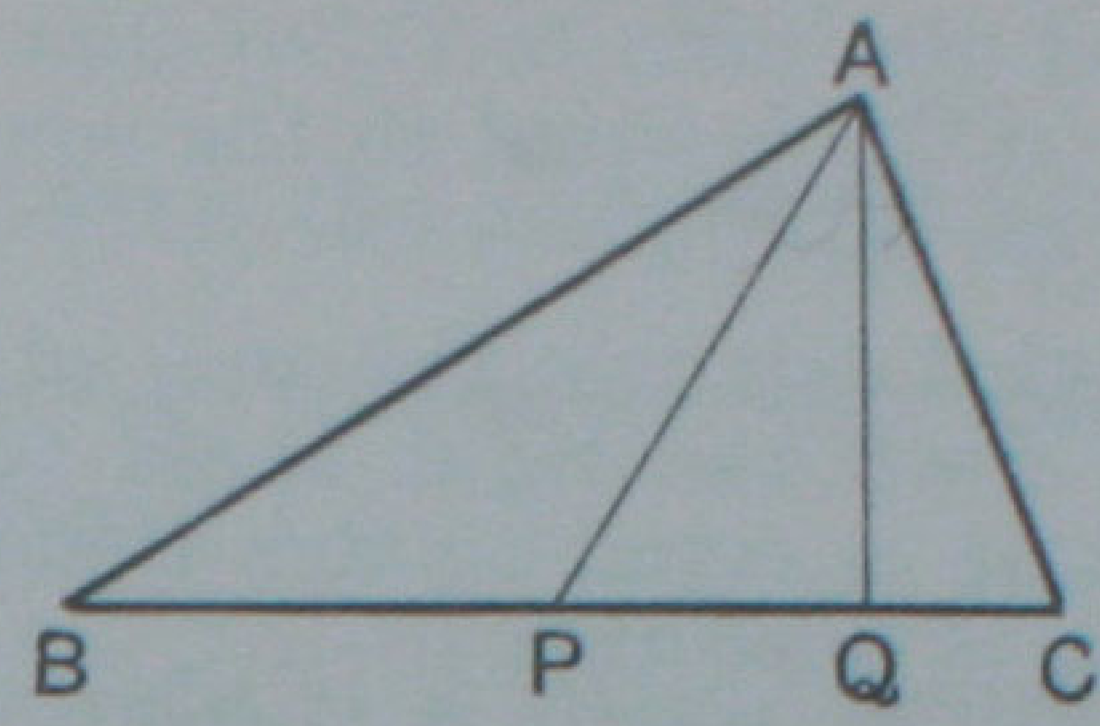
[Proceed as in II. 11, but instead of drawing EF, make EF′ equal to EB in the direction remote from A; and on AF′ describe the square AF′G′H′ on the side remote from AB. Then AB will be divided externally at H′ as required.]

ON II. 12 AND 13.

22. In a triangle ABC the angles at B and C are acute: if E and F are the feet of perpendiculars drawn from the opposite angles to the sides AC, AB, shew that the square on BC is equal to the sum of the rectangles AB, BF and AC, CE.

23. ABC is a triangle right-angled at C, and DE is drawn from a point D in AC perpendicular to AB: shew that the rectangle AB, AE is equal to the rectangle AC, AD.

24. *In any triangle the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.*



Let ABC be a triangle, and AP the median bisecting the side BC.

Then shall $AB^2 + AC^2 = 2BP^2 + 2AP^2$.

Draw AQ perp. to BC.

Consider the case in which AQ falls within the triangle, but does not coincide with AP.

Now of the angles APB, APC, one must be obtuse, and the other acute: let APB be obtuse.

Then in the △ APB, $AB^2 = BP^2 + AP^2 + 2BP . PQ$, II. 12.

Also in the △ APC, $AC^2 = CP^2 + AP^2 - 2CP . PQ$. II. 13.

But CP = BP,

∴ $CP^2 = BP^2$; and the rect. BP, PQ = the rect. CP, PQ,

Hence adding the above results,

$AB^2 + AC^2 = 2 . BP^2 + 2 . AP^2$. Q.E.D.

The student will have no difficulty in adapting this proof to the cases in which AQ falls without the triangle, or coincides with AP.

25. *The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.*

26. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides. [See Ex. 9, p. 105.]

27. If from any point within a rectangle straight lines are drawn to the angular points, the sum of the squares on one pair of the lines drawn to opposite angles is equal to the sum of the squares on the other pair.

28. The sum of the squares on the sides of a quadrilateral is greater than the sum of the squares on its diagonals by four times the square on the straight line which joins the middle points of the diagonals.

29. O is the middle point of a given straight line AB, and from O as centre, any circle is described: if P be any point on its circumference, shew that the sum of the squares on AP, BP *is constant.*

30. Given the base of a triangle, and the sum of the squares on the sides forming the vertical angle; find the locus of the vertex.

31. ABC is an isosceles triangle in which AB and AC are equal. AB is produced beyond the base to D, so that BD is equal to AB. Shew that the square on CD is equal to the square on AB together with twice the square on BC.

32. In a right-angled triangle the sum of the squares on the straight lines drawn from the right angle to the points of trisection of the hypotenuse is equal to five times the square on the line between the points of trisection.

33. Three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians.

34. ABC is a triangle, and O the point of intersection of its medians: shew that

$$AB^2+BC^2+CA^2=3(OA^2+OB^2+OC^2).$$

35. ABCD is a quadrilateral, and X the middle point of the straight line joining the bisections of the diagonals; with X as centre *any* circle is described, and P is any point upon this circle: shew that $PA^2+PB^2+PC^2+PD^2$ is *constant*, being equal to

$$XA^2+XB^2+XC^2+XD^2+4XP^2.$$

36. The squares on the diagonals of a trapezium are together equal to the sum of the squares on its two oblique sides, with twice the rectangle contained by its parallel sides.

PROBLEMS.

37. Construct a rectangle equal to the difference of two squares.

38. Divide a given straight line into two parts so that the rectangle contained by them may be equal to the square described on a given straight line which is less than half the straight line to be divided.

39. Given a square and one side of a rectangle which is equal to the square, find the other side.

40. Produce a given straight line so that the rectangle contained by the whole line thus produced and the part produced, may be equal to the square on another given line.

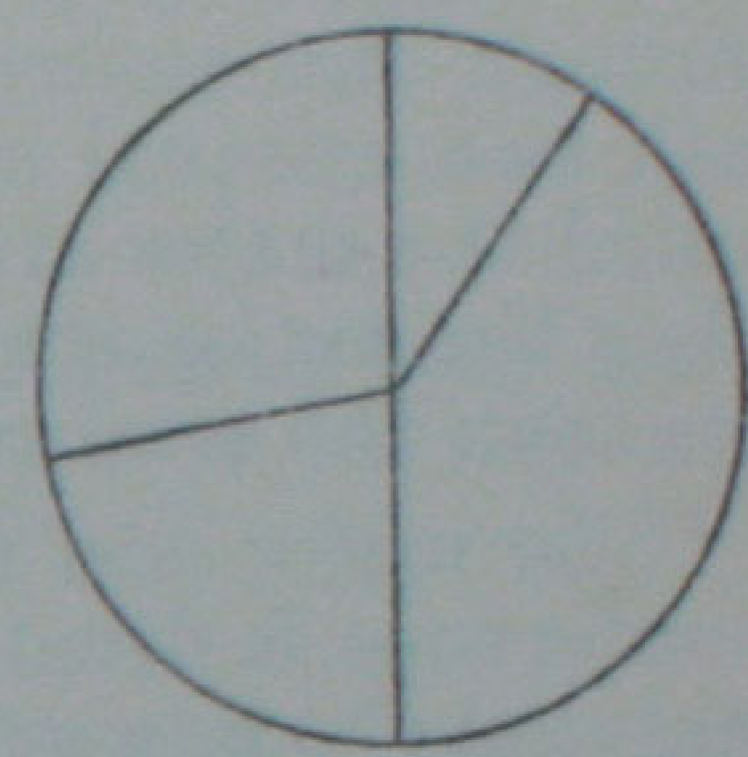
41. Produce a given straight line so that the rectangle contained by the whole line thus produced and the given line shall be equal to the square on the part produced.

42. Divide a straight line AB into two parts at C, such that the rectangle contained by BC and another line X may be equal to the square on AC.

BOOK III.

Book III. deals with the properties of Circles.

For convenience of reference the following definitions are repeated from Book I.



I. *Def.* 15. A **circle** is a plane figure bounded by one line, which is called the **circumference**, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another: this point is called the **centre** of the circle.

NOTE. Circles which have the same centre are said to be **concentric**.

I. *Def.* 16. A **radius** of a circle is a straight line drawn from the centre to the circumference.

I. *Def.* 17. A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

I. *Def.* 18. A **semicircle** is the figure bounded by a diameter of a circle and the part of the circumference cut off by the diameter.

NOTE. From these definitions we draw the following inferences:

(i) The distance of a point from the centre of a circle is less than the radius, if the point is within the circumference: and the distance of a point from the centre is greater than the radius, if the point is without the circumference.

(ii) A point is within a circle if its distance from the centre is less than the radius: and a point is without a circle if its distance from the centre is greater than the radius.

(iii) Circles of equal radius are equal in all respects; that is to say, their areas and circumferences are equal.

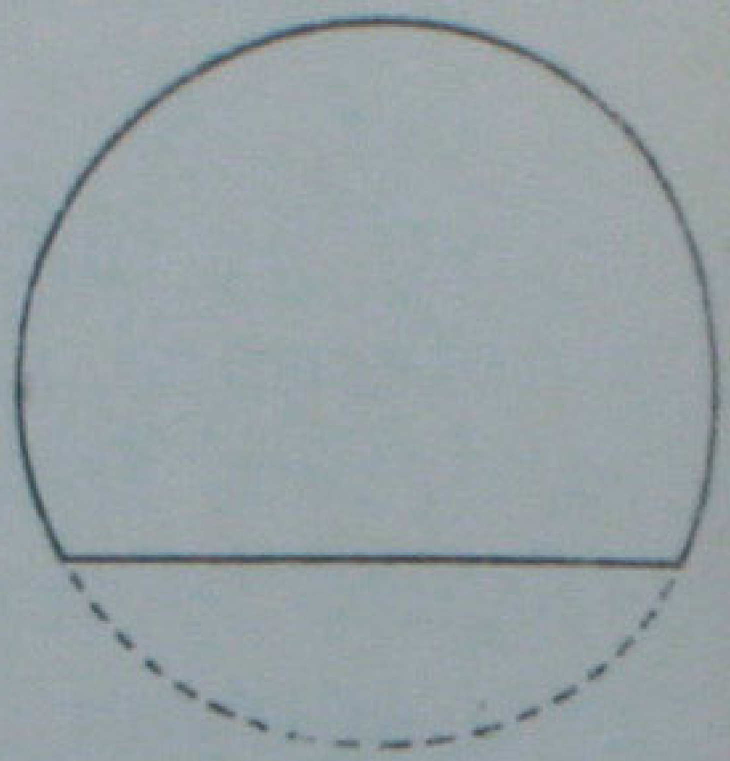
(iv) A circle is divided by any diameter into two parts which are equal in all respects.

DEFINITIONS TO BOOK III.

1. An **arc** of a circle is any part of the circumference.

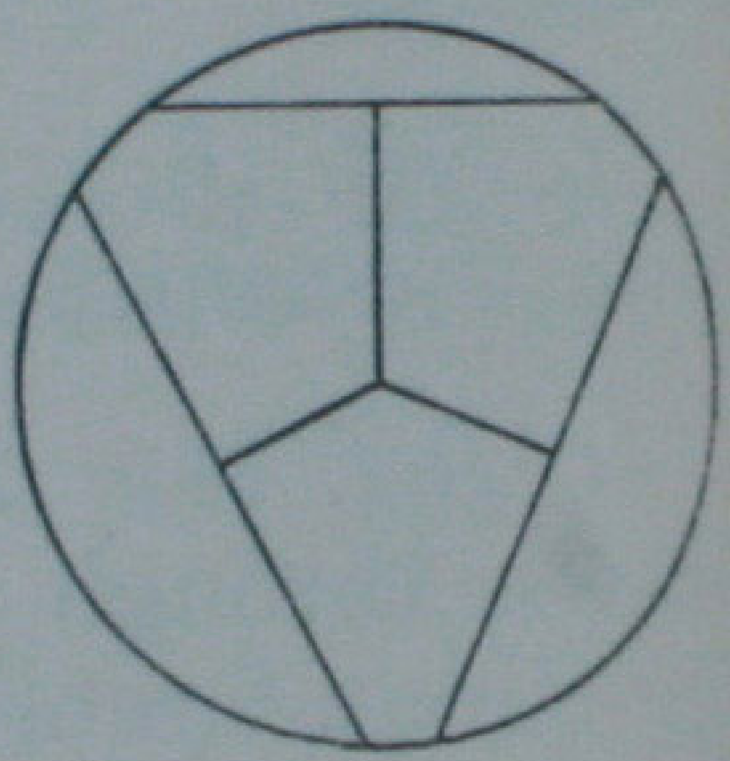
2. A **chord** of a circle is the straight line which joins any two points on the circumference.

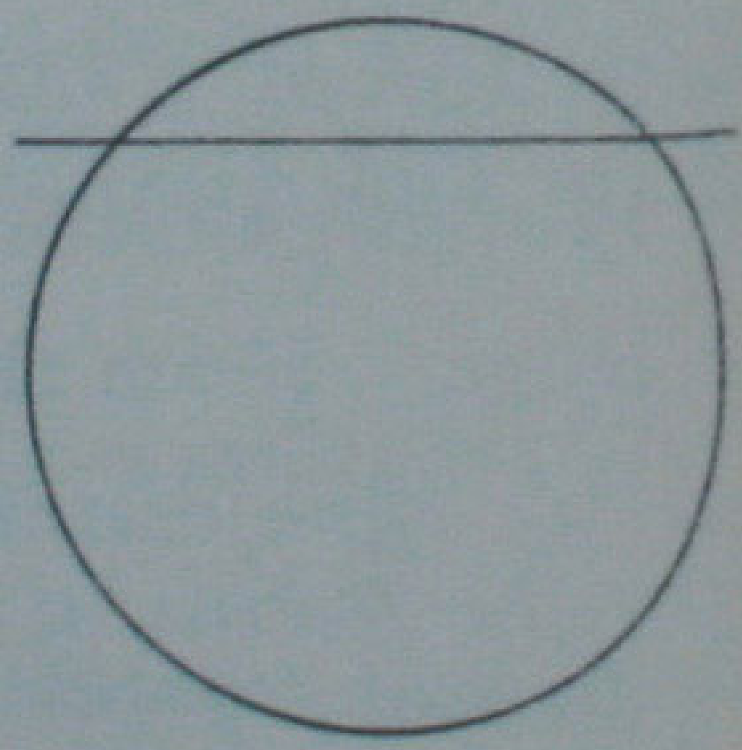
NOTE. From these definitions it may be seen that a chord of a circle, which does not pass through the centre, divides the circumference into two unequal arcs; of these, the greater is called the **major arc**, and the less the **minor arc**. Thus the major arc is *greater*, and the minor arc *less* than the semi-circumference.



The major and minor arcs, into which a circumference is divided by a chord, are said to be **conjugate** to one another.

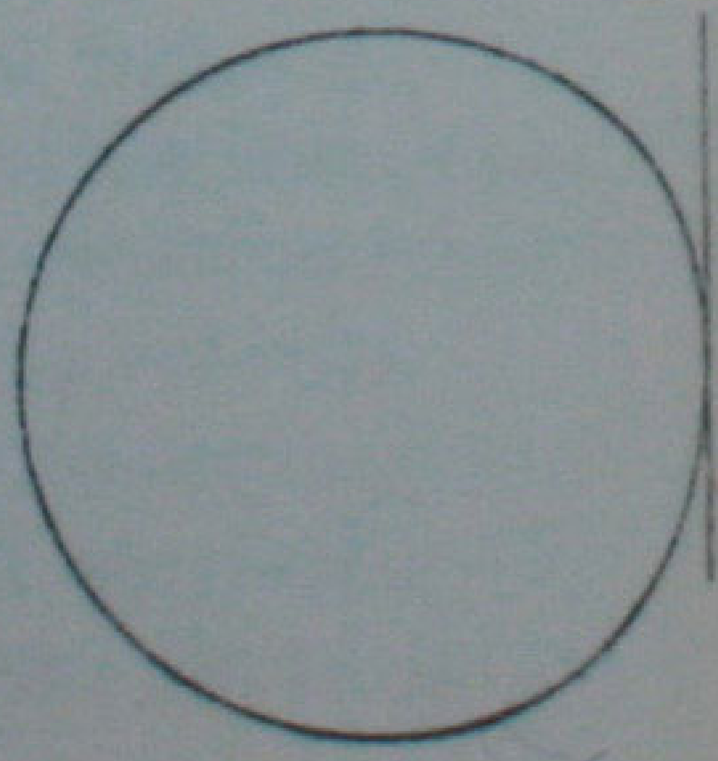
3. Chords of a circle are said to be **equidistant** from the centre, when the perpendiculars drawn to them from the centre are equal:
and one chord is said to be **further from the centre** than another, when the perpendicular drawn to it from the centre is greater than the perpendicular drawn to the other.

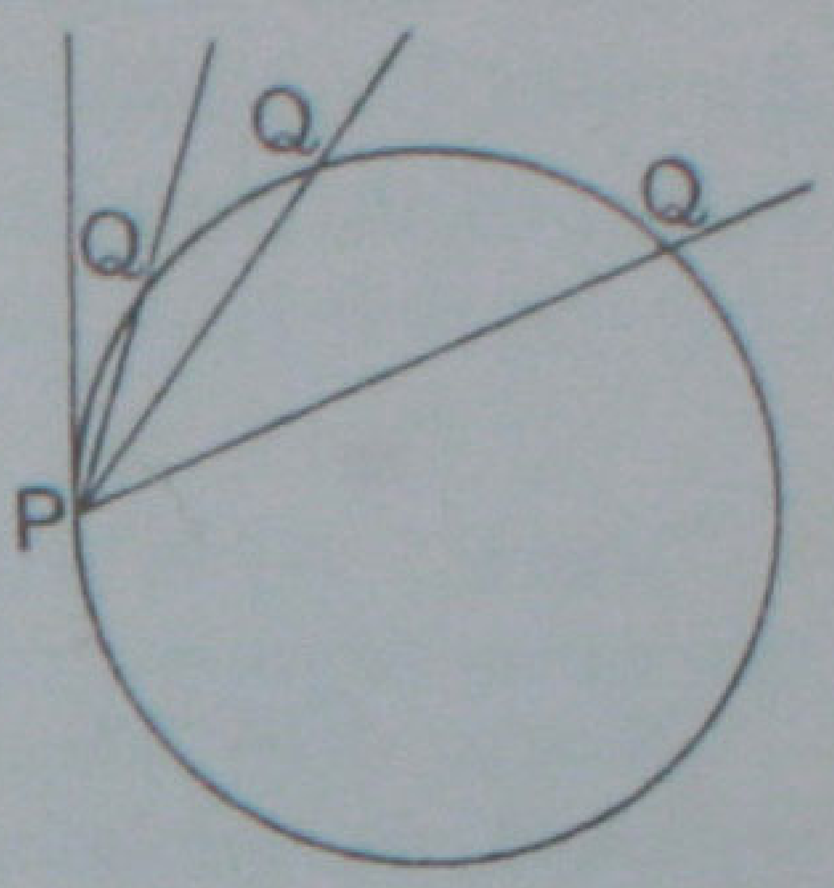




4. A **secant** of a circle is a straight line of indefinite length, which cuts the circumference in two points.

5. A **tangent** to a circle is a straight line which meets the circumference, but being produced, does not cut it. Such a line is said to **touch** the circle at a point; and the point is called the **point of contact.**

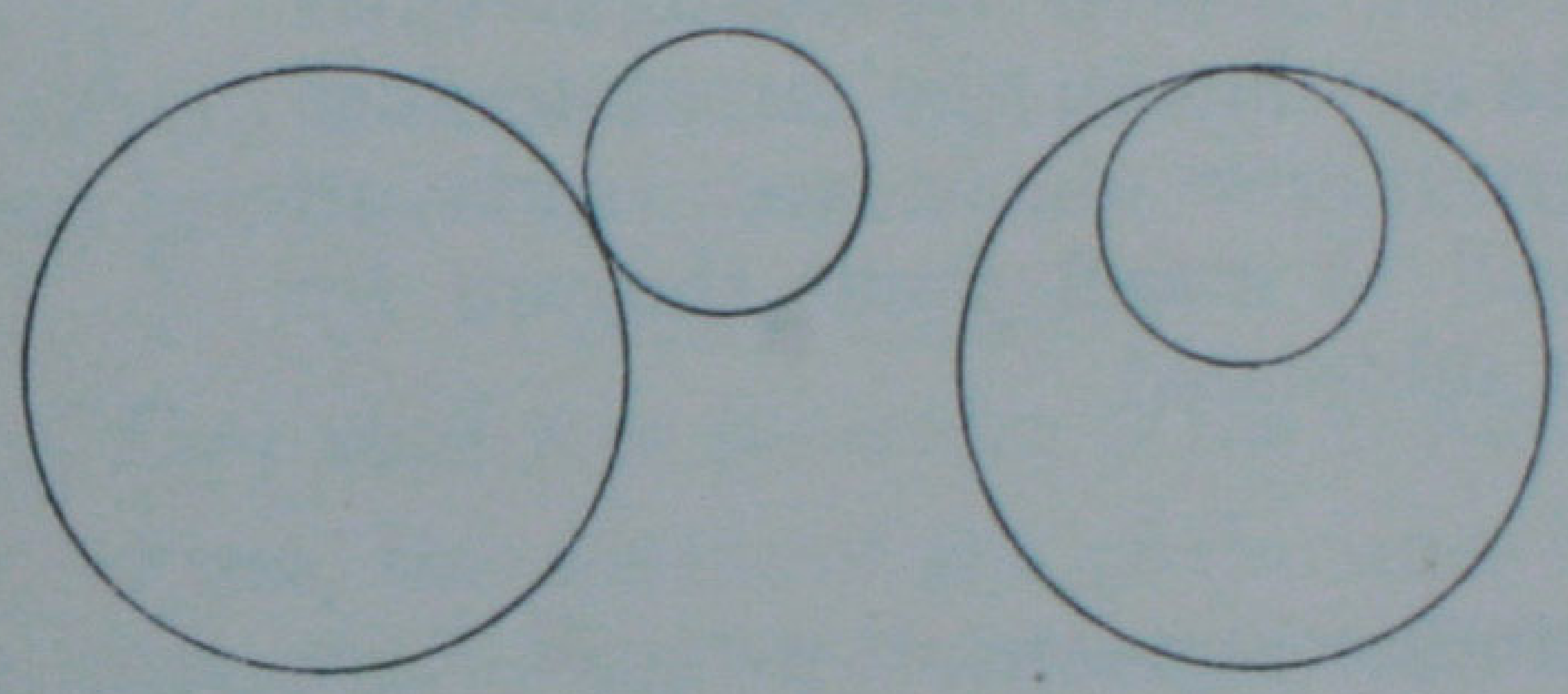




NOTE. If a secant, which cuts a circle at the points P and Q, gradually changes its position in such a way that P remains fixed, the point Q will ultimately approach the fixed point P, until at length these points may be made to coincide. When the straight line PQ reaches this limiting position, it becomes the *tangent* to the circle at the point P.

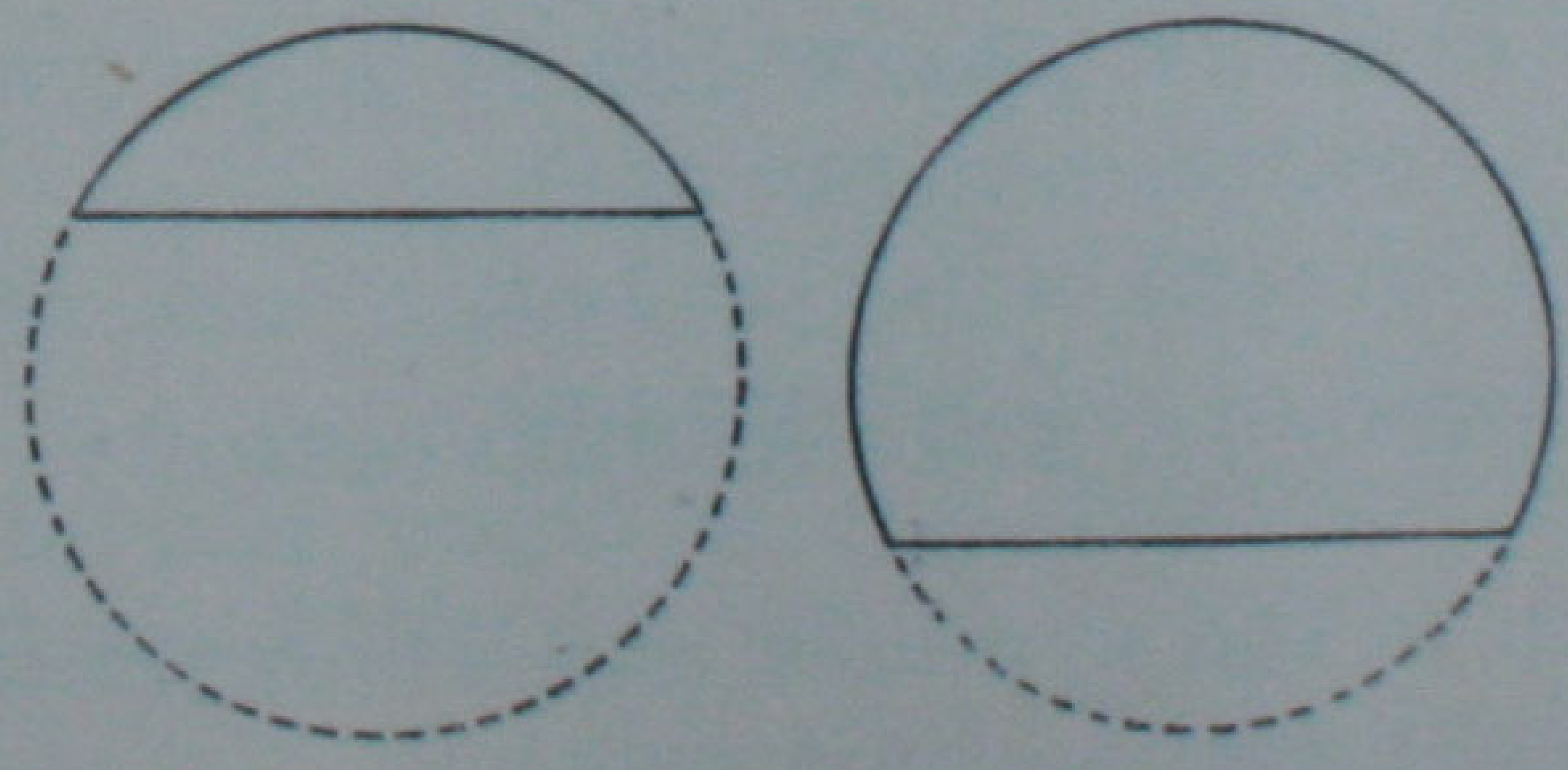
Hence a tangent may be defined as a straight line which passes through *two coincident points* on the circumference.

6. Circles are said to **touch one another** when they meet, but do not cut one another.



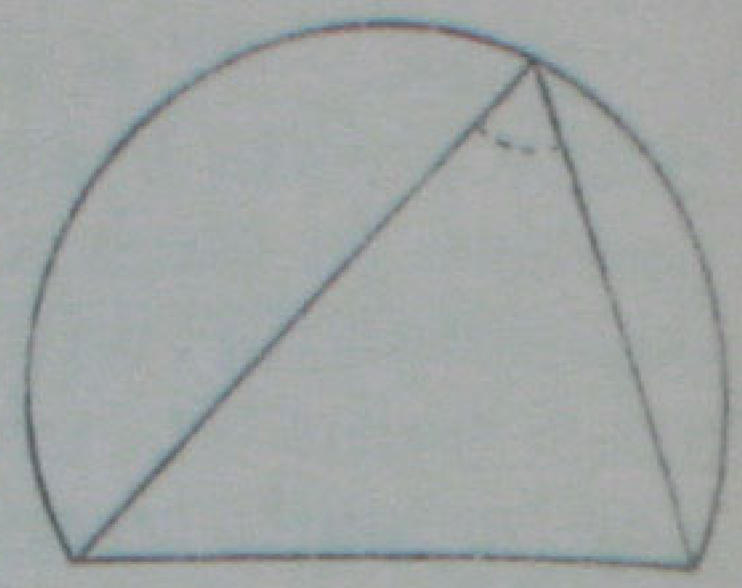
NOTE. When each of the circles which meet is *outside the other*, they are said to touch one another **externally**, or to have **external contact**: when one of the circles is *within the other*, the first is said to touch the other **internally**, or to have **internal contact** with it.

7. A **segment** of a circle is the figure bounded by a chord and one of the two arcs into which the chord divides the circumference.



NOTE. The chord of a segment is sometimes called its base.

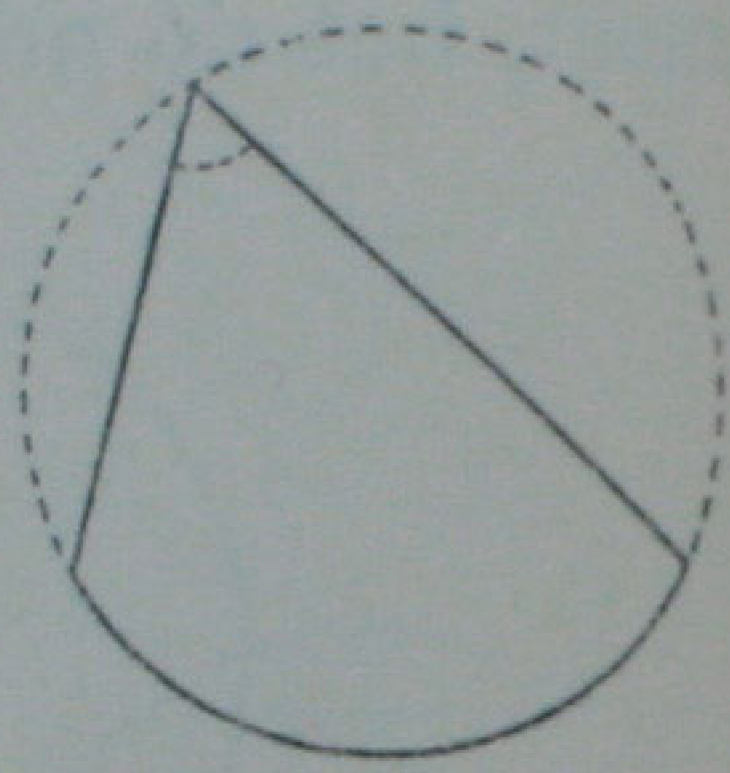
8. An **angle in a segment** is one formed by two straight lines drawn from any point in the arc of the segment to the extremities of its chord.



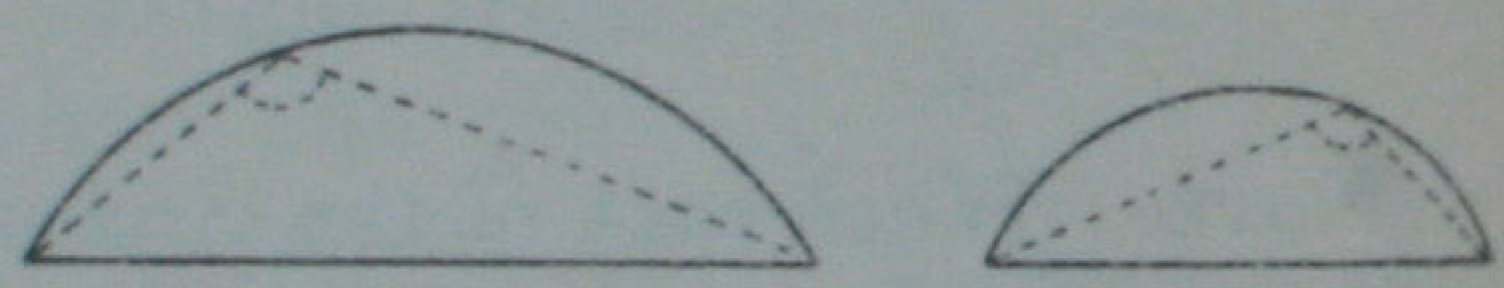
NOTE. (i) It will be shewn in Proposition 21, that all angles in the same segment of a circle are equal.

NOTE. (ii) The angle *of* a segment (as distinct from the angle *in* a segment) is sometimes defined as that which is contained between the *chord* and the *arc*; but this definition is not required in any proposition of Euclid.

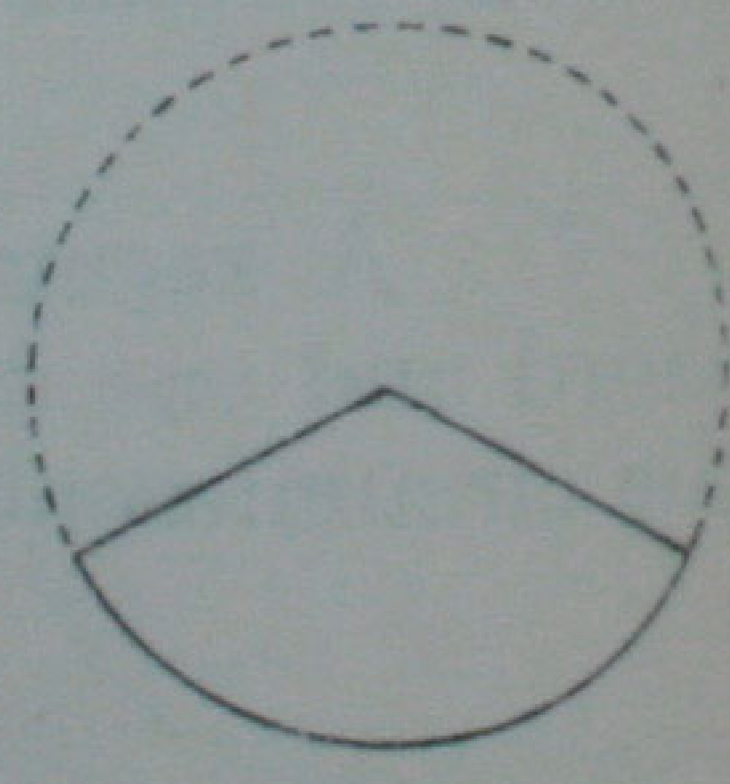
9. An **angle at the circumference** of a circle is one formed by straight lines drawn from a point on the circumference to the extremities of an arc: such an angle is said to **stand upon** the arc by which it is subtended.



10. **Similar segments** of circles are those which contain equal angles.



