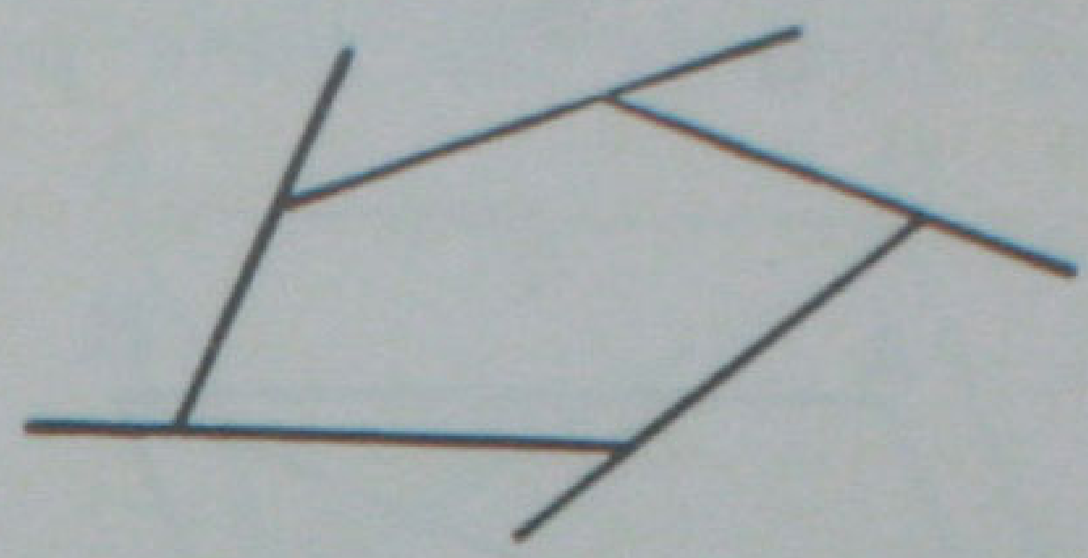
COROLLARY 2. *If the sides of a rectilineal figure, which has no re-entrant angle, are produced in order, then all the exterior angles so formed are together equal to four right angles.*



For at each angular point of the figure, the interior angle and the exterior angle are together equal to two right angles. I. 13.

Therefore all the interior angles, with all the exterior angles, are together equal to twice as many right angles as the figure has sides.

But all the interior angles, with four right angles, are together equal to twice as many right angles as the figure has sides. I. 32, *Cor.* 1.

Therefore all the interior angles, with all the exterior angles, are together equal to all the interior angles, with four right angles.

Therefore the exterior angles are together equal to four right angles. Q.E.D.

EXERCISES ON SIMSON'S COROLLARIES.

[A polygon is said to be **regular** when it has all its sides and all its angles equal.]

1. Express in terms of a right angle the magnitude of each angle of (i) a regular hexagon, (ii) a regular octagon.

2. If one side of a regular hexagon is produced, shew that the exterior angle is equal to the angle of an equilateral triangle.

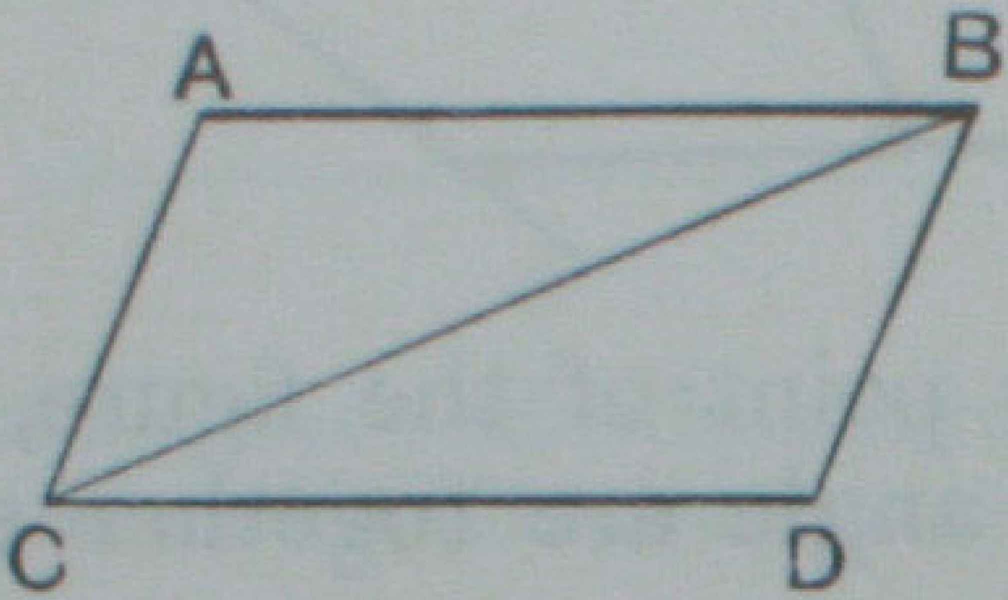
3. Prove Simson's first Corollary by joining one vertex of the rectilineal figure to each of the other vertices.

4. Find the magnitude of each angle of a regular polygon of *n* sides.

5. If the alternate sides of any polygon be produced to meet, the sum of the included angles, together with eight right angles, will be equal to twice as many right angles as the figure has sides.

Proposition 33. Theorem.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel.



Let AB and CD be equal and parallel straight lines; and let them be joined towards the same parts by the straight lines AC and BD.

Then shall AC *and* BD *be equal and parallel.*

Construction. Join BC.

Proof. Then because AB and CD are parallel, and BC meets them,

therefore the angle ABC is equal to the alternate angle DCB. I. 29.

Now in the triangles ABC, DCB,

Because { AB is equal to DC, *Hyp.*
and BC is common to both;
also the angle ABC is equal to the angle DCB; *Proved.*

therefore the triangle ABC is equal to the triangle DCB in all respects; I. 4.

so that the base AC is equal to the base DB,
and the angle ACB equal to the angle DBC.

But these are alternate angles.

Therefore AC and BD are parallel: I. 27.

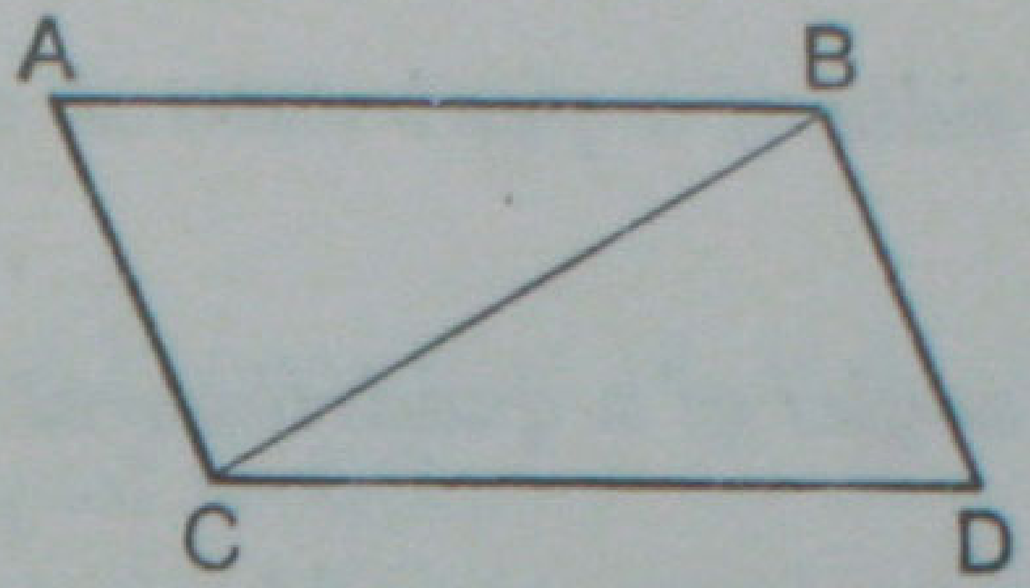
and it has been shewn that they are also equal.

Q.E.D.

Definition. A **Parallelogram** is a four-sided figure whose opposite sides are parallel.

PROPOSITION 34. THEOREM.

The opposite sides and angles of a parallelogram are equal to one another, and each diagonal bisects the parallelogram.



Let ACDB be a parallelogram, of which BC is a diagonal.

Then shall the opposite sides and angles of the figure be equal to one another; and the diagonal BC *shall bisect it*

Proof. Because AB and CD are parallel, and BC meets them,
therefore the angle ABC is equal to the alternate angle DCB; I. 29.

Again, because AC and BD are parallel, and BC meets them,
therefore the angle ACB is equal to the alternate angle DBC. I. 29.

Hence in the triangles ABC, DCB,

Because { the angle ABC is equal to the angle DCB,
and the angle ACB is equal to the angle DBC;
also the side BC is common to both;

therefore the triangle ABC is equal to the triangle DCB in all respects; I. 26.

so that AB is equal to DC, and AC to DB;
and the angle BAC is equal to the angle CDB.

Also, because the angle ABC is equal to the angle DCB,
and the angle CBD equal to the angle BCA,
therefore the whole angle ABD is equal to the whole angle DCA.

And the triangles ABC, DCB having been proved equal in all respects are equal in area.

Therefore the diagonal BC bisects the parallelogram ACDB.

Q.E.D.

EXERCISES ON PARALLELOGRAMS.

1. *If one angle of a parallelogram is a right angle, all its angles are right angles.*

2. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

3. *If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.*

4. *If a quadrilateral has all its sides equal and one angle a right angle, all its angles are right angles.*

5. *The diagonals of a parallelogram bisect each other.*

6. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

7. If two opposite angles of a parallelogram are bisected by the diagonal which joins them, the figure is equilateral.

8. If the diagonals of a parallelogram are equal, all its angles are right angles.

9. In a parallelogram which is not rectangular the diagonals are unequal.

10. Any straight line drawn through the middle point of a diagonal of a parallelogram and terminated by a pair of opposite sides, is bisected at that point.

11. *If two parallelograms have two adjacent sides of one equal to two adjacent sides of the other, each to each, and one angle of one equal to one angle of the other, the parallelograms are equal in all respects.*

12. *Two rectangles are equal if two adjacent sides of one are equal to two adjacent sides of the other, each to each.*

13. In a parallelogram the perpendiculars drawn from one pair of opposite angles to the diagonal which joins the other pair are equal.

14. If ABCD is a parallelogram, and X, Y respectively the middle points of the sides AD, BC; shew that the figure AYCX is a parallelogram.

MISCELLANEOUS EXERCISES ON SECTIONS I. AND II.

1. Shew that the construction in Proposition 2 may generally be performed in eight different ways. Point out the exceptional case.

2. The bisectors of two vertically opposite angles are in the same straight line.

3. In the figure of Proposition 16, if AF is joined, shew

(i) that AF is equal to BC;

(ii) that the triangle ABC is equal to the triangle CFA in all respects.

4. ABC is a triangle right-angled at B, and BC is produced to D: shew that the angle ACD is obtuse.

5. Shew that in any regular polygon of n sides each angle contains $\frac{2(n-2)}{n}$ right angles.

6. The angle contained by the bisectors of the angles at the base of any triangle is equal to the vertical angle together with half the sum of the base angles.

7. The angle contained by the bisectors of two exterior angles of any triangle is equal to half the sum of the two corresponding interior angles.

8. If perpendiculars are drawn to two intersecting straight lines from any point between them, shew that the bisector of the angle between the perpendiculars is parallel to (or coincident with) the bisector of the angle between the given straight lines.

9. If two points P, Q be taken in the equal sides of an isosceles triangle ABC, so that BP is equal to CQ, shew that PQ is parallel to BC.

10. ABC and DEF are two triangles, such that AB, BC are equal and parallel to DE, EF, each to each; shew that AC is equal and parallel to DF.

11. Prove the second Corollary to Prop. 32 by drawing through any angular point lines parallel to all the sides.

12. If two sides of a quadrilateral are parallel, and the remaining two sides equal but not parallel, shew that the opposite angles are supplementary; also that the diagonals are equal.

SECTION III.

THE AREAS OF PARALLELOGRAMS AND TRIANGLES.

Hitherto when two figures have been said to be *equal*, it has been implied that they are *identically* equal, that is, equal in all respects.

But figures may be equal *in area* without being equal in all respects, that is, without having the same shape.

The present section deals with parallelograms and triangles which are equal in area but not necessarily identically equal.

[The ultimate test of equality, as we have already seen, is afforded by Axiom 8, which asserts that magnitudes which *may be made to coincide with one another* are equal. Now figures which are not equal in all respects, cannot be made to coincide without first undergoing some change of form: hence the method of direct *superposition* is unsuited to the purposes of the present section.

We shall see however from Euclid's proof of Proposition 35, that two figures which are not identically equal, may nevertheless be so related to a third figure, that it is possible to infer the equality of their areas.]

DEFINITIONS.

1. The **Altitude** of a parallelogram with reference to a given side as base, is the perpendicular distance between the base and the opposite side.

2. The **Altitude** of a triangle with reference to a given side as base, is the perpendicular distance of the opposite vertex from the base.

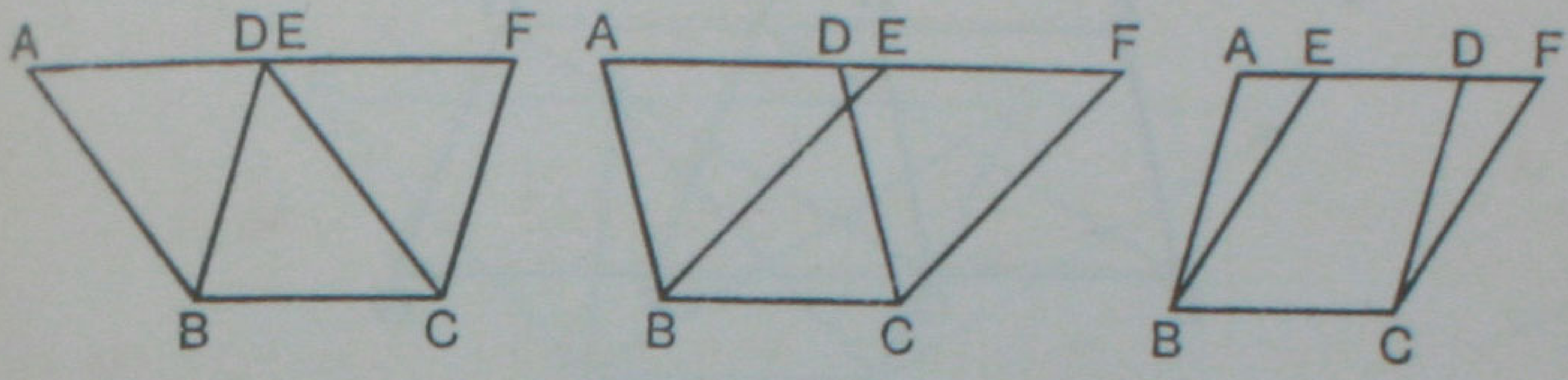
[From this point the following symbols will be introduced into the text:

$=$ for *is equal to*; $\therefore$ for *therefore*.

If it is thought desirable to shorten *written work* by the use of symbols and abbreviations, it is strongly recommended that only some well recognized system should be allowed, such, for example, as that given on page 11.]

PROPOSITION 35. THEOREM.

Parallelograms on the same base, and between the same parallels, are equal in area.



Let the parallelograms ABCD, EBCF be on the same base BC, and between the same parallels BC, AF.

Then shall the parallelogram ABCD *be equal in area to the parallelogram* EBCF.

CASE I. If the sides AD, EF, opposite to the base BC, are terminated at the same point D:
then each of the parallelograms ABCD, EBCF is double of the triangle BDC; I. 34.

∴ they are equal to one another. *Ax.* 6.

CASE II. But if the sides AD, EF are not terminated at the same point:

then because ABCD is a parallelogram,
∴ the side AD = the opposite side BC; I. 34.
similarly EF = BC;
∴ AD = EF. *Ax.* 1.

∴ the whole, or remainder, EA = the whole, or remainder, FD.

Then in the triangles FDC, EAB,

Because { FD = EA, *Proved.*
and the side DC = the opposite side AB, I. 34.
also the exterior angle FDC = the interior opposite angle EAB, I. 29.

∴ the triangle FDC = the triangle EAB. I. 4.

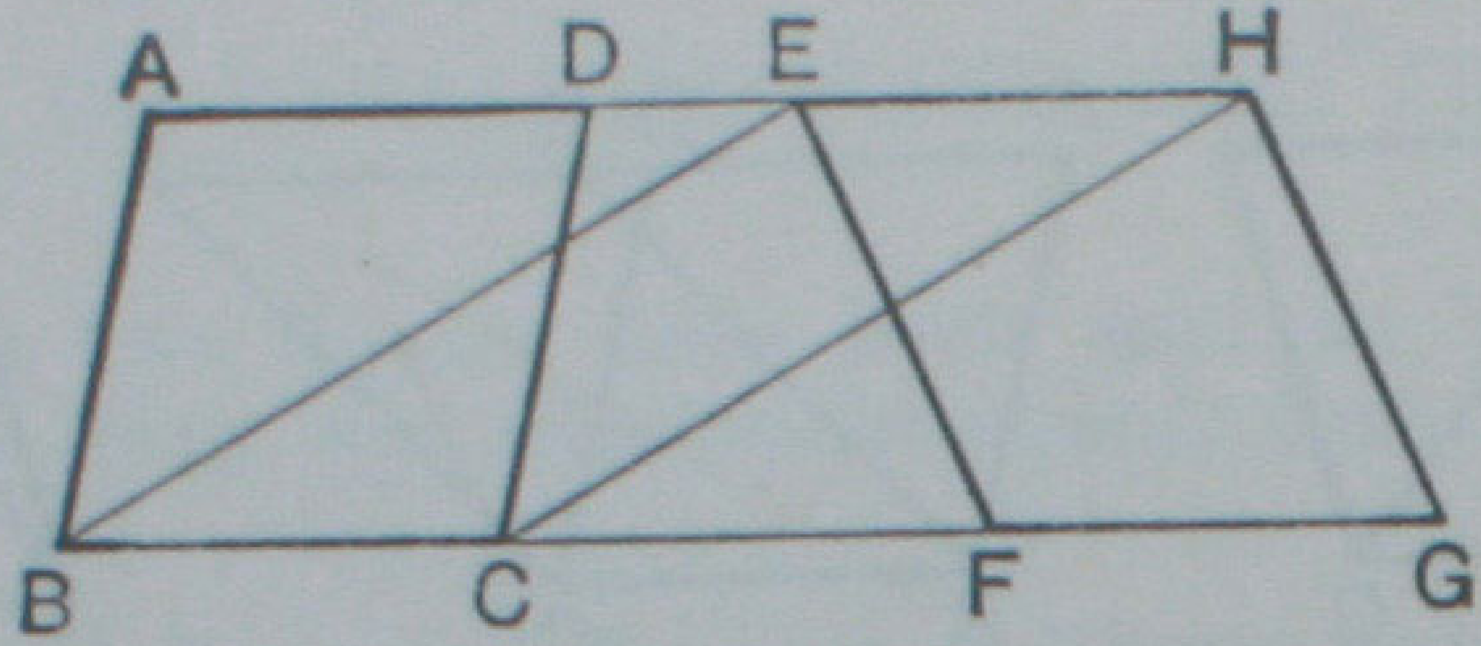
From the whole figure ABCF take the triangle FDC; and from the same figure take the equal triangle EAB;

then the remainders are equal. *Ax.* 3.