11. A **sector** of a circle is a figure bounded by two radii and the arc intercepted between them.



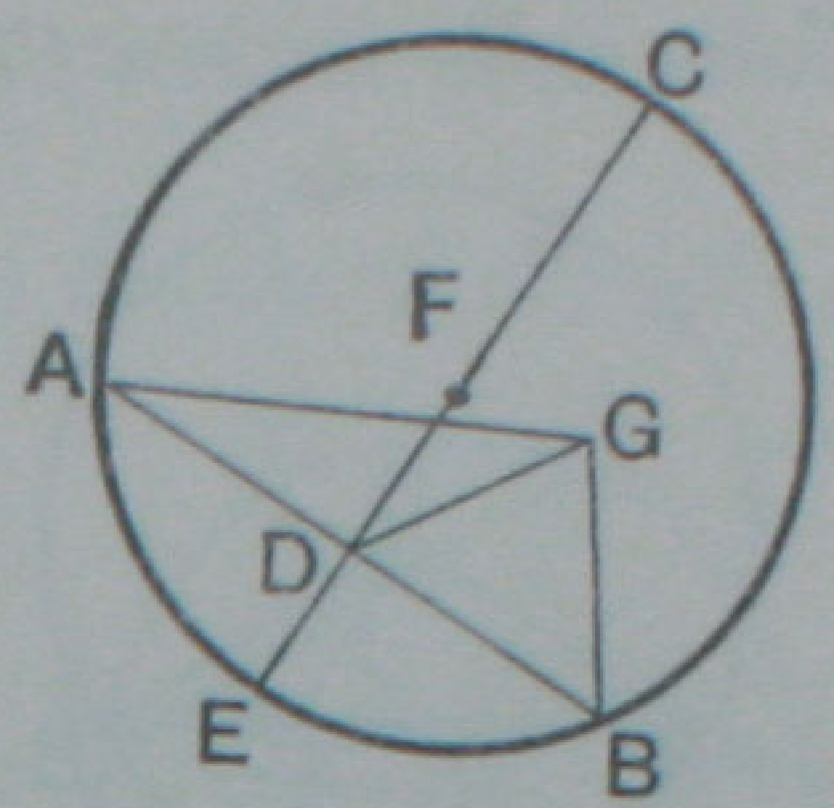
SYMBOLS AND ABBREVIATIONS.

In addition to the symbols and abbreviations given on page 11, we shall use the following.

⊙ *for* circle, ○ᶜᵉ *for* circumference.

PROPOSITION 1. PROBLEM.

To find the centre of a given circle.



Let ABC be a given circle.

It is required to find the centre of the ⊙ ABC.

Construction. In the given circle draw any chord AB, and bisect AB at D. I. 10.

From D draw DC at right angles to AB; I. 11.
and produce DC to meet the ○ᶜᵉ at E and C.

Bisect EC at F. I. 10.

Then shall F *be the centre of the* ⊙ ABC.

Proof. First, the centre of the circle must be in EC: for if not, suppose the centre to be at a point G outside EC.

Join AG, DG, BG.

Then in the △ˢ GDA, GDB,

Because {
DA = DB, *Constr.*
and GD is common;
and GA = GB, for by supposition they are radii;
}

∴ the ∠ GDA = the ∠ GDB; I. 8.

∴ these angles, being adjacent, are rt. angles.

But the ∠ CDB is a rt. angle; *Constr.*

∴ the ∠ GDB = the ∠ CDB, *Ax.* 11.

the part equal to the whole, which is impossible.

∴ G is not the centre.

So it may be shewn that no point outside EC is the centre;

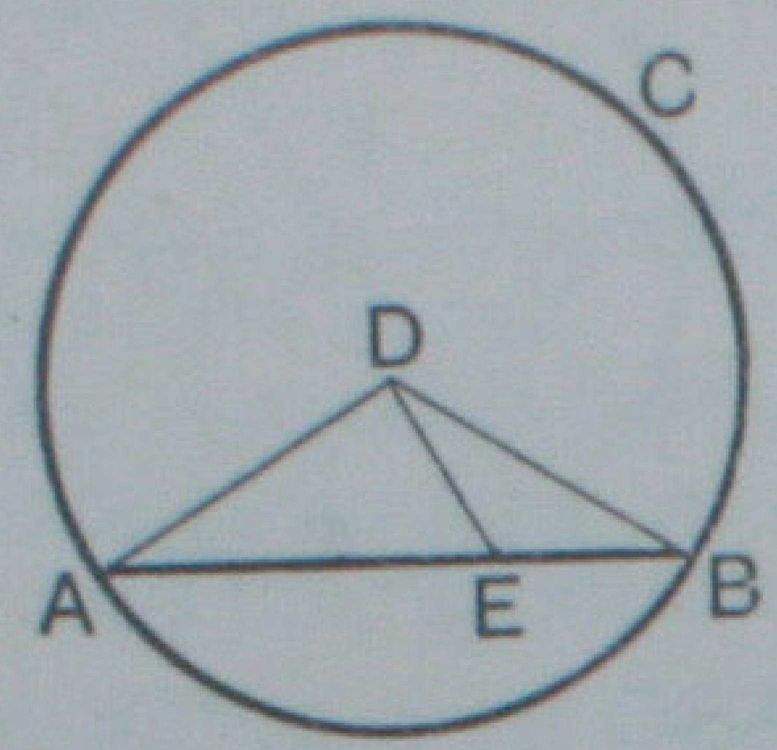
∴ the centre lies in EC.

∴ F, the middle point of the diameter EC, must be the centre of the ⊙ ABC. Q.E.F.

COROLLARY. *The straight line which bisects a chord of a circle at right angles passes through the centre.*

PROPOSITION 2. THEOREM.

If any two points are taken in the circumference of a circle, the chord which joins them falls within the circle.



Let ABC be a circle, and A and B any two points on its ⊙ce.

Then shall the chord AB *fall within the circle.*

Construction. Find D, the centre of the ⊙ ABC; III. 1.
and in AB take any point E.
Join DA, DE, DB.

Proof. In the △ DAB, because DA = DB, I. *Def.* 15.
∴ the ∠ DAB = the ∠ DBA. I. 5.

But the ext. ∠ DEB is greater than the int. opp. ∠ DAE; I. 16.

∴ the ∠ DEB is also greater than the ∠ DBE.

∴ in the △ DEB, the side DB, which is opposite the greater angle, is greater than DE which is opposite the less: I. 19.

that is to say, DE is less than DB, a radius of the circle;
∴ E falls within the circle.

Similarly, any other point between A and B may be shewn to fall within the circle.

∴ AB falls within the circle. Q.E.D.

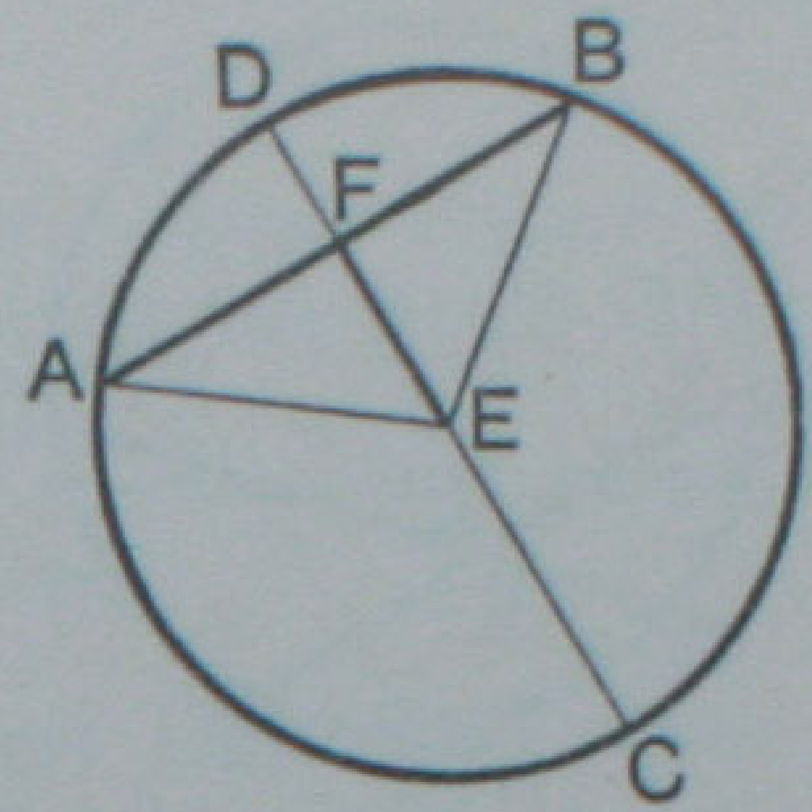
NOTE. A part of a curved line is said to be **concave** to a point, when for every chord (taken so as to lie between the point and the curve) all straight lines drawn from the given point to the intercepted arc are cut by the chord: if, when any chord whatever is taken, no straight line drawn from the given point to the intercepted arc is cut by the chord, the curve is said to be **convex** to that point.

Proposition 2 proves that the whole circumference of a circle is *concave to its centre.*

PROPOSITION 3. THEOREM.

If a straight line drawn through the centre of a circle bisects a chord which does not pass through the centre, it shall cut the chord at right angles.

Conversely, if it cuts the chord at right angles, it shall bisect it.



Let ABC be a circle; and let CD be a st. line drawn through the centre, and AB a chord which does not pass through the centre.

First. Let CD bisect the chord AB at F.
Then shall CD *cut* AB *at rt. angles.*

Construction. Find E the centre of the circle; III. 1.
and join EA, EB.

Proof. Then in the △^s AFE, BFE,

Because { AF = BF, *Hyp.*
and FE is common;
and AE = BE, being radii of the circle;

∴ the ∠ AFE = the ∠ BFE; I. 8.

∴ these angles, being adjacent, are rt. angles;
that is, DC cuts AB at rt. angles. Q.E.D.

Conversely. Let CD cut the chord AB at rt. angles.
Then shall CD *bisect* AB *at* F.

Construction. Find E the centre; and join EA, EB.

Proof. In the △ EAB, because EA = EB, I. *Def.* 15.
∴ the ∠ EAB = the ∠ EBA. I. 5.

Then in the △^s EFA, EFB,

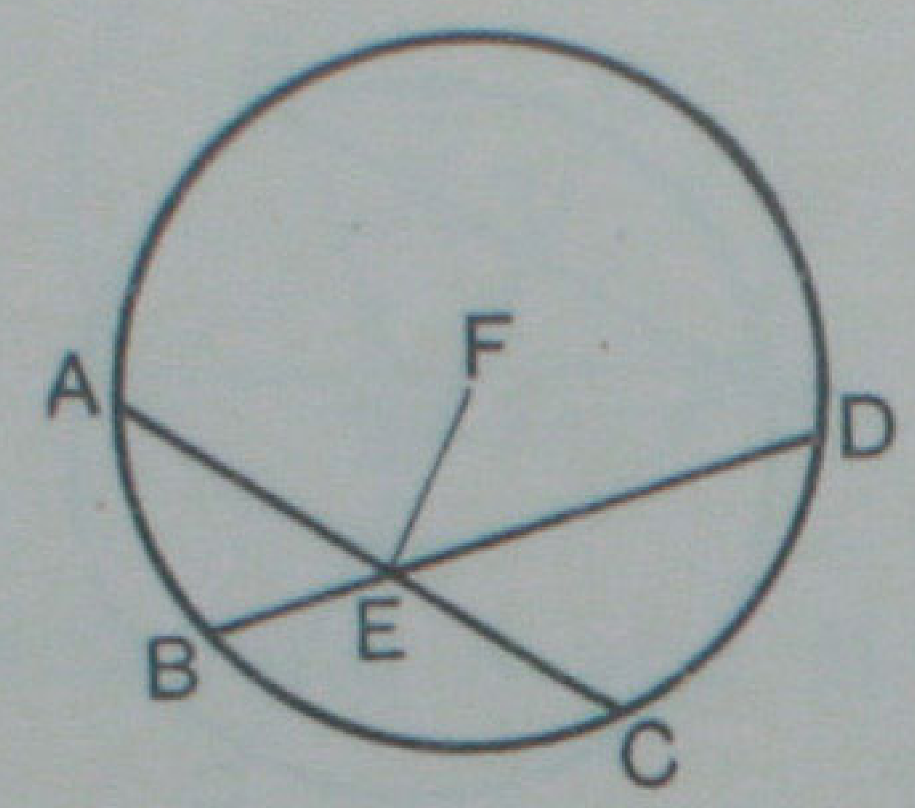
Because { the ∠ EAF = the ∠ EBF, *Proved.*
and the ∠ EFA = the ∠ EFB, being rt. angles; *Hyp.*
and EF is common;

∴ AF = BF; I. 26.

that is, CD bisects AB at F. Q.E.D.

PROPOSITION 4. THEOREM.

If in a circle two chords cut one another, which do not both pass through the centre, they cannot both be bisected at their point of intersection.



Let ABCD be a circle, and AC, BD two chords which intersect at E, but do not both pass through the centre.

Then AC *and* BD *shall not be* both *bisected at* E.

CASE I. If *one* chord passes through the centre, it is a diameter, and the centre is its middle point;

∴ it cannot be bisected by the other chord, which by hypothesis does not pass through the centre.

CASE II. If neither chord passes through the centre; then, if possible, suppose E to be the middle point of *both*; that is, let AE = EC; and BE = ED.

Construction. Find F, the centre of the circle. III. 1.
Join EF.

Proof. Because FE, which passes through the centre, bisects the chord AC, *Hyp.*

∴ the ∠ FEC is a rt. angle. III. 3.

And because FE, which passes through the centre, bisects the chord BD, *Hyp.*

∴ the ∠ FED is a rt. angle. III. 3.

∴ the ∠ FEC = the ∠ FED,

the whole equal to its part, which is impossible.

∴ AC and BD are not *both* bisected at E. Q.E.D.

EXERCISES.

ON PROPOSITION 1.

1. If two circles intersect at the points A, B, shew that the line which joins their centres bisects their common chord AB at right angles.

2. AB, AC are two equal chords of a circle; shew that the straight line which bisects the angle BAC passes through the centre.

3. *Two chords of a circle are given in position and magnitude: find the centre of the circle.*

4. *Describe a circle that shall pass through three given points, which are not in the same straight line.*

5. *Find the locus of the centres of circles which pass through two given points.*

6. Describe a circle that shall pass through two given points, and have a given radius. When is this impossible?

ON PROPOSITION 2.

7. *A straight line cannot cut a circle in more than two points.*

ON PROPOSITION 3.

8. Through a given point within a circle draw a chord which shall be bisected at that point.

9. The parts of a straight line intercepted between the circumferences of two concentric circles are equal.

10. The line joining the middle points of two parallel chords of a circle passes through the centre.

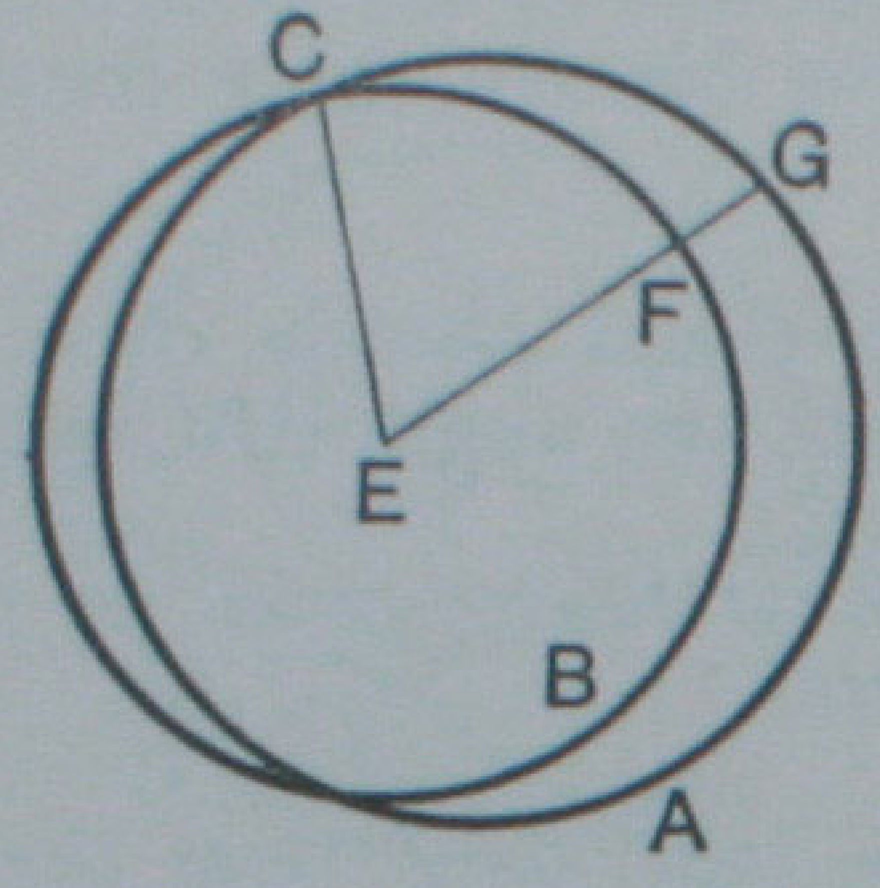
11. Find the locus of the middle points of a system of parallel chords drawn in a circle.

12. If two circles cut one another, any two parallel straight lines drawn through the points of intersection to cut the circles, are equal.

13. PQ and XY are two parallel chords in a circle: shew that the points of intersection of PX, QY, and of PY, QX, lie on the straight line which passes through the middle points of the given chords.

Proposition 5. Theorem.

If two circles cut one another, they cannot have the same centre.



Let the two ⊙s AGC, BFC cut one another at C.

Then they shall not have the same centre.

Construction. If possible, let the two circles have the same centre; and let it be called E.

Join EC;

and from E draw any st. line to meet the ○ces at F and G.

Proof. Because E is the centre of the ⊙ AGC, *Hyp.*

∴ EG = EC.

And because E is also the centre of the ⊙ BFC, *Hyp.*

∴ EF = EC.

∴ EG = EF,

the whole equal to its part, which is impossible.

Therefore the two circles have not the same centre.

Q.E.D.

EXERCISES.

on Propositions 4 and 5.

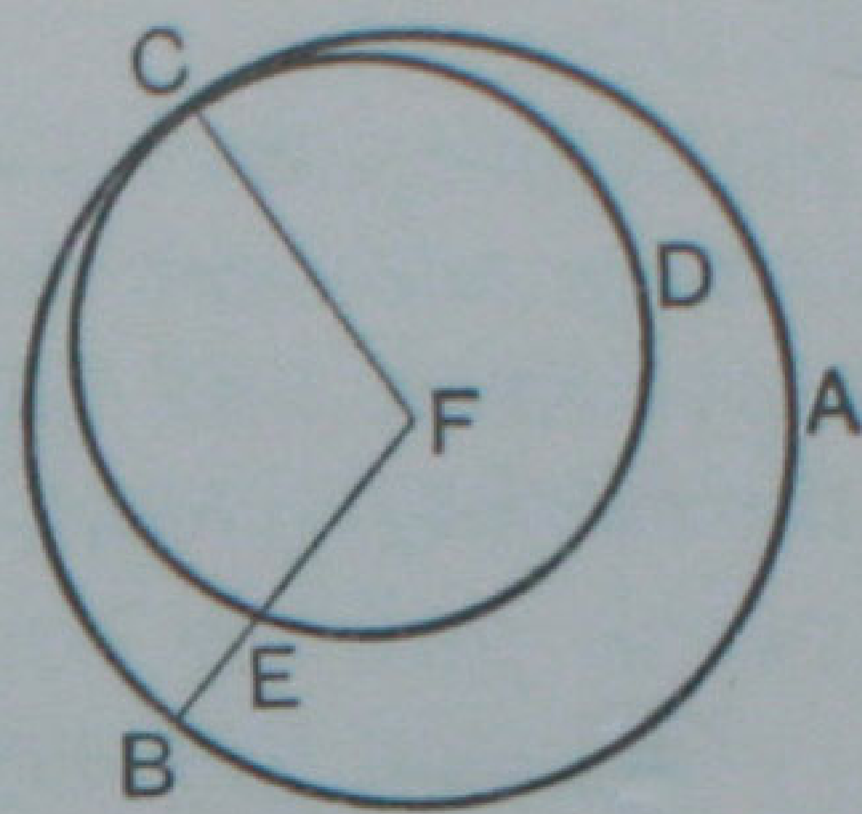
1. If a parallelogram can be inscribed in a circle, the point of intersection of its diagonals must be at the centre of the circle.

2. Rectangles are the only parallelograms that can be inscribed in a circle.

3. Two circles, which intersect at one point, must also intersect at another.

PROPOSITION 6. THEOREM.

If two circles touch one another internally, they cannot have the same centre.



Let the two ⊙ˢ ABC, DEC touch one another internally at C.

Then they shall not have the same centre.

Construction. If possible, let the two circles have the same centre; and let it be called F.

Join FC;

and from F draw any st. line to meet the ○ᶜᵉˢ at E and B.

Proof. Because F is the centre of the ⊙ ABC, *Hyp.*

∴ FB = FC.

And because F is the centre of the ⊙ DEC, *Hyp.*

∴ FE = FC.

∴ FB = FE,

the whole equal to its part, which is impossible.

Therefore the two circles have not the same centre.

Q.E.D.

NOTE. From Propositions 5 and 6 it is seen that circles, whose circumferences have any point in common, cannot be concentric, unless they coincide entirely.

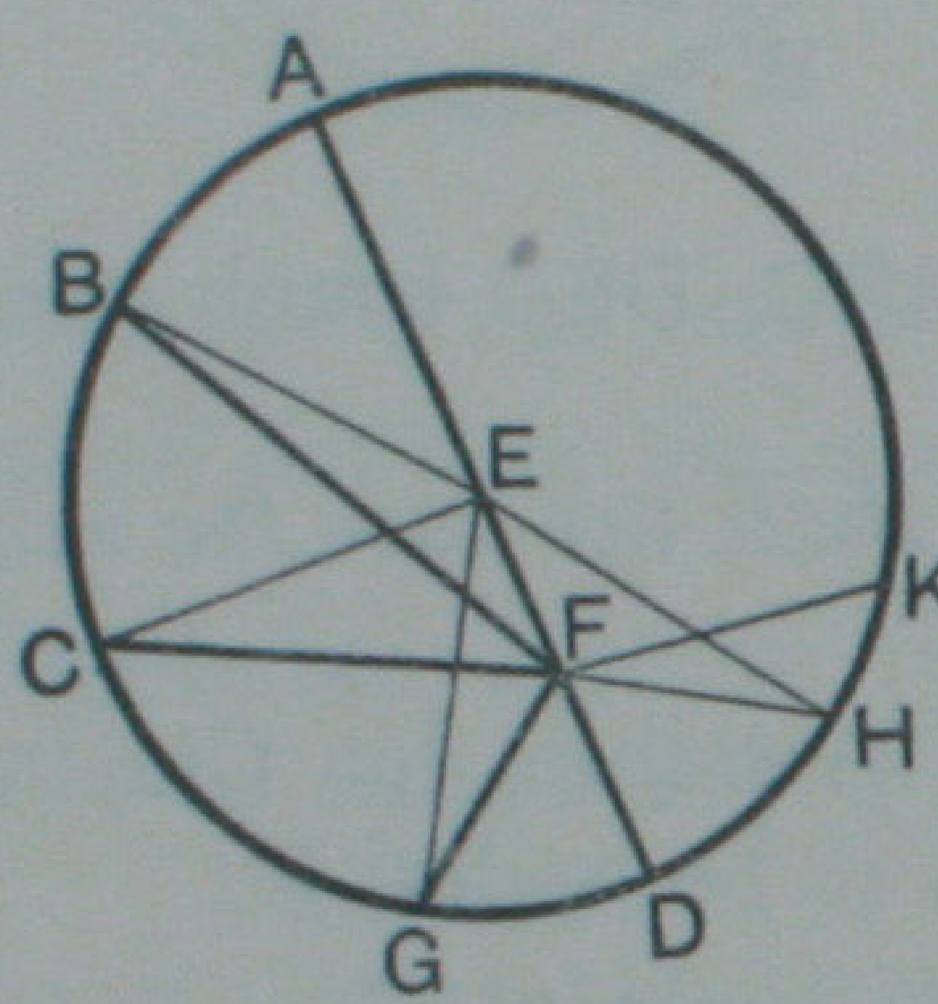
Conversely, the circumferences of concentric circles can have no point in common.

Proposition 7. Theorem.

If from any point within a circle which is not the centre, straight lines are drawn to the circumference, then the greatest is that which passes through the centre; and the least is the remaining part of the diameter.

And of all other such lines, that which is nearer to the greatest is always greater than one more remote.

And two equal straight lines, and only two, can be drawn from the given point to the circumference, one on each side of the diameter.



Let ABCD be a circle, and from F, any point within it which is not the centre, let FA, FB, FC, FG, and FD be drawn to the ○ce, of which FA passes through E the centre, and FD is the remaining part of the diameter.

Then of all these st. lines,

(i) FA *shall be the greatest;*

(ii) FD *shall be the least;*

(iii) FB, *which is nearer to* FA, *shall be greater than* FC, *which is more remote;*

(iv) *also two, and only two, equal st. lines can be drawn from* F *to the* ○ce.

Construction. Join EB, EC.

Proof. (i) In the △ FEB, the two sides FE, EB are together greater than the third side FB. I. 20.

But EB = EA, being radii of the circle;

∴ FE, EA are together greater than FB;

that is, FA is greater than FB.

Similarly FA may be shewn to be greater than any other st. line drawn from F to the ○ce;

∴ FA is the greatest of all such lines.

(ii) In the △ EFG, the two sides EF, FG are together greater than EG; I. 20.

and EG = ED, being radii of the circle;

∴ EF, FG are together greater than ED.

Take away the common part EF;

then FG is greater than FD.

Similarly any other st. line drawn from F to the ○ce may be shewn to be greater than FD;

∴ FD is the least of all such lines.

(iii) In the △s BEF, CEF,

Because { BE = CE, I. *Def.* 15.
and EF is common;
but the ∠ BEF is greater than the ∠ CEF;

∴ FB is greater than FC. I. 24.

Similarly it may be shewn that FC is greater than FG.

(iv) Join EG, and at E in FE make the ∠ FEH equal to the ∠ FEG. I. 23.

Join FH.

Then in the △s GEF, HEF,

Because { GE = HE, I. *Def.* 15.
and EF is common;
also the ∠ GEF = the ∠ HEF; *Constr.*

∴ FG = FH. I. 4.

And besides FH no other straight line can be drawn from F to the ○ce equal to FG.

For, if possible, let FK = FG.

Then, because FH = FG, *Proved.*

∴ FK = FH,

that is, a line nearer to FA, the greatest, is equal to a line which is more remote; which is impossible. *Proved.*

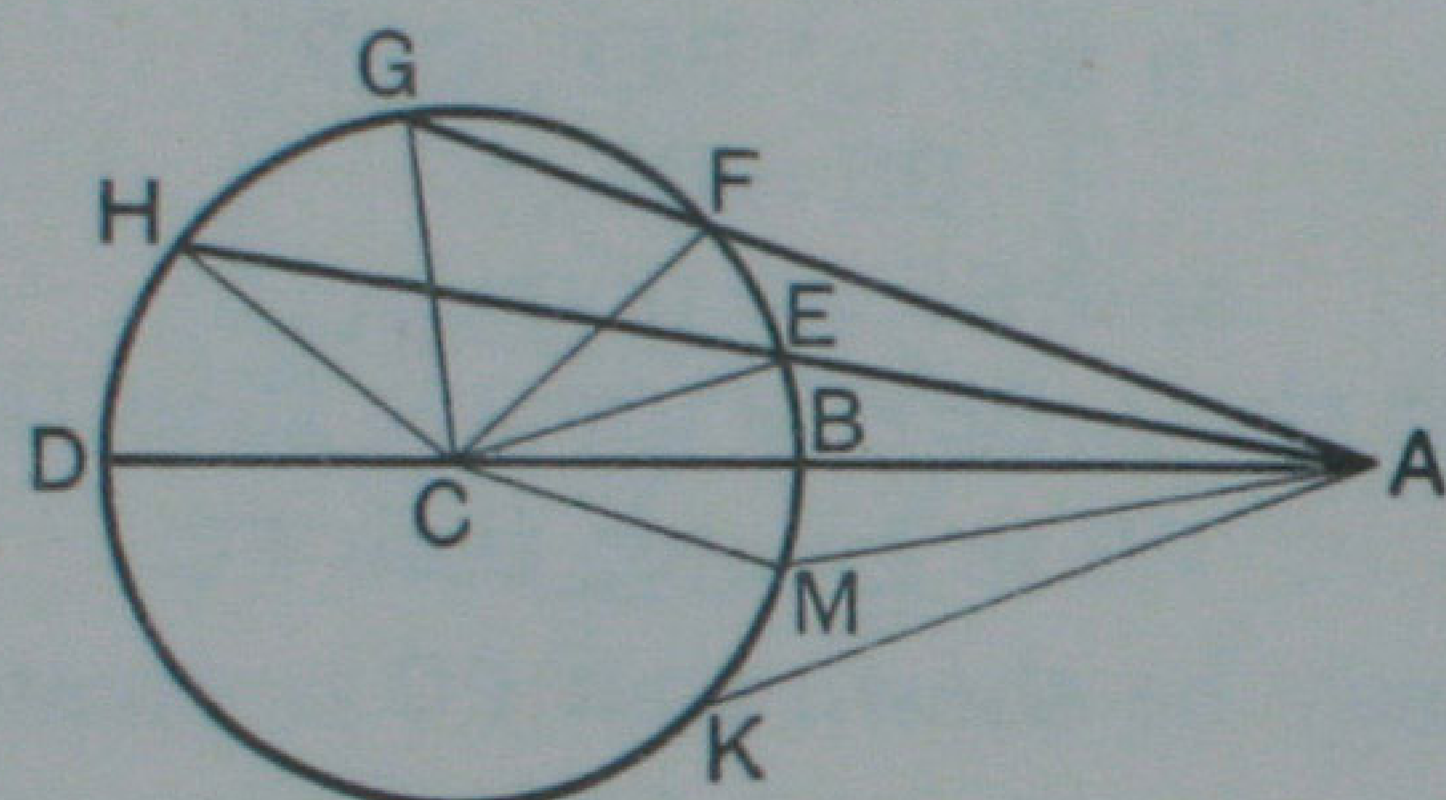
Therefore two, and only two, equal st. lines can be drawn from F to the ○ce. Q.E.D.

PROPOSITION 8. THEOREM.

If from any point without a circle straight lines are drawn to the circumference, of those which fall on the concave circumference, the greatest is that which passes through the centre; and of others, that which is nearer to the greatest is always greater than one more remote.

Of those which fall on the convex circumference, the least is that which, when produced, passes through the centre; and of others, that which is nearer to the least is always less than one more remote.

From the given point there can be drawn to the circumference two, and only two, equal straight lines, one on each side of the shortest line.



Let BGD be a circle; and from A, any point outside the circle, let ABD, AEH, AFG, be drawn, of which AD passes through C, the centre, and AH is nearer than AG to AD.

Then of st. lines drawn from A *to the concave* ○ᶜᵉ,

(i) AD *shall be the greatest, and* (ii) AH *greater than* AG.

And of st. lines drawn from A *to the convex* ○ᶜᵉ,

(iii) AB *shall be the least, and* (iv) AE *less than* AF.

(v) *Also two, and only two, equal st. lines can be drawn from* A *to the* ○ᶜᵉ.

Construction. Join CH, CG, CF, CE.

Proof. (i) In the △ ACH, the two sides AC, CH are together greater than AH: I. 20.

but CH = CD, being radii of the circle;

∴ AC, CD are together greater than AH:

that is, AD is greater than AH.

Similarly AD may be shewn to be greater than any other st. line drawn from A to the concave ○ᶜᵉ;

∴ AD is the greatest of all such lines.

(ii) In the $\triangle^s$ HCA, GCA,

Because { HC = GC, — I. *Def.* 15.
and CA is common;
but the $\angle$ HCA is greater than the $\angle$ GCA;

$\therefore$ AH is greater than AG. — I. 24.

(iii) In the $\triangle$ AEC, the two sides AE, EC are together greater than AC; — I. 20.

but EC = BC; — I. *Def.* 15.

$\therefore$ the remainder AE is greater than the remainder AB.

Similarly any other st. line drawn from A to the convex $\bigcirc^{ce}$ may be shewn to be greater than AB;

$\therefore$ AB is the least of all such lines.

(iv) In the $\triangle$ AFC, because AE, EC are drawn from the extremities of the base to a point E within the triangle,

$\therefore$ AF, FC are together greater than AE, EC., — I. 21.

But FC = EC; — I. *Def.* 15.

$\therefore$ the remainder AF is greater than the remainder AE.

(v) At C, in AC, make the $\angle$ ACM equal to the $\angle$ ACE.

Join AM.

Then in the two $\triangle^s$ ECA, MCA,

Because { EC = MC, — I. *Def.* 15.
and CA is common;
also the $\angle$ ECA = the $\angle$ MCA; — *Constr.*

$\therefore$ AE = AM. — I. 4.

And besides AM, no st. line can be drawn from A to the $\bigcirc^{ce}$, equal to AE.

For, if possible, let AK = AE:

then because AM = AE, — *Proved.*

$\therefore$ AM = AK;

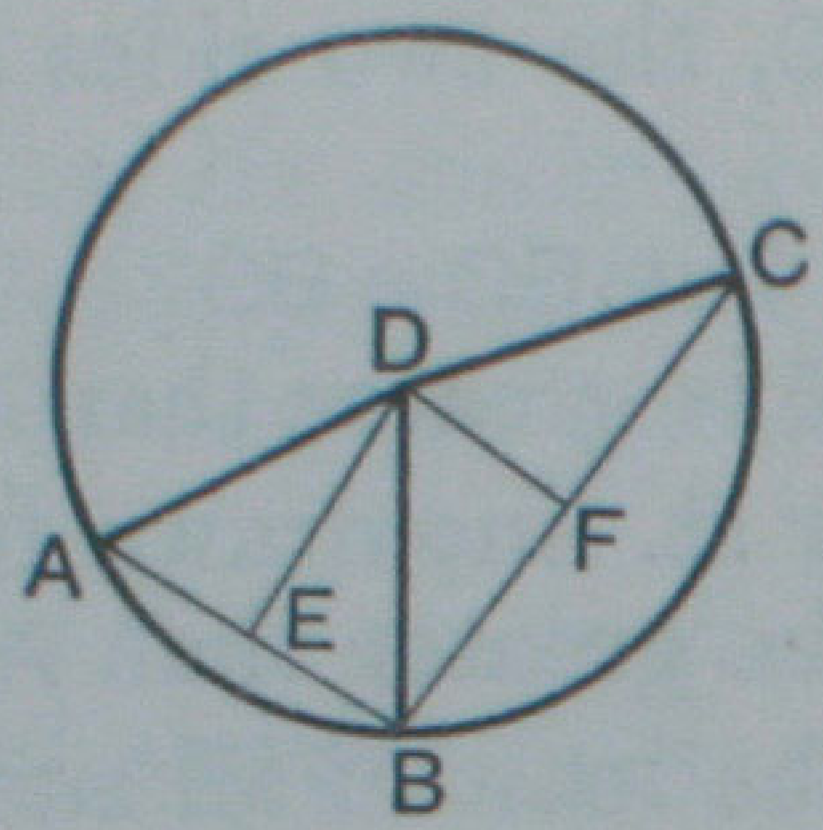
that is, a line nearer to AB, the shortest line, is equal to a line which is more remote; which is impossible. *Proved.*

Therefore two, and only two, equal st. lines can be drawn from A to the $\bigcirc^{ce}$. Q.E.D.

EXERCISE. Where are the limits of that part of the circumference which is concave to the point A?

Proposition 9. Theorem. [First Proof.]

If from a point within a circle more than two equal straight lines can be drawn to the circumference, that point is the centre of the circle.



Let ABC be a circle, and D a point within it, from which more than two equal st. lines are drawn to the ○[ce], namely DA, DB, DC.

Then D *shall be the centre of the circle* ABC.

Construction. Join AB, BC:
and bisect AB, BC at E and F respectively. I. 10.
Join DE, DF.

Proof. In the △[s] DEA, DEB,

Because { EA = EB, *Constr.*
and DE is common;
and DA = DB; *Hyp.*

∴ the ∠ DEA = the ∠ DEB; I. 8.

∴ these angles, being adjacent, are rt. angles.

Hence ED, which bisects the chord AB at rt. angles, must pass through the centre. III. 1. *Cor.*

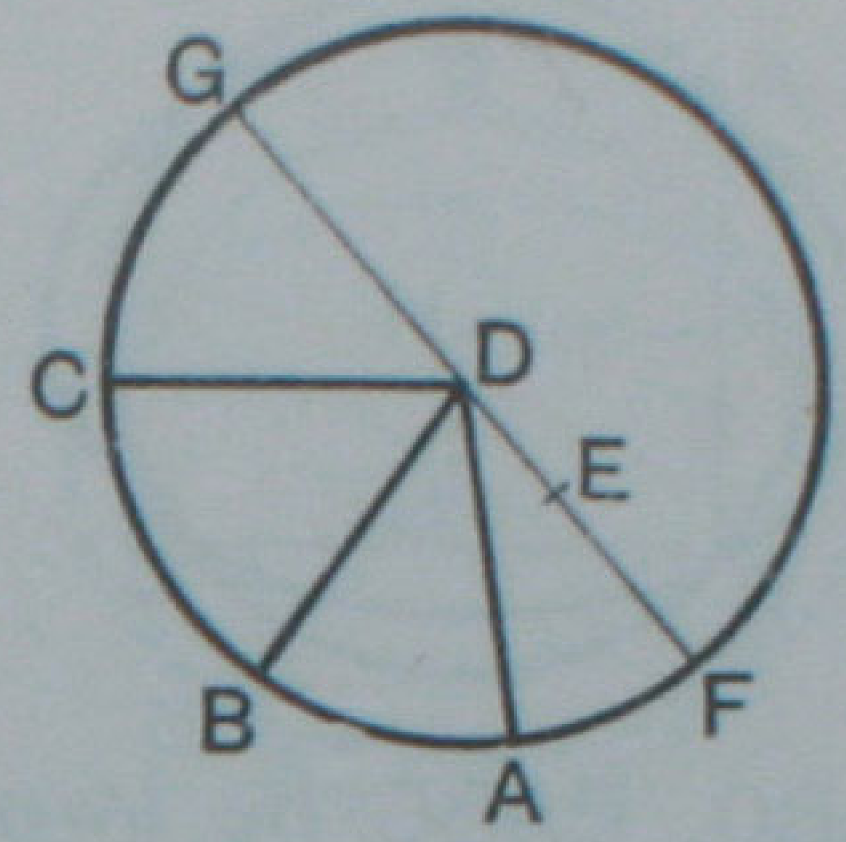
Similarly it may be shewn that FD passes through the centre.

∴ D, which is the only point common to ED and FD, must be the centre. Q.E.D.

Note. Of the two proofs of this proposition given by Euclid the first has the advantage of being *direct*.

PROPOSITION 9. THEOREM. [SECOND PROOF.]

If from a point within a circle more than two equal straight lines can be drawn to the circumference, that point is the centre of the circle.



Let ABC be a circle, and D a point within it, from which more than two equal st. lines are drawn to the ○ce, namely DA, DB, DC.

Then D *shall be the centre of the circle* ABC.

Construction. For if not, suppose, if possible, E to be the centre.

Join DE, and produce it to meet the ○ce at F, G.

Proof. Because D is a point within the circle, not the centre, and because DF passes through the centre E;

∴ DA, which is nearer to DF, is greater than DB, which is more remote: III. 7.

but this is impossible, since by hypothesis, DA, DB, are equal.

∴ E is not the centre of the circle.

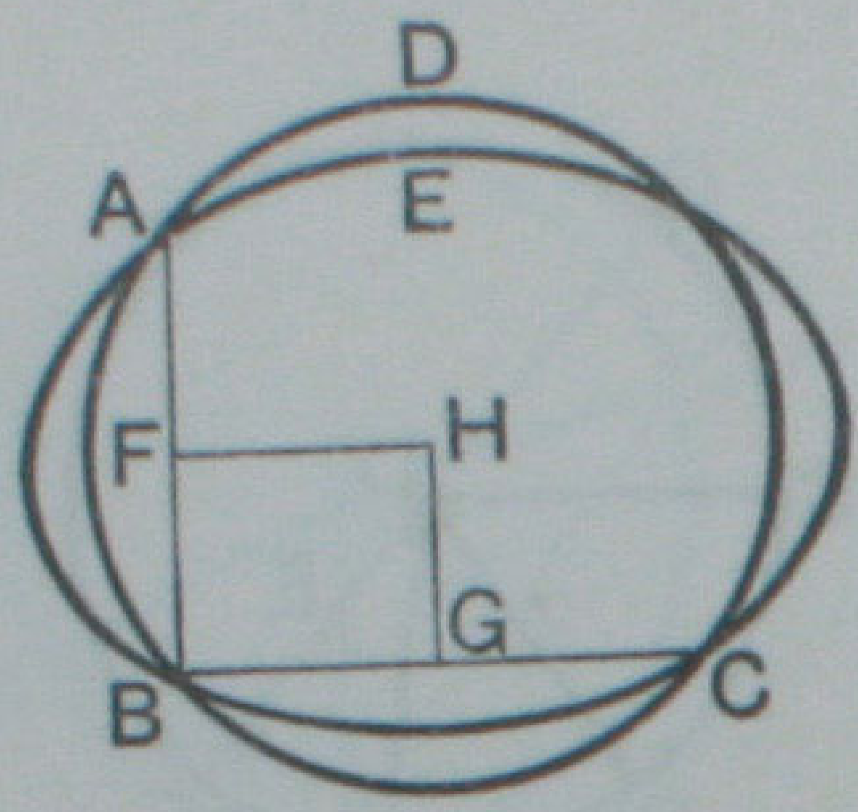
* And wherever we suppose the centre E to be, otherwise than at D, two at least of the st. lines DA, DB, DC may be shewn to be unequal, which is contrary to hypothesis.

∴ D is the centre of the ⊙ ABC. Q.E.D.

*NOTE. For example, if the centre E were supposed to be within the angle BDC, then DB would be greater than DA; if within the angle ADB, then DB would be greater than DC; if on one of the three straight lines, as DB, then DB would be greater than both DA and DC.

PROPOSITION 10. THEOREM. [FIRST PROOF.]

One circle cannot cut another at more than two points.



If possible, let DABC, EABC be two circles, cutting one another at more than two points, namely at A, B, C.

Construction. Join AB, BC.
Draw FH, bisecting AB at rt. angles; I. 10, 11.
and draw GH bisecting BC at rt. angles.

Proof. Because AB is a chord of *both* circles, and because FH bisects AB at rt. angles,
∴ the centre of both circles lies in FH. III. 1. *Cor.*

Again, because BC is a chord of both circles, and because GH bisects BC at right angles,
∴ the centre of both circles lies in GH. III. 1. *Cor.*

Hence H, the only point common to FH and GH, is the centre of both circles;
which is impossible, for circles which cut one another cannot have a common centre. III. 5.

Therefore one circle cannot cut another at more than two points. Q.E.D.

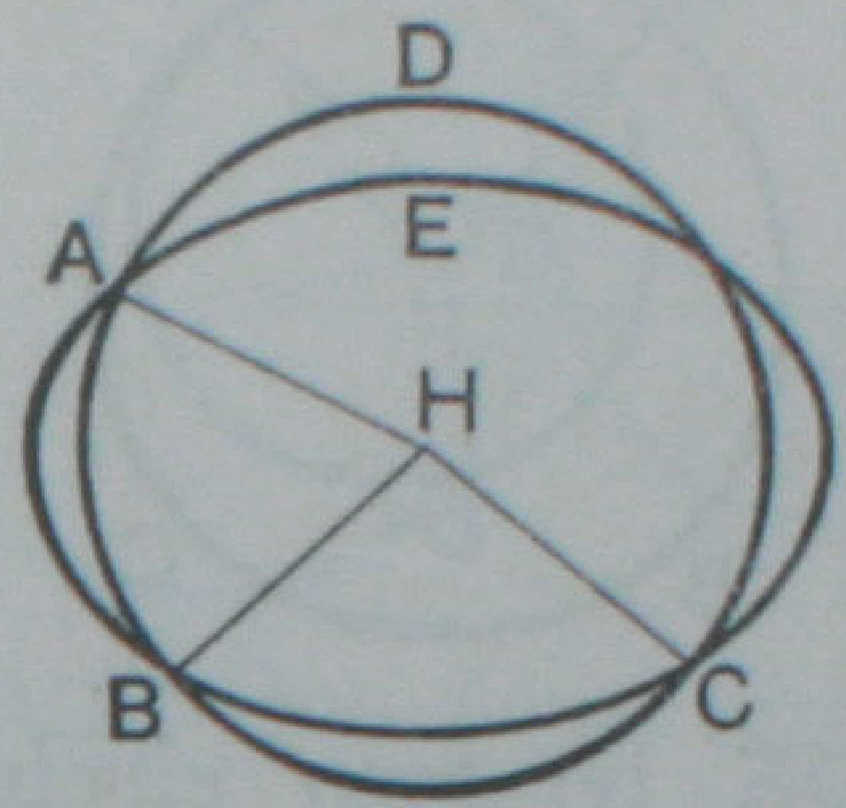
COROLLARIES. (i) *Two circles cannot have three points in common without coinciding entirely.*

(ii) *Two circles cannot have a common arc without coinciding entirely.*

(iii) *Only one circle can be described through three points, which are not in the same straight line.*

Proposition 10. Theorem. [Second Proof.]

One circle cannot cut another at more than two points.



If possible, let DABC, EABC be two circles, cutting one another at more than two points, namely at A, B, C.

Construction. Find H, the centre of the ⊙ DABC, III. 1.
and join HA, HB, HC.

Proof. Since H is the centre of the ⊙ DABC,
∴ HA, HB, HC are all equal. I. *Def.* 15.

And because H is a point within the ⊙ EABC, from which more than two equal st. lines, namely HA, HB, HC are drawn to the ○^{ce},

∴ H is the centre of the ⊙ EABC: III. 9.

that is to say, the two circles have a common centre H;
but this is impossible, since they cut one another. III. 5.

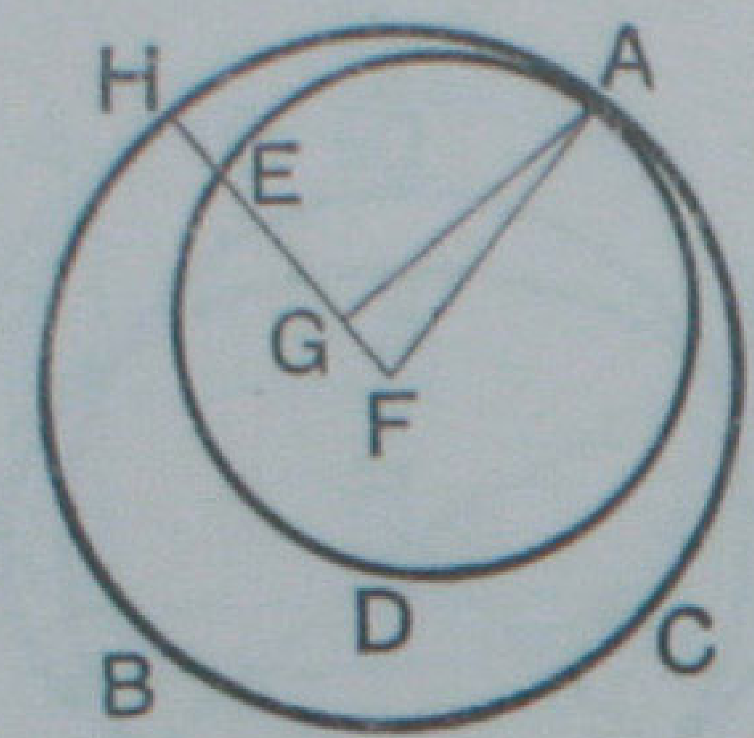
Therefore one circle cannot cut another in more than two points. Q.E.D.

Note. Both the proofs of Proposition 10 given by Euclid are indirect.

The second of these is imperfect, because it assumes that the centre of the circle DABC must fall within the circle EABC; whereas it may be conceived to fall either without the circle EABC, or on its circumference. Hence to make the proof complete, two additional cases are required.

Proposition 11. Theorem.

If two circles touch one another internally, the straight line which joins their centres, being produced, shall pass through the point of contact.



Let ABC and ADE be two circles which touch one another internally at A; let F be the centre of the ⊙ ABC, and G the centre of the ⊙ ADE.

Then shall FG *produced pass through* A.

Construction. For if not, suppose, if possible, FG to pass otherwise, as FGEH.

Join FA, GA.

Proof. In the △ FGA, the two sides FG, GA are together greater than FA: I. 20.

but FA = FH, being radii of the ⊙ ABC: *Hyp.*

FG, GA are together greater than FH.

Take away the common part FG:

then GA is greater than GH.

But GA = GE, being radii of the ⊙ ADE: *Hyp.*

∴ GE is greater than GH,

the part greater than the whole; which is impossible.

∴ FG, when produced, must pass through A.

Q.E.D.

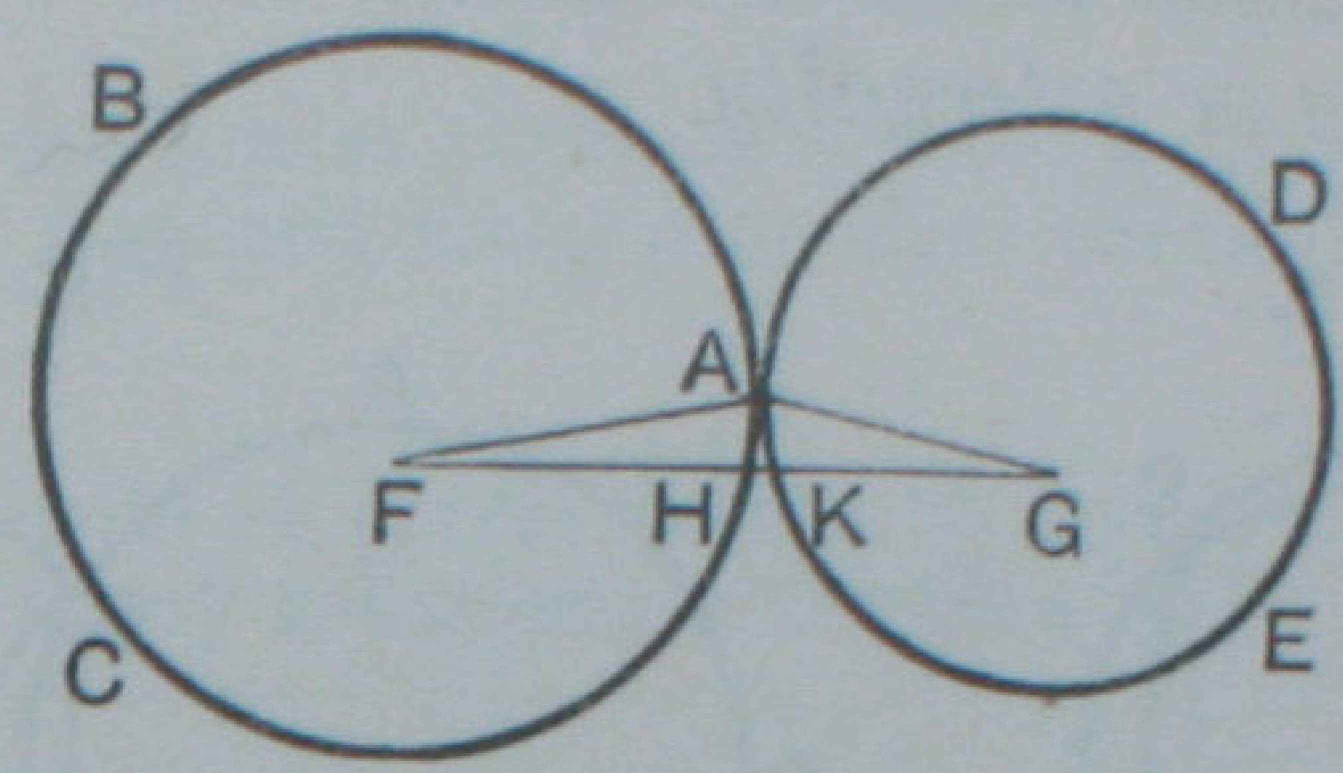
EXERCISES.

1. If the distance between the centres of two circles is equal to the difference of their radii, then the circles must meet in one point, but in no other; that is, they must touch one another.

2. *If two circles whose centres are* A *and* B *touch one another internally, and a straight line is drawn through their point of contact, cutting the circumferences at* P *and* Q; *shew that the radii* AP *and* BQ *are parallel.*

PROPOSITION 12. THEOREM.

If two circles touch one another externally, the straight line which joins their centres shall pass through the point of contact.



Let ABC and ADE be two circles which touch one another externally at A; let F be the centre of the ⊙ ABC, and G the centre of the ⊙ ADE.

Then shall FG *pass through* A.

Construction. For if not, suppose, if possible, FG to pass otherwise, as FHKG.

Join FA, GA.

Proof. In the △ FAG, the two sides FA, GA are together greater than FG: I. 20.

but FA = FH, being radii of the ⊙ ABC; *Hyp.*

and GA = GK, being radii of the ⊙ ADE; *Hyp.*

∴ FH and GK are together greater than FG;

which is impossible.

∴ FG must pass through A.

Q.E.D.

EXERCISES.

1. *Find the locus of the centres of all circles which touch a given circle at a given point.*

2. *Find the locus of the centres of all circles of given radius, which touch a given circle.*

3. If the distance between the centres of two circles is equal to the sum of their radii, then the circles meet in one point, but in no other; that is, they touch one another.

4. *If two circles whose centres are* A *and* B *touch one another externally, and a straight line is drawn through their point of contact cutting the circumferences at* P *and* Q; *shew that the radii* AP *and* BQ *are parallel.*

Proposition 13. Theorem.

Two circles cannot touch one another at more than one point, whether internally or externally.

Fig. 1.

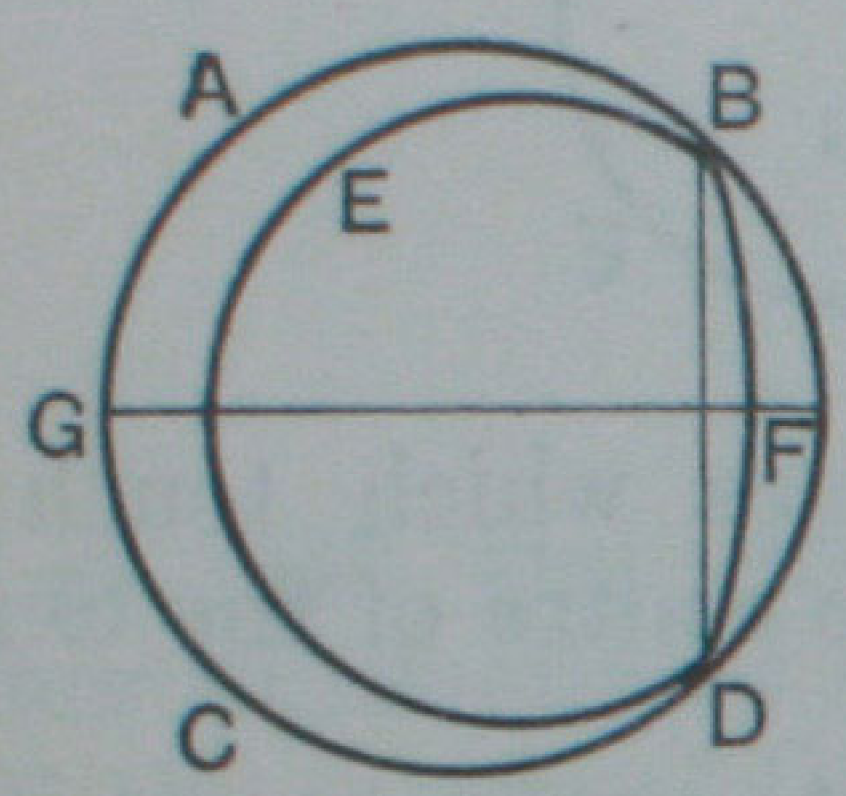
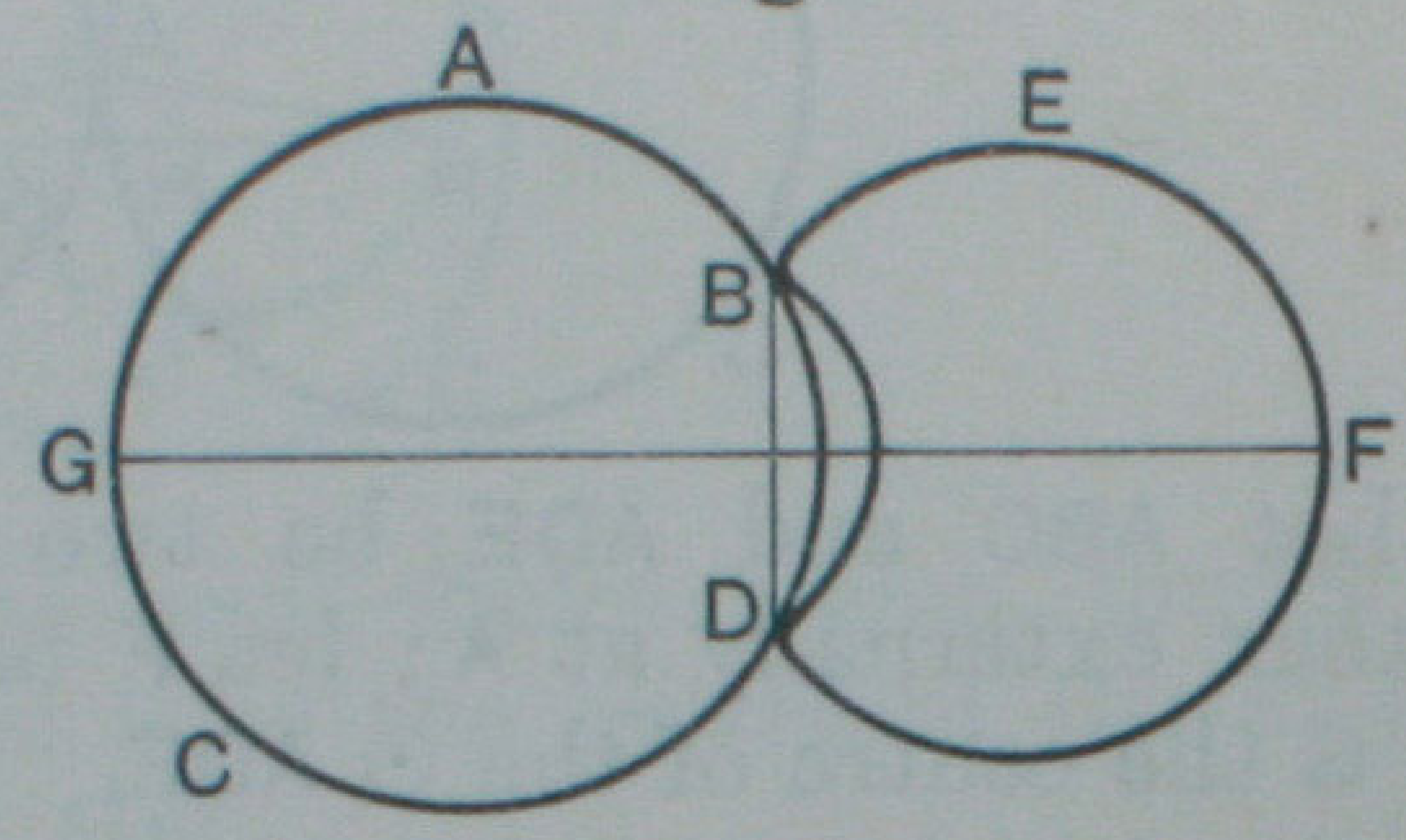


Fig. 2.



If possible, let ABC, EDF be two circles which touch one another at more than one point, namely at B and D.

Construction. Join BD;
and draw GF, bisecting BD at rt. angles. I. 10, 11.

Proof. Now, whether the circles touch one another internally, as in Fig 1 or externally as in Fig 2,

because B and D are on the ⨀^{ces} of both circles,

∴ BD is a chord of both circles:

∴ the centres of both circles lie in GF, which bisects BD at rt. angles. III. 1. *Cor.*

Hence GF which joins the centres must pass through a point of contact; III. 11, and 12.

∴ which is impossible, since B and D are outside GF.

Therefore two circles cannot touch one another at more than one point.

Q.E.D.

Note. It must be observed that the proof here given applies, by virtue of Propositions 11 and 12, to *both* the above figures: we have therefore omitted the separate discussion of Fig. 2, which finds a place in most editions based on Simson's text.

EXERCISES ON PROPOSITIONS 1–13.

1. Describe a circle to pass through two given points and have its centre on a given straight line. When is this impossible?

2. All circles which pass through a fixed point, and have their centres on a given straight line, pass also through a second fixed point.

3. Describe a circle of given radius to touch a given circle at a given point. How many solutions will there be? When will there be only one solution?

4. From a given point as centre describe a circle to touch a given circle. How many solutions will there be?

5. Describe a circle to pass through a given point, and touch a given circle at a given point. [See Ex. 1, p. 183, and Ex. 5, p. 171.] When is this impossible?

6. Describe a circle of given radius to touch two given circles. [See Ex. 2, p. 183.] How many solutions will there be?

7. Two parallel chords of a circle are six inches and eight inches in length respectively, and the perpendicular distance between them is one inch: find the radius.

8. If two circles touch one another externally, the straight lines, which join the extremities of parallel diameters towards opposite parts, must pass through the point of contact.

9. Find the greatest and least straight lines which have one extremity on each of two given circles, which do not intersect.

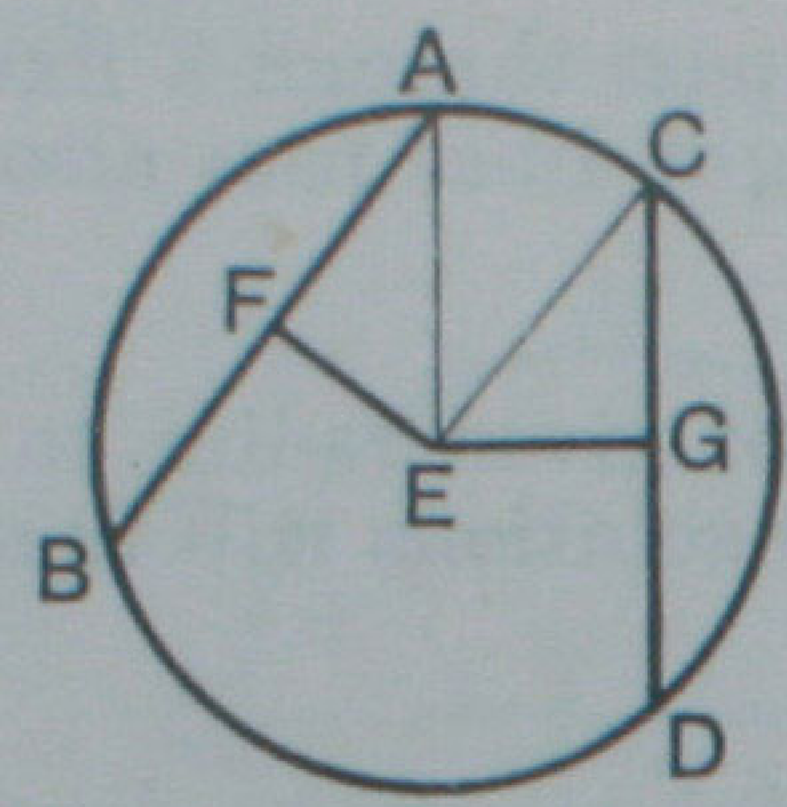
10. In any segment of a circle, of all straight lines drawn at right angles to the chord and intercepted between the chord and the arc, the greatest is that which passes through the middle point of the chord; and of others that which is nearer the greatest is greater than one more remote.

11. If from any point on the circumference of a circle straight lines be drawn to the circumference, the greatest is that which passes through the centre; and of others, that which is nearer to the greatest is greater than one more remote; and from this point there can be drawn to the circumference two, and only two, equal straight lines.

Proposition 14. Theorem.

Equal chords in a circle are equidistant from the centre.

Conversely, chords which are equidistant from the centre are equal.



Let ABC be a circle, and let AB and CD be chords, of which the perp. distances from the centre are EF and EG.

First. Let AB = CD.

Then shall AB *and* CD *be equidistant from the centre* E.

Construction. Join EA, EC.

Proof. Because EF, which passes through the centre, is perp. to the chord AB; *Hyp.*

∴ EF bisects AB; III. 3.

∴ AB is double of FA.

For a similar reason, CD is double of GC.

But AB = CD, *Hyp.*

∴ FA = GC. *Ax.* 7.

Now EA = EC, being radii of the circle;

∴ the sq. on EA = the sq. on EC.

But since the ∠ EFA is a rt. angle:

∴ the sq. on EA = the sqq. on EF, FA. I. 47.

And since the ∠ EGC is a rt. angle;

∴ the sq. on EC = the sqq. on EG, GC.

∴ the sqq. on EF, FA = the sqq. on EG, GC.

Now of these, the sq. on FA = the sq. on GC; for FA = GC.

∴ the sq. on EF = the sq. on EG;

∴ EF = EG;

that is, the chords AB, CD are equidistant from the centre.

Q.E.D.

Conversely. Let AB and CD be equidistant from the centre E;

that is, let EF = EG.
Then shall AB = CD.

Proof. The same construction being made, it may be shewn as before that AB is double of FA, and CD double of GC;

and that the sqq. on EF, FA = the sqq. on EG, GC.
Now of these, the sq. on EF = the sq. on EG,
for EF = EG: *Hyp.*
∴ the sq. on FA = the sq. on GC;
∴ FA = GC;
and doubles of these equals are equal; *Ax.* 6.
that is, AB = CD.

Q.E.D.

EXERCISES.

1. *Find the locus of the middle points of equal chords of a circle.*

2. If two chords of a circle cut one another, and make equal angles with the straight line which joins their point of intersection to the centre, they are equal.

3. *If two equal chords of a circle intersect, shew that the segments of the one are equal respectively to the segments of the other.*

4. In a given circle draw a chord which shall be equal to one given straight line (not greater than the diameter) and parallel to another.

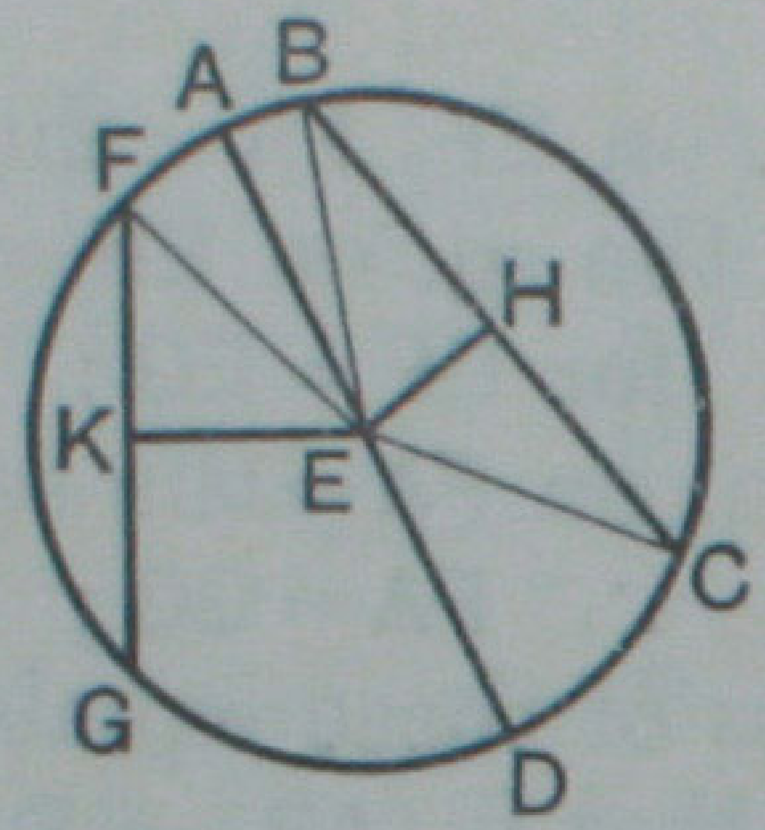
5. PQ is a fixed chord in a circle, and AB is any diameter: shew that the sum or difference of the perpendiculars let fall from A and B on PQ is constant. that is, the same for all positions of AB.

PROPOSITION 15. THEOREM.

The diameter is the greatest chord in a circle;

and of other chords, that which is nearer to the centre is greater than one more remote.

Conversely, the greater chord is nearer to the centre than the less.



Let ABCD be a circle of which AD is a diameter, and E the centre; and let BC and FG be any two chords, whose perp. distances from the centre are EH and EK.

Then (i) AD *shall be greater than* BC;

(ii) *if* EH *is less than* EK, BC *shall be greater than* FG:

(iii) *if* BC *is greater than* FG, EH *shall be less than* EK.

(i) **Construction.** Join EB, EC.

Proof. In the △ BEC, the two sides BE, EC are together greater than BC; I. 20.

but BE = AE, I. *Def.* 15.

and EC = ED;

∴ AE and ED together are greater than BC;

that is, AD is greater than BC.

Similarly AD may be shewn to be greater than any other chord, not a diameter.

(ii) Let EH be less than EK.

Then BC *shall be greater than* FG.

Construction. Join EF.

Proof. Since EH, passing through the centre, is perp. to the chord BC,

∴ EH bisects BC; III. 3.

∴ BC is double of HB.

For a similar reason FG is double of KF.

Now EB = EF, I. *Def.* 15.

∴ the sq. on EB = the sq. on EF.

But since the ∠ EHB is a rt. angle,

∴ the sq. on EB = the sqq. on EH, HB. I. 47.

And since the ∠ EKF is a rt. angle,

∴ the sq on EF = the sqq. on EK, KF;

∴ the sqq. on EH, HB = the sqq. on EK, KF.

But the sq. on EH is less than the sq. on EK,

for EH is less than EK; *Hyp.*

∴ the sq. on HB is greater than the sq. on KF;

∴ HB is greater than KF:

hence BC is greater than FG.

(iii) *Conversely.* Let BC be greater than FG.

Then EH *shall be less than* EK.