Therefore the parallelogram ABCD is equal to the parallelogram EBCF.

PROPOSITION 36. THEOREM.

Parallelograms on equal bases, and between the same parallels, are equal in area.



Let ABCD, EFGH be parallelograms on equal bases BC, FG, and between the same parallels AH, BG.

Then shall the parallelogram ABCD *be equal to the parallelogram* EFGH.

Construction. Join BE, CH.

Proof. Then because BC = FG; *Hyp.*
and the side FG = the opposite side EH; I. 34.
∴ BC = EH: *Ax.* 1.
and BC is parallel to EH; *Hyp.*
∴ BE and CH are also equal and parallel. I. 33.
Therefore EBCH is a parallelogram. *Def.* 36.

Now the parallelograms ABCD, EBCH are on the same base BC, and between the same parallels BC, AH;
∴ the parallelogram ABCD = the parallelogram EBCH. I. 35.

Also the parallelograms EFGH, EBCH are on the same base EH, and between the same parallels EH, BG;
∴ the parallelogram EFGH = the parallelogram EBCH. I. 35.

Therefore the parallelogram ABCD is equal to the parallelogram EFGH. *Ax.* 1.

Q.E.D.

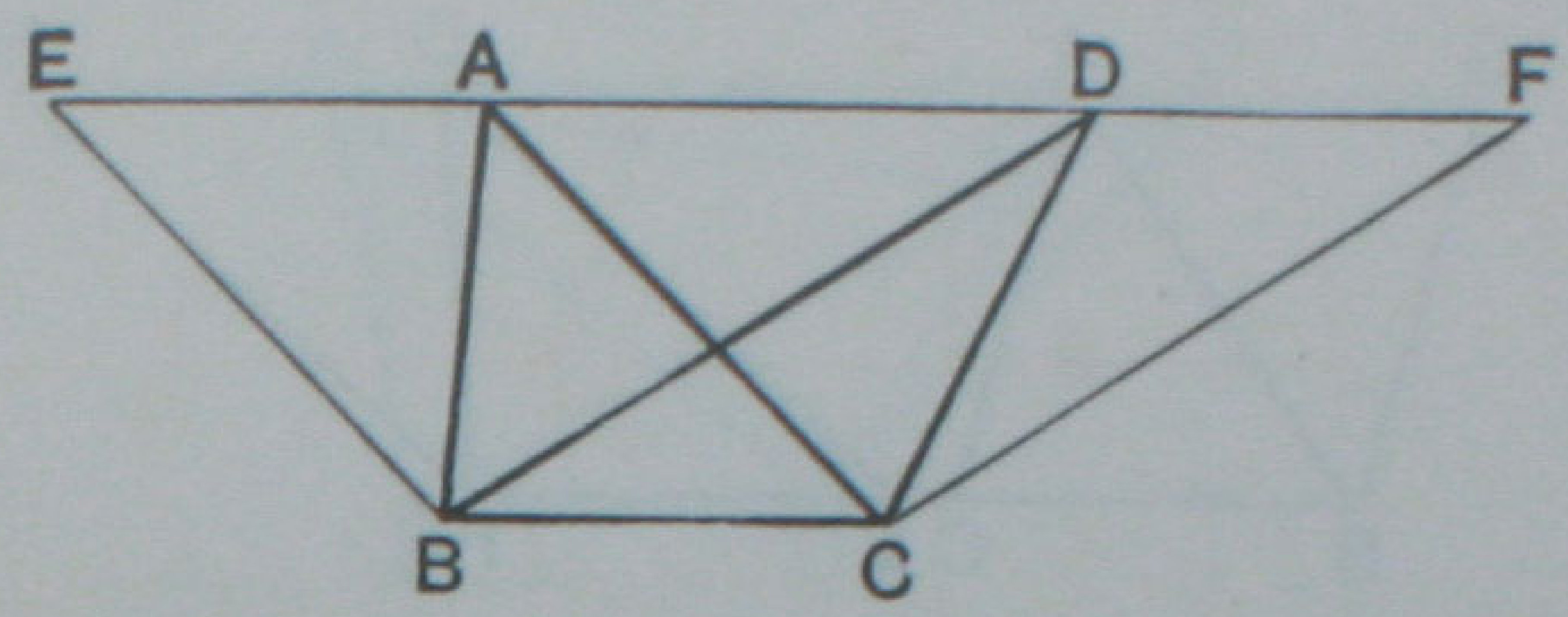
From the last two Propositions we infer that:

(i) *A parallelogram is equal in area to a rectangle of equal base and equal altitude.*

(ii) *Parallelograms on equal bases and of equal altitudes are equal in area.*

PROPOSITION 37. THEOREM.

Triangles on the same base, and between the same parallels, are equal in area.



Let the triangles ABC, DBC be upon the same base BC, and between the same parallels BC, AD.

Then shall the triangle ABC *be equal to the triangle* DBC.

Construction. Through B draw BE parallel to CA, to meet DA produced in E; I. 31.
through C draw CF parallel to BD, to meet AD produced in F.

Proof. Then, by construction, each of the figures EBCA, DBCF is a parallelogram. *Def.* 36.

And since they are on the same base BC, and between the same parallels BC, EF;

∴ the parallelogram EBCA = the parallelogram DBCF. I. 35.

Now the diagonal AB bisects EBCA; I. 34.

∴ the triangle ABC is half the parallelogram EBCA.

And the diagonal DC bisects DBCF; I. 34.

∴ the triangle DBC is half the parallelogram DBCF.

And the halves of equal things are equal. *Ax.* 7.

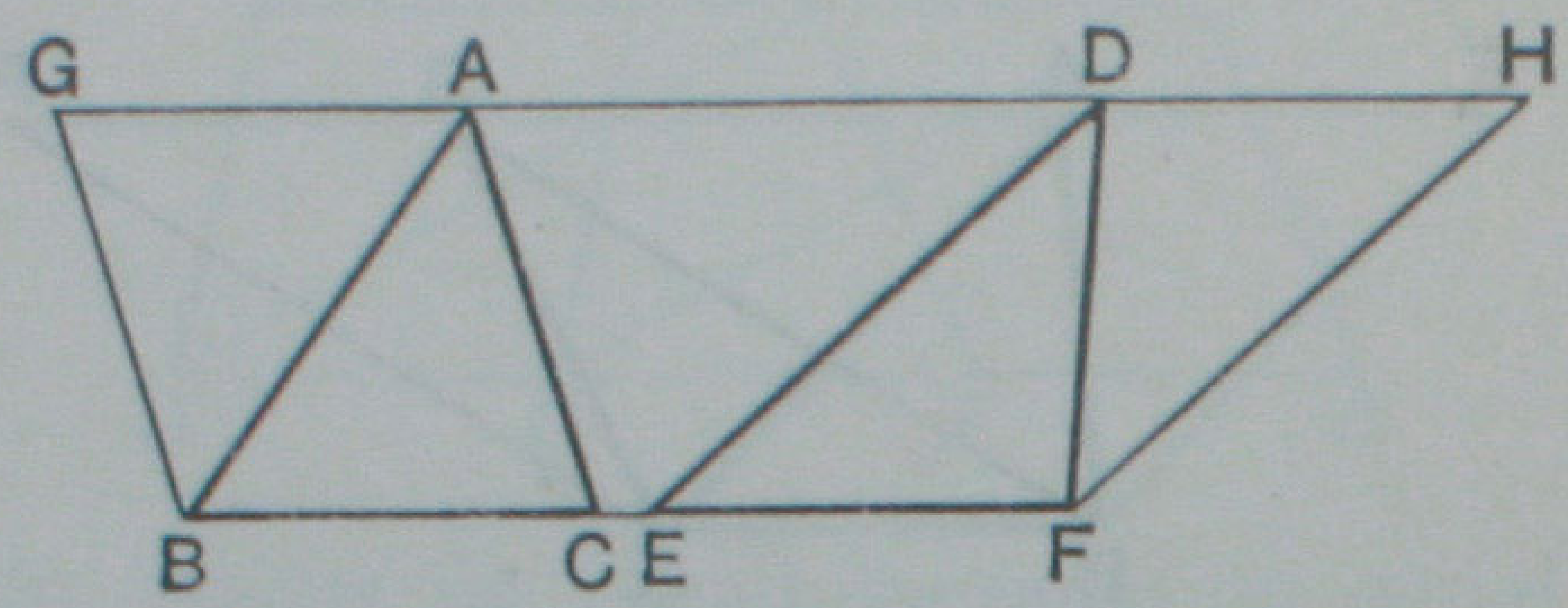
Therefore the triangle ABC is equal to the triangle DBC.

Q.E.D.

[For Exercises see page 79.]

PROPOSITION 38. THEOREM.

Triangles on equal bases, and between the same parallels, are equal in area.



Let the triangles ABC, DEF be on equal bases BC, EF, and between the same parallels BF, AD.

Then shall the triangle ABC *be equal to the triangle* DEF.

Construction. Through B draw BG parallel to CA, to meet DA produced in G; I. 31.
through F draw FH parallel to ED, to meet AD produced in H.

Proof. Then, by construction, each of the figures GBCA, DEFH is a parallelogram. *Def.* 36.

And since they are on equal bases BC, EF, and between the same parallels BF, GH;

∴ the parallelogram GBCA = the parallelogram DEFH. I. 36.

Now the diagonal DF bisects GBCA; I. 34.

∴ the triangle ABC is half the parallelogram GBCA.

And the diagonal DF bisects DEFH; I. 34.

∴ the triangle DEF is half the parallelogram DEFH.

And the halves of equal things are equal. *Ax.* 7.

Therefore the triangle ABC is equal to the triangle DEF.

Q.E.D.

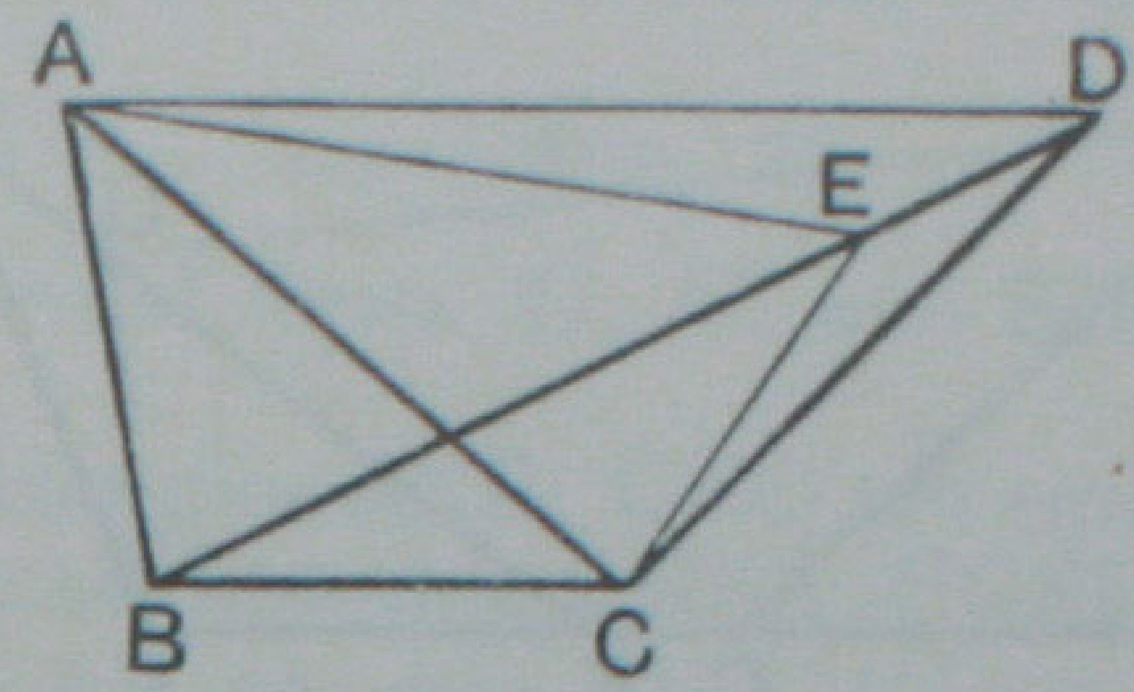
From this Proposition we infer that:

(i) *Triangles on equal bases and of equal altitude are equal in area.*

(ii) *Of two triangles of the same altitude, that is the greater which has the greater base; and of two triangles on the same base, or on equal bases, that is the greater which has the greater altitude.*

PROPOSITION 39. THEOREM.

Equal triangles on the same base, and on the same side of it, are between the same parallels.



Let the triangles ABC, DBC which stand on the same base BC, and on the same side of it be equal in area.

Then shall the triangles ABC, DBC *be between the same parallels; that is, if* AD *be joined,* AD *shall be parallel to* BC.

Construction. For if AD be not parallel to BC,
if possible, through A draw AE parallel to BC, I. 31.
meeting BD, or BD produced, in E.
Join EC.

Proof. Now the triangles ABC, EBC are on the same base BC, and between the same parallels BC, AE;
∴ the triangle ABC = the triangle EBC. I. 37.
But the triangle ABC = the triangle DBC; *Hyp.*
∴ the triangle DBC = the triangle EBC;
that is, the whole is equal to a part; which is impossible.
∴ AE is not parallel to BC.

Similarly it can be shewn that no other straight line through A, except AD, is parallel to BC.

Therefore AD is parallel to BC.

Q.E.D.

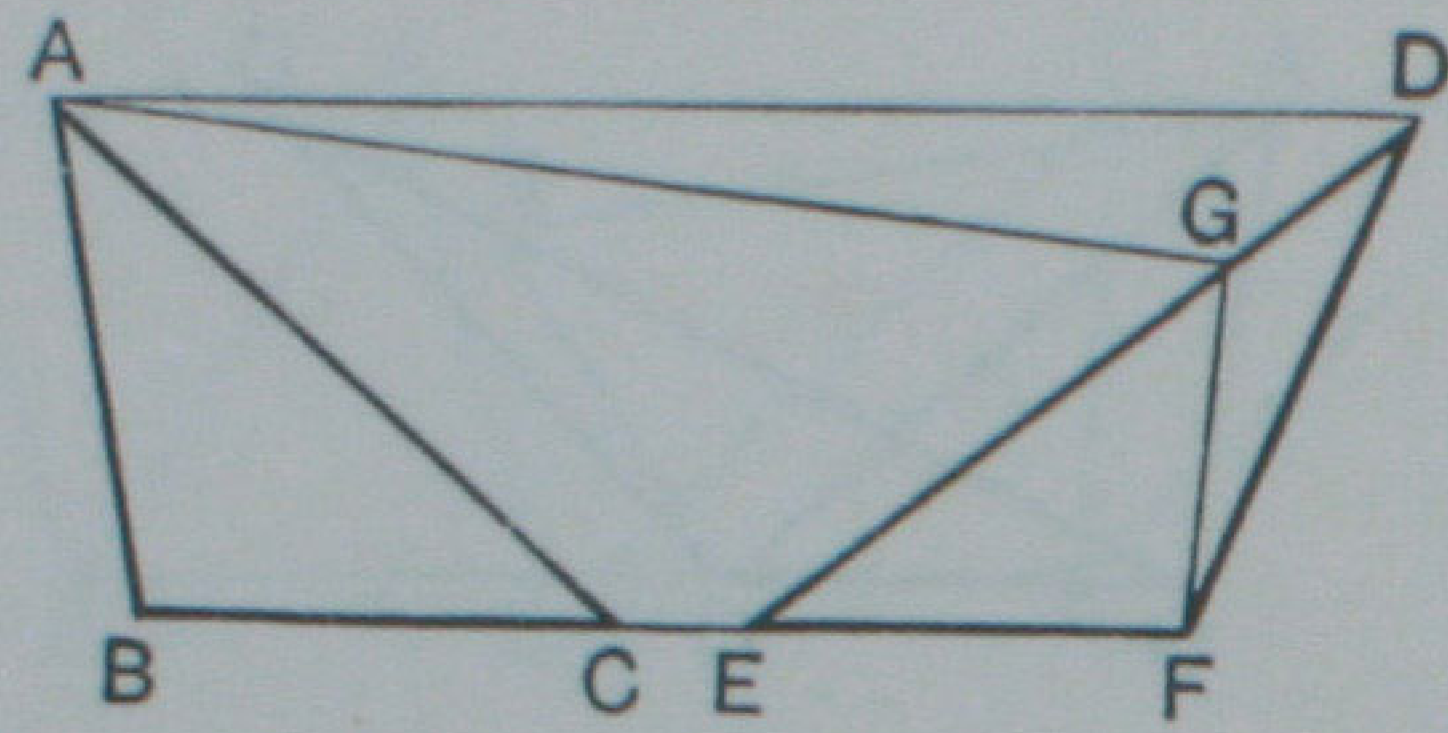
From this Proposition it follows that:

Equal triangles on the same base have equal altitudes.

[For Exercises see page 79.]

PROPOSITION 40. THEOREM.

Equal triangles, on equal bases in the same straight line, and on the same side of it, are between the same parallels.



Let the triangles ABC, DEF which stand on equal bases BC, EF, in the same straight line BF, and on the same side of it, be equal in area.

Then shall the triangles ABC, DEF *be between the same parallels; that is, if* AD *be joined,* AD *shall be parallel to* BF.

Construction. For if AD be not parallel to BF,
if possible, through A draw AG parallel to BF, I. 31.
meeting ED, or ED produced, in G.
Join GF.

Proof. Now the triangles ABC, GEF are on equal bases BC, EF, and between the same parallels BF, AG;

∴ the triangle ABC = the triangle GEF. I. 38.
But the triangle ABC = the triangle DEF: *Hyp.*
∴ the triangle DEF = the triangle GEF:
that is, the whole is equal to a part; which is impossible.
∴ AG is not parallel to BF.

Similarly it can be shewn that no other straight line through A, except AD, is parallel to BF.

Therefore AD is parallel to BF.

Q.E.D.

From this Proposition it follows that:

(i) *Equal triangles on equal bases have equal altitudes.*

(ii) *Equal triangles of equal altitudes have equal bases.*

EXERCISES ON PROPOSITIONS 37–40.

DEFINITION. Each of the three straight lines which join the angular points of a triangle to the middle points of the opposite sides is called a **Median** of the triangle.

ON PROP. 37.

1. If, in the figure of Prop. 37, AC and BD intersect in K, shew that

(i) the triangles AKB, DKC are equal in area.
(ii) the quadrilaterals EBKA, FCKD are equal.

2. In the figure of I. 16, shew that the triangles ABC, FBC are equal in area.

3. On the base of a given triangle construct a second triangle, equal in area to the first, and having its vertex in a given straight line.

4. Describe an isosceles triangle equal in area to a given triangle and standing on the same base.

ON PROP. 38.

5. *A triangle is divided by each of its medians into two parts of equal area.*

6. A parallelogram is divided by its diagonals into four triangles of equal area.

7. ABC is a triangle, and its base BC is bisected at X; if Y be any point in the median AX, shew that the triangles ABY, ACY are equal in area.

8. In AC, a diagonal of the parallelogram ABCD, any point X is taken, and XB, XD are drawn: shew that the triangle BAX is equal to the triangle DAX.

9. If two triangles have two sides of one respectively equal to two sides of the other, and the angles contained by those sides *supplementary*, the triangles are equal in area.

ON PROP. 39.

10. *The straight line which joins the middle points of two sides of a triangle is parallel to the third side.*

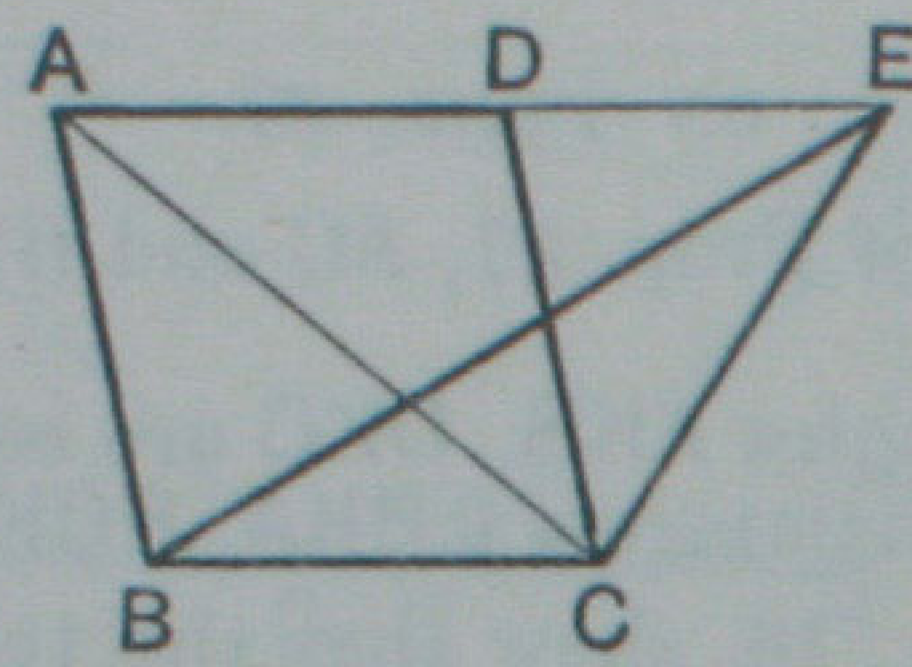
11. *If two straight lines* AB, CD *intersect in* O, *so that the triangle* AOC *is equal to the triangle* DOB, *shew that* AD *and* CB *are parallel.*

ON PROP. 40.

12. Deduce Prop. 40 from Prop. 39 by joining AE, AF in the figure of page 78.

PROPOSITION 41. THEOREM.

If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram shall be double of the triangle.



Let the parallelogram ABCD, and the triangle EBC be upon the same base BC, and between the same parallels BC, AE.

Then shall the parallelogram ABCD *be double of the triangle* EBC.

Construction. Join AC.

Proof. Now the triangles ABC, EBC are on the same base BC, and between the same parallels BC, AE;

∴ the triangle ABC = the triangle EBC. I. 37.

And since the diagonal AC bisects ABCD; I. 34.

∴ the parallelogram ABCD is double of the triangle ABC.

Therefore the parallelogram ABCD is also double of the triangle EBC.

Q.E.D.

EXERCISES.

1. ABCD is a parallelogram, and X, Y are the middle points of the sides AD, BC; if Z is any point in XY, or XY produced, shew that the triangle AZB is one quarter of the parallelogram ABCD.

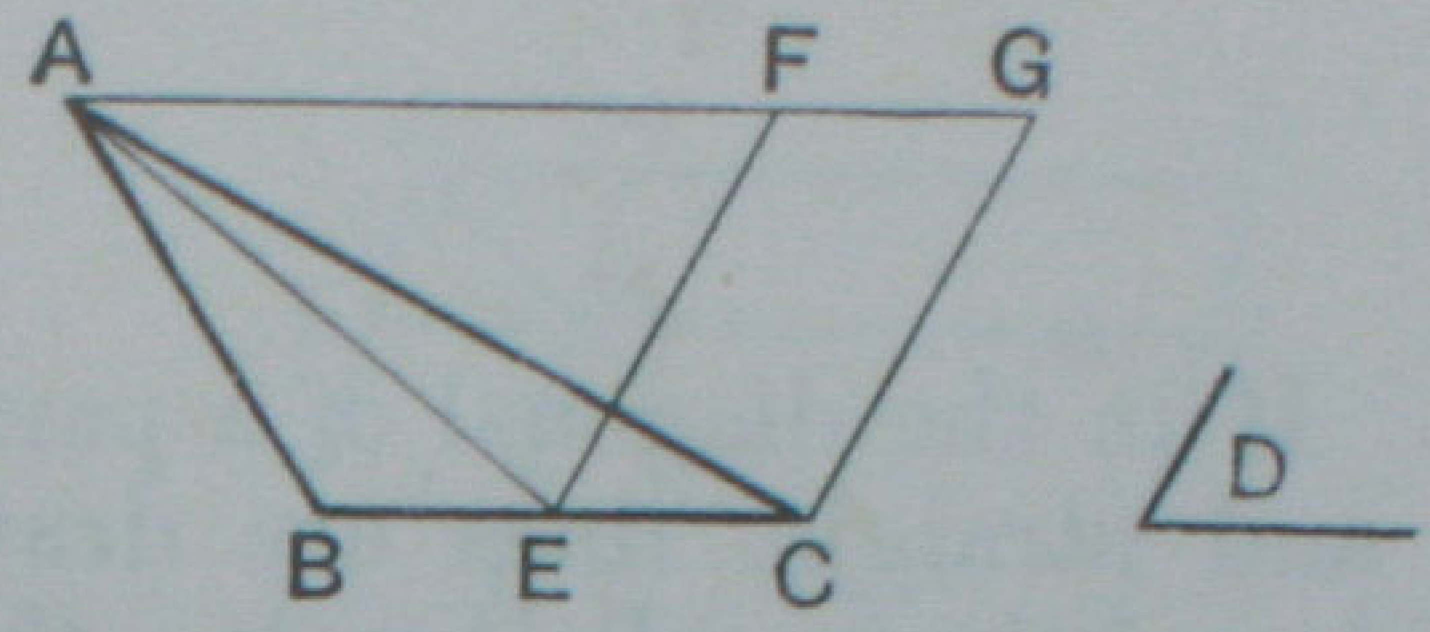
2. Describe a right-angled isosceles triangle equal to a given square.

3. If ABCD is a parallelogram, and X, Y any points in DC and AD respectively: shew that the triangles AXB, BYC are equal in area.

4. ABCD is a parallelogram, and P is any point within it; shew that the sum of the triangles PAB, PCD is equal to half the parallelogram.

PROPOSITION 42. PROBLEM.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given angle.



Let ABC be the given triangle, and D the given angle.

It is required to describe a parallelogram equal to ABC, *and having one of its angles equal to* D.

Construction. Bisect BC at E. I. 10.
At E in CE, make the angle CEF equal to D; I. 23.
through A draw AFG parallel to EC; I. 31.
and through C draw CG parallel to EF.
Then FECG *shall be the parallelogram required.*
Join AE.

Proof. Now the triangles ABE, AEC are on equal bases BE, EC, and between the same parallels;
∴ the triangle ABE = the triangle AEC; I. 38.
∴ the triangle ABC is double of the triangle AEC.
But FECG is a parallelogram by construction; *Def.* 36.
and it is double of the triangle AEC,
being on the same base EC, and between the same parallels EC and AG. I. 41.
Therefore the parallelogram FECG is equal to the triangle ABC;
and it has one of its angles CEF equal to the given angle D.

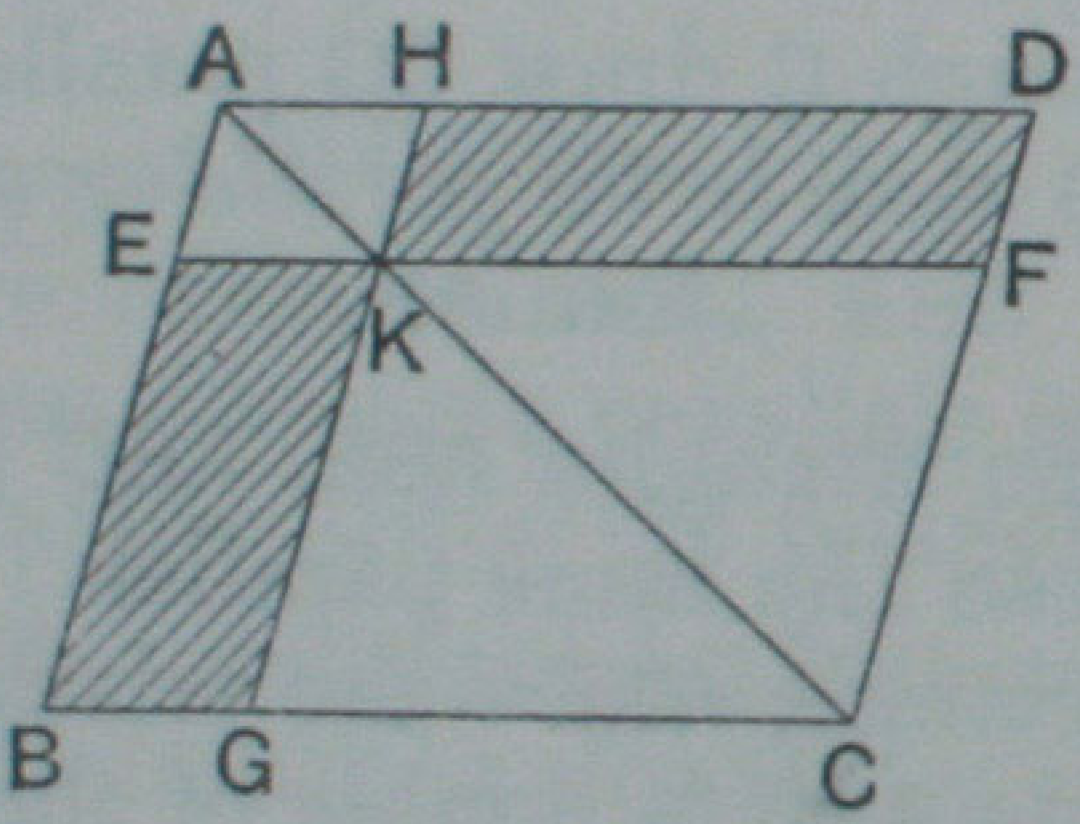
Q.E.F.

EXERCISES.

1. Describe a parallelogram equal to a given square standing on the same base, and having an angle equal to half a right angle.

2. Describe a rhombus equal to a given parallelogram and standing on the same base. When does the construction fail?

DEFINITION. If in the diagonal of a parallelogram any point is taken, and straight lines are drawn through it parallel to the sides of the parallelogram; then of the four parallelograms into which the whole figure is divided, the two through which the diagonal passes are called **Parallelograms about that diagonal**, and the other two, which with these make up the whole figure, are called the **complements** of the parallelograms about the diagonal.

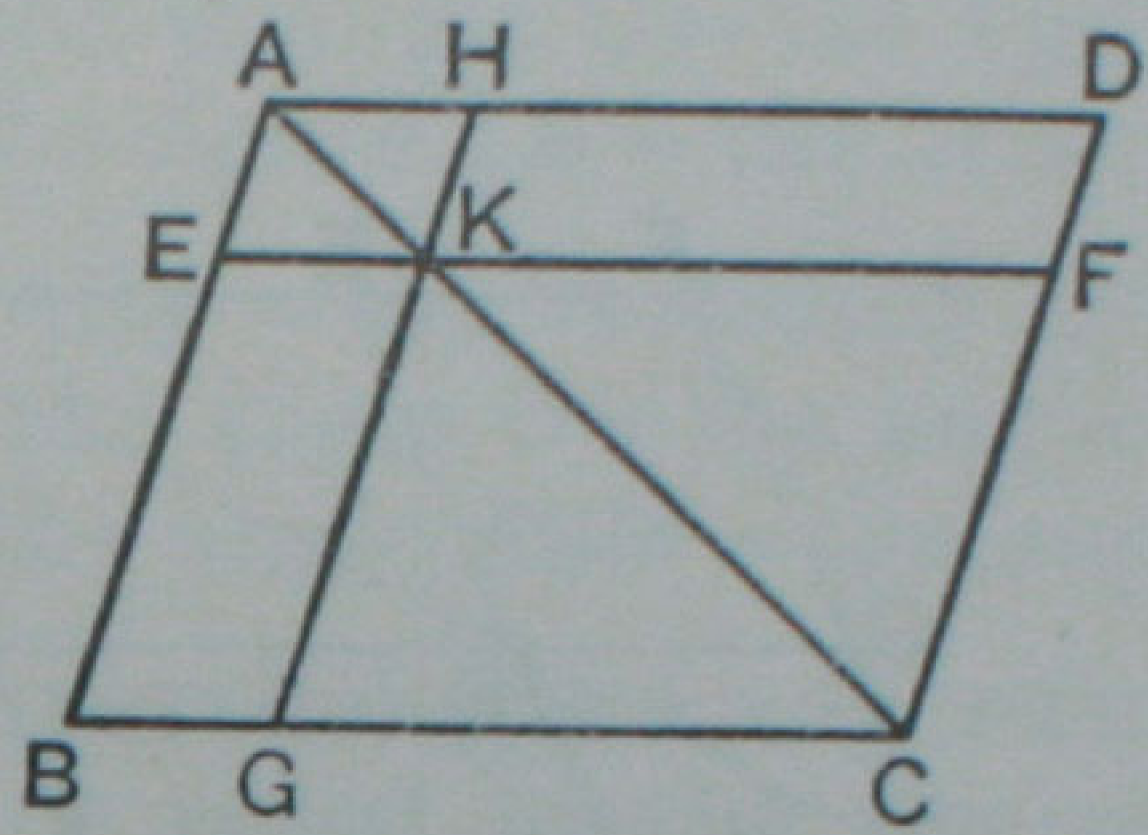


Thus in the figure given above, AEKH, KGCF are parallelograms about the diagonal AC; and the shaded figures HKFD, EBGK are the complements of those parallelograms.

NOTE. A parallelogram is often named by *two* letters only, these being placed at opposite angular points.

Proposition 43. Theorem.

The complements of the parallelograms about the diagonal of any parallelogram, are equal to one another.



Let ABCD be a parallelogram, and KD, KB the complements of the parallelograms EH, GF about the diagonal AC.

Then shall the complement BK *be equal to the complement* KD.

Proof. Because EH is a parallelogram, and AK its diagonal,

∴ the triangle AEK = the triangle AHK. I. 34.

Similarly the triangle KGC = the triangle KFC.

Hence the triangles AEK, KGC are together equal to the triangles AHK, KFC.

But since the diagonal AC bisects the parallelogram ABCD;

∴ the whole triangle ABC = the whole triangle ADC. I. 34.

Therefore the remainder, the complement BK, is equal to the remainder, the complement KD. Q.E.D.

EXERCISES.

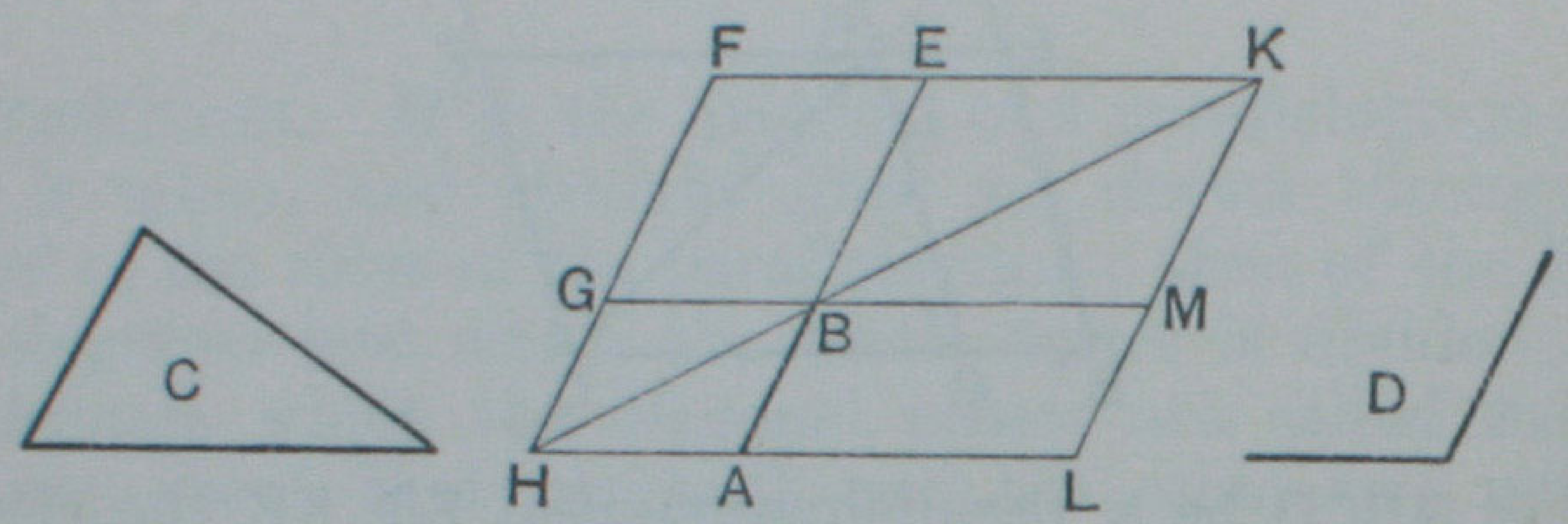
In the figure of Prop. 43, prove that

(i) The parallelogram ED is equal to the parallelogram BH.