Proof. The same construction being made, it may be shewn as before that BC is double of BH and FG double of FK; and that the sqq. on EH, HB = the sqq. on EK, KF.

But since BC is greater than FG, *Hyp.*

∴ HB is greater than KF;

∴ the sq. on HB is greater than the sq on KF.

∴ the sq. on EH is less than the sq. on EK;

∴ EH is less than EK. Q.E.D.

EXERCISES.

1. *Through a given point within a circle draw the least possible chord.*

2. AB is a fixed chord of a circle, and XY any other chord having its middle point Z on AB; what is the greatest, and what the least length that XY may have?

Shew that XY increases, as Z approaches the middle point of AB.

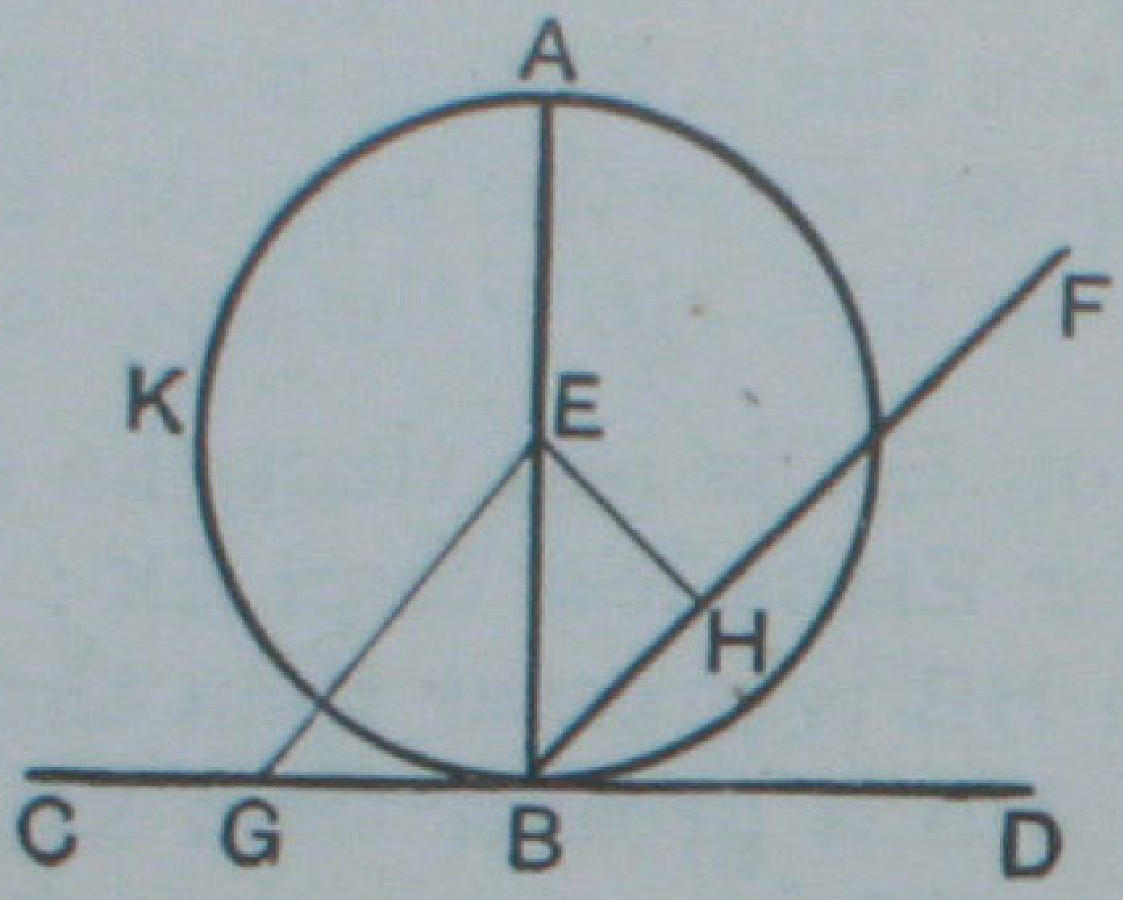
3. In a given circle draw a chord of given length, having its middle point on a given chord.

When is this problem impossible?

PROPOSITION 16. THEOREM. [ALTERNATIVE PROOF.]

The straight line drawn at right angles to a diameter of a circle at one of its extremities is a tangent to the circle:

and every other straight line drawn through this point cuts the circle.



Let AKB be a circle, of which E is the centre, and AB a diameter; and through B let the st. line CBD be drawn at rt. angles to AB.

Then (i) CBD *shall be a tangent to the circle*;

(ii) *any other st. line through* B, *such as* BF, *shall cut the circle.*

(i) **Construction.** In CD take any point G, and join EG

Proof. In the △ EBG, the ∠ EBG is a rt. angle; *Hyp.*

∴ the ∠ EGB is less than a rt. angle; I. 17.

∴ the ∠ EBG is greater than the ∠ EGB;

∴ EG is greater than EB: I. 19.

that is, EG is greater than a radius of the circle;

∴ the point G is without the circle.

Similarly any other point in CD, except B, may be shewn to be outside the circle.

Hence CD meets the circle at B, but being produced, does not cut it;

that is, CD is a tangent to the circle. III. *Def.* 5.

(ii) **Construction.** Draw EH perp. to BF. I. 12.

Proof. In the △ EHB, because the ∠ EHB is a rt. angle,

∴ the ∠ EBH is less than a rt. angle; I. 17.

∴ EB is greater than EH; I. 19.

that is, EH is less than a radius of the circle:

∴ H, a point in BF, is within the circle;

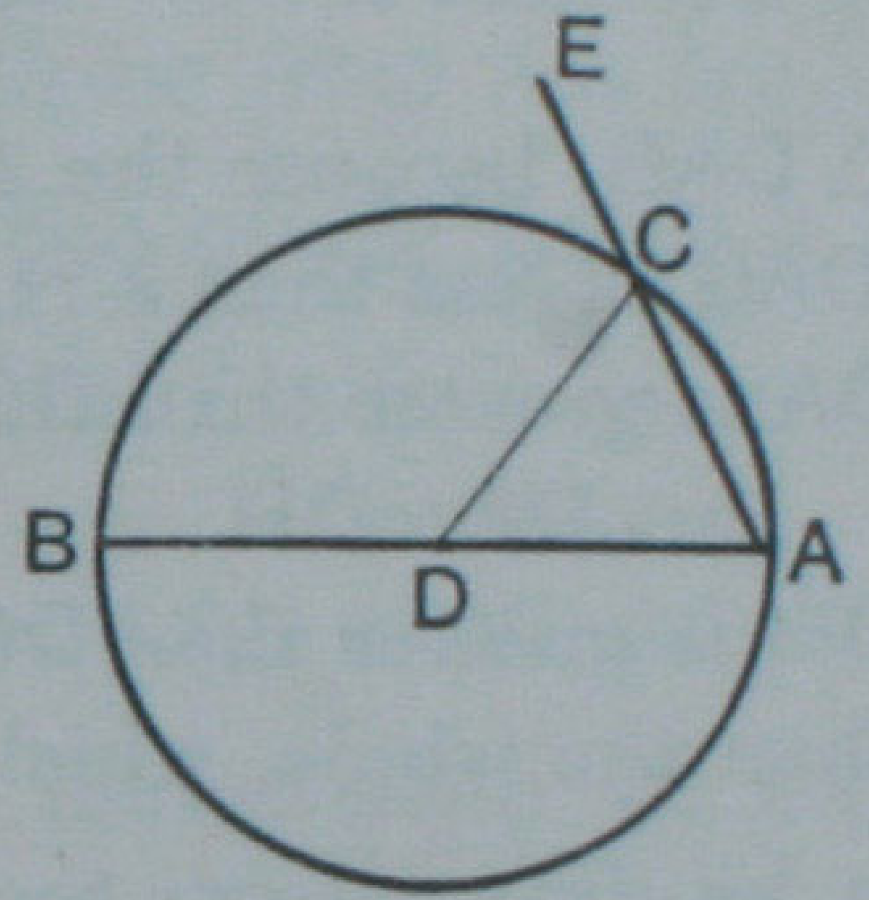
∴ BF must cut the circle. Q.E.D.

NOTE. The above proof of Proposition 16 is not that given by Euclid, but it is preferable as being *direct.* Euclid's proof by *Reductio ad Absurdum* is given below.

PROPOSITION 16. THEOREM. [EUCLID'S PROOF.]

The straight line drawn at right angles to a diameter of a circle at one of its extremities, is a tangent to the circle:

and no other straight line can be drawn through this point so as not to cut the circle.



Let ABC be a circle, of which D is the centre, and AB a diameter; let AE be drawn at rt. angles to BA, at its extremity A.

(i) *Then shall* AE *be a tangent to the circle.*

Construction.

For, if possible, suppose AE to cut the circle at C.

Join DC.

Proof. Then in the △ DAC, because DA = DC, I. *Def.* 15.

∴ the ∠ DAC = the ∠ DCA.

But the ∠ DAC is a rt. angle; *Hyp.*

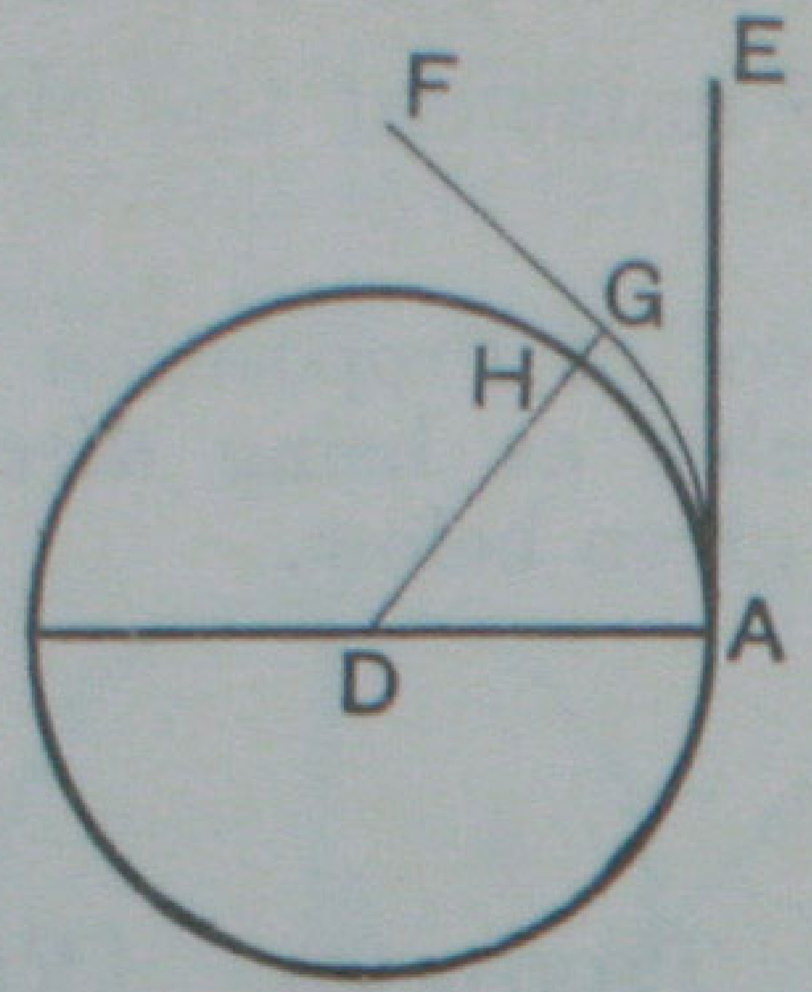
∴ the ∠ DCA is a rt. angle;

that is, two angles of the △ DAC are together equal to two rt. angles; which is impossible. I. 17.

Hence AE meets the circle at A, but being produced, does not cut it;

that is, AE is a tangent to the circle. III. *Def.* 5.

(ii) *Also through* A *no other straight line but* AE *can be drawn so as not to cut the circle.*



Construction. For, if possible, let AF be another st. line drawn through A so as not to cut the circle.

From D draw DG perp. to AF ; I. 12.
and let DG meet the ○ᶜᵉ at H.

Proof. Then in the △ DAG, because the ∠ DGA is a rt. angle,
∴ the ∠ DAG is less than a rt. angle ; I. 17.
∴ DA is greater than DG. I. 19.
But DA = DH, I. *Def.* 15.
∴ DH is greater than DG,
the part greater than the whole, which is impossible.

Therefore no st. line can be drawn from the point A, so as not to cut the circle, except AE.

COROLLARY. (i) *A tangent touches a circle at one point only.*

COROLLARY. (ii) *There can be but one tangent to a circle at a given point.*

PROPOSITION 17. PROBLEM.

To draw a tangent to a circle from a given point either on, or without the circumference.

Fig. 1.

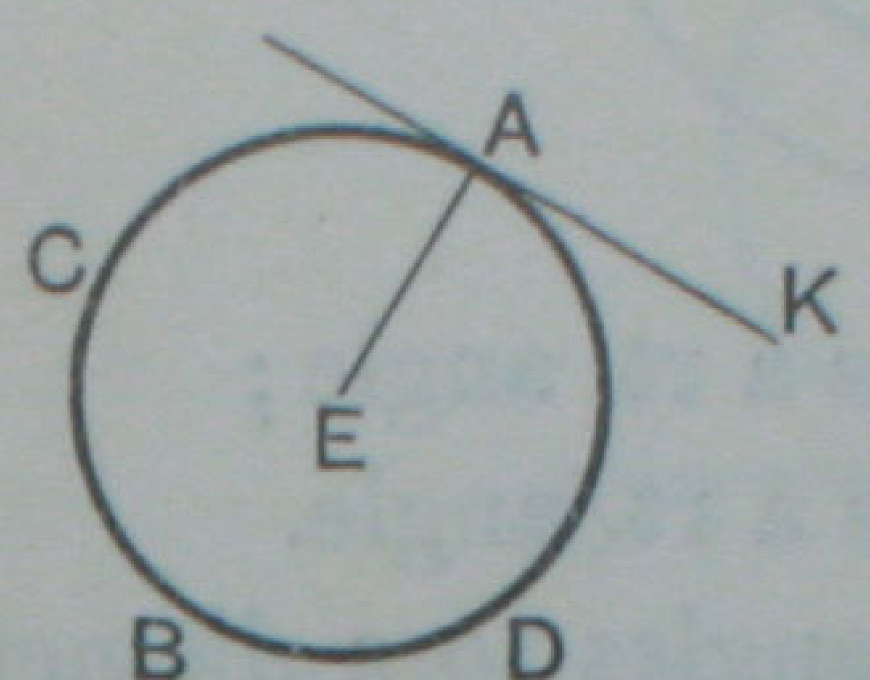
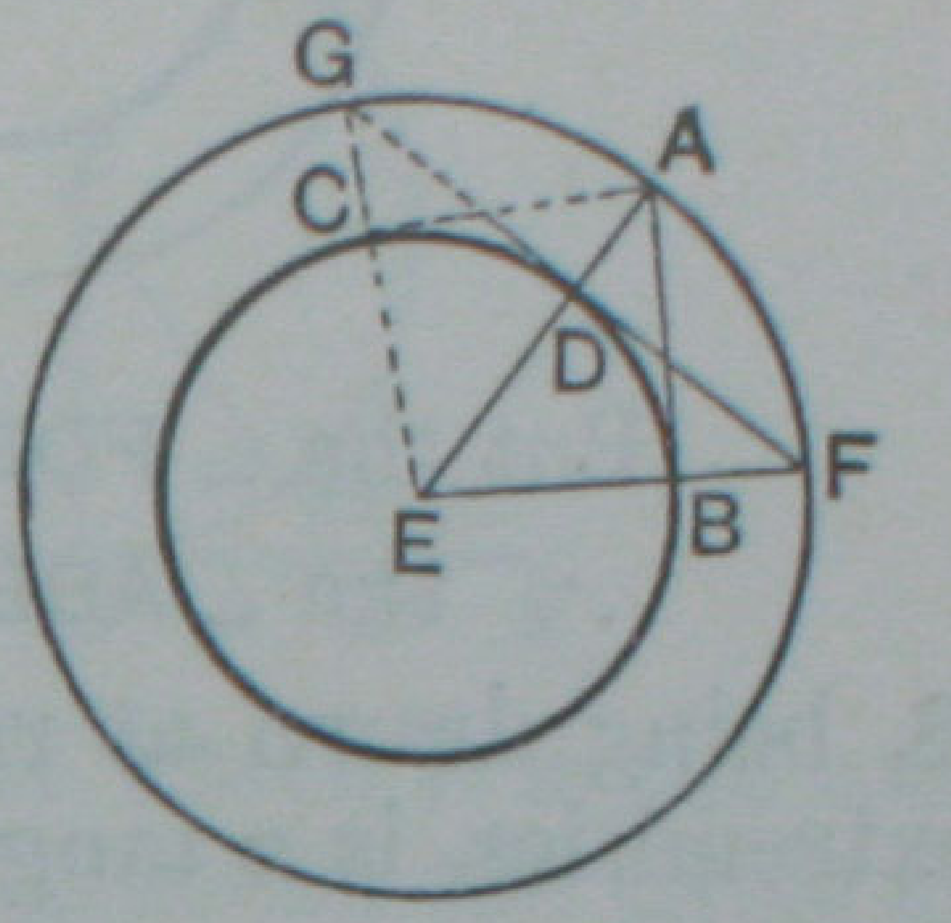


Fig. 2.



Let BCD be the given circle, and A the given point.

It is required to draw from A *a tangent to the* ⊙ CDB.

CASE I. When the given point A is on the ○ce.

Construction. Find E, the centre of the circle. III. 1.
Join EA
At A draw AK at rt. angles to EA. I. 11.

Proof. Then AK being perp to a diameter at one of its extremities, is a tangent to the circle. III. 16.

CASE II. When the given point A is without the ○ce.

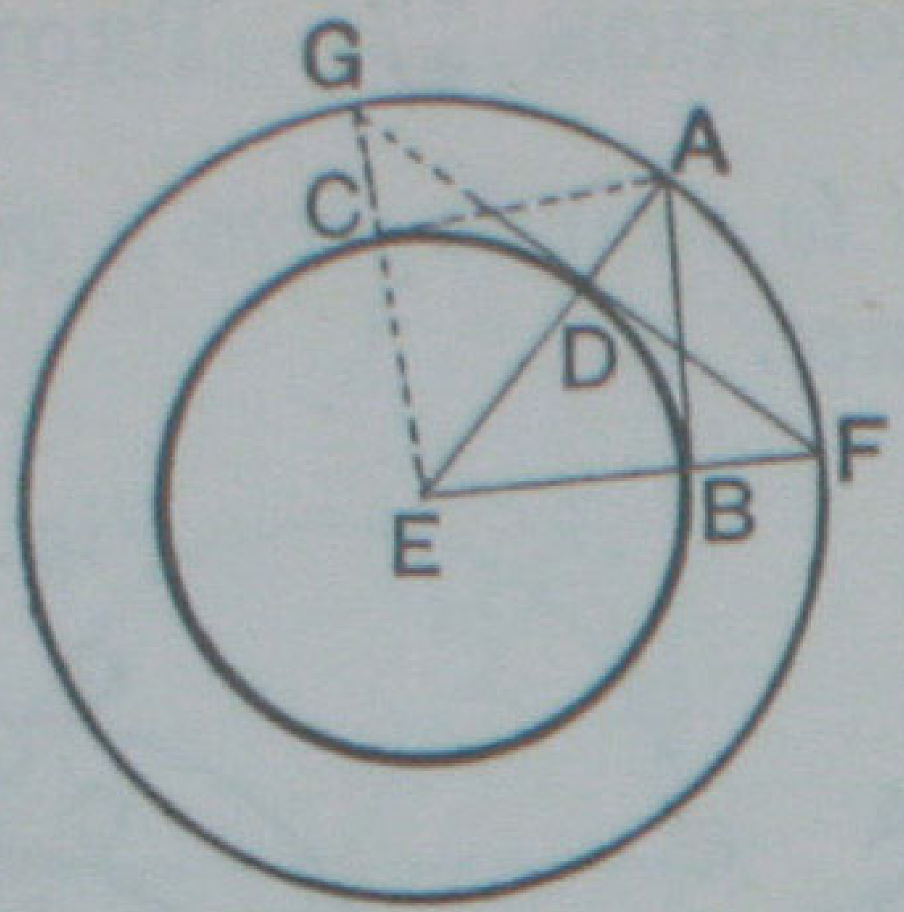
Construction. Find E, the centre of the circle; III. 1.
and join AE, cutting the ⊙ BCD at D.
With centre E and radius EA, describe the ⊙ AFG.
At D, draw GDF at rt. angles to EA, cutting the ⊙ AFG at F and G. I. 11.
Join EF, EG, cutting the ⊙ BCD at B and C.
Join AB, AC.

Then both AB *and* AC *shall be tangents to the* ⊙ CDB.

Proof. In the △s AEB, FED,

Because { AE = FE, being radii of the ⊙ GAF;
and EB = ED, being radii of the ⊙ BDC;
and the included angle AEF is common;

∴ the ∠ ABE = the ∠ FDE. I. 4.



But the ∠ FDE is a rt. angle; *Constr.*

∴ the ∠ ABE is a rt. angle.

Hence AB, being drawn at rt. angles to a diameter at one of its extremities, is a tangent to the ⊙ BCD. III. 16.

Similarly it may be shewn that AC is a tangent. Q.E.F.

COROLLARY. *If two tangents are drawn to a circle from an external point, then* (i) *they are equal;* (ii) *they subtend equal angles at the centre;* (iii) *they make equal angles with the straight line which joins the given point to the centre.*

For, in the above figure,

Since ED is perp. to FG, a chord of the ⊙ FAG, *Constr.*

∴ DF = DG. III. 3.

Then in the △s DEF, DEG,

Because { DE is common to both,
and EF = EG; I. *Def.* 15.
and DF = DG; *Proved.*

∴ the ∠ DEF = the ∠ DEG. I. 8.

Again in the △s AEB, AEC,

Because { AE is common to both,
and EB = EC,
and the ∠ AEB = the ∠ AEC; *Proved.*

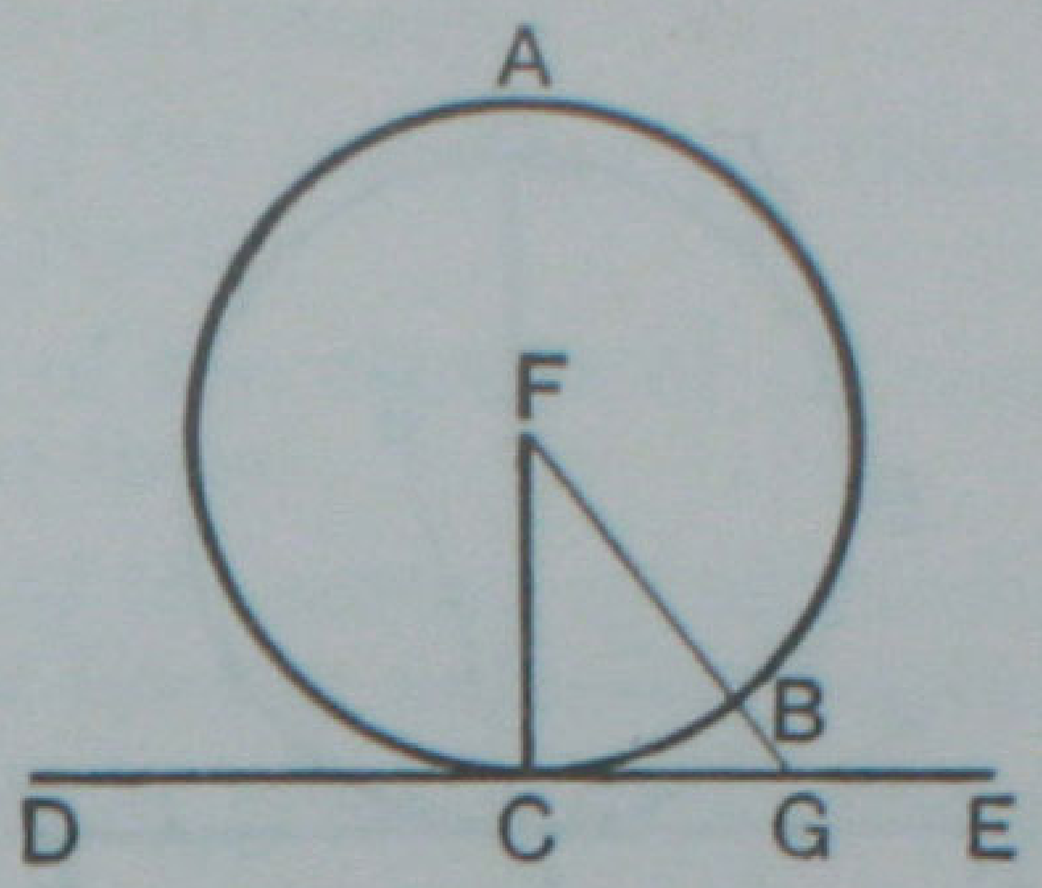
∴ AB = AC: I. 4.

and the ∠ EAB = the ∠ EAC. Q.E.D.

NOTE. If the given point A is within the circle, no solution is possible. Hence we see that this problem admits of *two* solutions, *one* solution, or *no* solution, according as the given point A is *without*, *on*, or *within* the circumference of a circle. For a simpler method of drawing a tangent to a circle from a given point, see page 218.

Proposition 18. Theorem.

The straight line drawn from the centre of a circle to the point of contact of a tangent is perpendicular to the tangent.



Let ABC be a circle, of which F is the centre;
and let the st. line DE touch the circle at C.

Then shall FC *be perp. to* DE.

For, if not, suppose, if possible, FG to be perp. to DE, I. 12.
and let FG meet the ○ce at B.

Proof.

In the △ FCG, because the ∠ FGC is a rt. angle, *Hyp.*
∴ the ∠ FCG is less than a rt. angle; I. 17.
∴ the ∠ FGC is greater than the ∠ FCG;
∴ FC is greater than FG: I. 19.
but FC = FB;
∴ FB is greater than FG,
the part greater than the whole, which is impossible.
∴ FC cannot be otherwise than perp. to DE:
that is, FC is perp. to DE. Q.E.D.

EXERCISES.

1. Draw a tangent to a circle (i) parallel to, (ii) at right angles to a given straight line.

2. *Tangents drawn to a circle from the extremities of a diameter are parallel.*

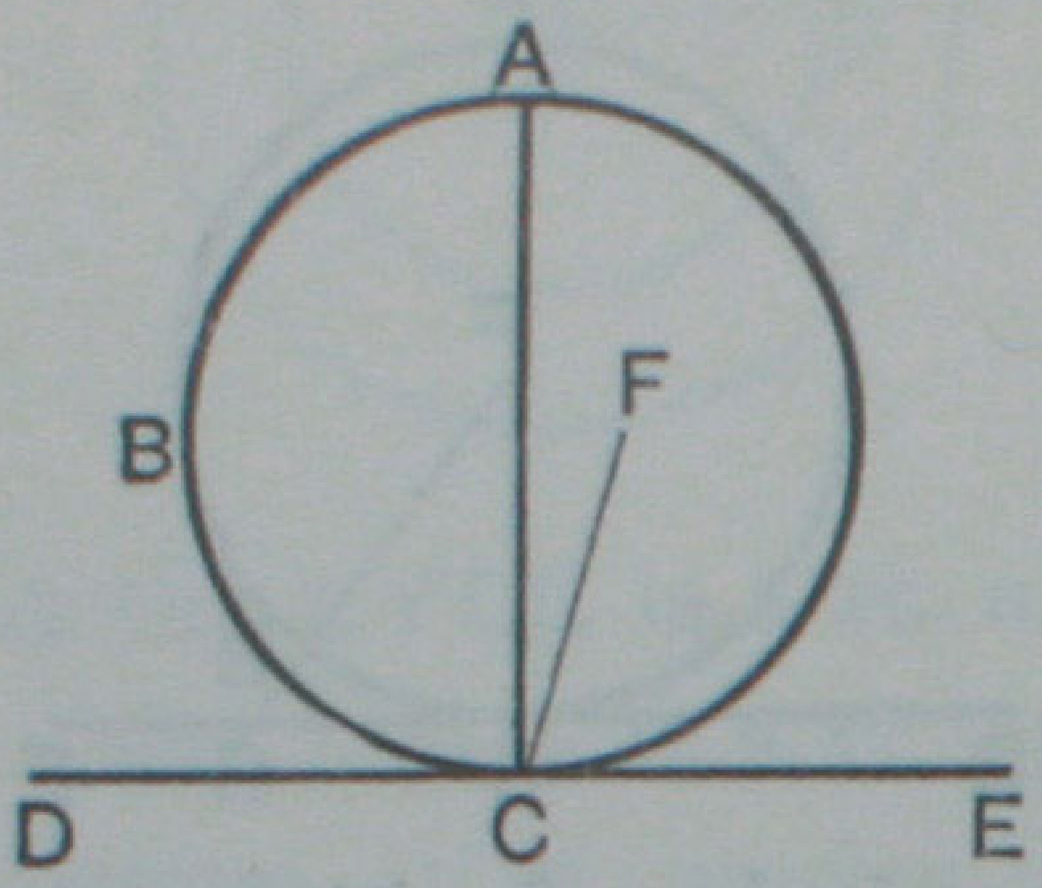
3. *Circles which touch one another internally or externally have a common tangent at their point of contact.*

4. *In two concentric circles, any chord of the outer circle which touches the inner, is bisected at the point of contact.*

5. *In two concentric circles, all chords of the outer circle which touch the inner, are equal.*

PROPOSITION 19. THEOREM.

The straight line drawn perpendicular to a tangent to a circle from the point of contact passes through the centre.



Let ABC be a circle, and DE a tangent to it at the point C; and let CA be drawn perp. to DE.

Then shall CA *pass through the centre.*

Construction. For if not, suppose, if possible, the centre F to be outside CA.

Join CF.

Proof. Because DE is a tangent to the circle, and FC is drawn from the centre F to the point of contact,

∴ the ∠ FCE is a rt. angle. III. 18.

But the ∠ ACE is a rt. angle; *Hyp.*

∴ the ∠ FCE = the ∠ ACE;

the part equal to the whole, which is impossible.

∴ the centre cannot be otherwise than in CA;

that is, CA passes through the centre.

Q.E.D.

EXERCISES ON THE TANGENT.

PROPOSITIONS 16, 17, 18, 19.

1. *The centre of any circle which touches two intersecting straight lines must lie on the bisector of the angle between them.*

2. AB and AC are two tangents to a circle whose centre is O; shew that AO bisects the chord of contact BC at right angles.

3. If two circles are concentric all tangents drawn from points on the circumference of the outer to the inner circle are equal.

4. The diameter of a circle bisects all chords which are parallel to the tangent at either extremity.

5. *Find the locus of the centres of all circles which touch a given straight line at a given point.*

6. Find the locus of the centres of all circles which touch each of two parallel straight lines.

7. *Find the locus of the centres of all circles which touch each of two intersecting straight lines of unlimited length.*

8. Describe a circle of given radius to touch two given straight lines.

9. Through a given point, within or without a circle, draw a chord equal to a given straight line.

In order that the problem may be possible, between what limits must the given line lie, when the given point is (i) without the circle, (ii) within it?

10. Two parallel tangents to a circle intercept on any third tangent a segment which subtends a right angle at the centre.

11. *In any quadrilateral circumscribed about a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.*

12. Any parallelogram which can be circumscribed about a circle, must be equilateral.

13. If a quadrilateral is described about a circle, the angles subtended at the centre by any two opposite sides are together equal to two right angles.

14. AB is any chord of a circle, AC the diameter through A, and AD the perpendicular on the tangent at B: shew that AB bisects the angle DAC.

15. Find the locus of the extremities of tangents of fixed length drawn to a given circle.

16. In the diameter of a circle produced, determine a point such that the tangent drawn from it shall be of given length.

17. In the diameter of a circle produced, determine a point such that the two tangents drawn from it may contain a given angle.

18. Describe a circle that shall pass through a given point, and touch a given straight line at a given point. [See page 197. Ex. 5.]

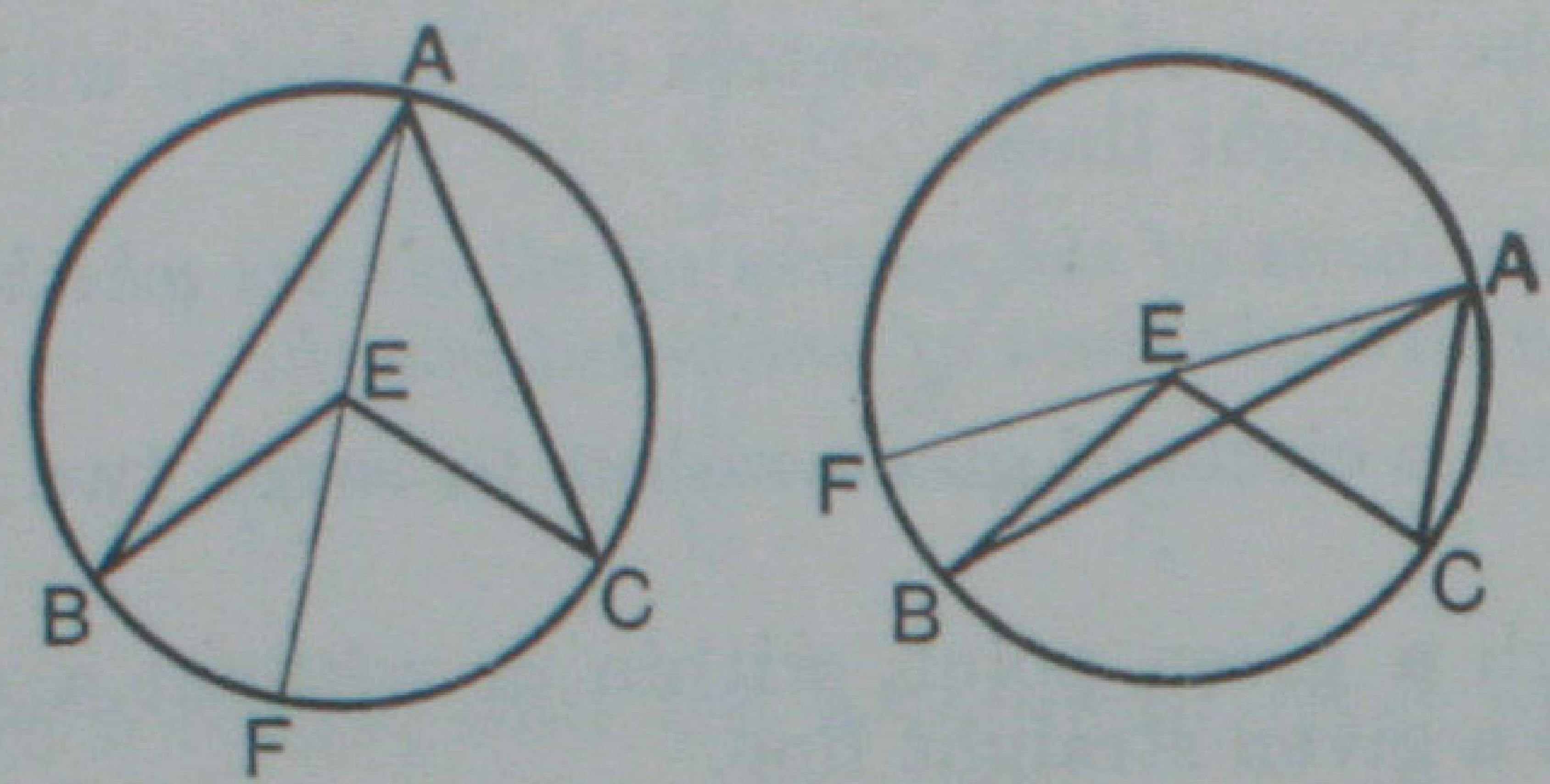
19. Describe a circle of given radius, having its centre on a given straight line, and touching another given straight line.