(ii) If KБ, KD are joined, the triangle AKB is equal to the triangle AKD.

Proposition 44. Problem.

To a given straight line to apply a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given angle.



Let AB be the given straight line, C the given triangle, and D the given angle.

It is required to apply to the straight line AB *a parallelogram equal to the triangle* C, *and having an angle equal to the angle* D.

Construction. On AB produced describe a parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D. I. 22 *and* I. 42*.

Through A draw AH parallel to BG or EF, to meet FG produced in H. I. 31.

Join HB.

Then because AH and EF are parallel, and HF meets them,

∴ the angles AHF, HFE together = two right angles. I. 29.

Hence the angles BHF, HFE are together less than two right angles;

∴ HB and FE will meet if produced towards B and E. *Ax.* 12.

Produce HB and FE to meet at K.

Through K draw KL parallel to EA or FH; I. 31.

and produce HA, GB to meet KL in the points L and M.

Then shall BL *be the parallelogram required.*

Proof. Now FHLK is a parallelogram, *Constr.*
and LB, BF are the complements of the parallelograms about the diagonal HK:

∴ the complement LB = the complement BF. I. 43.

But the triangle C = the figure BF; *Constr.*
∴ the figure LB = the triangle C.

Again the angle ABM = the vertically opposite angle GBE;
also the angle D = the angle GBE; *Constr.*
∴ the angle ABM = the angle D.

Therefore the parallelogram LB, which is applied to the straight line AB, is equal to the triangle C, and has the angle ABM equal to the angle D. Q.E.F.

*This step of the construction is effected by first describing on AB produced a triangle whose sides are respectively equal to those of the triangle C (I. 22); and by then making a parallelogram equal to the triangle so drawn, and having an angle equal to D (I. 42).

QUESTIONS FOR REVISION.

1. Quote Euclid's Twelfth Axiom. What objections have been raised to it, and what substitute for it has been suggested?

2. Which of Euclid's Propositions, dealing with parallel straight lines, depends on Axiom 12? Furnish an alternative proof.

3. *Straight lines which are parallel to the same straight line are parallel to one another* [Prop. 30]. Deduce this from Playfair's Axiom.

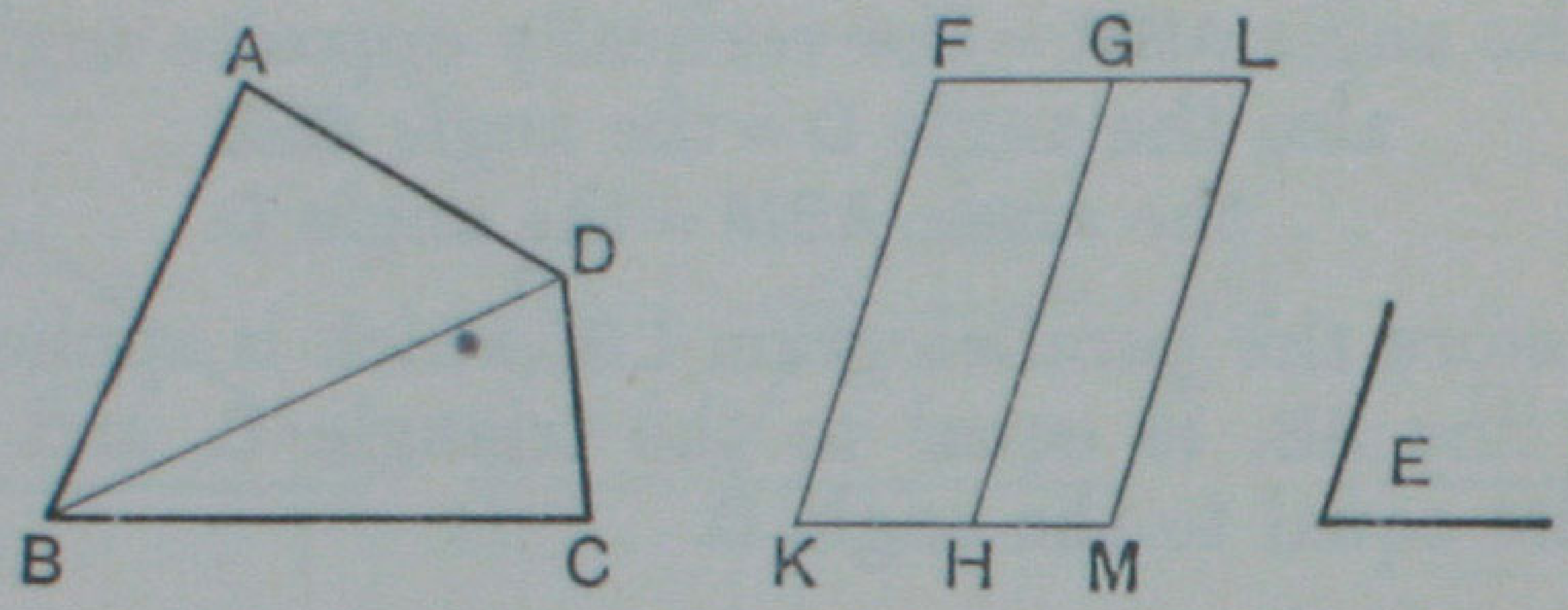
4. Define *a parallelogram*, *an altitude* of a triangle, *a median* of a triangle, *parallelograms about the diagonal of a parallelogram.*

5. What is meant by *superposition*? On what Axiom does this method depend? Give instances of figures which are equal *in area*, but which cannot be superposed.

6. In fig. 2 of Prop. 35 shew how one parallelogram may be cut into pieces, which, when fitted together in other positions, make up the other parallelogram.

Proposition 45. Problem.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given angle.



Let ABCD be the given rectilineal figure, and E the given angle.

It is required to describe a parallelogram equal to ABCD, *and having an angle equal to* E.

Suppose the given rectilineal figure to be a quadrilateral.

Construction. Join BD.

Describe the parallelogram FH equal to the triangle ABD, and having the angle FKH equal to the angle E. I. 42.

To GH apply the parallelogram GM, equal to the triangle DBC, and having the angle GHM equal to E. I. 44.

Then shall FKML *be the parallelogram required.*

Proof. Because each of the angles GHM, FKH = the angle E;

∴ the angle FKH = the angle GHM.

To each of these equals add the angle GHK;

then the angles FKH, GHK together = the angles GHM, GHK.

But since FK, GH are parallel, and KH meets them;

∴ the angles FKH, GHK together = two right angles; I. 29.

∴ also the angles GHM, GHK together = two right angles;

∴ KH, HM are in the same straight line. I. 14.

Again, because KM, FG are parallel, and HG meets them,
∴ the angle MHG = the alternate angle HGF. I. 29.

To each of these equals add the angle HGL;
then the angles MHG, HGL together = the angles HGF, HGL.

But because HM, GL are parallel, and HG meets them,
∴ the angles MHG, HGL together = two right angles: I. 29.
∴ also the angles HGF, HGL together = two right angles:
∴ FG, GL are in the same straight line. I. 14.

And because KF and ML are each parallel to HG, *Constr.*
therefore KF is parallel to ML; I. 30.
and KM, FL are parallel; *Constr.*
∴ FKML is a parallelogram. *Def.* 36.

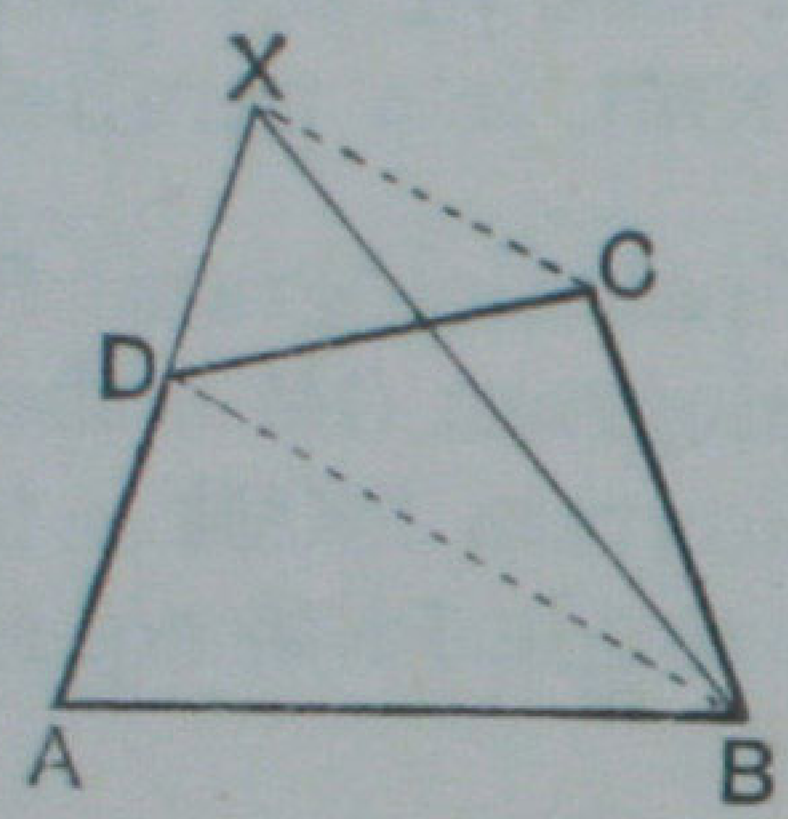
Again, because the parallelogram FH = the triangle ABD,
and the parallelogram GM = the triangle DBC; *Constr.*
∴ the whole parallelogram FKML = the whole figure ABCD;
and it has the angle FKM equal to the angle E.

By a series of similar steps, a parallelogram may be constructed equal to a rectilineal figure of more than four sides. Q.E.F.

The following Problem is important, and furnishes a useful application of the principles of the foregoing propositions.

ADDITIONAL PROBLEM.

To describe a triangle equal in area to a given quadrilateral.



Let ABCD be the given quadrilateral.

It is required to describe a triangle equal to ABCD *in area.*

Construction. Join BD.

Through C draw CX parallel to BD, meeting AD produced in X.

Join BX.

Then XAB shall be the required triangle.

Proof. Now the triangles XDB, CDB are on the same base DB and between the same parallels DB, XC;

∴ the triangle XDB = the triangle CDB in area. I. 37.

To each of these equals add the triangle ADB;

then the triangle XAB = the figure ABCD.

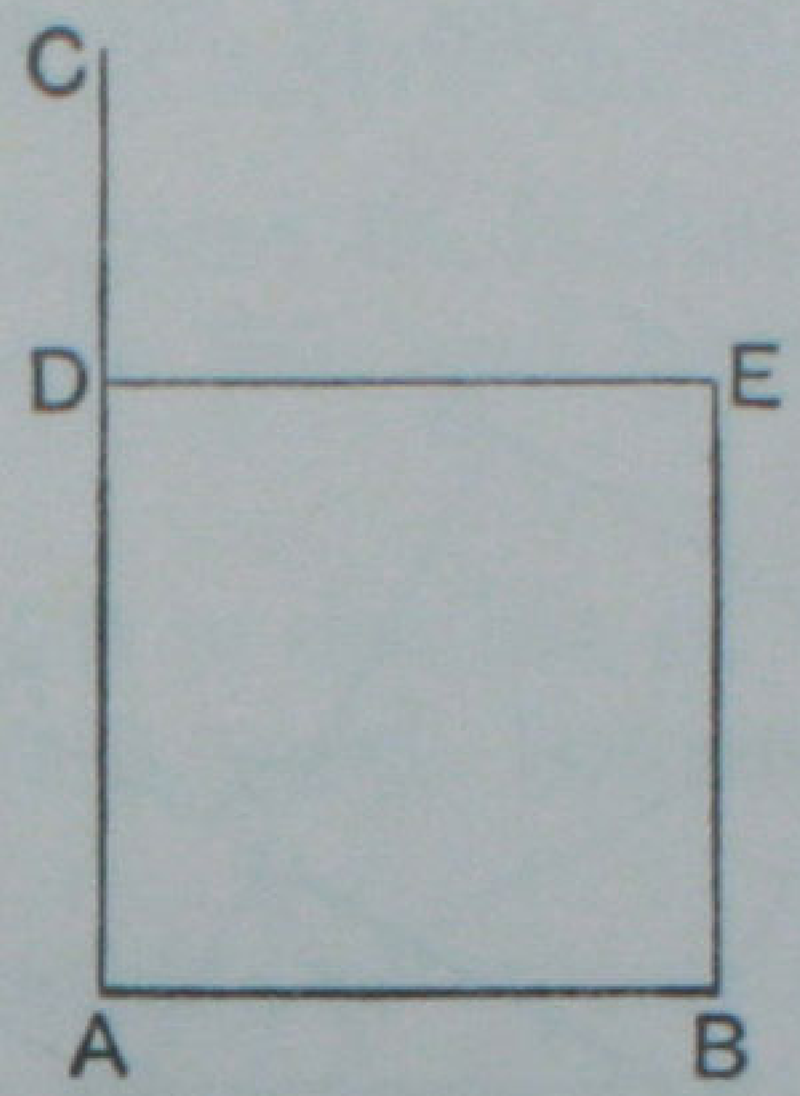
EXERCISE.

Construct a rectilineal figure equal to a given rectilineal figure, and having fewer sides by one than the given figure.

Hence shew how to construct a triangle equal to a given rectilineal figure.

PROPOSITION 46. PROBLEM.

To describe a square on a given straight line.



Let AB be the given straight line.
It is required to describe a square on AB.

Constr. From A draw AC at right angles to AB ; I. 11.
and make AD equal to AB. I. 3.
Through D draw DE parallel to AB ; I. 31.
and through B draw BE parallel to AD, meeting DE in E.
Then shall ADEB *be a square.*

Proof. For, by construction, ADEB is a parallelogram :
∴ AB = DE, and AD = BE. I. 34.
But AD = AB ; *Constr.*
∴ the four straight lines AB, AD, DE, EB are all equal ;
that is, the figure ADEB is equilateral.

Again, since AB, DE are parallel, and AD meets them,
∴ the angles BAD, ADE together = two right angles ; I. 29.
but the angle BAD is a right angle ; *Constr.*
∴ also the angle ADE is a right angle.

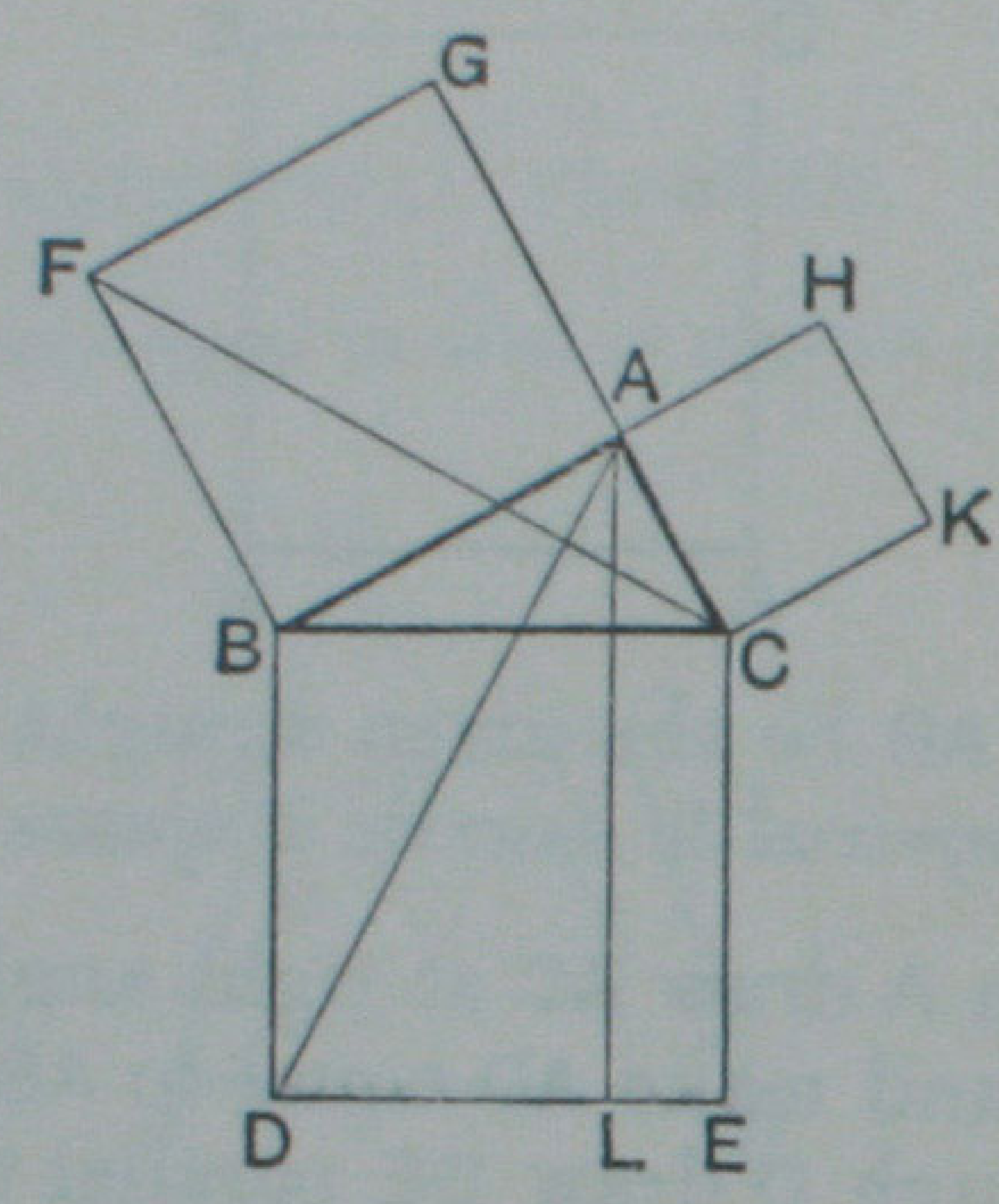
And the opposite angles of a parallelogram are equal ; I. 34.
∴ each of the angles DEB, EBA is a right angle :
that is the figure ADEB is rectangular.
Hence it is a square, and it is described on AB.

Q.E.F.

COROLLARY. *If one angle of a parallelogram is a right angle, all its angles are right angles.*

PROPOSITION 47. THEOREM.

In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the other two sides.



Let ABC be a right-angled triangle, having the angle BAC a right angle.

Then shall the square described on the hypotenuse BC *be equal to the sum of the squares described on* BA, AC.

Construction. On BC describe the square BDEC; I. 46.
and on BA, AC describe the squares BAGF, ACKH.
Through A draw AL parallel to BD or CE; I. 31.
and join AD, FC.

Proof. Then because each of the angles BAC, BAG is a right angle,

∴ CA and AG are in the same straight line. I. 14.

Now the angle CBD = the angle FBA,
for each of them is a right angle.
Add to each the angle ABC:
then the whole angle ABD = the whole angle FBC.

Then in the triangles ABD, FBC,

Because { AB = FB,
and BD = BC,
also the angle ABD = the angle FBC; *Proved.*

∴ the triangle ABD = the triangle FBC. I. 4.

Now the parallelogram BL is double of the triangle ABD, being on the same base BD, and between the same parallels BD, AL. I. 41.

And the square GB is double of the triangle FBC, being on the same base FB, and between the same parallels FB, GC. I. 41.

But doubles of equals are equal: *Ax.* 6.

therefore the parallelogram BL = the square GB.

Similarly, by joining AE, BK it can be shewn that the parallelogram CL = the square CH.

Therefore the whole square BE = the sum of the squares GB, HC:

that is, the square described on the hypotenuse BC is equal to the sum of the squares described on the two sides BA, AC. Q.E.D.

NOTE. It is not necessary to the proof of this Proposition that the three squares should be described *external* to the triangle ABC; and since *each* square may be drawn either *towards* or *away from* the triangle, it may be shewn that there are $2 \times 2 \times 2$, or *eight*, possible constructions.

Obs. The following properties of a square, though not formally enunciated by Euclid, are employed in subsequent proofs. [See I. 48.]

(i) *The squares on equal straight lines are equal.*

(ii) *Equal squares stand upon equal straight lines.*

EXERCISES ON PROPOSITION 47.

1. In the figure of this Proposition, shew that

   (i) If BG, CH are joined, these straight lines are parallel;

   (ii) The points F, A, K are in one straight line;

   (iii) FC and AD are at right angles to one another;

   (iv) If GH, KE, FD are joined, the triangle GAH is equal to the given triangle in all respects; and the triangles FBD, KCE are each equal in area to the triangle ABC.
   [See Ex. 9, p. 79.]

2. On the sides AB, AC of *any* triangle ABC, squares ABFG, ACKH are described both toward the triangle, or both on the side remote from it: shew that the straight lines BH and CG are equal.

3. On the sides of any triangle ABC, equilateral triangles BCX, CAY, ABZ are described, all externally, or all towards the triangle: shew that AX, BY, CZ are all equal.

4. *The square described on the diagonal of a given square, is double of the given square.*

5. ABC *is an equilateral triangle, and* AX *is the perpendicular drawn from* A *to* BC*: shew that the square on* AX *is three times the square on* BX.

6. Describe a square equal to the sum of two given squares.

7. From the vertex A of a triangle ABC, AX is drawn perpendicular to the base: shew that the difference of the squares on the sides AB and AC, is equal to the difference of the squares on BX and CX, the segments of the base.

8. If from any point O within a triangle ABC, perpendiculars OX, OY, OZ are drawn to the sides BC, CA, AB respectively: shew that the sum of the squares on the segments AZ, BX, CY is equal to the sum of the squares on the segments AY, CX, BZ.

9. ABC is a triangle right-angled at A; and the sides AB, AC are intersected by a straight line PQ, and BQ, PC are joined. Prove that the sum of the squares on BQ, PC is equal to the sum of the squares on BC, PQ.

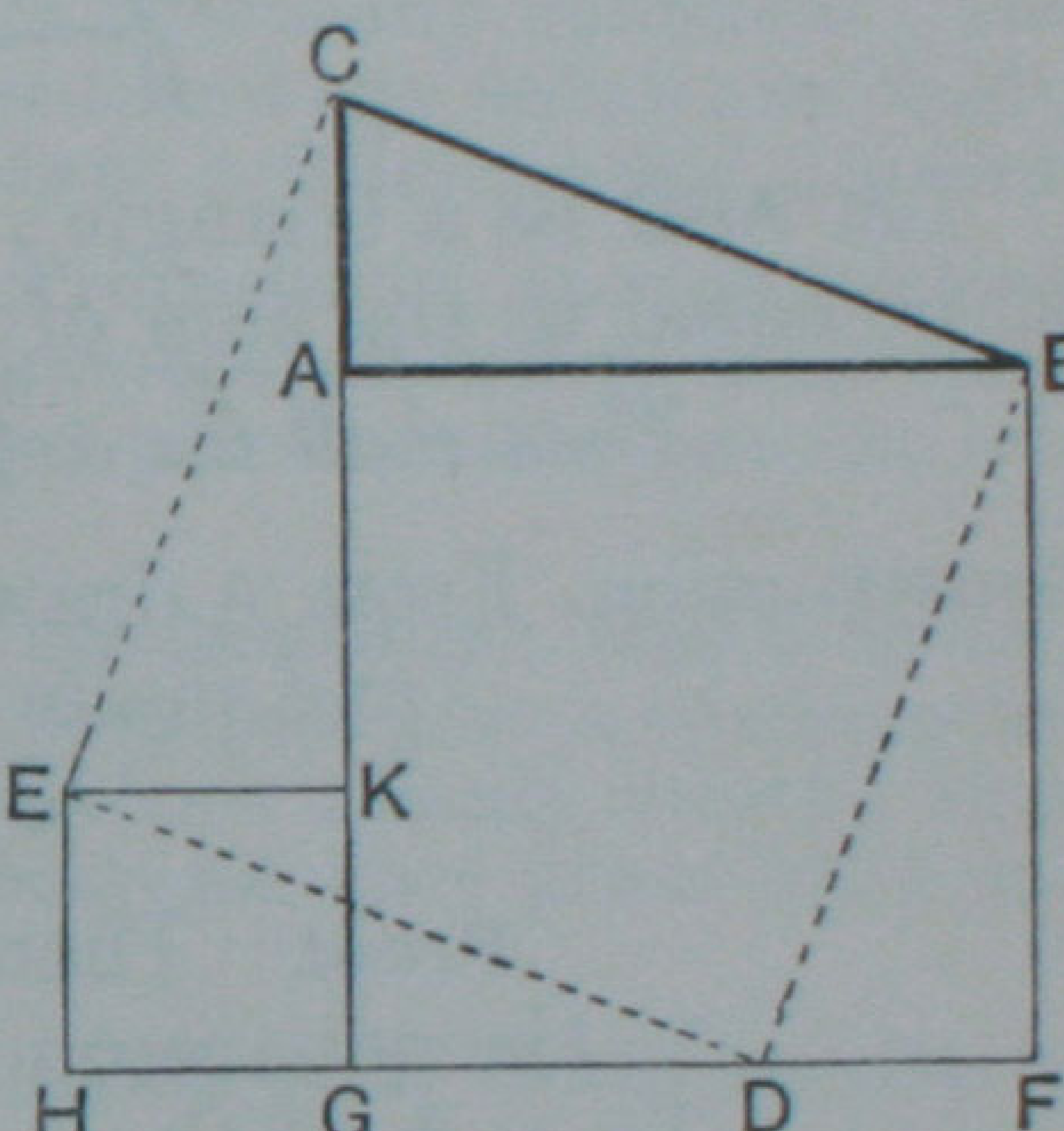
10. In a right-angled triangle four times the sum of the squares on the two medians drawn from the acute angles is equal to five times the square on the hypotenuse.

NOTES ON PROPOSITION 47.

It is believed that Proposition 47 is due to Pythagoras, a Greek philosopher and mathematician, who lived about two centuries before Euclid.

Many experimental proofs of this theorem have been given by means of actual *dissection*: that is to say, it has been shewn how the squares on the sides containing the right angle may be cut up into pieces which, when fitted together in other positions, exactly make up the square on the hypotenuse. Two of these methods of dissection are given below.

I. In the adjoining diagram ABC is the given right-angled triangle, and the figures AF, HK are the squares on AB, AC, placed side by side.

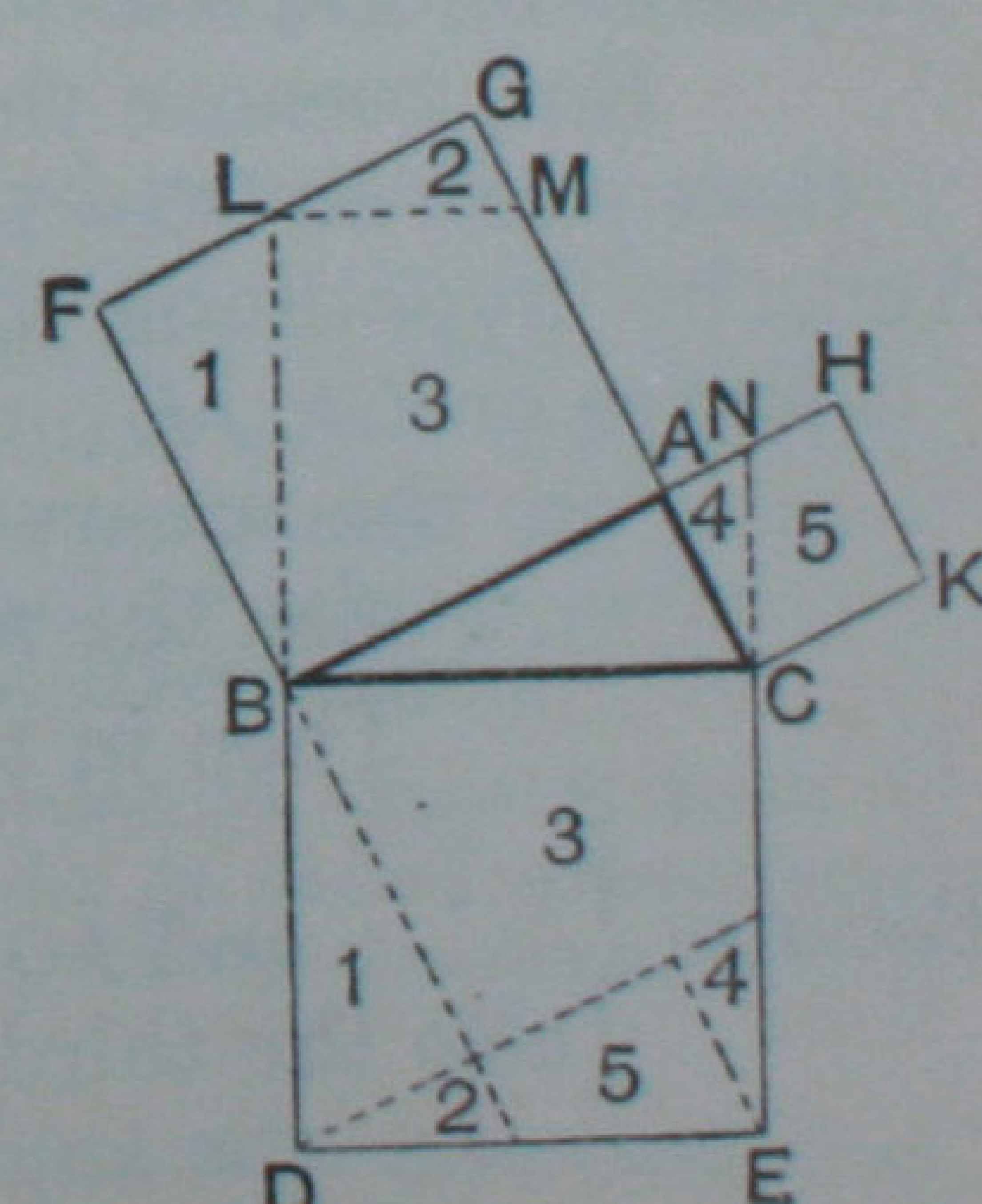


FD is made equal to EH or AC;

and the two squares AF, HK are cut along the lines ED, DB.

Then it will be found that the triangle EHD may be placed so as to fill up the space CAB; and the triangle BFD may be made to fill the space CKE.

Hence the two squares AF, HK may be fitted together so as to form the single figure CBDE, which will be found to be a perfect square, namely the square on the hypotenuse BC.

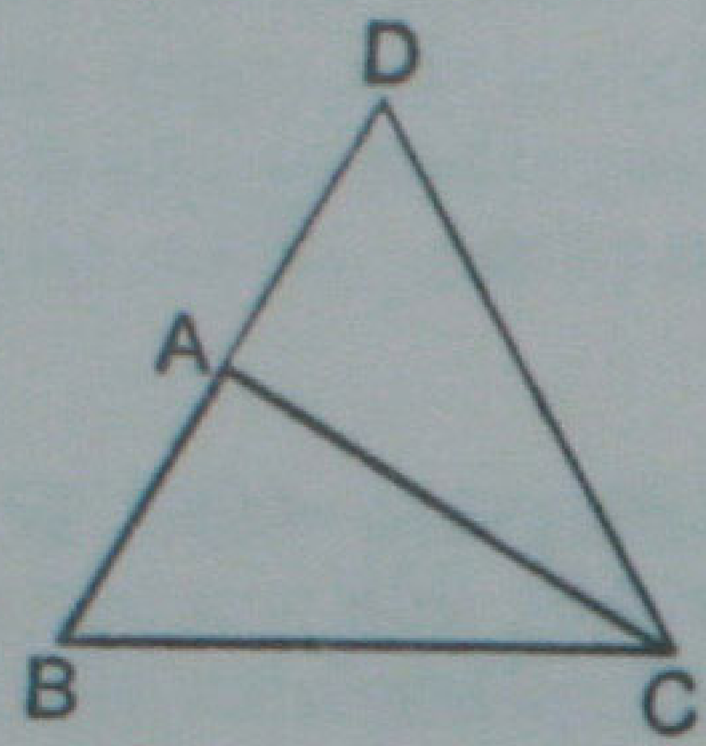


II. In the figure of I. 47, let DB and EC be produced to meet FG and AH in L and N respectively; and let LM be drawn parallel to BC.

Then it will be found that the several parts of the two squares FA, AK can be fitted together (in the places bearing corresponding numbers) so as exactly to fill up the square DC.

PROPOSITION 48. THEOREM.

If the square described on one side of a triangle be equal to the sum of the squares described on the other two sides, then the angle contained by these two sides shall be a right angle.



Let ABC be a triangle; and let the square described on BC be equal to the sum of the squares described on BA, AC.

Then shall the angle BAC *be a right angle.*

Construction. From A draw AD at right angles to AC; I. 11.
and make AD equal to AB. I. 3.
Join DC.

Proof. Then, because AD = AB, *Constr.*
∴ the square on AD = the square on AB.
To each of these add the square on CA;
then the sum of the squares on CA, AD = the sum of the squares on CA, AB.

But, because the angle DAC is a right angle, *Constr.*
∴ the square on DC = the sum of the squares on CA, AD. I. 47.
And, by hypothesis, the square on BC = the sum of the squares on CA, AB;
∴ the square on DC = the square on BC:
∴ also the side DC = the side BC.

Then in the triangles DAC, BAC,
Because { DA = BA, *Constr.*
and AC is common to both;
also the third side DC = the third side BC; *Proved.*
∴ the angle DAC = the angle BAC. I. 8.
But DAC is a right angle. *Constr.*
Therefore also BAC is a right angle. Q.E.D.

THEOREMS AND EXAMPLES ON BOOK I.

INTRODUCTORY.

HINTS TOWARDS THE SOLUTION OF GEOMETRICAL EXERCISES. ANALYSIS. SYNTHESIS.

It is commonly found that exercises in Pure Geometry present to a beginner far more difficulty than examples in any other branch of Elementary Mathematics. This seems to be due to the following causes:

(i) The variety of such exercises is practically unlimited; and it is impossible to lay down for their treatment any definite methods, such for example as the rules of Elementary Arithmetic and Algebra.

(ii) The arrangement of Euclid's Propositions, though perhaps the most *convincing* of all forms of argument, affords in most cases little clue as to the way in which the proof or construction *was discovered*.

Euclid's propositions are arranged **synthetically**: that is to say, starting from the hypothesis or data, they first give a construction in accordance with postulates, and problems already solved; then by successive steps based on known theorems, they prove what was required in the enunciation.

Thus Geometrical Synthesis is a *building up* of *known* results, in order to obtain a *new* result.

But as this is not the way in which constructions or proofs are usually discovered, we draw the student's attention to the following hints.

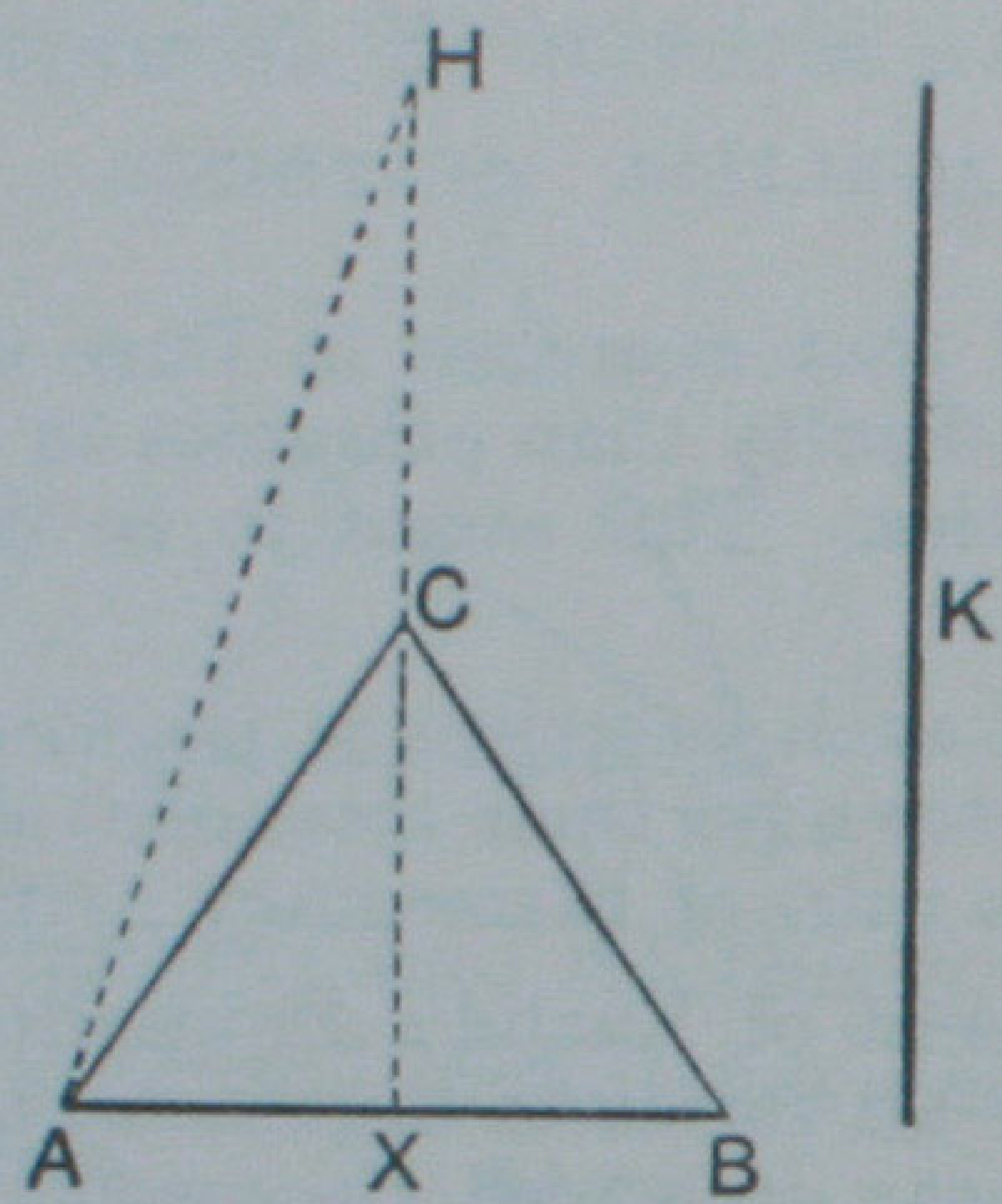
Begin by *assuming* the result it is desired to establish; then by working backwards, trace the consequences of the assumption, and try to ascertain its dependence on some simpler theorem which is already known to be true, or on some condition which suggests the necessary construction. If this attempt is successful, the steps of the argument may in general be re-arranged in reverse order, and the construction and proof presented in a synthetic form.