20. Describe a circle that shall have a given radius, and touch a given circle and a given straight line. How many such circles can be drawn?

Proposition 20. Theorem.

The angle at the centre of a circle is double of an angle at the circumference, standing on the same arc.

Fig. 1. Fig. 2.



Let ABC be a circle, of which E is the centre; and let BEC be the angle at the centre, and BAC an angle at the ○ce, standing on the same arc BC.

Then shall the ∠ BEC *be double of the* ∠ BAC.

Construction. Join AE, and produce it to F.

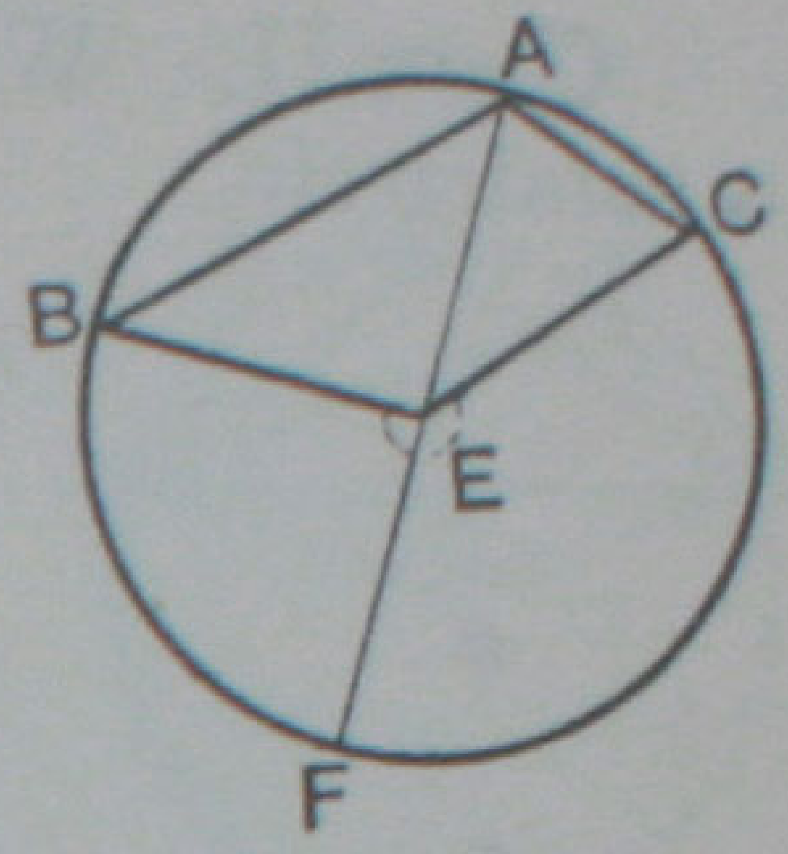
Case I. When the centre E is within the angle BAC.

Proof. In the △ EAB, because EA = EB, I. *Def.* 15.
∴ the ∠ EAB = the ∠ EBA; I. 5.
∴ the sum of the ∠s EAB, EBA = twice the ∠ EAB.
But the ext. ∠ BEF = the sum of the ∠s EAB, EBA; I. 32.
∴ the ∠ BEF = twice the ∠ EAB.
Similarly the ∠ FEC = twice the ∠ EAC.
∴ the sum of the ∠s BEF, FEC = twice the sum of the ∠s EAB, EAC;
that is, the ∠ BEC = twice the ∠ BAC.

Case II. When the centre E is without the ∠ BAC.

As before, it may be shewn that the ∠ FEB = twice the ∠ FAB;
also the ∠ FEC = twice the ∠ FAC;
∴ the difference of the ∠s FEC, FEB = twice the difference of the ∠s FAC, FAB:
that is, the ∠ BEC = twice the ∠ BAC. Q.E.D.

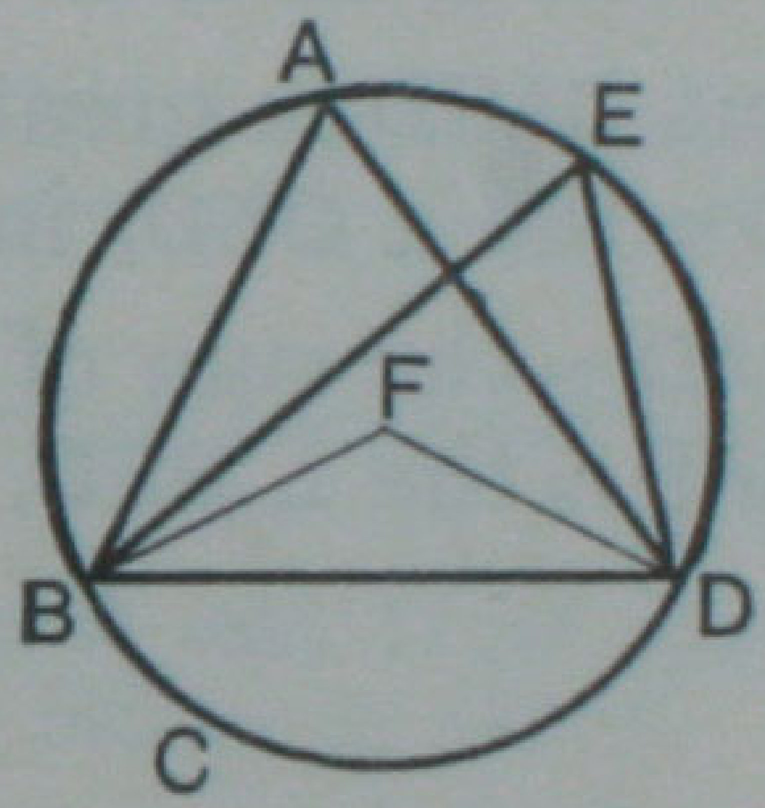
NOTE 1. The case in which the centre E falls *on* AB or AC needs no proof beyond that given under Case I.



NOTE 2. If the arc BFC, on which the angles stand, is greater than a semi-circumference, the angle BEC at the centre will be *reflex*: but it may still be shewn as, in Case I., that the reflex ∠ BEC is double of the ∠ BAC at the ○ce, standing on the same arc BFC.

PROPOSITION 21. THEOREM.

Angles in the same segment of a circle are equal.



Let ABCD be a circle, and let BAD, BED be angles in the same segment BAED.

Then shall the ∠ BAD = *the* ∠ BED.

Construction. Find F, the centre of the circle. III. 1.

CASE I. When the segment BAED is greater than a semicircle.

Join BF, DF.

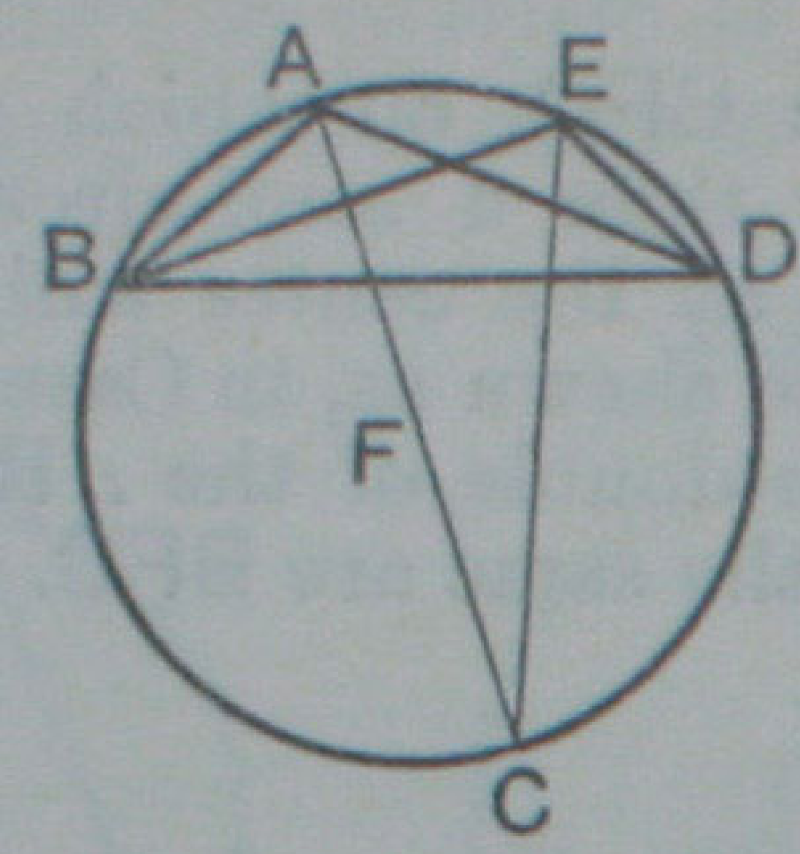
Proof. Because the ∠ BFD is at the centre, and the ∠ BAD at the ○ce, standing on the same arc BD,

∴ the ∠ BFD = twice the ∠ BAD. III. 20.

Similarly the ∠ BFD = twice the ∠ BED. III. 20.

∴ the ∠ BAD = the ∠ BED. *Ax.* 7.

CASE II. When the segment BAED is not greater than a semicircle.



Construction.

Join AF, and produce it to meet the ○ᶜᵉ at C.
Join EC.

Proof. Then since AEDC is a semicircle;
∴ the segment BAEC is greater than a semicircle:
∴ the ∠ BAC = the ∠ BEC, in this segment. *Case* 1

Similarly the segment CAED is greater than a semicircle;
∴ the ∠ CAD = the ∠ CED, in this segment.
∴ the ∠ˢ BAC, CAD = the sum of the ∠ˢ BEC, CED.
that is, the ∠ BAD = the ∠ BED. Q.E.D.

EXERCISES.

1. P is any point on the arc of a segment of which AB is the chord. Shew that the sum of the angles PAB, PBA is constant.

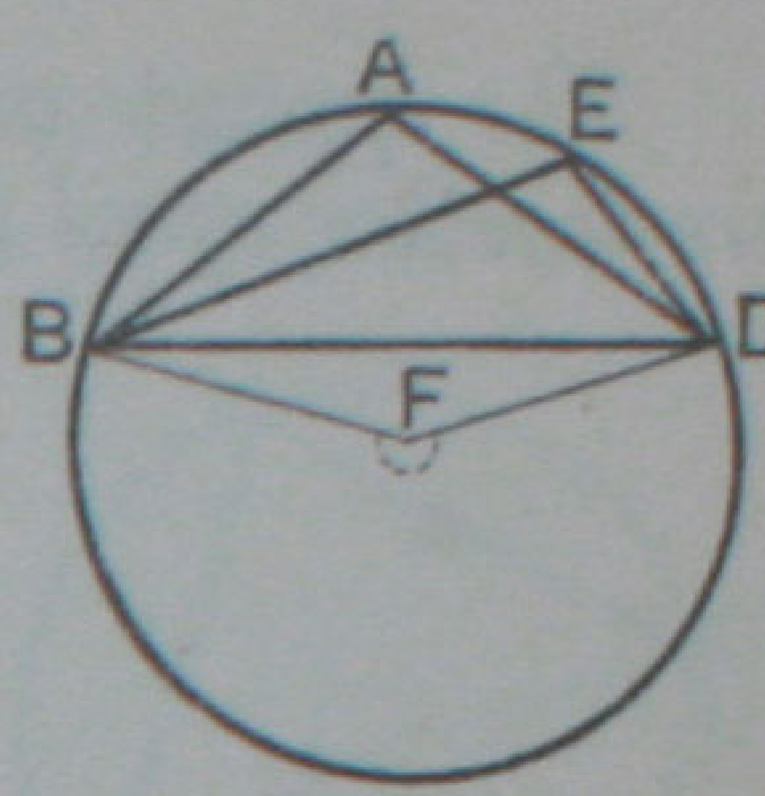
2. PQ and RS are two chords of a circle intersecting at X: prove that the triangles PXS, RXQ are equiangular.

3. Two circles intersect at A and B; and through A any straight line PAQ is drawn terminated by the circumferences: shew that PQ subtends a constant angle at B.

4. Two circles intersect at A and B; and through A any two straight lines PAQ, XAY are drawn terminated by the circumferences; shew that the arcs PX, QY subtend equal angles at B.

5. P is any point on the arc of a segment whose chord is AB: and the angles PAB, PBA are bisected by straight lines which intersect at O. Find the locus of the point O.

NOTE. If the extension of Proposition 20, given in Note 2 on page 199, is adopted, a separate treatment of the second case of the present proposition is unnecessary.



For, as in Case I.,

the reflex ∠ BFD = twice the ∠ BAD; III. 20.

also the reflex ∠ BFD = twice the ∠ BED;

∴ the ∠ BAD = the ∠ BED.

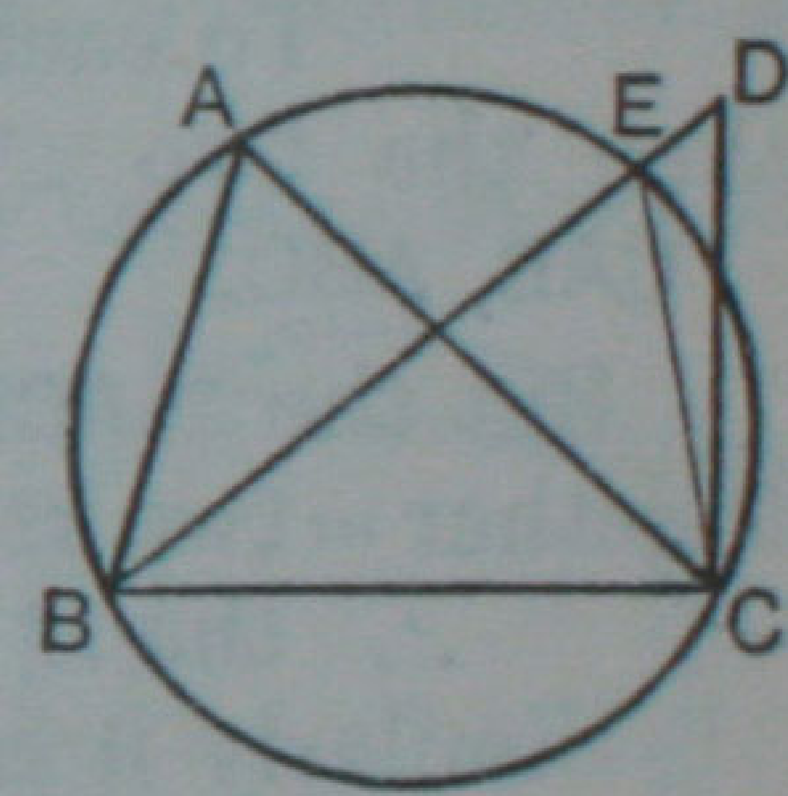
Obs. The converse of Prop. 21 is important. For the construction used, viz. *To describe a circle about a given triangle*, see Book IV., Prop. 5, or Theorems and Examples on Book I, page 111, No. 1.

CONVERSE OF PROPOSITION 21.

Equal angles standing on the same base, and on the same side of it, have their vertices on an arc of a circle, of which the given base is the chord.

Let BAC, BDC be two equal angles standing on the same base BC.

Then shall the vertices A *and* D *lie upon a segment of a circle having* BC *as its chord.*



Describe a circle about the △ BAC. IV. 5.

Then this circle shall pass through D.

For, if not, it must cut BD, or BD produced, at some other point E.

Join EC.

Then the ∠ BAC = the ∠ BEC, in the same segment: III. 21.

but the ∠ BAC = the ∠ BDC, by hypothesis;

∴ the ∠ BEC = the ∠ BDC;

that is, an ext. angle of a triangle = an int. opp. angle;

which is impossible. I. 16.

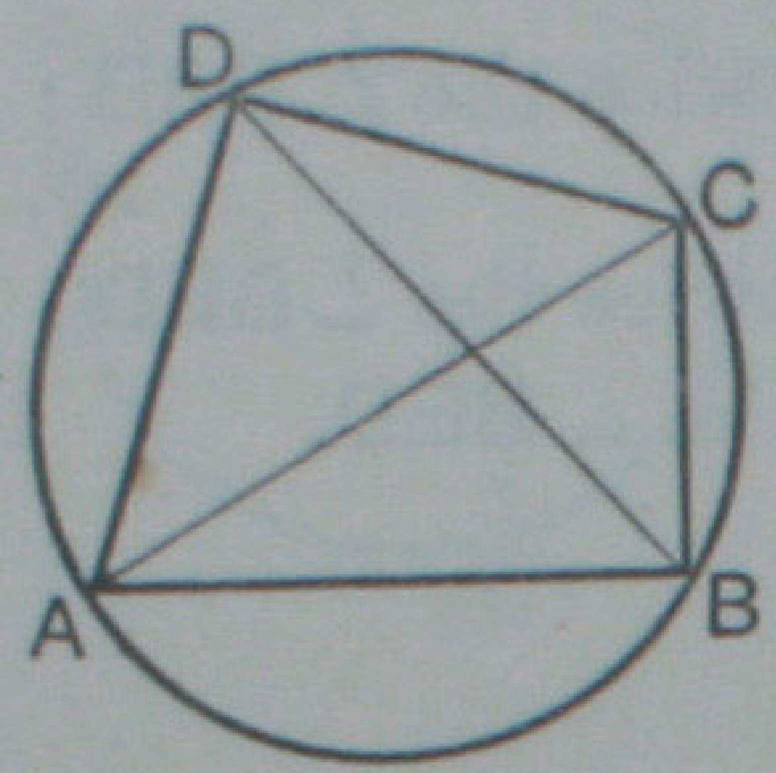
∴ the circle which passes through B, A, C, cannot pass otherwise than through D.

That is, the vertices A and D are on an arc of a circle of which the chord is BC. Q.E.D.

COROLLARY. *The locus of the vertices of triangles drawn on the same base and on the same side of it with equal vertical angles is an arc of a circle.*

PROPOSITION 22. THEOREM.

The opposite angles of any quadrilateral inscribed in a circle are together equal to two right angles.



Let ABCD be a quadrilateral inscribed in the ⊙ ABC.

Then shall (i) *the* ∠s ADC, ABC *together = two rt. angles*;

(ii) *the* ∠s BAD, BCD *together = two rt. angles.*

Construction. Join AC, BD.

Proof.

Since the ∠ ADB = the ∠ ACB, in the segment ADCB; III. 21.

and the ∠ CDB = the ∠ CAB, in the segment CDAB;

∴ the ∠ ADC = the sum of the ∠s ACB, CAB.

To each of these equals add the ∠ ABC:

then the two ∠s ADC, ABC together = the three ∠s ACB, CAB, ABC.

But the ∠s ACB, CAB, ABC, being the angles of a triangle, together = two rt. angles; I. 32.

∴ the ∠s ADC, ABC together = two rt. angles.

Similarly it may be shewn that

the ∠s BAD, BCD together = two rt. angles. Q.E.D.

EXERCISES.

1. If a circle can be described about a parallelogram, the parallelogram must be rectangular.

2. ABC is an isosceles triangle, and XY is drawn parallel to the base BC cutting the sides in X and Y: shew that the four points B, C, X, Y lie on a circle.

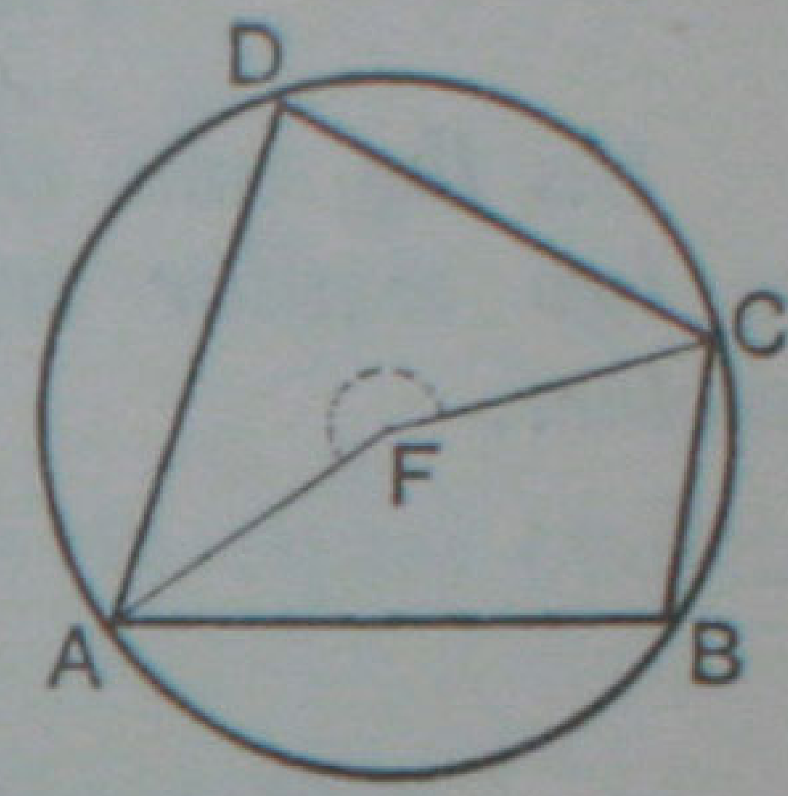
3. *If one side of a quadrilateral inscribed in a circle is produced, the exterior angle is equal to the opposite interior angle of the quadrilateral.*

PROPOSITION 22. [ALTERNATIVE PROOF.]

Let ABCD be a quadrilateral inscribed in the ⊙ ABC.

Then shall the ∠ˢ ADC, ABC *together = two rt. angles.*

Join FA, FC.



Since the ∠ AFC at the centre = twice the ∠ ADC at the Oᶜᵉ, standing on the same arc ABC; III. 20.

and the reflex angle AFC at the centre = twice the ∠ ABC at the Oᶜᵉ, standing on the same arc ADC; III. 20.

∴ the ∠ˢ ADC, ABC are together half the sum of the ∠ AFC and the reflex angle AFC;

but these make up four rt. angles: I. 15. *Cor.* 2.

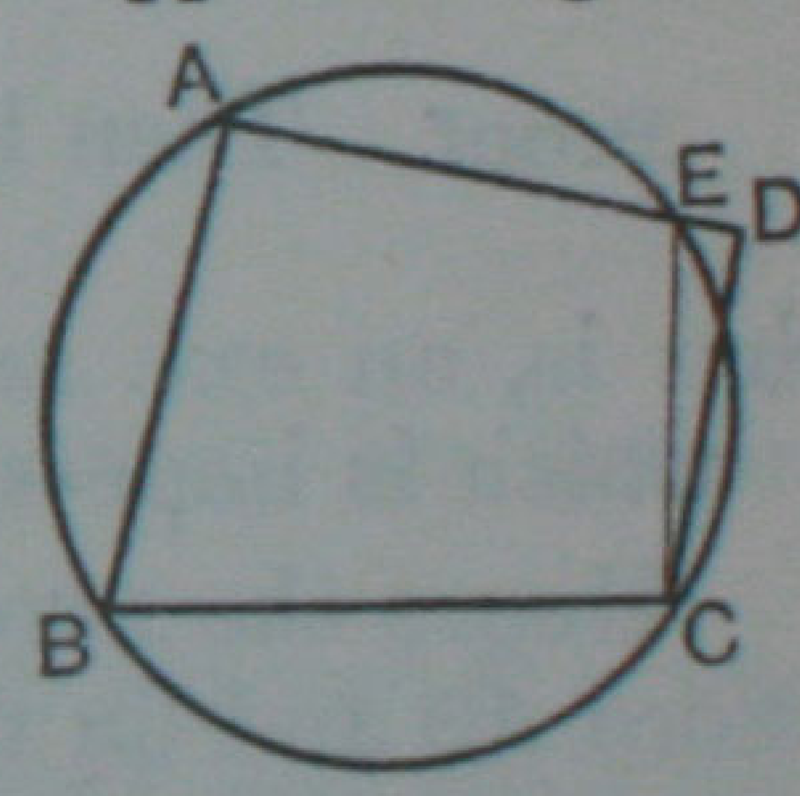
∴ the ∠ˢ ADC, ABC together = two rt. angles. Q.E.D.

DEFINITION. Four or more points through which a circle may be described are said to be **concyclic.**

CONVERSE OF PROPOSITION 22.

If a pair of opposite angles of a quadrilateral are together equal to two right angles, its vertices are concyclic.

Let ABCD be a quadrilateral, in which the opposite angles at B and D together = two rt. angles.



Then shall the four points A, B, C, D *be concyclic.*

Through the three points A, B, C describe a circle. IV. 5.

Then this circle must pass through D.

For, if not, it will cut AD, or AD produced, at some other point E.

Join EC.

Then since the quadrilateral ABCE is inscribed in a circle,

∴ the ∠ˢ ABC, AEC together = two rt. angles. III. 22.

But the ∠ˢ ABC, ADC together = two rt. angles; *Hyp.*

hence the ∠ˢ ABC, AEC = the ∠ˢ ABC, ADC.

Take from these equals the ∠ ABC;

then the ∠ AEC = the ∠ ADC;

that is, an ext. angle of a triangle = an int. opp. angle; which is impossible. I. 16.

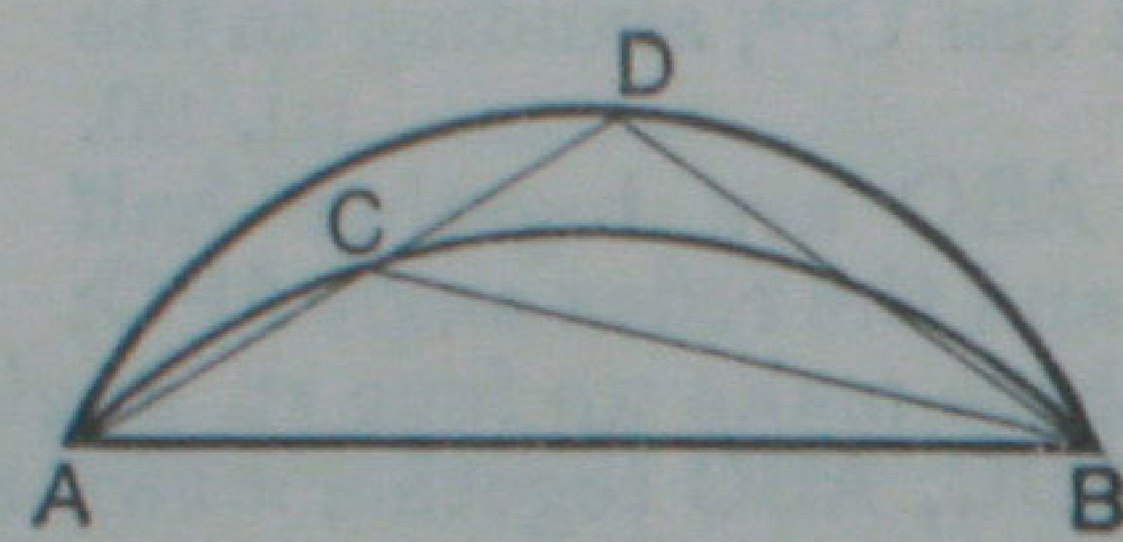
∴ the circle which passes through A, B, C cannot pass otherwise than through D:

that is the four vertices A, B, C, D are concyclic. Q.E.D.

DEFINITION. **Similar segments** of circles are those which contain equal angles. [Book III., Def. 10.]

PROPOSITION 23. THEOREM.

On the same chord and on the same side of it, there cannot be two similar segments of circles, not coinciding with one another.



If possible, on the same chord AB, and on the same side of it, let there be two similar segments of circles ACB, ADB, not coinciding with one another.

Then since the arcs ADB, ACB intersect at A and B,
∴ they cannot cut one another again; III. 10.
∴ one segment falls within the other.

Construction. In the inner arc take any point C.
Join AC, producing it to meet the outer arc at D:
join CB, DB.

Proof. Then because the segments are similar,
∴ the ∠ ACB = the ∠ ADB; III. *Def.* 10.
that is, an ext. angle of the △ CDB = an int. opp. angle;
which is impossible. I. 16.

Hence the two similar segments ACB, ADB, on the same chord AB and on the same side of it, must coincide. Q.E.D.

EXERCISES ON PROPOSITION 22.

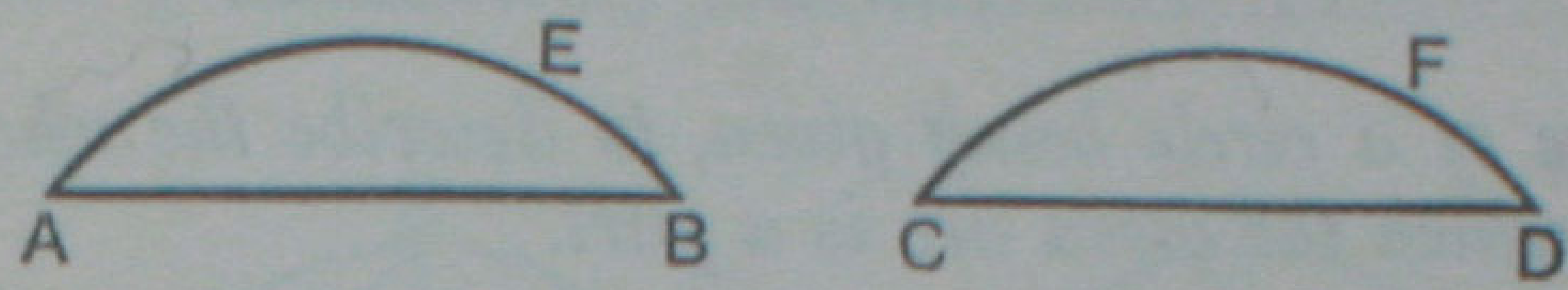
1. The straight lines which bisect any angle of a quadrilateral figure inscribed in a circle and the opposite exterior angle, meet on the circumference.

2. A triangle is inscribed in a circle: shew that the sum of the angles in the three segments exterior to the triangle is equal to four right angles.

3. Divide a circle into two segments, so that the angle contained by the one shall be double of the angle contained by the other.

PROPOSITION 24. THEOREM.

Similar segments of circles on equal chords are equal to one another.



Let AEB and CFD be similar segments on equal chords AB, CD.

Then shall the segment AEB = *the segment* CFD.

Proof. If the segment AEB be applied to the segment CFD, so that A falls on C, and AB falls along CD;

then since AB = CD, *Hyp.*

∴ B must coincide with D.

∴ the segment AEB must coincide with the segment CFD; for if not, on the same chord and on the same side of it there would be two similar segments of circles, not coinciding with one another: which is impossible. III. 23.

∴ the segment AEB = the segment CFD. Q.E.D.

EXERCISES.

1. Of two segments standing on the same chord, the greater segment contains the smaller angle.

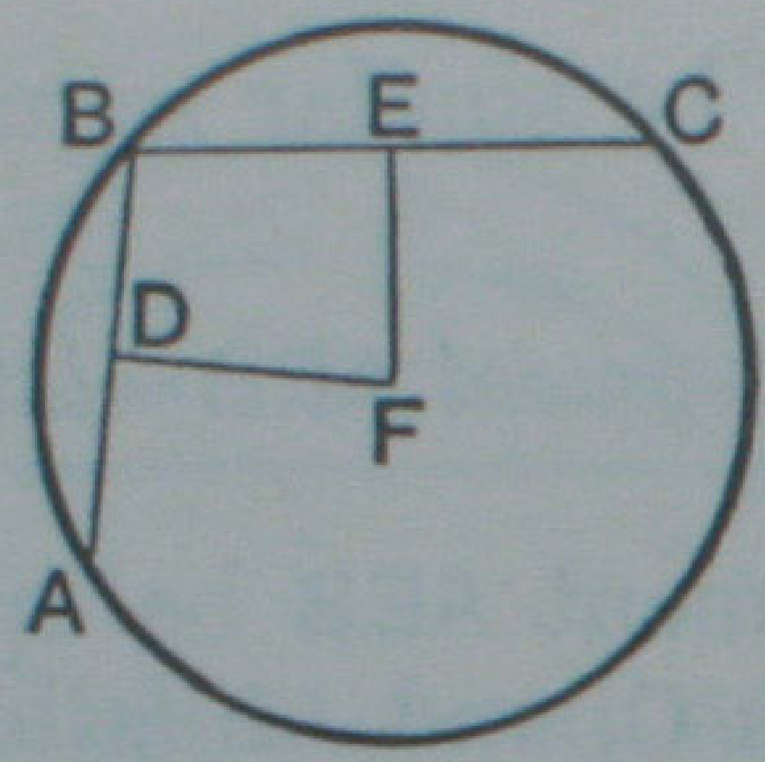
2. A segment of a circle stands on a chord AB, and P is any point on the same side of AB as the segment: shew that the angle APB is greater or less than the angle in the segment, according as P is within or without the segment.

3. P, Q, R *are the middle points of the sides of a triangle, and* X *is the foot of the perpendicular let fall from one vertex on the opposite side: shew that the four points* P, Q, R, X *are concyclic.*
[See page 104, Ex. 2: also page 108, Ex. 2.]

4. *Use the preceding exercise to shew that the middle points of the sides of a triangle and the feet of the perpendiculars let fall from the vertices on the opposite sides, are concyclic.*

Proposition 25. Problem.

An arc of a circle being given, to describe the whole circumference of which the given arc is a part.



Let ABC be an arc of a circle.

It is required to describe the whole ○ᶜᵉ *of which the arc* ABC *is a part.*

Construction.

In the given arc take any three points A, B, C.

Join AB, BC.

Draw DF bisecting AB at rt. angles, I. 10. 11.

and draw EF bisecting BC at rt. angles.

Proof.

Then because DF bisects the chord AB at rt. angles,

∴ the centre of the circle lies in DF. III. 1 *Cor.*

Again, because EF bisects the chord BC at rt. angles,

∴ the centre of the circle lies in EF. III. 1 *Cor.*

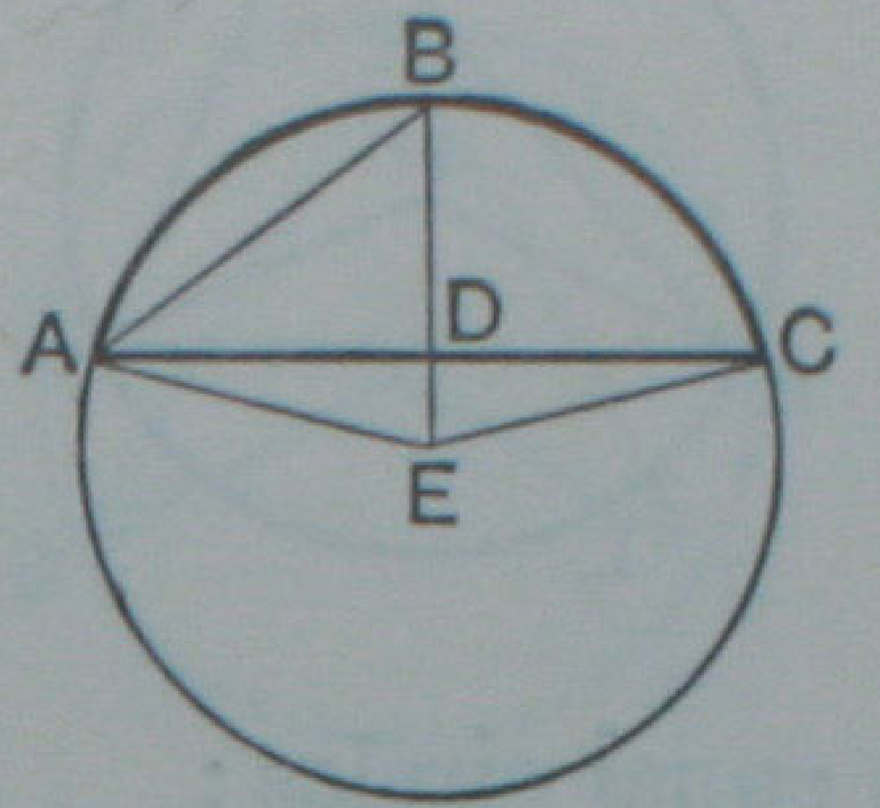
∴ the centre of the circle is F, the only point common to DF and EF.

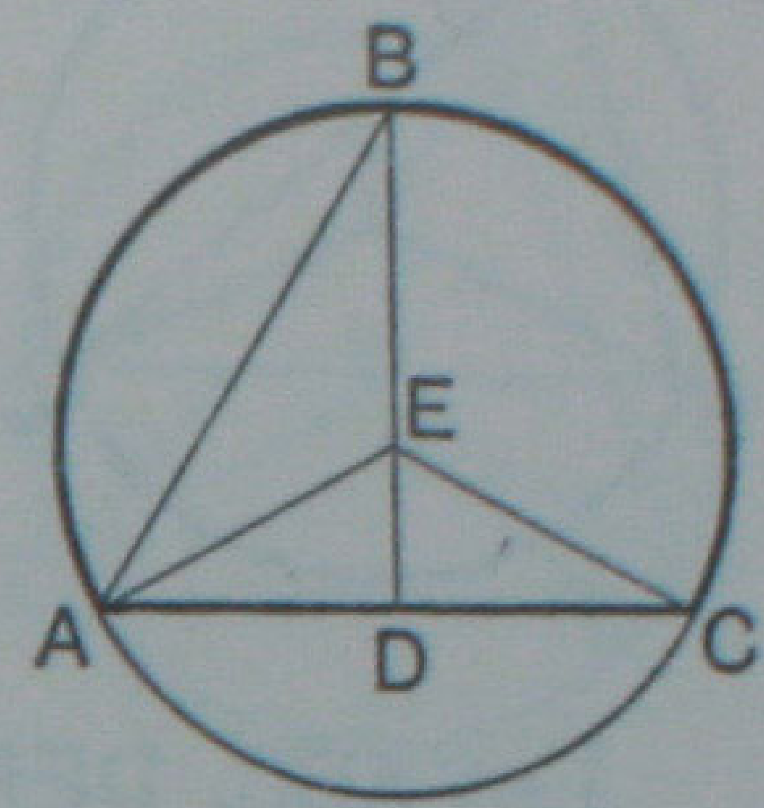
Hence the ○ᶜᵉ of a circle described with centre F, and radius FA, is that of which the given arc is a part. Q.E.F.

NOTE. Euclid gave to this proposition a somewhat different form, as follows:

PROPOSITION 25. [EUCLID'S METHOD.]

A segment of a circle being given, to describe the circle of which it is a segment.





Let ABC be the given segment of a circle, standing on the chord AC.

It is required to describe the circle of which ABC *is a segment.*

Construction. Draw DB, bisecting AC at rt. angles, and meeting the $\bigcirc^{ce}$ at B.

Join AB.

CASE I. When the $\angle$ DAB is not equal to the $\angle$ ABD.

At A, in BA, make the $\angle$ BAE equal to the $\angle$ ABD; I. 23.

and let AE meet BD, or BD produced, at E.

Join EC.

Then E *shall be the centre of the required circle.*

Proof. Since the $\angle$ EAB = the $\angle$ EBA, *Constr.*

$\therefore$ EA = EB. I. 6.

And in the $\triangle^s$ EDA, EDC,

Because { DA = DC, *Constr.*
and ED is common;
also the $\angle$ EDA = the $\angle$ EDC, being rt. angles;

$\therefore$ EA = EC. I. 4.

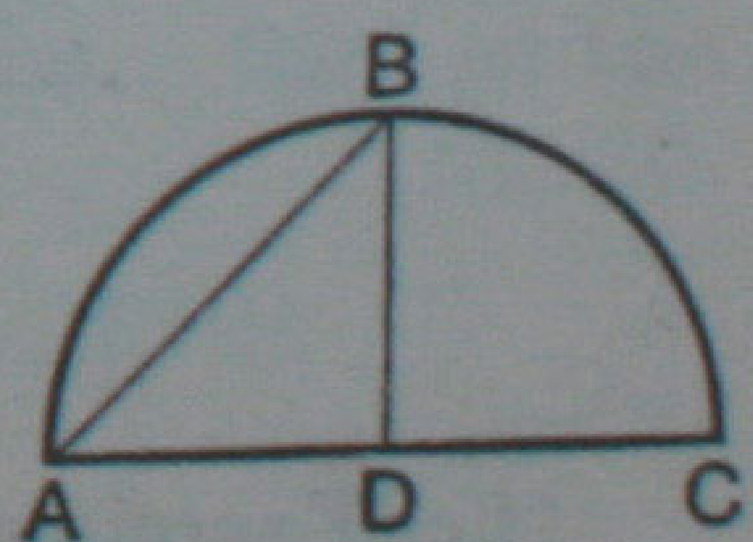
Hence EA, EB, and EC are all equal;

$\therefore$ E is the centre of the required circle, and EA, EB, EC are radii.

CASE II. When the $\angle$ DAB = the $\angle$ ABD.

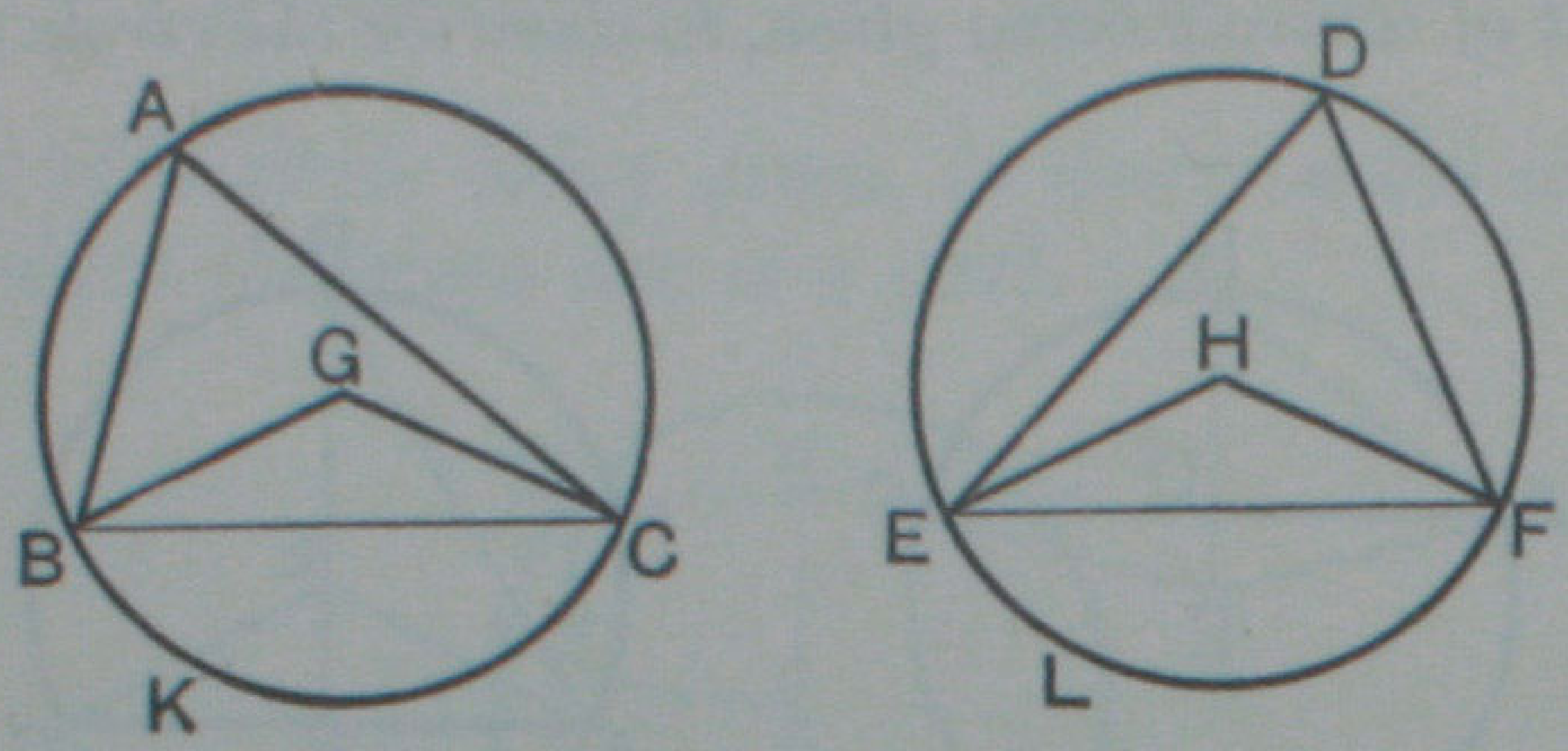
In this case it follows that DB = DA; I. 6.

$\therefore$ DB, DA, DC are all equal, so that D is the centre of the required circle.



PROPOSITION 26. THEOREM.

In equal circles the arcs which subtend equal angles, whether at the centres or at the circumferences, shall be equal.



Let ABC, DEF be equal circles;
and let the $\angle^s$ BGC, EHF at the centres be equal,
and consequently the $\angle^s$ BAC, EDF at the $\bigcirc^{ces}$ equal. III. 20.
Then shall the arc BKC = *the arc* ELF.

Construction. Join BC, EF.

Proof. Because the $\odot^s$ ABC, DEF are equal,
∴ their radii are equal.
Hence in the $\triangle^s$ BGC, EHF,

Because { BG = EH,
and GC = HF,
and the $\angle$ BGC = the $\angle$ EHF; } *Hyp.*
∴ BC = EF. I. 4.

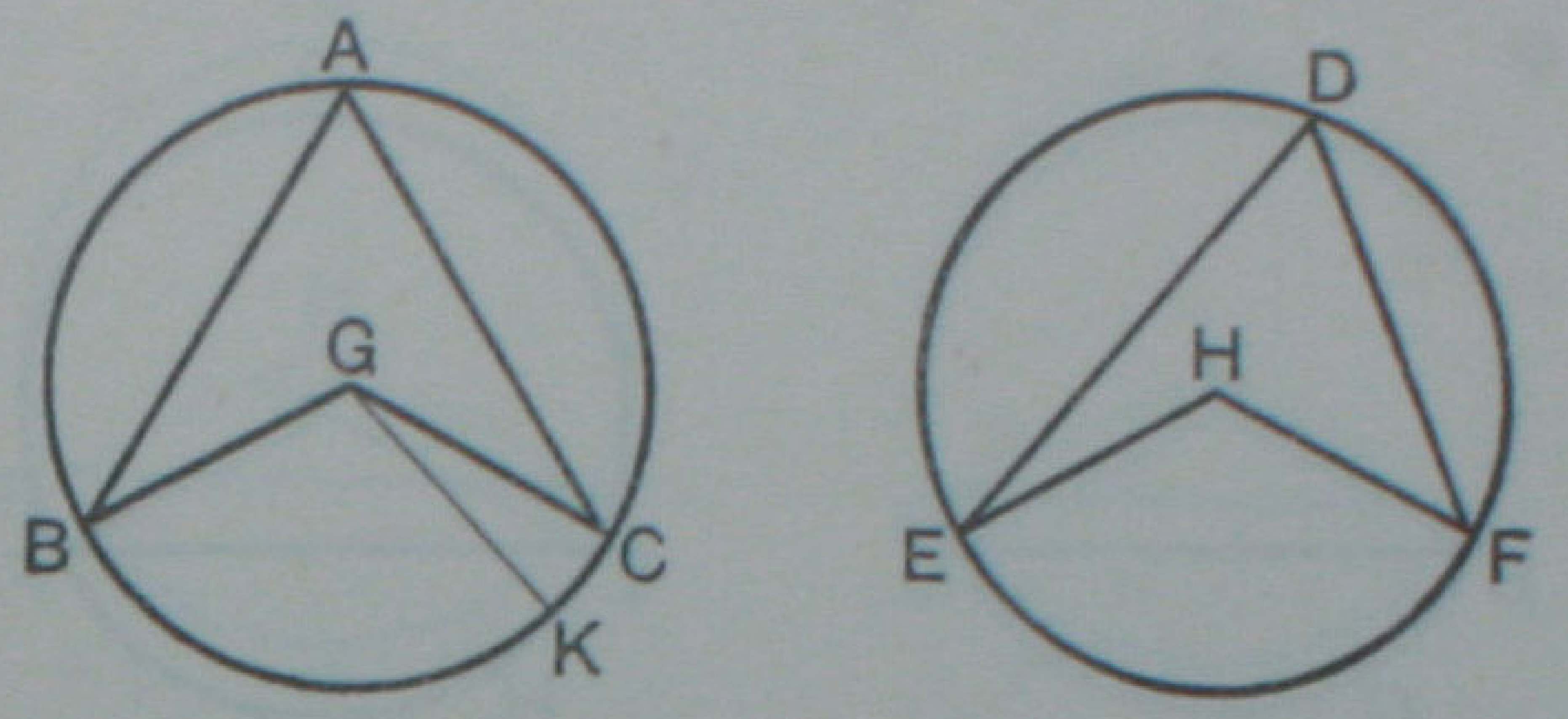
Again, because the $\angle$ BAC = the $\angle$ EDF, *Hyp.*
∴ the segment BAC is similar to the segment EDF;
III. *Def.* 10.
and these segments are on equal chords BC, EF;
∴ the segment BAC = the segment EDF. III. 24.
But the whole $\odot$ ABC = the whole $\odot$ DEF;
∴ the remaining segment BKC = the remaining segment ELF;
∴ the arc BKC = the arc ELF.

Q.E.D.

[For Exercises and an Alternative Proof see pp. 212, 213.]

PROPOSITION 27. THEOREM.

In equal circles the angles, whether at the centres or the circumferences, which stand on equal arcs, shall be equal.



Let ABC, DEF be equal circles;
and let the arc BC = the arc EF.

Then shall the ∠ BGC = *the* ∠ EHF, *at the centres;*
and also the ∠ BAC = *the* ∠ EDF, *at the* ○ᶜᵉˢ.

Construction. If the ∠ˢ BGC, EHF are not equal, one must be the greater.

If possible, let the ∠ BGC be the greater.
At G, in BG, make the ∠ BGK equal to the ∠ EHF. I. 23.

Proof. In the equal ⊙ˢ ABC, DEF,
because the ∠ BGK = the ∠ EHF, at the centres; *Constr.*
∴ the arc BK = the arc EF. III. 26.

But the arc BC = the arc EF; *Hyp.*
∴ the arc BK = the arc BC,
a part equal to the whole, which is impossible.

∴ the ∠ BGC is not unequal to the ∠ EHF;
that is, the ∠ BGC = the ∠ EHF.

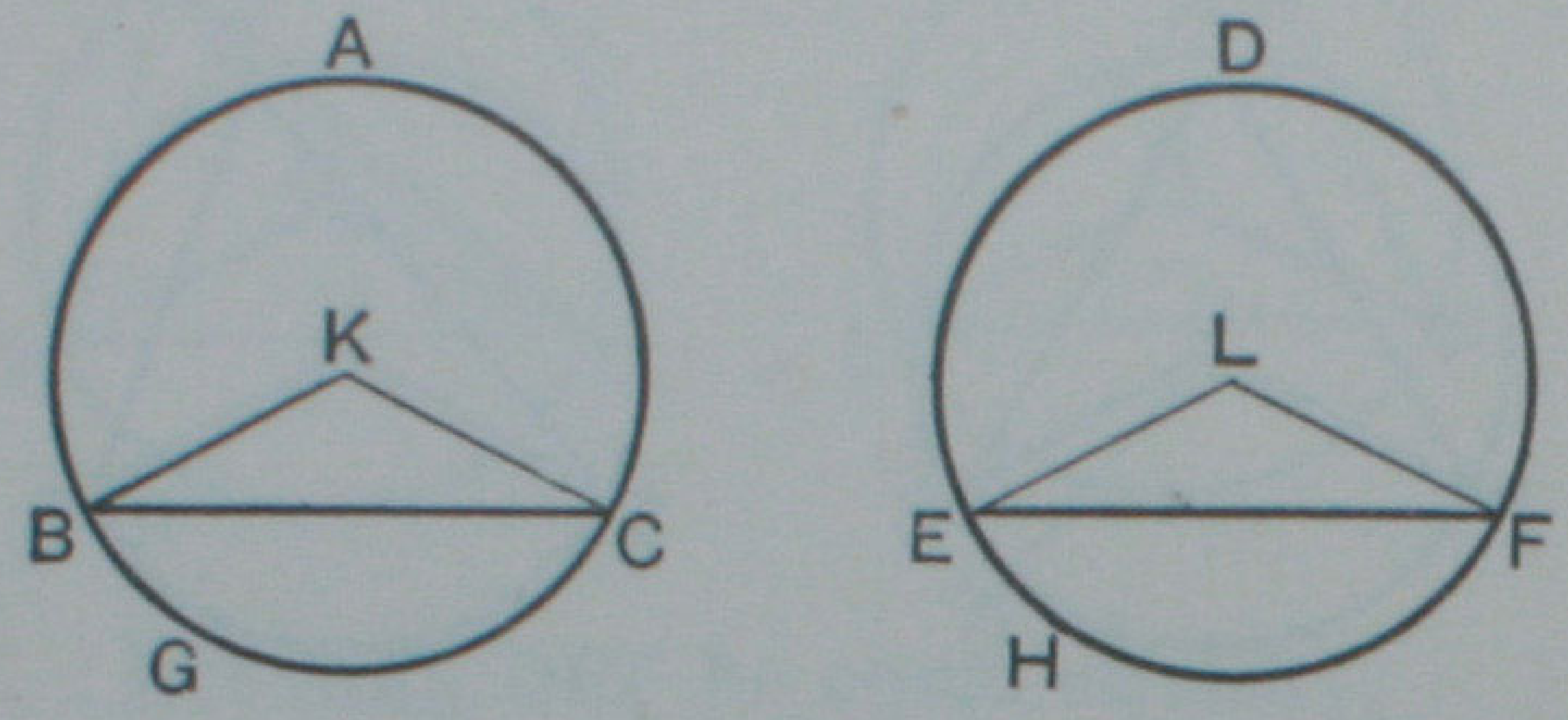
And since the ∠ BAC at the ○ᶜᵉ is half the ∠ BGC at the centre, III. 20.

and likewise the ∠ EDF is half the ∠ EHF,
∴ the ∠ BAC = the ∠ EDF. *Ax.* 7.

Q.E.D.

Proposition 28. Theorem.

In equal circles the arcs, which are cut off by equal chords, shall be equal, the major arc equal to the major arc, and the minor to the minor.



Let ABC, DEF be equal circles ;
and let the chord BC = the chord EF.

Then shall the major arc BAC = *the major arc* EDF ;
and the minor arc BGC = *the minor arc* EHF.

Construction.
Find K and L the centres of the ⊙ˢ ABC, DEF ; III. 1.
and join BK, KC, EL, LF.

Proof. Because the ⊙ˢ ABC, DEF are equal,
∴ their radii are equal.

Hence in the △ˢ BKC, ELF,

Because { BK = EL,
KC = LF,
and BC = EF ; *Hyp.*

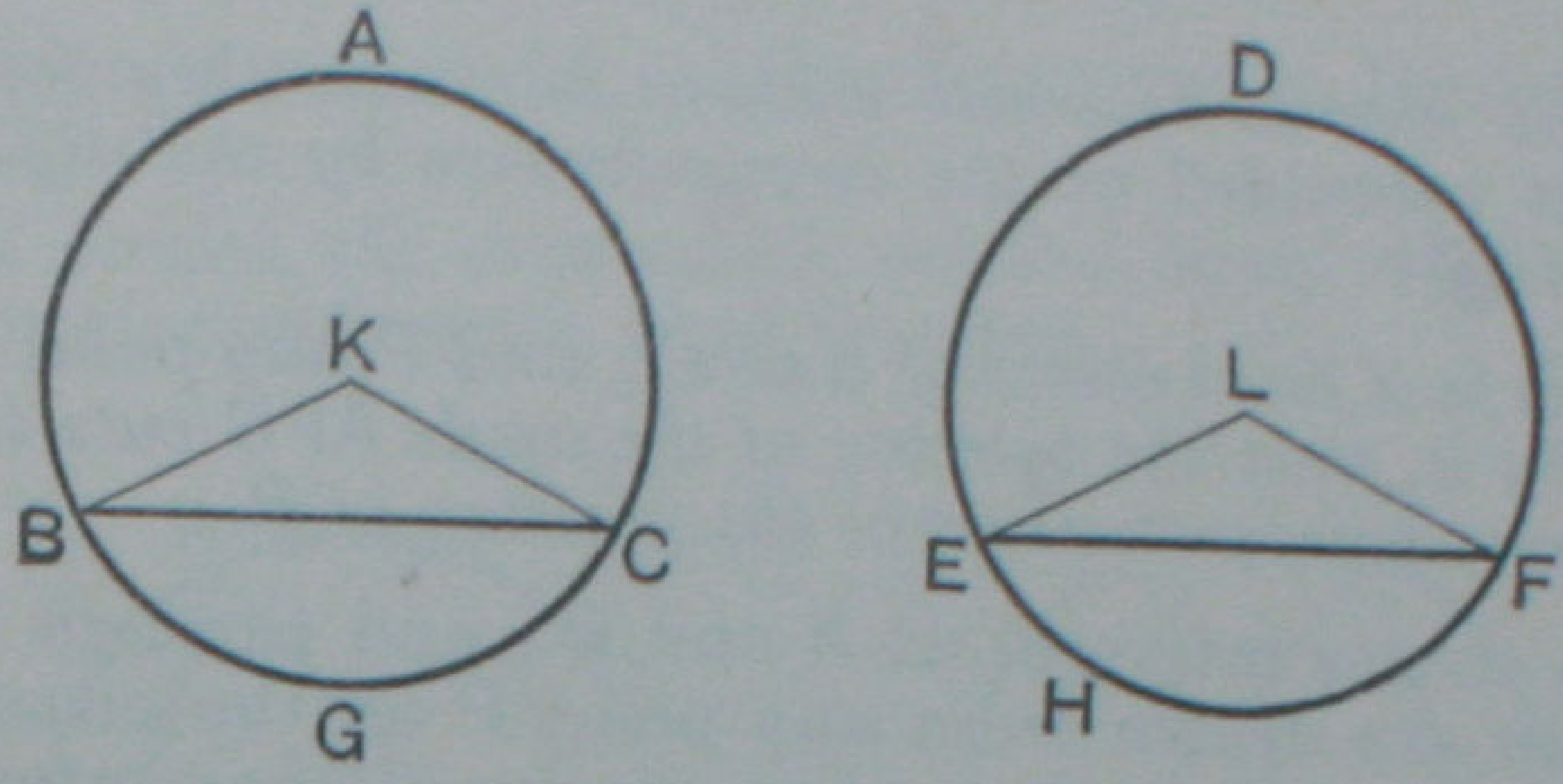
∴ the ∠ BKC = the ∠ ELF. I. 8.
∴ the arc BGC = the arc EHF ;
for these arcs subtend equal angles at the centre ; III. 26.
and they are the minor arcs.

But the whole ○ᶜᵉ ABGC = the whole ○ᶜᵉ DEHF ; *Hyp.*
∴ the remaining arc BAC = the remaining arc EDF :
and these are the major arcs. Q.E.D.

[For Exercises see p. 212.]

PROPOSITION 29. THEOREM.

In equal circles the chords, which cut off equal arcs, shall be equal.



Let ABC, DEF be equal circles;
and let the arc BGC = the arc EHF.
Then shall the chord BC = *the chord* EF.

Construction. Find K, L the centres of the circles.
Join BK, KC, EL, LF.

Proof. In the equal ⊙s ABC, DEF,
because the arc BGC = the arc EHF,
∴ the ∠ BKC = the ∠ ELF, at the centres. III. 27.

Hence in the △s BKC, ELF,
Because { BK = EL, being radii of equal circles;
KC = LF, for the same reason,
and the ∠ BKC = the ∠ ELF; *Proved.*
∴ BC = EF. I. 4.

Q.E.D.

EXERCISES.

ON PROPOSITIONS 26, 27.

1. *If two chords of a circle are parallel, they intercept equal arcs.*

2. The straight lines, which join the extremities of two equal arcs of a circle towards the same parts, are parallel.

3. *In a circle, or in equal circles, sectors are equal if their angles at the centres are equal.*

4. If two chords of a circle intersect at right angles, the opposite arcs are together equal to a semi-circumference.

5. *If two chords intersect within a circle, they form an angle equal to that subtended at the circumference by the sum of the arcs they cut off.*

6. *If two chords intersect without a circle, they form an angle equal to that subtended at the circumference by the difference of the arcs they cut off.*

7. *If* AB *is a fixed chord of a circle, and* P *any point on one of the arcs cut off by it, then the bisector of the angle* APB *cuts the conjugate arc in the same point, whatever be the position of* P.

8. Two circles intersect at A and B; and through these points straight lines are drawn from any point P on the circumference of one of the circles: shew that when produced they intercept on the other circumference an arc which is constant for all positions of P.

9. A triangle ABC is inscribed in a circle, and the bisectors of the angles meet the circumference at X, Y, Z. Find each angle of the triangle XYZ in terms of those of the original triangle.

ON PROPOSITIONS 28, 29.

10. The straight lines which join the extremities of parallel chords in a circle (i) towards the same parts, (ii) towards opposite parts, are equal.

11. Through A, a point of intersection of two equal circles, two straight lines PAQ, XAY are drawn: shew that the chord PX is equal to the chord QY.

12. Through the points of intersection of two circles two parallel straight lines are drawn terminated by the circumferences: shew that the straight lines which join their extremities towards the same parts are equal.

13. Two equal circles intersect at A and B; and through A any straight line PAQ is drawn terminated by the circumferences: shew that BP=BQ.

14. ABC is an isosceles triangle inscribed in a circle, and the bisectors of the base angles meet the circumference at X and Y. Shew that the figure BXAYC must have four of its sides equal.

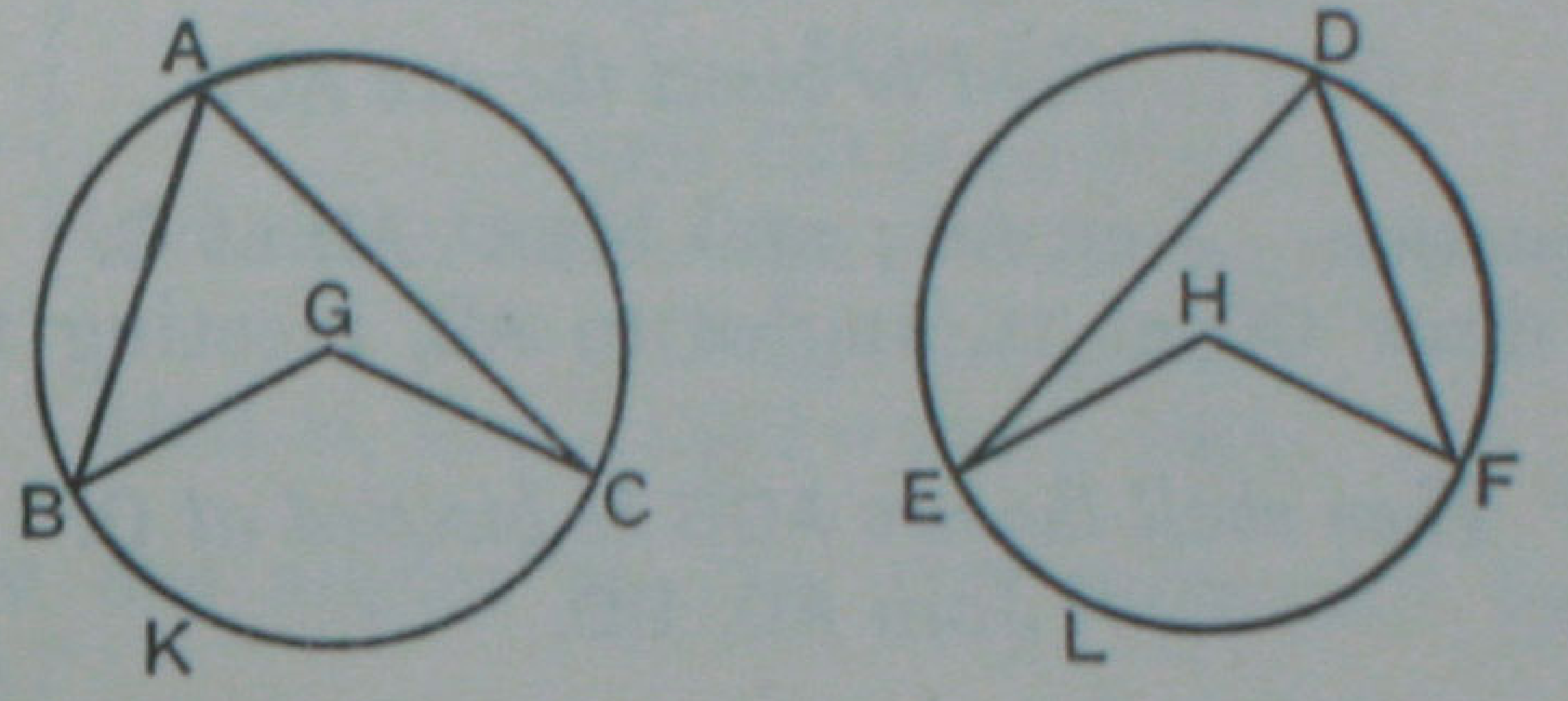
What relation must subsist among the angles of the triangle ABC, in order that the figure BXAYC may be equilateral?

NOTE. We have given Euclid's demonstrations of Propositions 26, 27; but it should be noticed that these propositions also admit of proof by the method of *superposition*.

To illustrate this method we will apply it to Proposition 26.

PROPOSITION 26. [ALTERNATIVE PROOF.]

In equal circles, the arcs which subtend equal angles, whether at the centres or circumferences, shall be equal.



Let ABC, DEF be equal circles;
and let the ∠ˢ BGC, EHF at the centres be equal,
and consequently the ∠ˢ BAC, EDF at the ○ᶜᵉˢ equal. III. 20.
Then shall the arc BKC = *the arc* ELF.

Proof. For if the ⊙ ABC be applied to the ⊙ DEF, so that the centre G may fall on the centre H,

then because the circles are equal, *Hyp.*
∴ their ○ᶜᵉˢ must coincide;

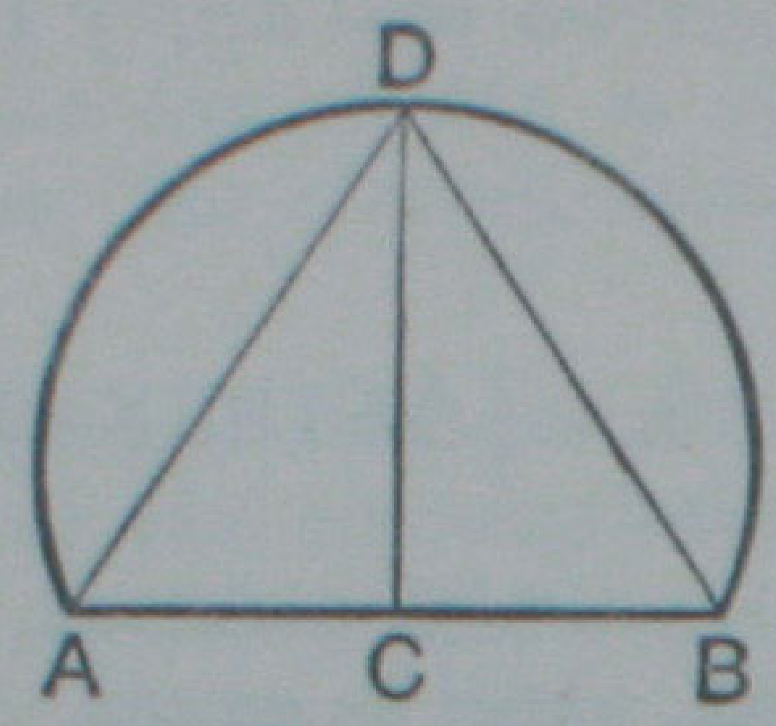
hence by revolving the upper circle about its centre, the lower circle remaining fixed,

B may be made to coincide with E,
and consequently GB with HE.
And because the ∠ BGC = the ∠ EHF, *Hyp.*
∴ GC must coincide with HF:
and since GC = HF, *Hyp.*
∴ C must fall on F.
Now B coincides with E, and C with F,
and the ○ᶜᵉ of the ⊙ ABC with the ○ᶜᵉ of the ⊙ DEF;
∴ the arc BKC must coincide with the arc ELF.
∴ the arc BKC = the arc ELF.

Q. E. D.

PROPOSITION 30. PROBLEM.

To bisect a given arc.



Let ADB be the given arc.

It is required to bisect the arc ADB.

Construction. Join AB; and bisect AB at C. I. 10.

At C draw CD at rt. angles to AB, meeting the given arc at D. I. 11.

Then shall the arc ADB *be bisected at* D.

Join AD, BD.

Proof. In the $\triangle^s$ ACD, BCD,

Because { AC = BC, *Constr.*
and CD is common;
and the ∠ ACD = the ∠ BCD, being rt. angles:

∴ AD = BD. I. 4.

And since in the ⊙ ADB, the chords AD, BD are equal,
∴ the arcs cut off by them are equal, the minor arc equal to the minor, and the major arc to the major: III. 28.
and the arcs AD, BD are both minor arcs,
for each is less than a semi-circumference, since DC, bisecting the chord AB at rt. angles, must pass through the centre of the circle. III. 1. *Cor.*

∴ the arc AD = the arc BD:
that is, the arc ADB is bisected at D. Q.E.F.

EXERCISES.

1. If a tangent to a circle is parallel to a chord, the point of contact will bisect the arc cut off by the chord.

2. Trisect a quadrant, or the fourth part of the circumference, of a circle.

NOTE. The following alternative proof of Proposition 30 removes the necessity of distinguishing between the major and minor arcs cut off by the chords AD, BD.

PROPOSITION 30. [ALTERNATIVE PROOF.]