This unravelling of a proposition in order to trace it back to some earlier principle on which it depends, is called **geometrical analysis**: it is the natural way of attacking many theorems, and it is especially useful in solving *problems*.

Although the above directions do not amount to a *method*, they often furnish a mode of *searching for a suggestion*. Geometrical Analysis however can only be used with success when a thorough grasp of the chief propositions of Euclid has been gained.

The practical application of the foregoing hints is illustrated by the following examples.

1. *Construct an isosceles triangle having given the base, and the sum of one of the equal sides and the perpendicular drawn from the vertex to the base.*



Let AB be the given base, and K the sum of one side and the perpendicular drawn from the vertex to the base.

ANALYSIS. *Suppose* ABC *to be the required triangle.*

From C draw CX perpendicular to AB:
then AB is bisected at X. I. 26.
Now if we produce XC to H, making XH equal to K,
it follows that CH = CA;
and if AH is joined,
we notice that the angle CAH = the angle CHA. I. 5.

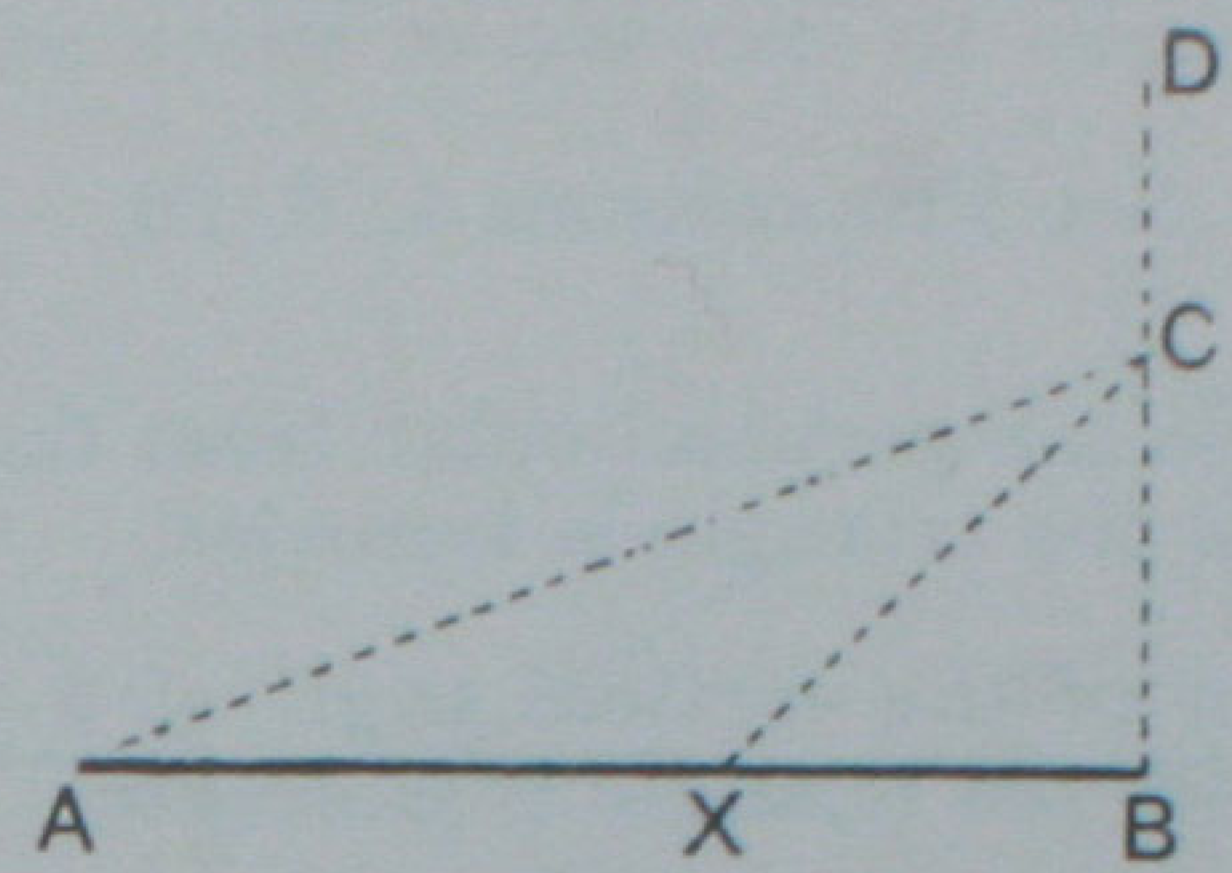
Now the straight lines XH and AH can be drawn *before the position of* C *is known*;

Hence we have the following construction, which we arrange synthetically.

SYNTHESIS. Bisect AB at X:
from X draw XH perpendicular to AB, making XH equal to K.
Join AH.
At the point A in HA, *make the angle* HAC *equal to the angle* AHX.
Join CB.
Then ACB *shall be the triangle required.*

First the triangle is isosceles, for AC = BC. I. 4.
Again, since the angle HAC = the angle AHC, *Constr.*
∴ HC = AC. I. 6.
To each add CX;
then the sum of AC, CX = the sum of HC, CX
= HX.
That is, the sum of AC, CX = K. Q.E.F.

2. *To divide a given straight line so that the square on one part may be double of the square on the other.*



Let AB be the given straight line.

ANALYSIS. *Suppose* AB *to be divided as required at* X: *that is, suppose the square on* AX *to be double of the square on* XB.

Now we remember that in an isosceles right-angled triangle, the square on the hypotenuse is double of the square on either of the equal sides.

This suggests to us to draw BC perpendicular to AB, to make BC equal to BX, and to join XC.

Then the square on XC is double of the square on XB; I. 47.

∴ XC = AX.

Hence when we join AC, we notice that

the angle XAC = the angle XCA. I. 5.

Thus the exterior angle CXB is double of the angle XAC. I. 32.

But the angle CXB is half of a right angle: I. 32.

∴ *the angle* XAC *is one-fourth of a right angle.*

This supplies the clue to the following construction :—

SYNTHESIS. From B draw BD perpendicular to AB;

and from A draw AC, *making* BAC *one-fourth of a right angle.*

From C, the intersection of AC and BD, draw CX, making the angle ACX equal to the angle BAC. I. 23.

Then AB *shall be divided as required at* X.

For since the angle XCA = the angle XAC,

∴ XA = XC. I. 6.

And because the angle BXC = the sum of the angles BAC, ACX, I. 32.

∴ the angle BXC is half a right angle.

And the angle at B is a right angle;

∴ the angle BCX is half a right angle; I. 32.

∴ the angle BXC = the angle BCX;

∴ BX = BC.

Hence the square on XC is double of the square on XB: I. 47.

that is, the square on AX is double of the square on XB. Q.E.F.

I. ON THE IDENTICAL EQUALITY OF TRIANGLES.

See Propositions 4, 8, 26.

1. If in a triangle the perpendicular from the vertex on the base bisects the base, then the triangle is isosceles.

2. If the bisector of the vertical angle of a triangle is also perpendicular to the base, the triangle is isosceles.

3. If the bisector of the vertical angle of a triangle also bisects the base, the triangle is isosceles.

[Produce the bisector, and complete the construction after the manner of I. 16.]

4. If in a triangle a pair of straight lines drawn from the extremities of the base, making equal angles with the remaining sides, are equal, the triangle is isosceles.

5. If in a triangle the perpendiculars drawn from the extremities of the base to the opposite sides are equal, the triangle is isosceles.

6. Two triangles ABC, ABD on the same base AB, and on opposite sides of it, are such that AC is equal to AD, and BC is equal to BD: shew that the line joining the points C and D is perpendicular to AB.

7. If from the extremities of the base of an isosceles triangle perpendiculars are drawn to the opposite sides, shew that the straight line joining the vertex to the intersection of these perpendiculars bisects the vertical angle.

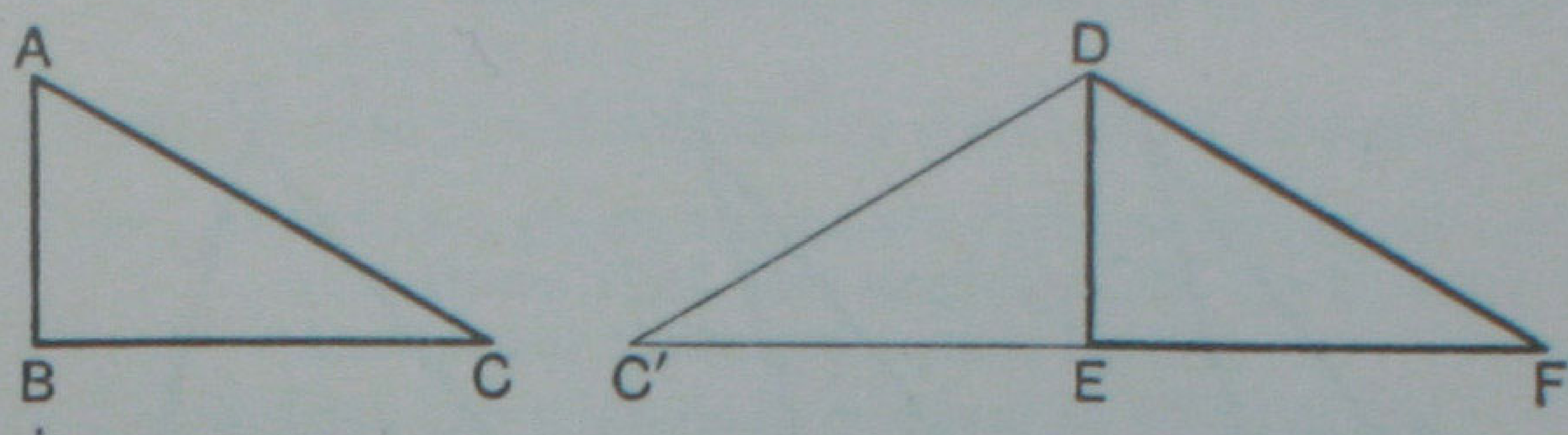
8. ABC is a triangle in which the vertical angle BAC is bisected by the straight line AX: from B draw BD perpendicular to AX, and produce it to meet AC, or AC produced, in E; then shew that BD is equal to DE.

9. In a quadrilateral ABCD, AB is equal to AD, and BC is equal to DC: shew that the diagonal AC bisects each of the angles which it joins.

10. In a quadrilateral ABCD the opposite sides AD, BC are equal, and also the diagonals AC, BD are equal: if AC and BD intersect at K, shew that each of the triangles AKB, DKC is isosceles.

11. If one angle of a triangle be equal to the sum of the other two, the greatest side is double of the distance of its middle point from the opposite angle.

12. *Two right-angled triangles which have their hypotenuses equal, and one side of one equal to one side of the other, are equal in all respects.*



Let ABC, DEF be two △ˢ right-angled at B and E, having AC equal to DF, and AB equal to DE.

Then shall the △ ABC *be equal to the* △ DEF *in all respects.*

For apply the △ ABC to the △ DEF, so that AB may coincide with the equal line DE, and C may fall on the side of DE remote from F. Let C′ be the point on which C falls.

Then DEC′ represents the △ ABC in its new position.
Now each of the ∠ˢ DEF, DEC′ is a rt. ∠; *Hyp.*
∴ EF and EC′ are in one st. line. I. 14.

Then in the △ C′DF, because DF = DC′ (*i.e.* AC), *Hyp.*
∴ the ∠ DFC′ = the ∠ DC′F. I. 5.

Hence in the two △ˢ DEF, DEC′,
Because { the ∠ DEF = the ∠ DEC′, being rt. ∠ˢ;
and the ∠ DFE = the ∠ DC′E; *Proved.*
also the side DE is common to both;
∴ the △ˢ DEF, DEC′ are equal in all respects; I. 26.
that is, the △ˢ DEF, ABC are equal in all respects. Q.E.D.

Alternative Proof. Since the ∠ ABC is a rt. angle;
∴ the sq. on AC = the sqq. on AB, BC. I. 47.

Similarly, the sq. on DF = the sqq. on DE, EF; I. 47.

But the sq. on AC = the sq. on DF, since AC = DF;
∴ the sqq. on AB, BC = the sqq. on DE, EF.

And of these, the sq. on AB = the sq. on DE, since AB = DE;
∴ the sq. on BC = the sq. on EF; *Ax.* 3.
∴ BC = EF.

Hence the three sides of the △ ABC are respectively equal to the three sides of the △ DEF;
∴ the △ ABC = the △ DEF in all respects. I. 8.

13. *If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles opposite to one pair of equal sides equal, then the angles opposite to the other pair of equal sides shall be either equal or supplementary, and in the former case the triangles shall be equal in all respects.*

Fig. 1.

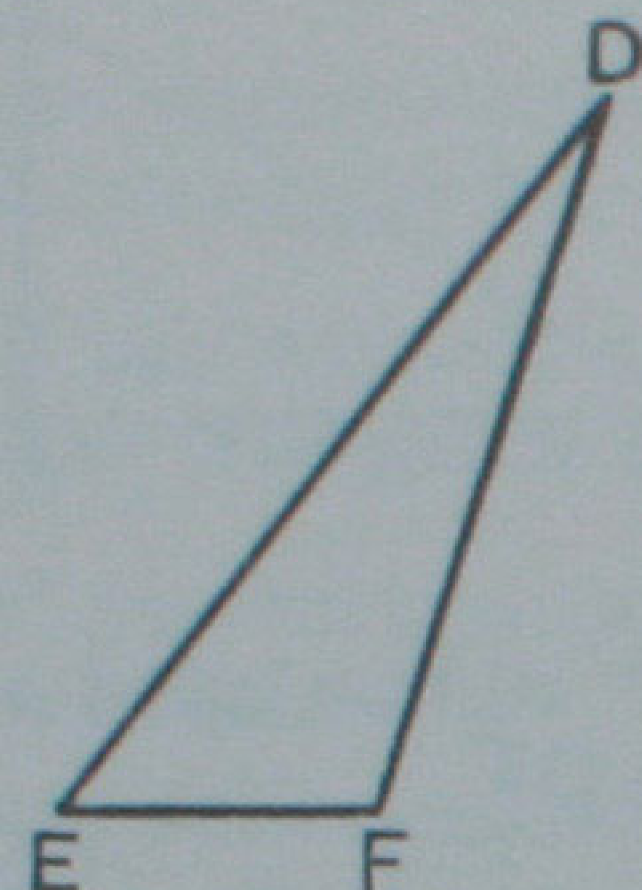


Fig. 2.

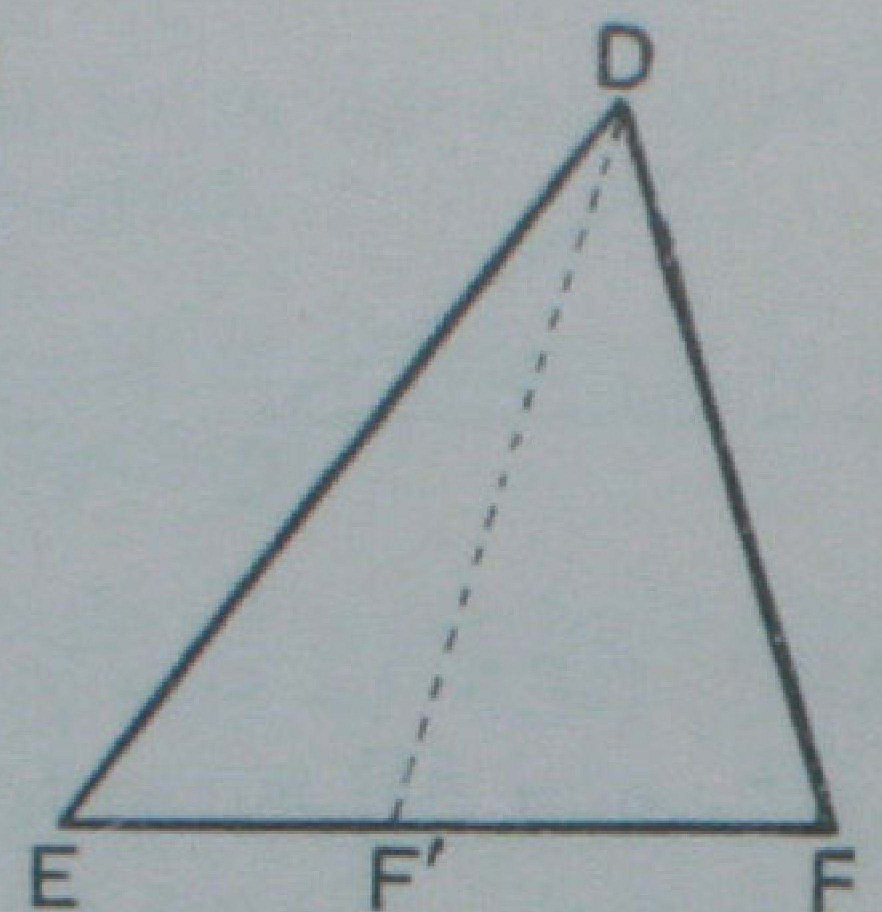


Fig. 3.

Let ABC, DEF be two triangles, in which
the side $AB =$ the side DE,
the side $AC =$ the side DF,
and the $\angle ABC =$ the $\angle DEF$.