The construction being made as before, we may proceed thus:

Proof. In the $\triangle^s$ ACD, BCD,

Because { AC = BC, *Constr.*
and CD is common;
and the ∠ ACD = the ∠ BCD, being rt. angles:

∴ the ∠ DAC = the ∠ DBC: I. 4.

that is, the ∠ DAB = the ∠ DBA.

But these are angles at the $\bigcirc^{ce}$ subtended by the arcs DB, DA;

∴ the arc DB = the arc DA: III. 26.

that is, the arc ADB is bisected at D. Q.E.F.

QUESTIONS FOR REVISION.

1. When is a straight line said (i) *to meet*, (ii) *to cut*, (iii) *to touch*, the circumference of a circle?

2. When are circles said *to touch* one another? Distinguish between *internal* and *external* contact.

3. What theorems have been so far proved by Euclid regarding (i) *circles which cut one another*, (ii) *circles which touch one another*?

4. If two unequal circles are concentric, shew that one must lie wholly within the other.

5. Shew how to divide the circumference of a circle into *three*, *four*, or *six* equal parts.

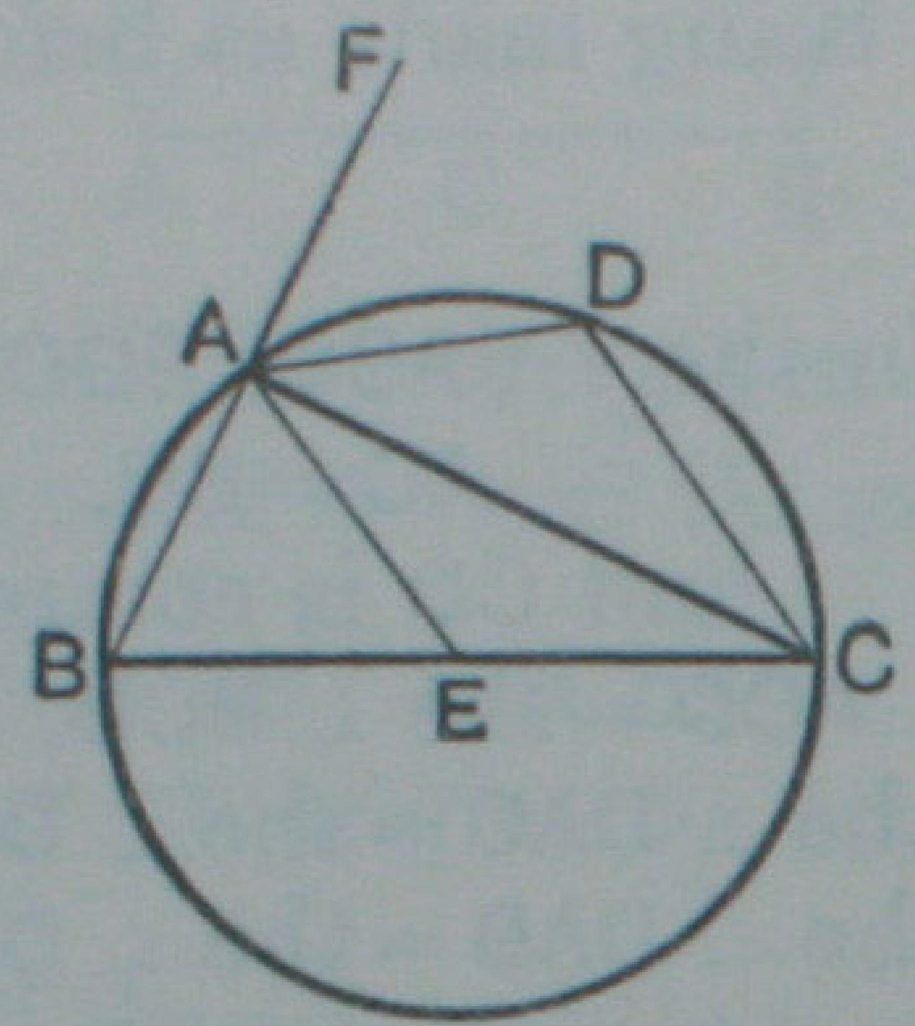
6. Enunciate the propositions so far proved by Euclid relating to the properties of *a tangent to a circle*.

Proposition 31. Theorem.

The angle in a semicircle is a right angle.

The angle in a segment greater than a semicircle is less than a right angle.

The angle in a segment less than a semicircle is greater than a right angle.



Let ABCD be a circle, of which BC is a diameter, and E the centre; and let AC be a chord dividing the circle into the segments ABC, ADC, of which the segment ABC is greater, and the segment ADC is less than a semicircle.

Then (i) *the angle in the semicircle* BAC *shall be a right angle;*

(ii) *the angle in the segment* ABC *shall be less than a rt. angle;*

(iii) *the angle in the segment* ADC *shall be greater than a rt. angle.*

Construction. In the arc ADC take any point D;

Join BA, AD, DC, AE; and produce BA to F.

Proof. (i) Because EA = EB, I. *Def.* 15.

∴ the ∠ EAB = the ∠ EBA. I. 5.

And because EA = EC,

∴ the ∠ EAC = the ∠ ECA.

∴ the whole ∠ BAC = the sum of the ∠ˢ EBA, ECA:

but the ext. ∠ FAC = the sum of the two int. ∠ˢ CBA, BCA;

∴ the ∠ BAC = the ∠ FAC;

∴ these angles, being adjacent, are rt. angles.

∴ the ∠ BAC, in the semicircle BAC, is a rt. angle.

(ii) In the △ ABC, because the sum of the ∠s ABC, BAC is less than two rt. angles; I. 17.

and of these, the ∠ BAC is a rt. angle; *Proved.*

∴ the ∠ ABC, which is the angle in the segment ABC, is less than a rt. angle.

(iii) Because ABCD is a quadrilateral inscribed in the ⊙ ABC,

∴ the opp. ∠s ABC, ADC together = two rt. angles; III. 22.

and of these, the ∠ ABC is less than a rt. angle: *Proved.*

∴ the ∠ ADC, which is the angle in the segment ADC, is greater than a rt. angle. Q.E.D.

EXERCISES.

1. *A circle described on the hypotenuse of a right-angled triangle as diameter, passes through the opposite angular point.*

2. A system of right-angled triangles is described upon a given straight line as hypotenuse; find the locus of the opposite angular points.

3. A straight rod of given length slides between two straight rulers placed at right angles to one another; find the locus of its middle point.

4. Two circles intersect at A and B; and through A two diameters AP, AQ are drawn, one in each circle: shew that the points P, B, Q are collinear. [See Def. p. 110.]

5. A circle is described on one of the equal sides of an isosceles triangle as diameter. Shew that it passes through the middle point of the base.

6. Of two circles which have internal contact, the diameter of the inner is equal to the radius of the outer. Shew that any chord of the outer circle, drawn from the point of contact, is bisected by the circumference of the inner circle.

7. Circles described on any two sides of a triangle as diameters intersect on the third side, or the third side produced.

8. *Find the locus of the middle points of chords of a circle drawn through a fixed point. Distinguish between the cases when the given point is within, on, or without the circumference.*

9. Describe a square equal to the difference of two given squares.

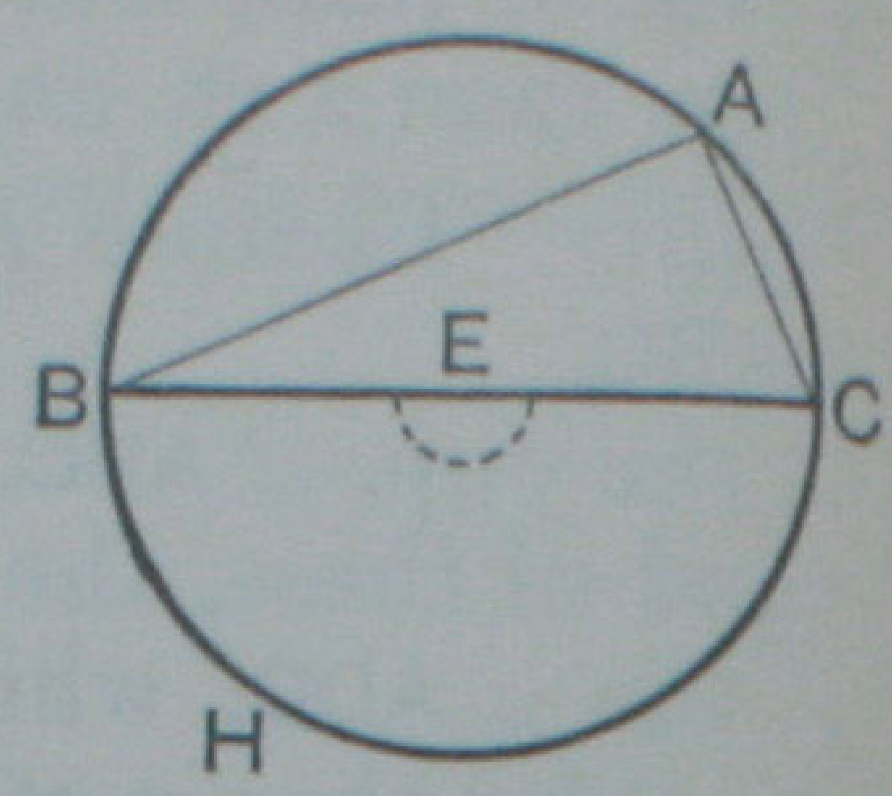
10. Through one of the points of intersection of two circles draw a chord of one circle which shall be bisected by the other.

11. On a given straight line as base a system of equilateral four-sided figures is described: find the locus of the intersection of their diagonals.

NOTES ON PROPOSITION 31.

Note 1. The extension of Proposition 20 to *straight* and *reflex* angles furnishes a simple alternative proof of the first theorem contained in Proposition 31, namely,

The angle in a semicircle is a right angle.



For, in the adjoining figure, the angle at the centre, standing on the arc BHC, is double the angle BAC at the $\bigcirc^{ce}$, standing on the same arc.

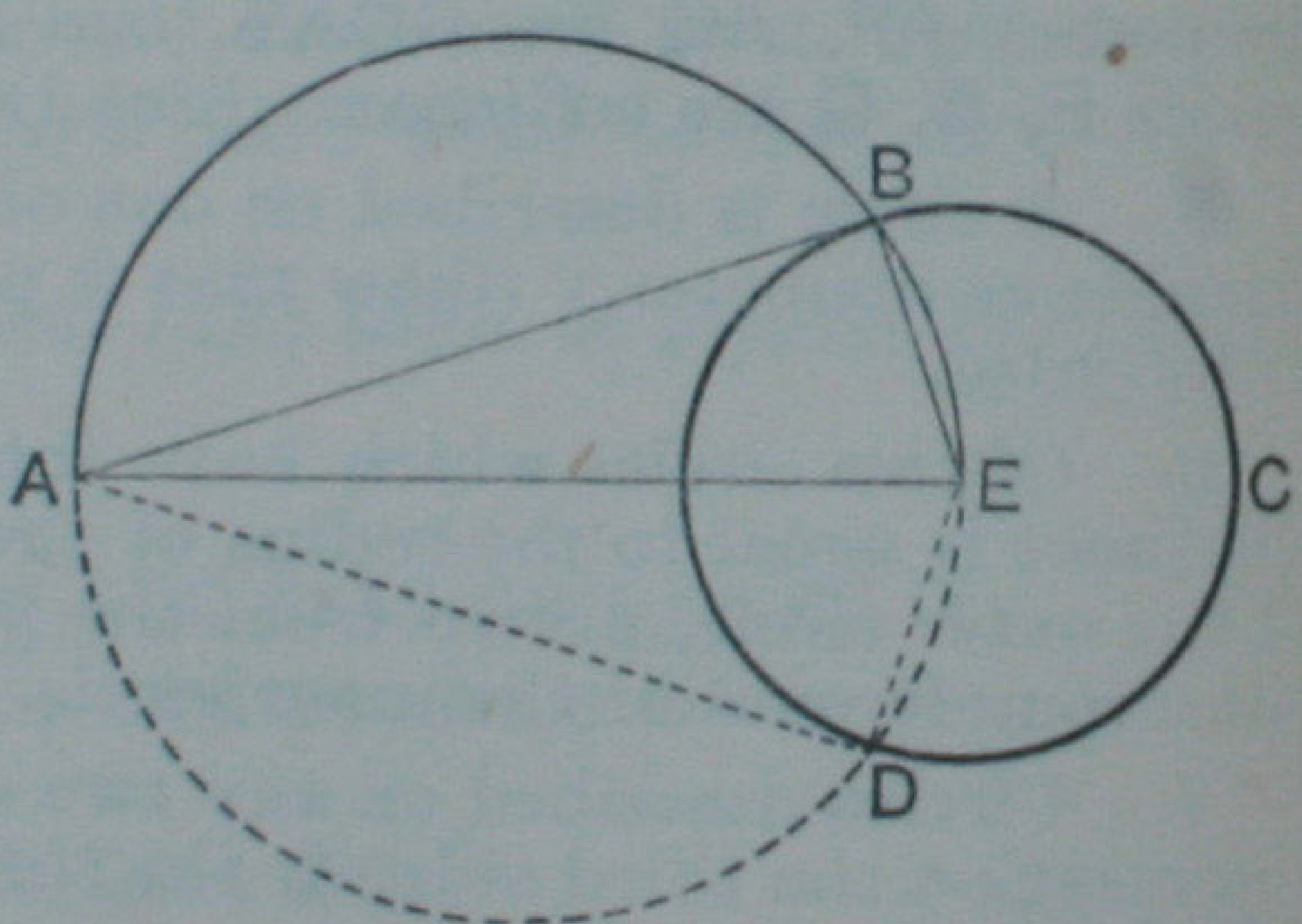
Now the angle at the centre is the *straight angle* BEC;
∴ the ∠ BAC is half of the *straight angle* BEC:
and a straight angle = two rt. angles;
∴ the ∠ BAC = one half of two rt. angles,
= one rt. angle. Q.E.D.

Note 2. From Proposition 31 we may derive a simple practical solution of Proposition 17, namely,

To draw a tangent to a circle from a given external point.

Let BCD be the given circle, and A the given external point.

It is required to draw from A *a tangent to the* ⊙ BCD.



Find E, the centre of the given circle, and join AE.

On AE describe the semicircle ABE, to cut the given circle at B.

Join AB.

Then AB *shall be a tangent to the* ⊙ BCD.

For the ∠ ABE, being in a semicircle, is a rt. angle. III. 31.
∴ AB is drawn at rt. angles to the radius EB, from its extremity B;
∴ AB is a tangent to the circle. III. 16.
Q.E.F.

Since the semicircle might be described on either side of AE, it is clear that there will be a second solution of the problem, as shewn by the dotted lines of the figure.

QUESTIONS FOR REVISION AND NUMERICAL EXERCISES.

1. Define an *arc*, a *chord*, a *segment* of a circle. When are segments of circles said to be *similar* to one another?

2. Enunciate propositions which give the properties of *chords* of a circle in relation to the *centre*.

3. Prove that in a circle whose diameter is 34 inches, a chord 30 inches in length is at a distance of 8 inches from the centre.

4. In a circle a chord 2 feet in length stands at a distance of 5 inches from the centre: shew that the diameter of the circle is 2 inches longer than the chord.

5. What must be the length of a chord which is 1 foot distant from the centre of a circle, if the diameter is 2 yards 2 inches?

6. Two parallel chords of a circle, whose diameter is 13 inches, are respectively 5 inches and 1 foot in length: shew that the distance between them is $8\frac{1}{2}$ inches, or $3\frac{1}{2}$ inches.

7. Two circles, whose radii are respectively 26 inches and 25 inches, intersect at two points which are 4 feet apart. Shew that the distance between their centres is 17 inches.

8. The diameters of two concentric circles are respectively 50 inches and 48 inches: shew that any chord of the outer circle which touches the inner must be 14 inches in length.

9. Of two concentric circles the diameter of the greater is 74 inches, and any chord of it which touches the smaller circle is 70 inches in length: shew that the diameter of the smaller circle is 2 feet.

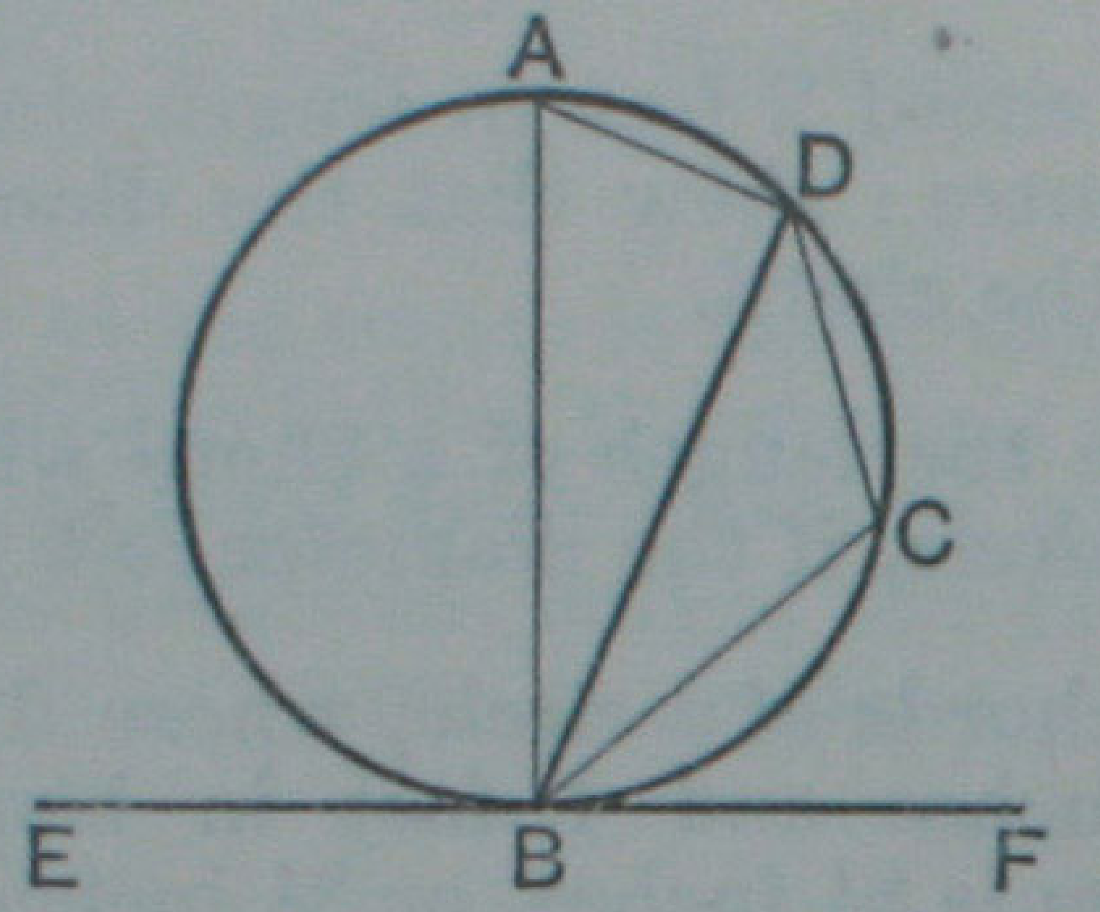
10. Two circles of diameters 74 and 40 inches respectively have a common chord 2 feet in length: shew that the distance between their centres is 51 inches.

11. The chord of an arc is 24 inches in length, and the height of the arc is 8 inches; shew that the diameter of the circle is 26 inches.

12. AB is a line 20 inches in length, and C is its middle point. On AB, AC, CB semicircles are described. Shew that if a circle is inscribed in the space enclosed by the three semicircles its radius must be $3\frac{1}{3}$ inches.

PROPOSITION 32. THEOREM.

If a straight line touches a circle, and from the point of contact a chord is drawn, the angles which this chord makes with the tangent shall be equal to the angles in the alternate segments of the circle.



Let EF touch the given ⊙ ABC at B, and let BD be a chord drawn from B, the point of contact.

Then shall

(i) *the* ∠ DBF = *the angle in the alternate segment* BAD :

(ii) *the* ∠ DBE = *the angle in the alternate segment* BCD.

Construction. From B draw BA perp. to EF. I. 11.
Take any point C in the arc BD;
and join AD, DC, CB.

(i) **Proof.** Because BA is drawn perp. to the tangent EF, at its point of contact B,

∴ BA passes through the centre of the circle : III. 19.

∴ the ∠ ADB, being in a semicircle, is a rt. angle : III. 31.

∴ in the △ ABD, the other ∠ˢ ABD, BAD together = a rt. angle ; I. 32.

that is, the ∠ˢ ABD, BAD together = the ∠ ABF.

From these equals take the common ∠ ABD ;

∴ the ∠ DBF = the ∠ BAD, which is in the alternate segment.

(ii) Because ABCD is a quadrilateral inscribed in a circle,

the opp. $\angle^s$ BCD, BAD together = two rt. angles: III. 22.

but the $\angle^s$ DBE, DBF together = two rt. angles; I. 13.

$\therefore$ the $\angle^s$ DBE, DBF together = the $\angle^s$ BCD, BAD;

and of these the $\angle$ DBF = the $\angle$ BAD; *Proved.*

$\therefore$ the $\angle$ DBE = the $\angle$ BCD, which is in the alternate segment.

Q.E.D.

EXERCISES.

1. State and prove the converse of Proposition 32.

2. Use this proposition to shew that the tangents drawn to a circle from an external point are equal.

3. If two circles touch one another, any straight line drawn through the point of contact cuts off similar segments.

Prove this for (i) internal, (ii) external contact.

4. If two circles touch one another, and from A, the point of contact, two chords APQ, AXY are drawn: then PX and QY are parallel.

Prove this for (i) internal, (ii) external contact.

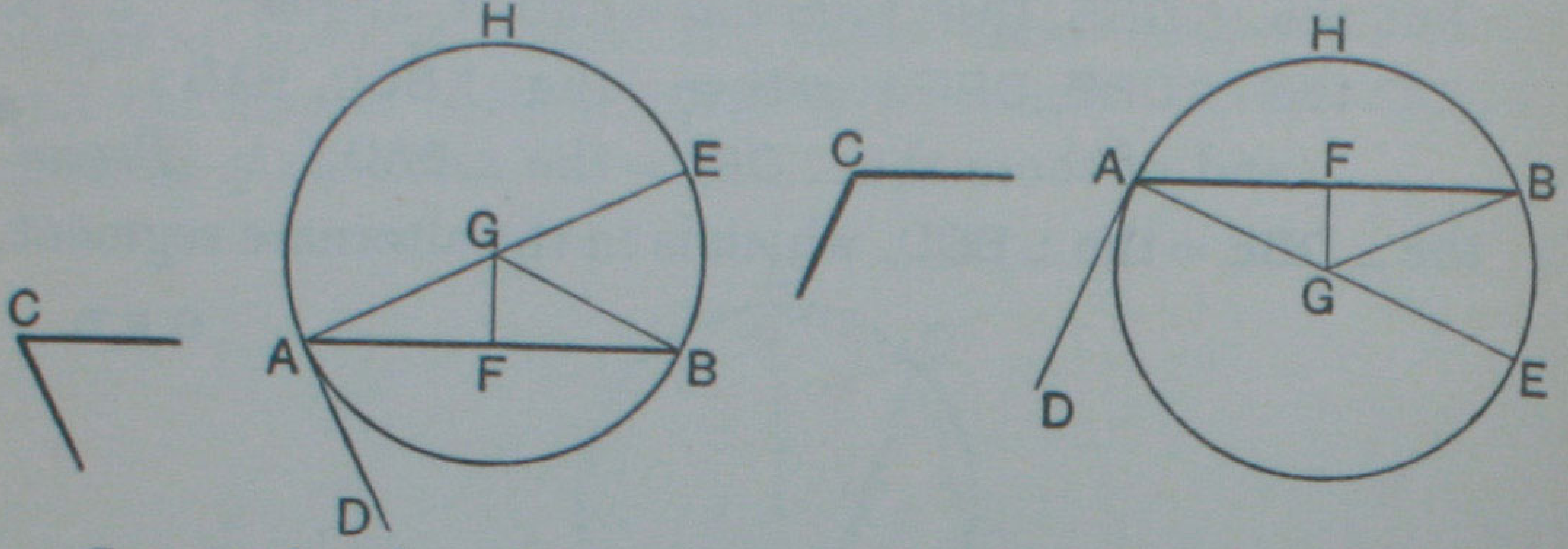
5. Two circles intersect at the points A, B: and one of them passes through O, the centre of the other: prove that OA bisects the angle between the common chord and the tangent to the first circle at A.

6. Two circles intersect at A and B; and through P, any point on the circumference of one of them, straight lines PAC, PBD are drawn to cut the other circle at C and D: shew that CD is parallel to the tangent at P.

7. If from the point of contact of a tangent to a circle, a chord is drawn, the perpendiculars dropped on the tangent and chord from the middle point of either arc cut off by the chord are equal.

Proposition 33. Problem.

On a given straight line to describe a segment of a circle which shall contain an angle equal to a given angle.



Let AB be the given st. line, and C the given angle.

It is required to describe on AB *a segment of a circle which shall contain an angle equal to* C.

Construction.

At A in BA, make the ∠ BAD equal to the ∠ C. I. 23.

From A draw AE at rt. angles to AD. I. 11.

Bisect AB at F. I. 10.

From F draw FG at rt. angles to AB, cutting AE at G.

Join GB.

Then in the △ˢ AFG, BFG,

Because { AF = BF, *Constr.*
and FG is common,
and the ∠ AFG = the ∠ BFG, being rt. angles;

∴ GA = GB: I. 4.

∴ the circle described with centre G, and radius GA, will pass through B.

Describe this circle, and call it ABH.

Then the segment AHB *shall contain an angle equal to* C.

Proof. Because AD is drawn at rt. angles to the radius GA from its extremity A,

∴ AD is a tangent to the circle; III. 16.

and from A, its point of contact, a chord AB is drawn;

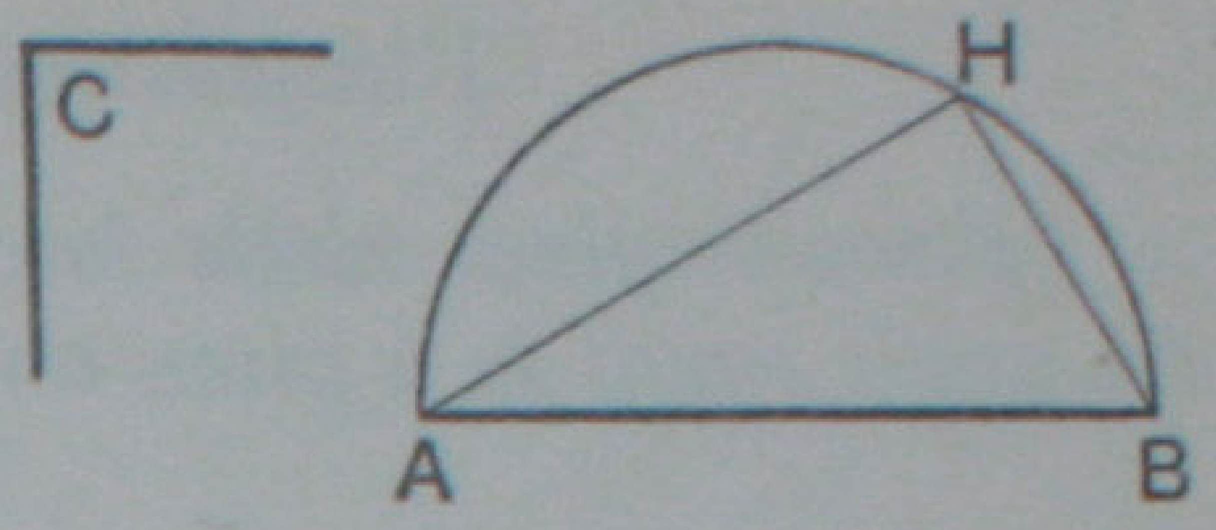
∴ the ∠ BAD = the angle in the alt. segment AHB. III. 32.

But the ∠ BAD = the ∠ C: *Constr.*

∴ the angle in the segment AHB = the ∠ C.

∴ AHB is the segment required. Q.E.F.

NOTE. In the particular case when the given angle C is a rt. angle, the segment required will be the semicircle described on the given st. line AB; for the angle in a semicircle is a rt. angle. III. 31.



EXERCISES.

[The following exercises depend on the corollary to the Converse of Proposition 21 given on page 201, namely

The locus of the vertices of triangles which stand on the same base and have a given vertical angle, is the arc of the segment standing on this base, and containing an angle equal to the given angle.

Exercises 1 and 2 afford good illustrations of the method of finding required points by the *Intersection of Loci*. See page 125.]

1. *Describe a triangle on a given base, having a given vertical angle, and having its vertex on a given straight line.*

2. *Construct a triangle, having given the base, the vertical angle and* (i) *one other side.*
   (ii) *the altitude.*
   (iii) *the length of the median which bisects the base.*
   (iv) *the point at which the perpendicular from the vertex meets the base.*

3. *Construct a triangle having given the base, the vertical angle, and the point at which the base is cut by the bisector of the vertical angle.*

[Let AB be the base, X the given point in it, and K the given angle. On AB describe a segment of a circle containing an angle equal to K; complete the ⊙ce by drawing the arc APB. Bisect the arc APB at P: join PX, and produce it to meet the ⊙ce at C. Then ABC shall be the required triangle.]

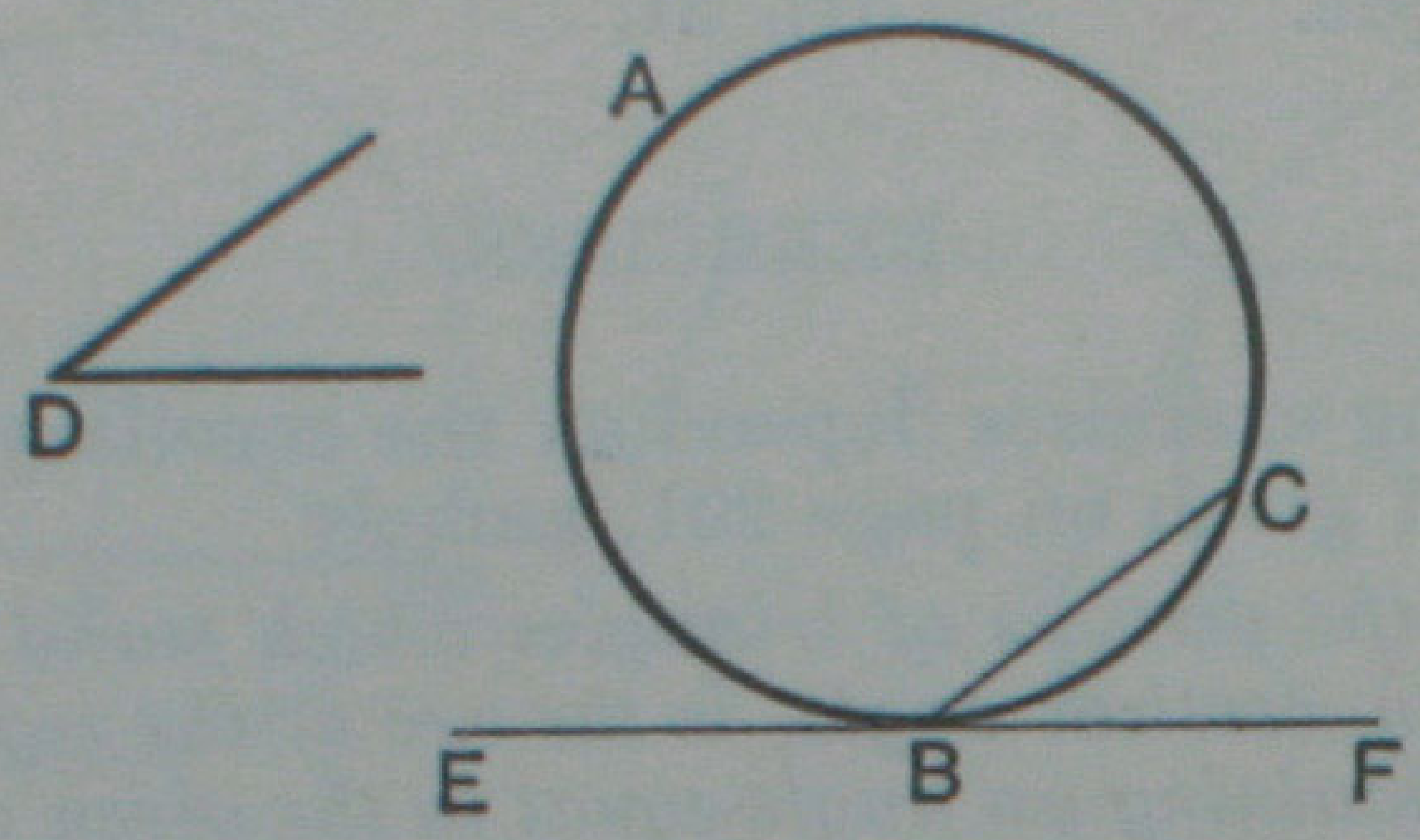
4. *Construct a triangle having given the base, the vertical angle, and the sum of the remaining sides.*

[Let AB be the given base, K the given angle, and H the given line equal to the sum of the sides. On AB describe a segment containing an angle equal to K, also another segment containing an angle equal to half the ∠ K. From centre A, with radius H, describe a circle cutting the arc of the last drawn segment at X and Y. Join AX (or AY) cutting the arc of the first segment at C. Then ABC shall be the required triangle.]

5. Construct a triangle having given the base, the vertical angle, and the difference of the remaining sides.

PROPOSITION 34. PROBLEM.

From a given circle to cut off a segment which shall contain an angle equal to a given angle.



Let ABC be the given circle, and D the given angle.

It is required to cut off from the ⊙ ABC *a segment which shall contain an angle equal to* D.

Construction. Take any point B on the ○ce,
and at B draw the tangent EBF. III. 17.

At B, in FB, make the ∠ FBC equal to the ∠ D. I. 23.

Then the segment BAC *shall contain an angle equal to* D.

Proof. Because EF is a tangent to the circle, and from B, its point of contact, a chord BC is drawn,

∴ the ∠ FBC = the angle in the alternate segment BAC. III. 32.

But the ∠ FBC = the ∠ D; *Constr.*

∴ the angle in the segment BAC = the ∠ D.

Hence from the given ⊙ ABC a segment BAC has been cut off, containing an angle equal to D. Q.E.F.

EXERCISES.

1. The chord of a given segment of a circle is produced to a fixed point: on this straight line so produced draw a segment of a circle similar to the given segment.

2. Through a given point without a circle draw a straight line that will cut off a segment capable of containing an angle equal to a given angle.

QUESTIONS FOR REVISION.

1. Enunciate the propositions from which we infer that a straight line and a circle must either

(i) intersect in two points; or
(ii) touch at one point; or
(iii) have no point in common.

2. Give two independent constructions for drawing a tangent to a circle from an external point.

Shew that the two tangents so drawn

(i) are equal;
(ii) subtend equal angles at the centre;
(iii) make equal angles with the straight line which joins the given point to the centre.

3. Enunciate propositions relating to

(i) angles in a segment of a circle;
(ii) similar segments of circles.

4. What are *conjugate arcs* of a circle?

The angles in conjugate segments of a circle are supplementary. How does Euclid enunciate this theorem? State and prove its converse.