Then shall the $\angle^s$ ACB, DFE *be either equal* (*as in Figs.* 1 *and* 2) *or supplementary* (*as in Figs.* 1 *and* 3) *; and in the former case the triangles shall be equal in all respects.*

If the $\angle BAC =$ the $\angle EDF$. [*Figs.* 1 *and* 2.]
then the $\angle ACB =$ the $\angle DFE$, and the triangles are equal in all respects. I. 4.

But if the $\angle BAC$ be not equal to the $\angle EDF$, [*Figs.* 1 *and* 3.]
let the $\angle EDF$ be greater than the $\angle BAC$.

At D in ED make the $\angle EDF'$ equal to the $\angle BAC$.
Then the $\triangle^s$ BAC, EDF′ are equal in all respects. I. 26.
$\therefore AC = DF'$;
but $AC = DF$; *Hyp.*
$\therefore DF = DF'$,
$\therefore$ the angle $DFF' =$ the $\angle DF'F$. I. 5.
But the $\angle^s$ DF′F, DF′E are supplementary, I. 13.
$\therefore$ the $\angle^s$ DFF′, DF′E are supplementary :
that is, the $\angle^s$ DFE, ACB are supplementary.

Q.E.D.

Corollaries. Three cases of this theorem deserve special attention.

It has been proved that if the angles ACB, DFE are not *supplementary* they are *equal* :

Hence, in addition to the hypothesis of this theorem,

(i) If the angles ACB, DFE opposite to the two equal sides AB, DE are both acute or both obtuse they cannot be supplementary, and are therefore equal; or if one of them is a right angle, the other must also be a right angle (whether considered as supplementary or equal to it):

in either case the triangles are equal in all respects.

(ii) If the two given angles are right angles or obtuse angles, it follows that the angles ACB, DFE must be both acute, and therefore equal, by (i):

so that the triangles are equal in all respects.

(iii) If in each triangle the side opposite the given angle is not less than the other given side; that is, if AC and DF are not less than AB and DE respectively) then the angles ACB, DFE cannot be greater than the angles ABC, DEF respectively;

therefore the angles ACB, DFE are both acute;
hence, as above, they are equal;

and the triangles ABC, DEF are equal in all respects.

II. ON INEQUALITIES.

See Propositions 16, 17, 18, 19, 20, 21, 24, 25.

1. In a triangle ABC, if AC is not greater than AB, shew that any straight line drawn through the vertex A, and terminated by the base BC, is less than AB.

2. ABC *is a triangle, and the vertical angle* BAC *is bisected by a straight line which meets the base* BC *in* X; *shew that* BA *is greater than* BX, *and* CA *greater than* CX. *Hence obtain a proof of* I. 20.

3. *The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line; and of others, that which is nearer to the perpendicular is less than the more remote; and two, and only two equal straight lines can be drawn from the given point to the given straight line, one on each side of the perpendicular.*

4. The sum of the distances of any point from the three angular points of a triangle is greater than half its perimeter.

5. The sum of the distances of any point within a triangle from its angular points is less than the perimeter of the triangle.

6. The perimeter of a quadrilateral is greater than the sum of its diagonals.

7. The sum of the diagonals of a quadrilateral is less than the sum of the four straight lines drawn from the angular points to any given point. Prove this, and point out the exceptional case.

8. *In a triangle any two sides are together greater than twice the median which bisects the remaining side.* [See Def. p. 79.]

[Produce the median, and complete the construction after the manner of I. 16.]

9. *In any triangle the sum of the medians is less than the perimeter.*

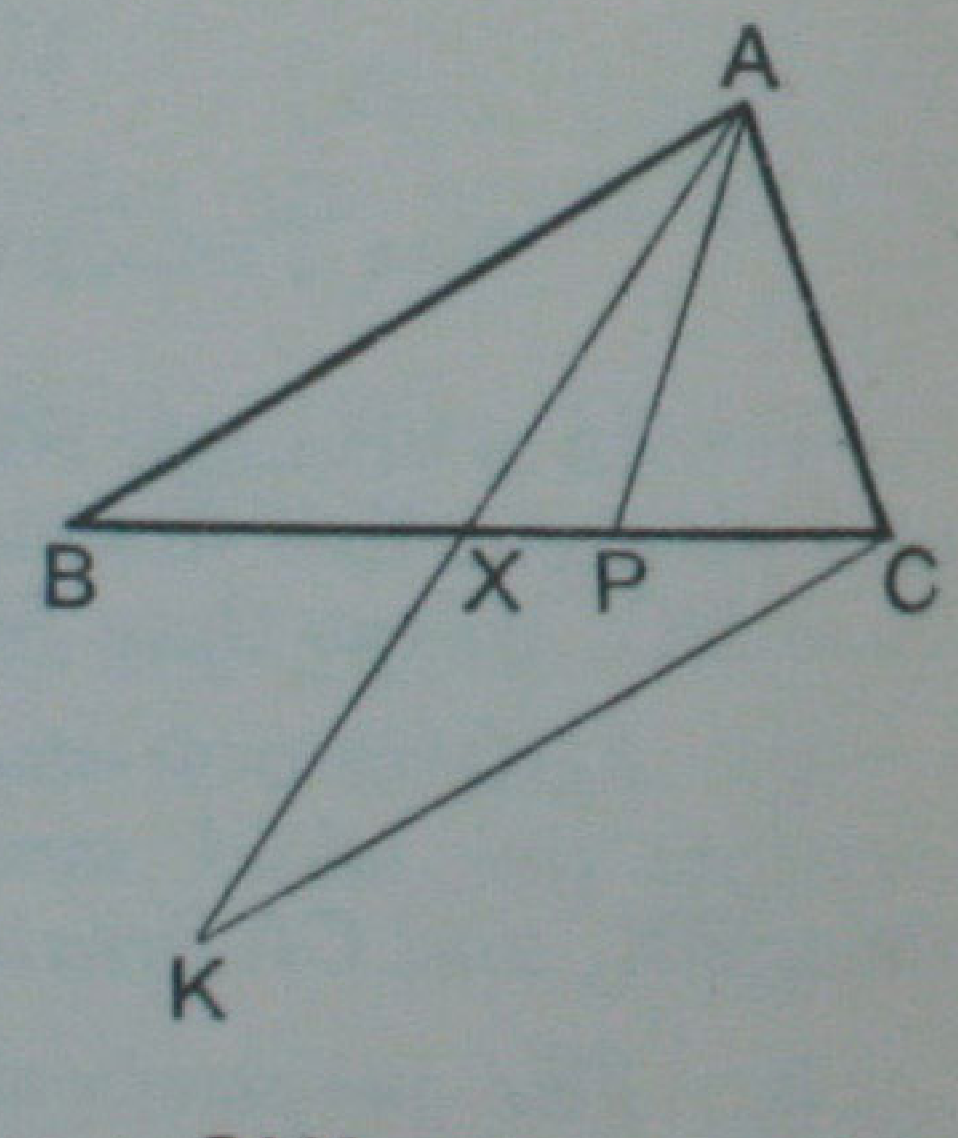
10. In a triangle an angle is acute, obtuse, or a right angle, according as the median drawn from it is greater than, less than, or equal to half the opposite side. [See Ex. 4, p. 65.]

11. The diagonals of a rhombus are unequal.

12. *If the vertical angle of a triangle is contained by unequal sides, and if from the vertex the median and the bisector of the angle are drawn, then the median lies within the angle contained by the bisector and the longer side.*

Let ABC be a △, in which AB is greater than AC; let AX be the median drawn from A, and AP the bisector of the vertical ∠ BAC.

Then shall AX *lie between* AP *and* AB.



Produce AX to K, making XK equal to AX. Join KC.

Then the △s BXA, CXK may be shewn to be equal in all respects; I. 4.

hence BA = CK, and the ∠ BAX = the ∠ CKX.

But since BA is greater than AC, *Hyp.*

∴ CK is greater than AC;

∴ the ∠ CAK is greater than the ∠ CKA: I. 18.

that is, the ∠ CAX is greater than the ∠ BAX:

∴ the ∠ CAX must be more than half the vert. ∠ BAC;

hence AX lies within the angle BAP. Q.E.D.

13. *If the vertical angle of a triangle is contained by two unequal sides, and if from the vertex there are drawn the bisector of the vertical angle, the median, and the perpendicular to the base, the first of these lines is intermediate in position and magnitude to the other two.*

III. ON PARALLELS.

See Propositions 27—31.

1. If a straight line meets two parallel straight lines, and the two interior angles on the same side are bisected; shew that the bisectors meet at right angles. [I. 29, I. 32.]

2. The straight lines drawn from any point in the bisector of an angle parallel to the arms of the angle, and terminated by them, are equal; and the resulting figure is a rhombus.

3. AB and CD are two straight lines intersecting at D, and the adjacent angles so formed are bisected: if through any point X in DC a straight line YXZ be drawn parallel to AB and meeting the bisectors in Y and Z, shew that XY is equal to XZ.

4. If two straight lines are parallel to two other straight lines, each to each; and if the acute angles contained by each pair are bisected; shew that the bisecting lines are parallel.

5. The middle point of any straight line which meets two parallel straight lines, and is terminated by them, is equidistant from the parallels.

6. A straight line drawn between two parallels and terminated by them, is bisected; shew that any other straight line passing through the middle point and terminated by the parallels, is also bisected at that point.

7. If through a point equidistant from two parallel straight lines, two straight lines are drawn cutting the parallels, the portions of the latter thus intercepted are equal.

PROBLEMS.

8. AB and CD are two given straight lines, and X is a given point in AB: find a point Y in AB such that YX may be equal to the perpendicular distance of Y from CD.

9. ABC is an isosceles triangle: required to draw a straight line DE parallel to the base BC, and meeting the equal sides in D and E, so that BD, DE, EC may be all equal.

10. ABC is any triangle: required to draw a straight line DE parallel to the base BC, and meeting the other sides in D and E, so that DE may be equal to the sum of BD and CE.

11. ABC is any triangle: required to draw a straight line parallel to the base BC, and meeting the other sides in D and E, so that DE may be equal to the difference of BD and CE.

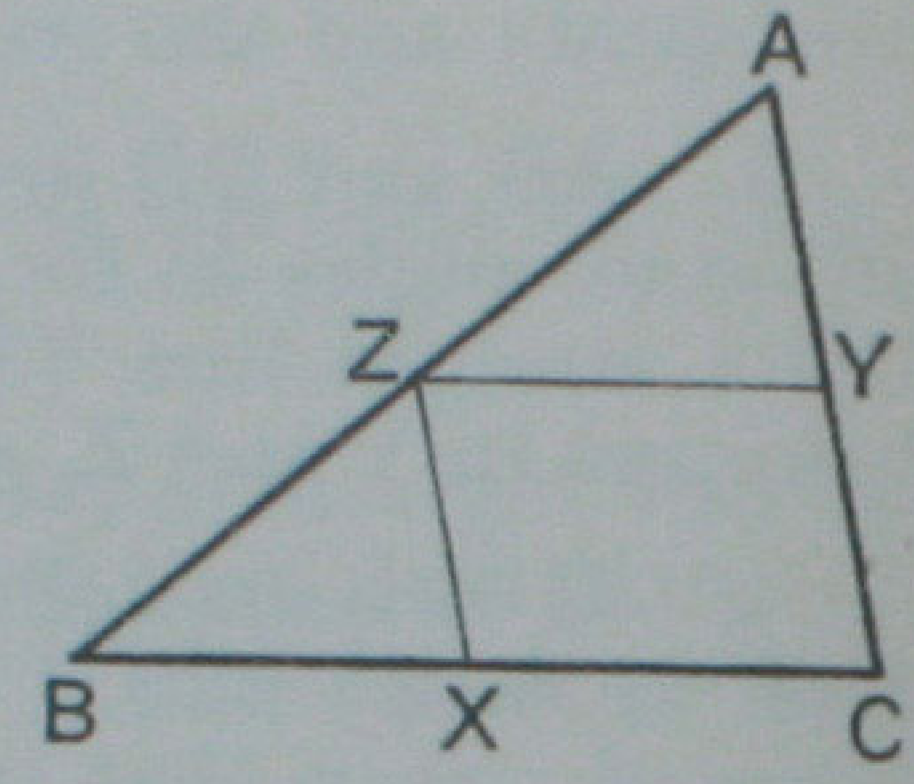
IV. ON PARALLELOGRAMS.

See Propositions 33, 34, and the deductions from these Props. given on page 70.

1. *The straight line drawn through the middle point of a side of a triangle parallel to the base, bisects the remaining side.*

Let ABC be a △, and Z the middle point of the side AB. Through Z, ZY is drawn par[l] to BC.

Then shall Y *be the middle point of* AC.



Through Z draw ZX par[l] to AC. I. 31.

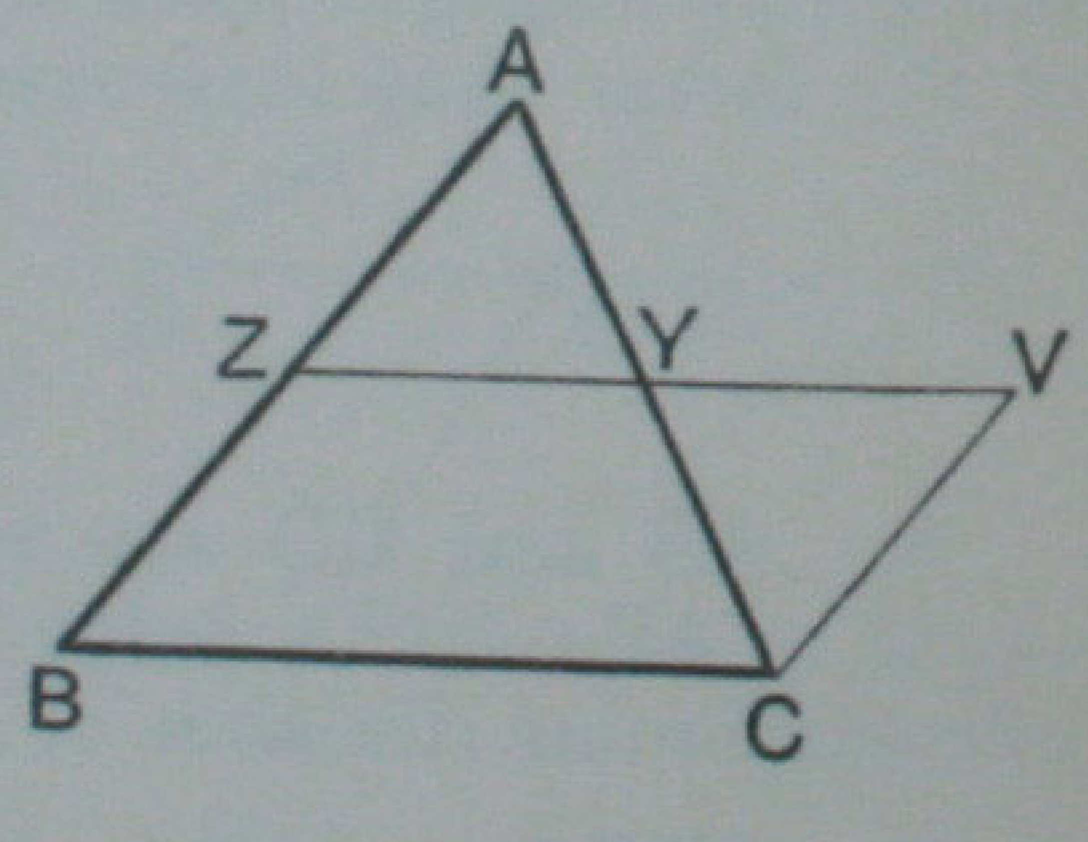
Then in the △[s] AZY, ZBX,
because ZY and BC are par[l],
∴ the ∠ AZY = the ∠ ZBX; I. 29.
and because ZX and AC are par[l],
∴ the ∠ ZAY = the ∠ BZX; I. 29.
also AZ = ZB: *Hyp.*
∴ AY = ZX. I. 26.

But ZXCY is a par[m] by construction;
∴ ZX = YC. I. 34.
Hence AY = YC;
that is, AC is bisected at Y. Q.E.D.

2. *The straight line which joins the middle points of two sides of a triangle, is parallel to the third side.*

Let ABC be a △, and Z, Y the middle points of the sides AB, AC.

Then shall ZY *be par[l] to* BC.



Produce ZY to V, making YV equal to ZY.

Join CV.
Then in the △[s] AYZ, CYV,
Because { AY = CY, *Hyp.*
and YZ = YV, *Constr.*
and the ∠ AYZ = the vert. opp. ∠ CYV; I. 15.
∴ AZ = CV, I. 4.
and the ∠ ZAY = the ∠ VCY;
hence CV is par[l] to AZ. I. 27.
But CV is equal to AZ, that is, to BZ: *Hyp.*
∴ CV is equal and par[l] to BZ:
∴ ZV is equal and par[l] to BC: I. 33.
that is, ZY is par[l] to BC. Q.E.D.

[A second proof of this proposition may be derived from I. 38, 39.]

3. *The straight line which joins the middle points of two sides of a triangle is equal to half the third side.*

4. *Shew that the three straight lines which join the middle points of the sides of a triangle, divide it into four triangles which are identically equal.*

5. *Any straight line drawn from the vertex of a triangle to the base is bisected by the straight line which joins the middle points of the other sides of the triangle.*

6. Given the three middle points of the sides of a triangle, construct the triangle.

7. AB, AC are two given straight lines, and P is a given point between them; required to draw through P a straight line terminated by AB, AC, and bisected by P.

8. ABCD is a parallelogram, and X, Y are the middle points of the opposite sides AD, BC: shew that BX and DY trisect the diagonal AC.

9. *If the middle points of adjacent sides of any quadrilateral are joined, the figure thus formed is a parallelogram.*

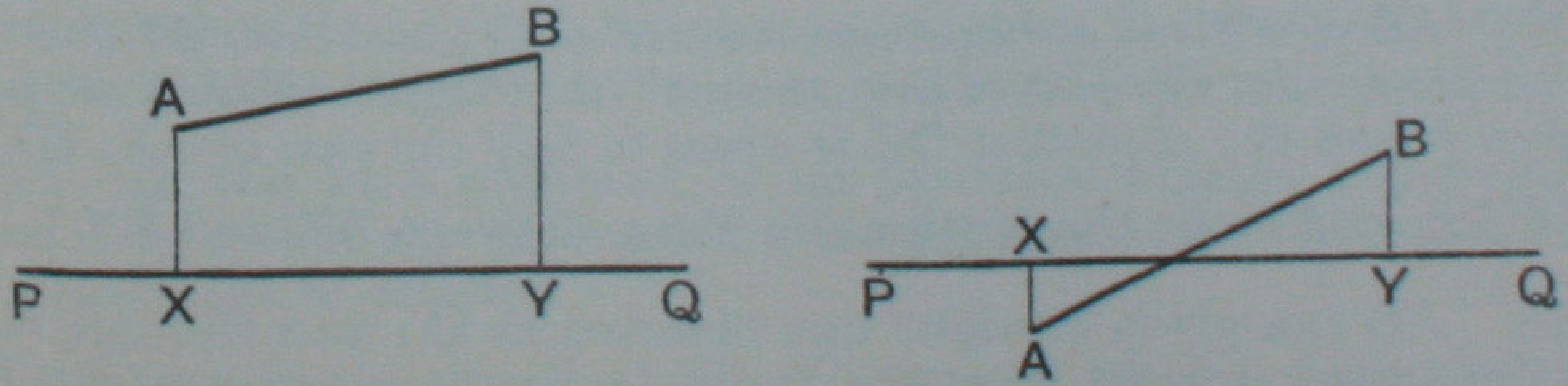
10. Shew that the straight lines which join the middle points of opposite sides of a quadrilateral, bisect one another.