5. Explain what is meant by a *reflex angle.* What simplifications may be made in the proofs of Third Book Propositions if reflex angles are admitted?

6. If the circumference of a circle is divided into six equal arcs, shew that the chords joining successive points of division are all equal to the radius of the circle.

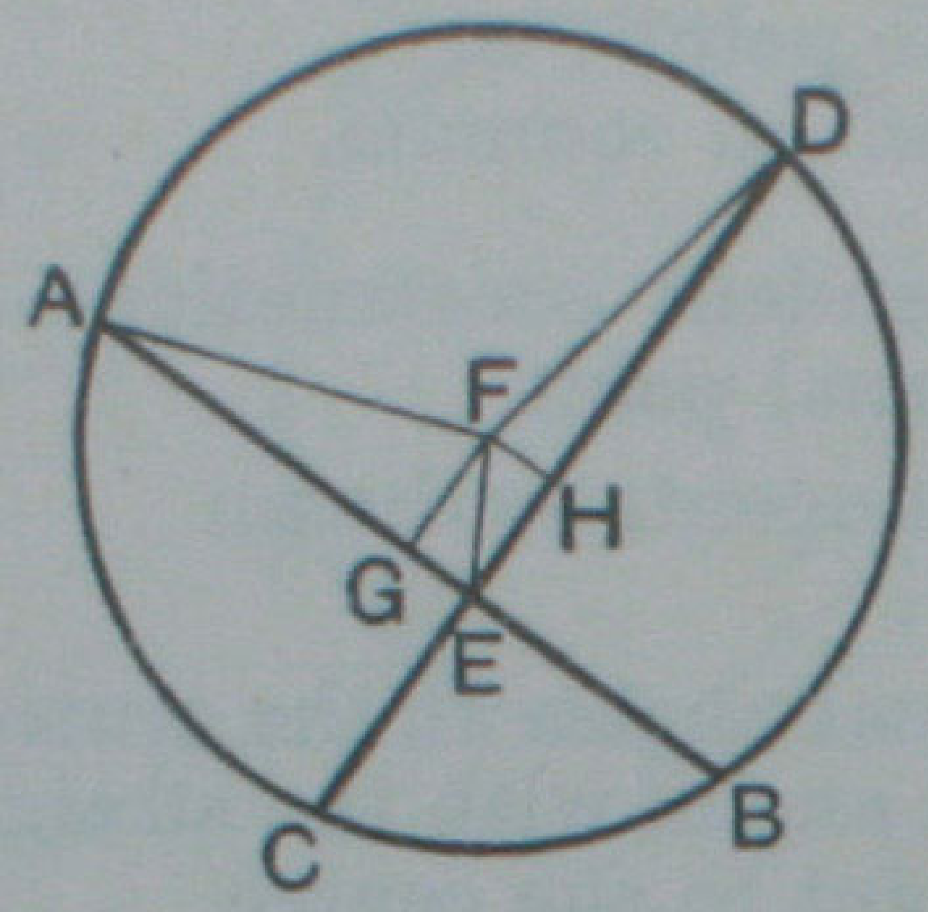
7. Find the locus of the centres of all circles

(i) which pass through two given points;
(ii) which touch a given circle at a given point;
(iii) which are of given radius, and touch a given circle;
(iv) which are of given radius, and pass through a given point;
(v) which touch a given straight line at a given point;
(vi) which touch each of two parallel straight lines;
(vii) which touch each of two intersecting straight lines of unlimited length.

8. If a system of triangles stand on the same base and on the same side of it, and have equal vertical angles, shew that the locus of their vertices is the arc of a circle. Prove this theorem, having first enunciated the proposition of which it is the converse.

PROPOSITION 35. THEOREM.

If two chords of a circle cut one another, the rectangle contained by the segments of one shall be equal to the rectangle contained by the segments of the other.



Let AB, CD, two chords of the ⊙ ACBD, cut one another at E.

Then shall the rect. AE, EB = *the rect.* CE, ED.

Construction. Find F, the centre of the ⊙ ACB; III. 1.
From F draw FG, FH perp. respectively to AB, CD. I. 12.
Join FA, FE, FD.

Proof. Because FG is drawn from the centre F perp. to AB,
∴ AB is bisected at G. III. 3.
For a similar reason CD is bisected at H.
Again, because AB is divided equally at G, and unequally at E,
∴ the rect. AE, EB with the sq. on EG = the sq. on AG. II. 5.
To each of these equals add the sq. on GF;
then the rect. AE, EB with the sqq. on EG, GF = the sum of the sqq. on AG, GF.
But the sqq. on EG, GF = the sq. on FE; I. 47.
and the sqq. on AG, GF = the sq. on AF;
for the angles at G are rt. angles.
∴ the rect. AE, EB with the sq. on FE = the sq. on AF.
Similarly it may be shewn that
the rect. CE, ED with the sq. on FE = the sq. on FD.
But the sq. on AF = the sq. on FD; for AF = FD.
∴ the rect. AE, EB with the sq. on FE = the rect. CE, ED with the sq. on FE.
From these equals take the sq. on FE:
then the rect. AE, EB = the rect. CE, ED. Q.E.D.

COROLLARY. *If through a fixed point within a circle any number of chords are drawn, the rectangles contained by their segments are all equal.*

NOTE. The following special cases of this proposition deserve notice:

(i) when the given chords both pass through the centre:
(ii) when one chord passes through the centre, and cuts the other at right angles:
(iii) when one chord passes through the centre, and cuts the other obliquely.

In each of these cases the general proof requires some modification, which may be left as an exercise to the student.

EXERCISES.

1. *Two straight lines* AB, CD *intersect at* E, *so that the rectangle* AE, EB *is equal to the rectangle* CE, ED; *shew that the four points* A, B, C, D *are concyclic.*

2. The rectangle contained by the segments of any chord drawn through a given point within a circle is equal to the square on half the shortest chord which may be drawn through that point.

3. ABC is a triangle right-angled at C; and from C a perpendicular CD is drawn to the hypotenuse: shew that the square on CD is equal to the rectangle AD, DB.

4. ABC is a triangle; and AP, BQ, the perpendiculars dropped from A and B on the opposite sides, intersect at O: shew that the rectangle AO, OP is equal to the rectangle BO, OQ.

5. Two circles intersect at A and B, and through any point in AB their common chord two chords are drawn, one in each circle; shew that their four extremities are concyclic.

6. A and B are two points within a circle such that the rectangle contained by the segments of any chord drawn through A is equal to the rectangle contained by the segments of any chord through B: shew that A and B are equidistant from the centre.

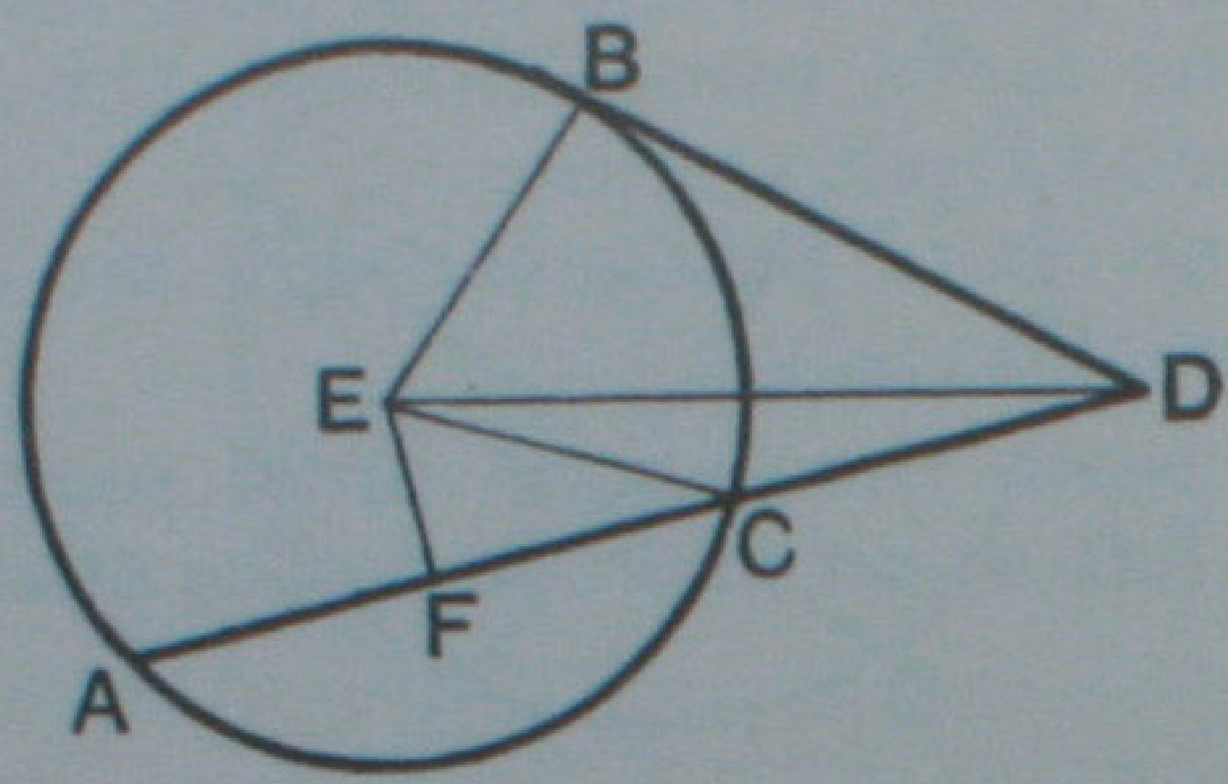
7. *If through* E, *a point without a circle, two secants,* EAB, ECD *are drawn; shew that the rectangle* EA, EB *is equal to the rectangle* EC, ED.

[Proceed as in III. 35, using II. 6.]

8. Through A, a point of intersection of two circles, two straight lines CAE, DAF are drawn, each passing through a centre and terminated by the circumferences: shew that the rectangle CA, AE is equal to the rectangle DA, AF.

PROPOSITION 36. THEOREM.

If from any point without a circle a tangent and a secant are drawn, then the rectangle contained by the whole secant and the part of it without the circle shall be equal to the square on the tangent.



Let ABC be a circle; and from D, a point without it, let there be drawn the secant DCA, and the tangent DB.

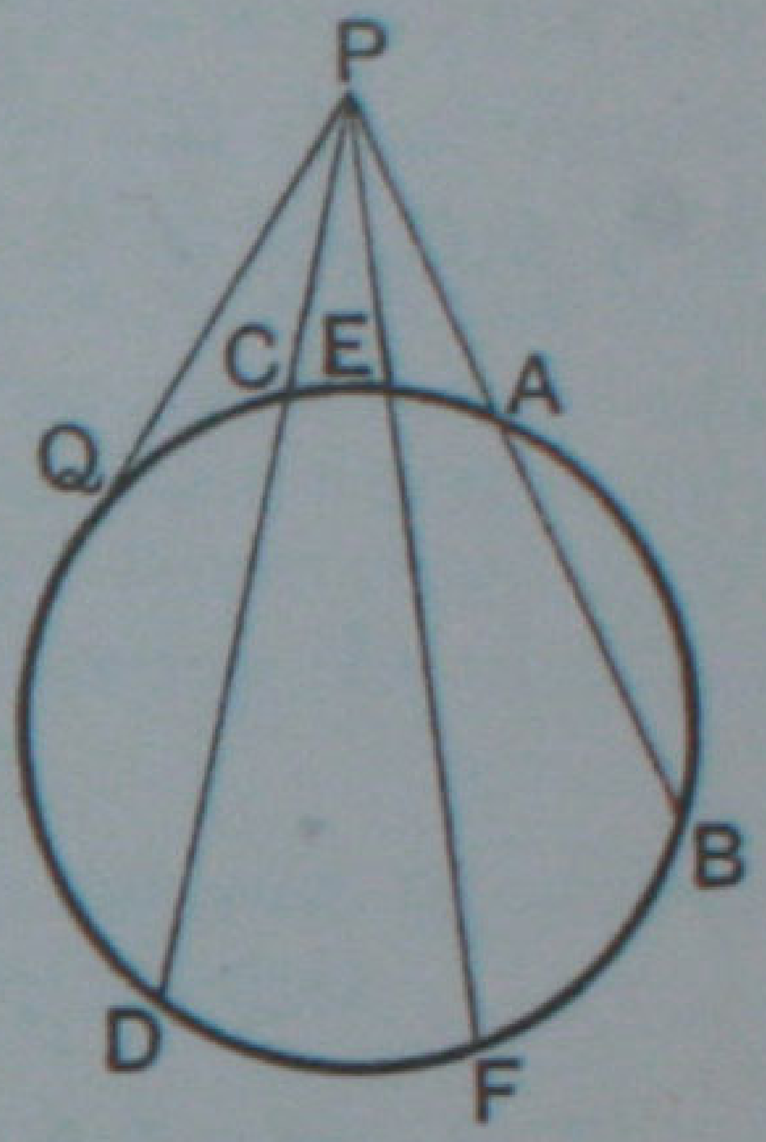
Then the rect. DA, DC *shall be equal to the sq. on* DB.

Construction. Find E, the centre of the ⊙ ABC: III. 1.
and from E, draw EF perp. to AD. I. 12.
Join EB, EC, ED.

Proof. Because EF, passing through the centre, is perp. to the chord AC,
∴ AC is bisected at F. III. 3.
And since AC is bisected at F and produced to D,
∴ the rect. DA, DC with the sq. on FC = the sq. on FD. II. 6.
To each of these equals add the sq. on EF:
then the rect. DA, DC with the sqq. on EF, FC = the sqq. on EF, FD.
But the sqq. on EF, FC = the sq. on EC; for EFC is a rt. angle;
= the sq. on EB.
And the sqq. on EF, FD = the sq. on ED; for EFD is a rt. angle;
= the sqq. on EB, BD; for EBD is a rt. angle. III. 18.
∴ the rect. DA, DC with the sq. on EB = the sqq. on EB, BD.
From these equals take the sq. on EB:
then the rect. DA, DC = the sq. on DB. Q.E.D.

NOTE. This proof may easily be adapted to the case when the secant passes through the centre of the circle.

COROLLARY. *If from a given point without a circle any number of secants are drawn, the rectangles contained by the whole secants and the parts of them without the circle are all equal; for each of these rectangles is equal to the square on the tangent drawn from the given point to the circle.*



For instance, in the adjoining figure, each of the rectangles PB, PA and PD, PC and PF, PE is equal to the square on the tangent PQ:

∴ the rect. PB, PA

= the rect. PD, PC

= the rect. PF, PE.

NOTE. Remembering that the segments into which the chord AB is divided at P, are the lines PA, PB, (see Def., page 139) we are enabled to include the corollaries of Propositions 35 and 36 in a single enunciation.

If any number of chords of a circle are drawn through a given point within or without a circle, the rectangles contained by the segments of the chords are equal.

EXERCISES.

1. Use this proposition to shew that tangents drawn to a circle from an external point are equal.

2. If two circles intersect, tangents drawn to them from any point in their common chord produced are equal.

3. If two circles intersect at A and B, and PQ is a tangent to both circles; shew that AB produced bisects PQ.

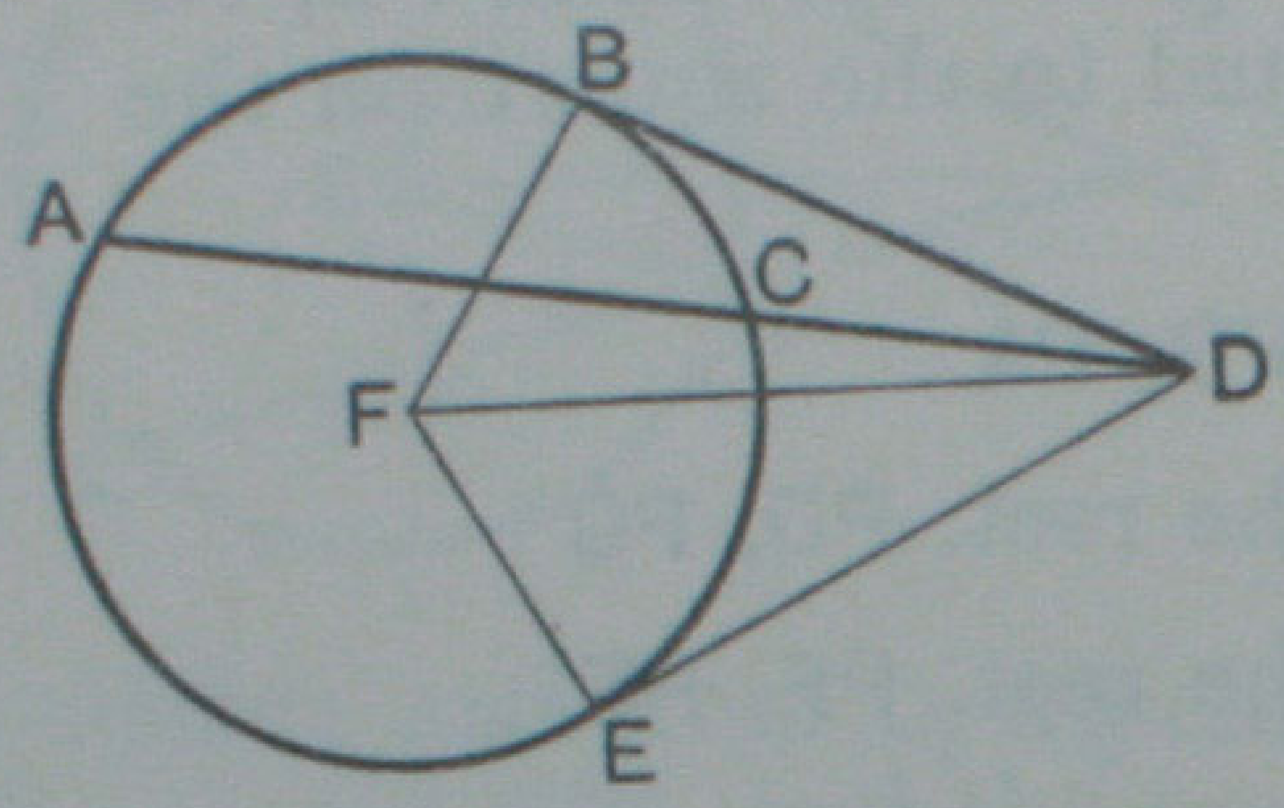
4. If P is any point on the straight line AB produced, shew that the tangents drawn from P to all circles which pass through A and B are equal.

5. ABC is a triangle right-angled at C, and from any point P in AC, a perpendicular PQ is drawn to the hypotenuse: shew that the rectangle AC, AP is equal to the rectangle AB, AQ.

6. ABC is a triangle right-angled at C, and from C a perpendicular CD is drawn to the hypotenuse: shew that the rect. AB, AD is equal to the square on AC.

PROPOSITION 37. THEOREM.

If from a point without a circle there are drawn two straight lines, one of which cuts the circle, and the other meets it, and if the rectangle contained by the whole line which cuts the circle and the part of it without the circle is equal to the square on the line which meets the circle, then the line which meets the circle shall be a tangent to it.



Let ABC be a circle; and from D, a point without it, let there be drawn two st lines DCA and DB, of which DCA cuts the circle at C and A, and DB meets it; and let the rect. DA, DC = the sq. on DB.

Then shall DB *be a tangent to the circle.*

Construction. From D draw DE to touch the ⊙ ABC: III. 17.
let E be the point of contact.
Find the centre F, and join FB, FD, FE. III. 1.

Proof. Since DCA is a secant, and DE a tangent to the circle,
∴ the rect. DA, DC = the sq. on DE, III. 36.
But, *by hypothesis*, the rect. DA, DC = the sq. on DB;
∴ the sq. on DE = the sq. on DB;
∴ DE = DB.

Hence in the △ˢ DBF, DEF,

Because { DB = DE, *Proved.*
and BF = EF; I. *Def.* 15.
and DF is common;

∴ the ∠ DBF = the ∠ DEF. I. 8.

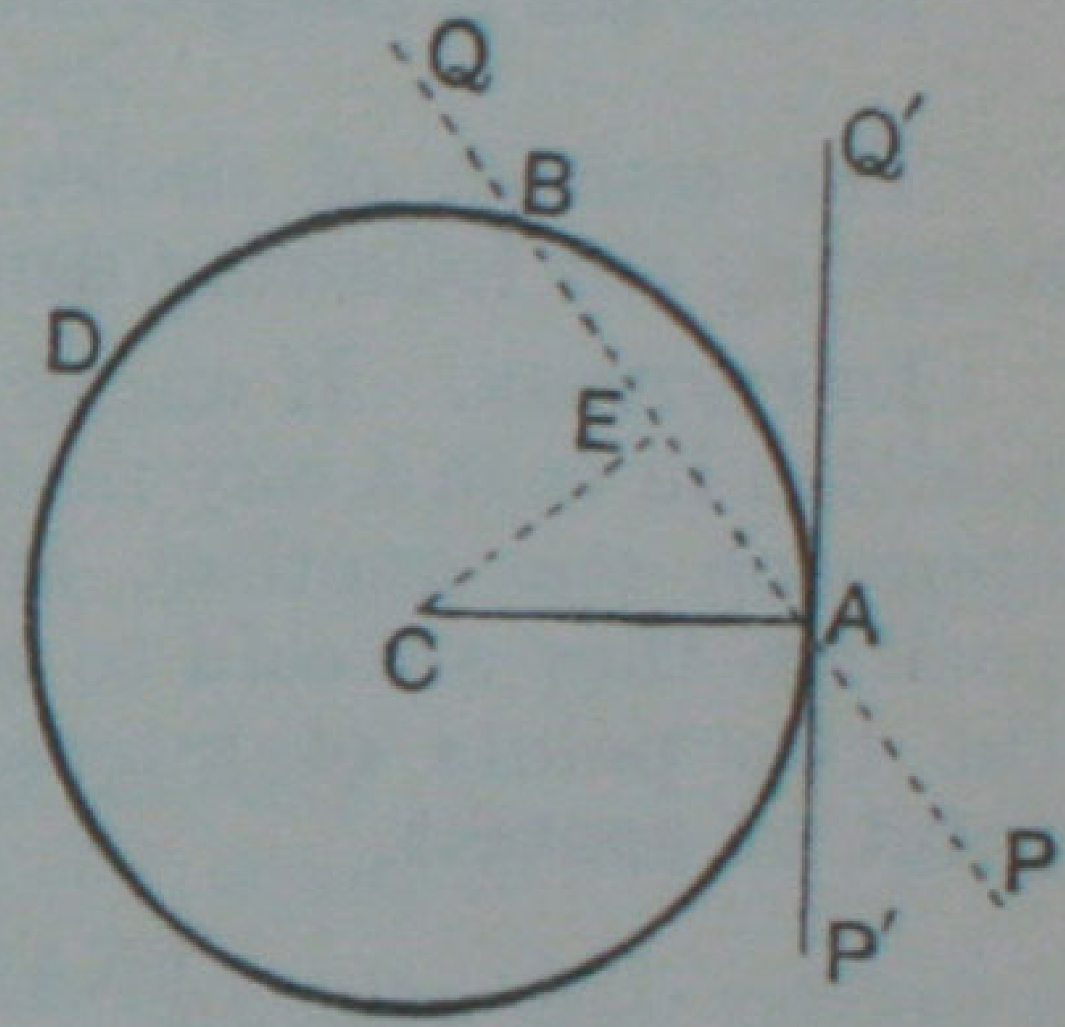
But DEF is a rt. angle, for DE is a tangent; III. 18.
∴ DBF is also a rt. angle;
and since BF is a radius,
∴ DB touches the ⊙ ABC at the point B. Q.E.D.

NOTE ON THE METHOD OF LIMITS AS APPLIED TO TANGENCY.

Euclid defines a tangent to a circle as *a straight line which meets the circumference, but being produced, does not cut it:* and from this definition he deduces the fundamental theorem that a tangent is perpendicular to the radius drawn to the point of contact. III. Prop. 16.

But this result may also be established by the Method of Limits, which regards the tangent as *the ultimate position of a secant when its two points of intersection with the circumference are brought into coincidence* [See Note on page 165]: and it may be shewn that every theorem relating to the tangent may be derived from some more general proposition relating to the secant, by considering the ultimate case when the two points of intersection coincide.

1. *To prove by the Method of Limits that a tangent to a circle is at right angles to the radius drawn to the point of contact.*



Let ABD be a circle, whose centre is C; and PABQ a secant cutting the ⊙ce in A and B; and let P′AQ′ be the limiting position of PQ when the point B is brought into coincidence with A.

Then shall CA *be perp. to* P′Q′.

Bisect AB at E and join CE:

then CE is perp. to PQ. III. 3.

Now let the secant PABQ change its position in such a way that while the point A remains fixed, the point B continually approaches A, and ultimately coincides with it;

then, *however near* B *approaches to* A, the st. line CE is always perp. to PQ, since it joins the centre to the middle point of the chord AB.

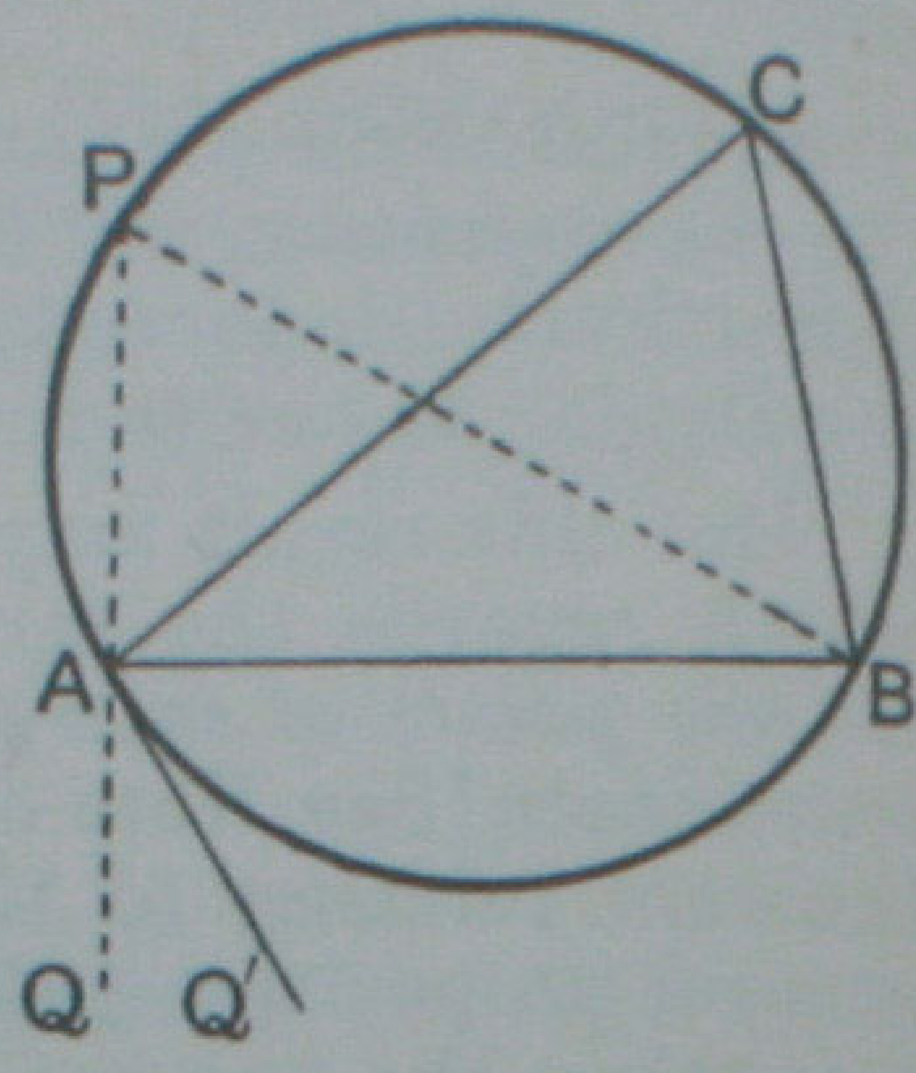
But in the limiting position, when B coincides with A, and the secant PQ becomes the tangent P′Q′, it is clear that the point E will also coincide with A; and the perpendicular CE becomes the radius CA. Hence CA is perp. to the tangent P′Q′ at its point of contact A. Q.E.D.

NOTE. It follows from Proposition 2 that *a straight line cannot cut the circumference of a circle at more than two points.* Now when the two points in which a secant cuts a circle move towards coincidence, the secant ultimately becomes a tangent to the circle: we infer therefore that a tangent cannot meet a circle otherwise than at its point of contact. Thus Euclid's definition of a tangent may be deduced from that given by the Method of Limits.

2. *By this method Proposition* 32 *may be derived as a special case from Proposition* 21.

For let A and B be two points on the ○^{ce} of the ⊙ ABC;
and let BCA, BPA be any two angles in the segment BCPA:
then the ∠ BPA = the ∠ BCA. III. 21.

Produce PA to Q.



Now let the point P continually approach the fixed point A, and ultimately coincide with it;
then, *however near* P *may approach to* A,
the ∠ BPQ = the ∠ BCA. III. 21.

But in the limiting position when P coincides with A,
and the secant PAQ becomes the tangent AQ′,
it is clear that BP will coincide with BA,
and the ∠ BPQ becomes the ∠ BAQ′.

Hence the ∠ BAQ′ = the ∠ BCA, in the alternate segment. Q.E.D.

The contact of circles may be treated in a similar manner by adopting the following definition.

DEFINITION. If one or other of two intersecting circles alters its position in such a way that the two points of intersection continually approach one another, and ultimately coincide; in the limiting position they are said to **touch** one another, and the point in which the two points of intersection ultimately coincide is called the **point of contact**.

EXAMPLES ON LIMITS.

1. Deduce Proposition 19 from the Corollary of Proposition 1 and Proposition 3.

2. Deduce Propositions 11 and 12 from Ex. 1, page 171.

3. Deduce Proposition 6 from Proposition 5.

4. Deduce Proposition 13 from Proposition 10.

5. Shew that a straight line cuts a circle in two different points, two coincident points, or not at all, according as its distance from the centre is less than, equal to, or greater than a radius.

6. Deduce Proposition 32 from Ex. 3, page 202.

7. Deduce Proposition 36 from Ex. 7, page 227.

8. *The angle in a semi-circle is a right angle.*

To what Theorem is this statement reduced, when the vertex of the right angle is brought into coincidence with an extremity of the diameter?

9. From Ex. 1, page 204, deduce the corresponding property of a triangle inscribed in a circle.

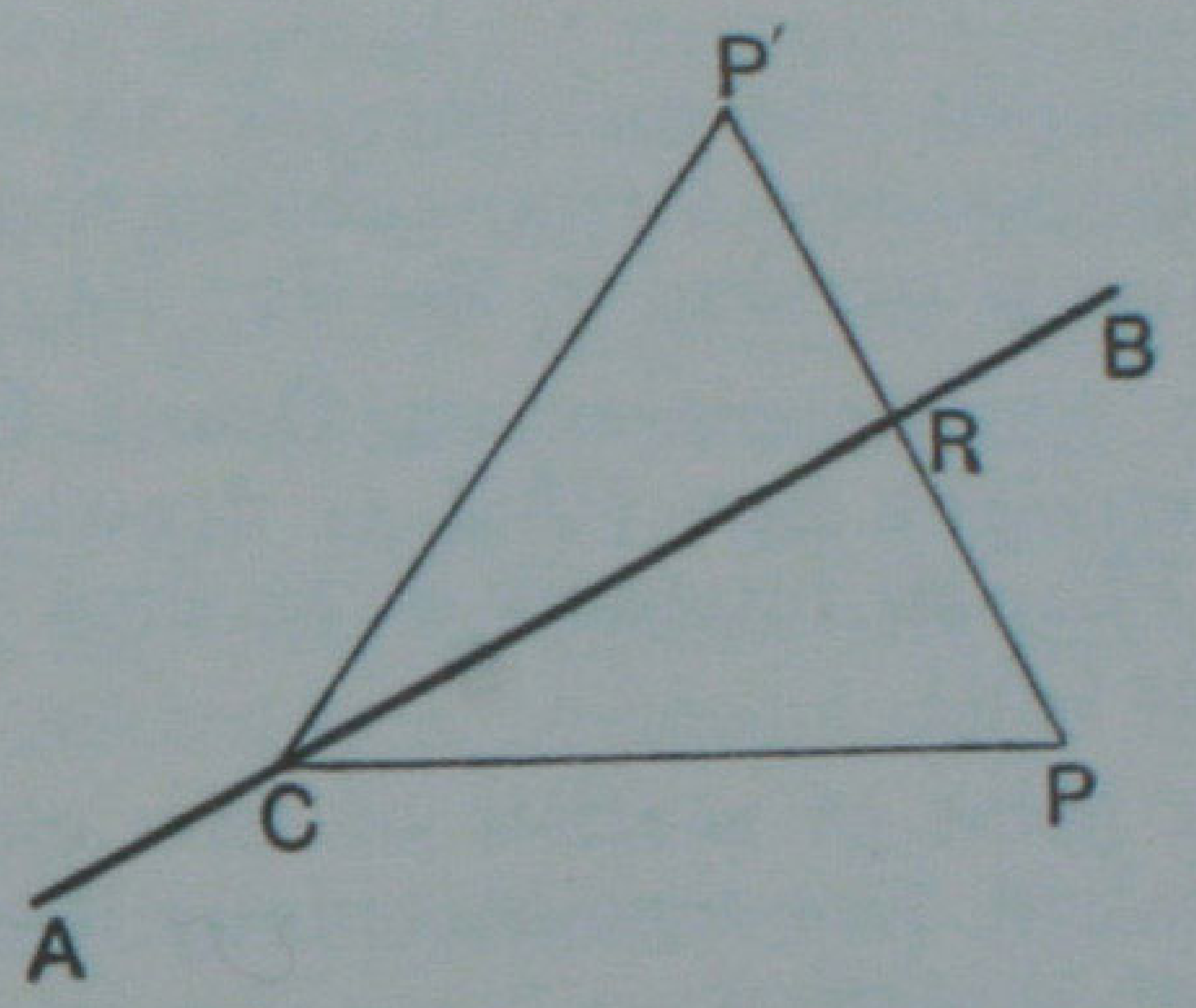
THEOREMS AND EXAMPLES ON BOOK III.

I. ON THE CENTRE AND CHORDS OF A CIRCLE.

[See Propositions 1, 3, 14, 15, 25.]

1. *All circles which pass through a fixed point, and have their centres on a given straight line, pass also through a second fixed point.*

Let AB be the given st. line, and P the given point.



From P draw PR perp. to AB;
and produce PR to P′, making RP′ equal to PR.

Then all circles which pass through P, *and have their centres on* AB, *shall pass also through* P′.

For let C be the centre of *any one* of these circles.
Join CP, CP′.

Then in the △s CRP, CRP′,

Because { CR is common,
and RP=RP′, *Constr.*
and the ∠ CRP=the ∠ CRP′, being rt. angles;
∴ CP=CP′; I. 4.

∴ the circle whose centre is C, and which passes through P, must pass also through P′.

But C is the centre of *any* circle of the system;

∴ all circles, which pass through P, and have their centres in AB, pass also through P′. Q. E. D.

2. *Describe a circle that shall pass through three given points not in the same straight line.*