11. The straight line which joins the middle points of the oblique sides of a trapezium, is parallel to the two parallel sides, and passes through the middle points of the diagonals.

12. *The straight line which joins the middle points of the oblique sides of a trapezium is equal to half the* sum *of the parallel sides; and the portion intercepted between the diagonals is equal to half the* difference *of the parallel sides.*

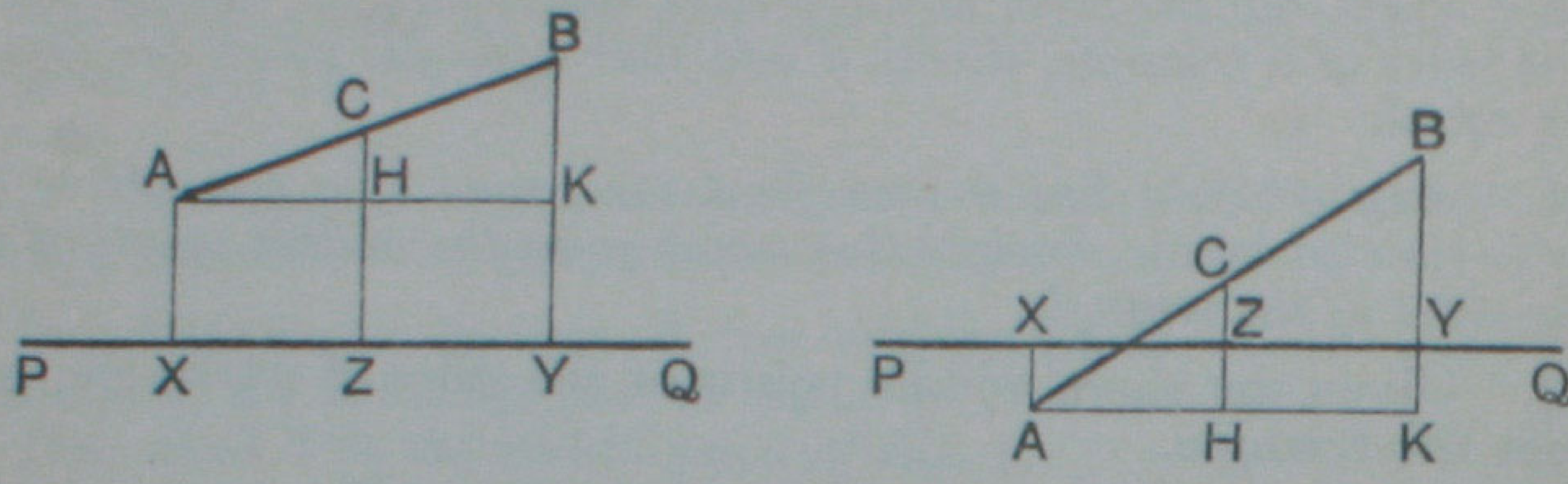
DEFINITION.

If from the extremities of one straight line perpendiculars are drawn to another, the portion of the latter intercepted between the perpendiculars is said to be the **Orthogonal Projection** of the first line upon the second.



Thus in the adjoining figures, if from the extremities of the straight line AB the perpendiculars AX, BY are drawn to PQ, then XY is the *orthogonal projection* of AB on PQ.

13. *A given straight line* AB *is bisected at* C; *shew that the projections of* AC, CB *on any other straight line are equal.*



Let XZ, ZY be the projections of AC, CB on any straight line PQ.
Then XZ *and* ZY *shall be equal.*

Through A draw a straight line parallel to PQ, meeting CZ, BY or these lines produced in H, K. I. 31.

Now AX, CZ, BY are parallel, for they are perp. to PQ; I. 28.
∴ the figures XH, HY are par^{ms};
∴ AH = XZ, and HK = ZY. I. 34.

But through C, the middle point of AB, a side of the △ ABK, CH has been drawn parallel to the side BK;
∴ CH bisects AK: Ex. 1, p. 104.
that is, AH = HK;
∴ XZ = ZY. Q.E.D.

14. *If three parallel straight lines make equal intercepts on a fourth straight line which meets them, they will also make equal intercepts on any other straight line which meets them.*

15. *Equal and parallel straight lines have equal projections on any other straight line.*

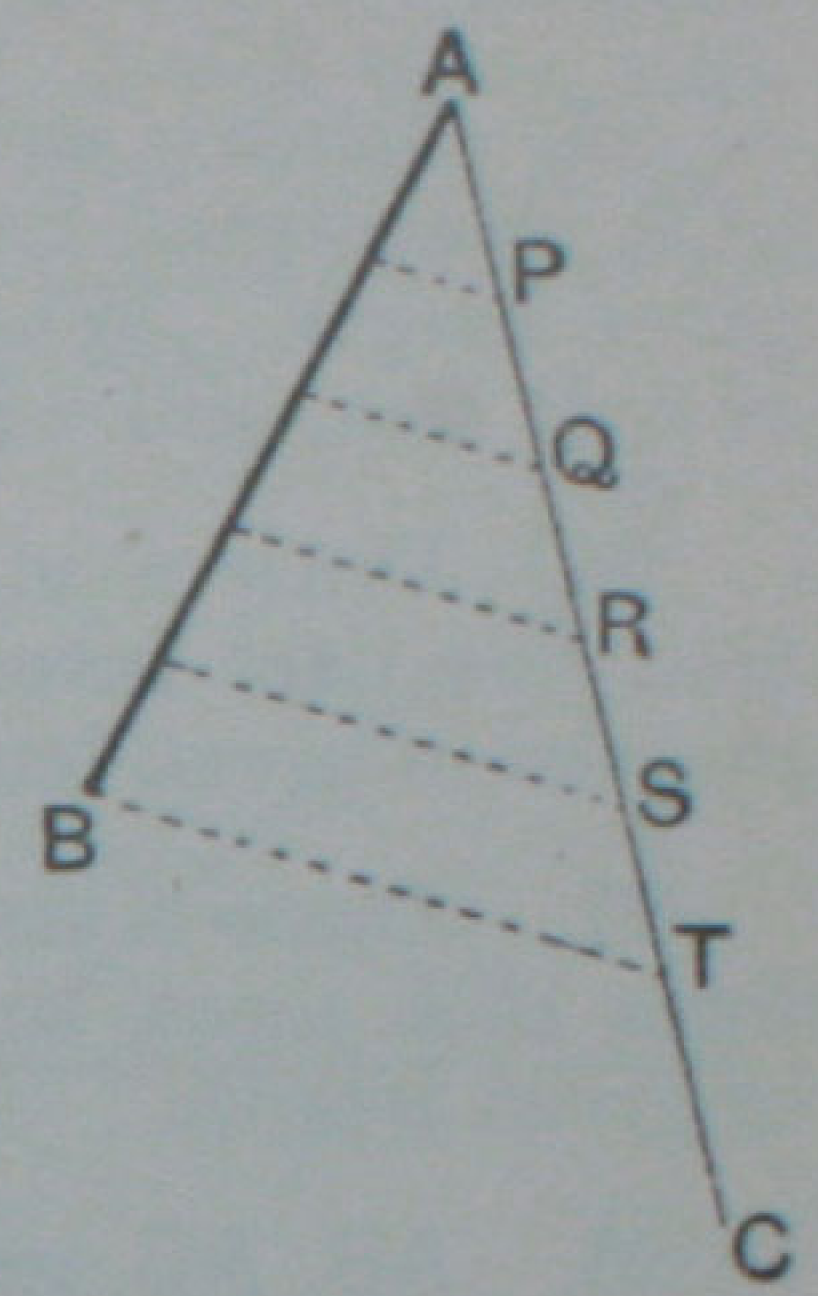
16. AB is a given straight line bisected at O; and AX, BY are perpendiculars drawn from A and B on any other straight line: shew that OX is equal to OY.

17. AB *is a given straight line bisected at* O: *and* AX, BY *and* OZ *are perpendiculars drawn to any straight line* PQ, *which does not pass between* A *and* B: *shew that* OZ *is equal to half the sum of* AX, BY.

[OZ is said to be the **Arithmetic Mean** between AX and BY.]

18. AB is a given straight line bisected at O; and through A, B and O parallel straight lines are drawn to meet a given straight line PQ in X, Y, Z: shew that OZ is equal to half the *sum*, or half the *difference* of AX and BY, according as A and B lie on the *same* side or on *opposite* sides of PQ.

19. *To divide a given finite straight line into any number of equal parts.*



[For example: *required to divide the straight line* AB *into five equal parts.*

From A draw AC, a straight line of unlimited length, making any angle with AB.

In AC take *any* point P; and by marking off successive parts PQ, QR, RS, ST each equal to AP, make AT to contain AP *five* times.

Join BT; and through P, Q, R, S draw parallels to BT.

It may be shewn by Ex. 14, p. 106, that these parallels divide AB into five equal parts.]

20. *If through an angle of a parallelogram any straight line is drawn, the perpendicular drawn to it from the opposite angle is equal to the sum or difference of the perpendiculars drawn to it from the two remaining angles, according as the given straight line falls without the parallelogram, or intersects it.*

[Through the opposite angle draw a straight line parallel to the given straight line, so as to meet the perpendicular from one of the remaining angles, produced if necessary; then apply I. 34, I. 26. Or proceed as in the following example.]

21. From the angular points of a parallelogram perpendiculars are drawn to any straight line which is without the parallelogram: shew that the sum of the perpendiculars drawn from one pair of opposite angles is equal to the sum of those drawn from the other pair.

[Draw the diagonals, and from their point of intersection let fall a perpendicular upon the given straight line. See Ex. 17, p. 106.]

22. The sum of the perpendiculars drawn from any point in the base of an isosceles triangle to the equal sides is equal to the perpendicular drawn from either extremity of the base to the opposite side.

[It follows that the sum of the distances of *any* point in the base of an isosceles triangle from the equal sides is **constant**, that is, the same whatever point in the base is taken.]

23. In the base produced of an isosceles triangle any point is taken: shew that the difference of its perpendicular distances from the equal sides is *constant*.

24. The sum of the perpendiculars drawn from any point within an equilateral triangle to the three sides is equal to the perpendicular drawn from any one of the angular points to the opposite side, and is therefore constant.

PROBLEMS.

25. Draw a straight line through a given point, so that the part of it intercepted between two given parallel straight lines may be of given length. When does this problem admit of two solutions, when of only one, and when is it impossible?

26. Draw a straight line parallel to a given straight line, so that the part intercepted between two other given straight lines may be of given length.

27. Draw a straight line equally inclined to two given straight lines that meet, so that the part intercepted between them may be of given length.

28. AB, AC are two given straight lines, and P is a given point *without* the angle contained by them. It is required to draw through P a straight line to meet the given lines, so that the part intercepted between them may be equal to the part between P and the nearer line.

V. MISCELLANEOUS THEOREMS AND EXAMPLES.

Chiefly on I. 32.

1. A *is the vertex of an isosceles triangle* ABC, *and* BA *is produced to* D, *so that* AD *is equal to* BA; *if* DC *is drawn, shew that* BCD *is a right angle.*

2. *The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is equal to half the hypotenuse.*

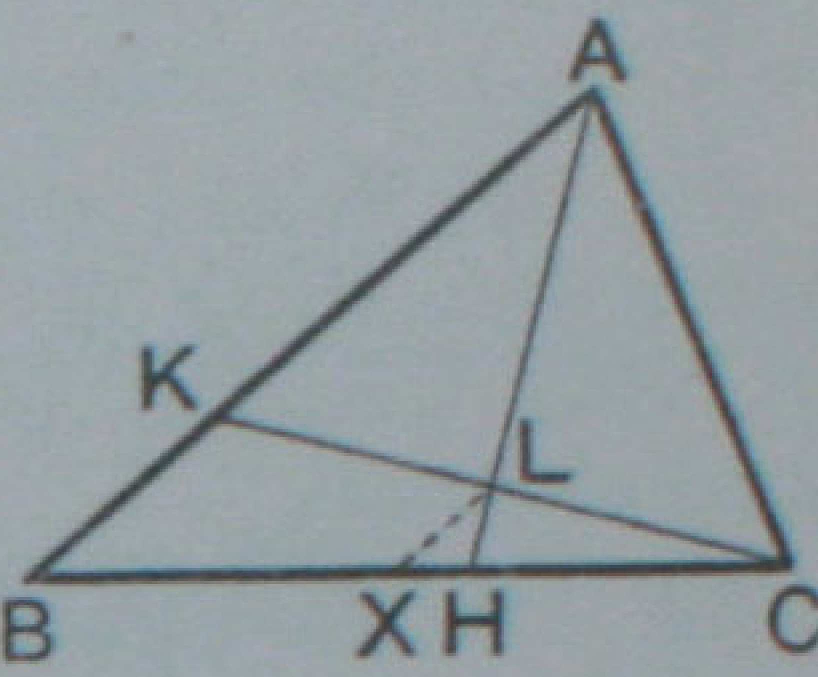
3. From the extremities of the base of a triangle perpendiculars are drawn to the opposite sides (produced if necessary); shew that the straight lines which join the middle point of the base to the feet of the perpendiculars are equal.

4. *In a triangle* ABC, AD *is drawn perpendicular to* BC; *and* X, Y, Z *are the middle points of the sides* BC, CA, AB *respectively: shew that each of the angles* ZXY, ZDY *is equal to the angle* BAC.

5. *In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the two triangles thus formed are equiangular to one another.*

6. *In a right-angled triangle two straight lines are drawn from the right angle, one bisecting the hypotenuse, the other perpendicular to it: shew that they contain an angle equal to the difference of the two acute angles of the triangle.* [See above, Ex. 2 and Ex. 5.]

7. *In a triangle if a perpendicular is drawn from one extremity of the base to the bisector of the vertical angle,* (i) *it will make with either of the sides containing the vertical angle an angle equal to half the sum of the angles at the base;* (ii) *it will make with the base an angle equal to half the difference of the angles at the base.*



Let ABC be the given △, and AH the bisector of the vertical ∠ BAC.

Let CLK meet AH at right angles.

(i) *Then shall each of the* ∠s AKC, ACK *be equal to half the sum of the* ∠s ABC, ACB.

In the △s AKL, ACL,

Because { the ∠ KAL = the ∠ CAL, *Hyp.*
also the ∠ ALK = the ∠ ALC, being rt. ∠s;
and AL is common to both △s;

∴ the ∠ AKL = the ∠ ACL. I. 26.

Again, the ∠ AKC = the sum of the ∠s KBC, KCB; I. 32.
∴ the ∠ ACK = the sum of the ∠s KBC, KCB.
To each add the ∠ ACK:
then twice the ∠ ACK = the sum of the ∠s ABC, ACB;
∴ the ∠ ACK = half the sum of the ∠s ABC, ACB.

(ii) *The* ∠ KCB *shall be equal to half the difference of the* ∠s ACB, ABC.

As before, the ∠ ACK = the sum of the ∠s KBC, KCB.
To each of these add the ∠ KCB:
then the ∠ ACB = the ∠ KBC together with twice the ∠ KCB.
∴ twice the ∠ KCB = the difference of the ∠s ACB, KBC;
that is, the ∠ KCB = half the difference of the ∠s ACB, ABC.

COROLLARY. *If* X *is the middle point of the base, and* XL *is joined, it may be shewn by Ex. 3, p. 105, that* XL *is half* BK; *that is, that* XL *is half the difference of the sides* AB, AC.

8. *In any triangle the angle contained by the bisector of the vertical angle and the perpendicular from the vertex to the base is equal to half the difference of the angles at the base.* [See Ex. 3, p. 65.]

9. In a triangle ABC the side AC is produced to D, and the angles BAC, BCD are bisected by straight lines which meet at F; shew that they contain an angle equal to half the angle at B.

10. If in a right-angled triangle one of the acute angles is double of the other, shew that the hypotenuse is double of the shorter side.

11. If in a diagonal of a parallelogram any two points equidistant from its extremities are joined to the opposite angles, the figure thus formed will be also a parallelogram.

12. ABC is a given equilateral triangle, and in the sides BC, CA, AB the points X, Y, Z are taken respectively, so that BX, CY and AZ are all equal. AX, BY, CZ are now drawn, intersecting in P, Q, R: shew that the triangle PQR is equilateral.

13. If in the sides AB, BC, CD, DA of a parallelogram ABCD four points P, Q, R, S are taken in order, one in each side, so that AP, BQ, CR, DS are all equal; shew that the figure PQRS is a parallelogram.

14. In the figure of I. 1, if the circles intersect at F, and if CA and CB are produced to meet the circles in P and Q respectively; shew that the points P, F, Q are in the same straight line; and shew also that the triangle CPQ is equilateral.

[Problems marked (*) admit in general of more than one solution.]

15. Through two given points draw two straight lines forming with a straight line given in position, an equilateral triangle.

*16. From a given point it is required to draw to two parallel straight lines two equal straight lines at right angles to one another.

*17. Three given straight lines meet at a point; draw another straight line so that the two portions of it intercepted between the given lines may be equal to one another.

18. From a given point draw three straight lines of given lengths, so that their extremities may be in the same straight line, and intercept equal distances on that line. [See Fig. to I. 16.]

19. Use the properties of the equilateral triangle to trisect a given finite straight line.

20. In a given triangle inscribe a rhombus, having one of its angles coinciding with an angle of the triangle.

VI. ON THE CONCURRENCE OF STRAIGHT LINES IN A TRIANGLE.

DEFINITIONS. (i) Three or more straight lines are said to be **concurrent** when they meet in one point.

(ii) Three or more points are said to be **collinear** when they lie upon one straight line.

Obs. We here give some propositions relating to the concurrence of certain groups of straight lines drawn in a triangle: the importance of these theorems will be more fully appreciated when the student is familiar with Books III. and IV.

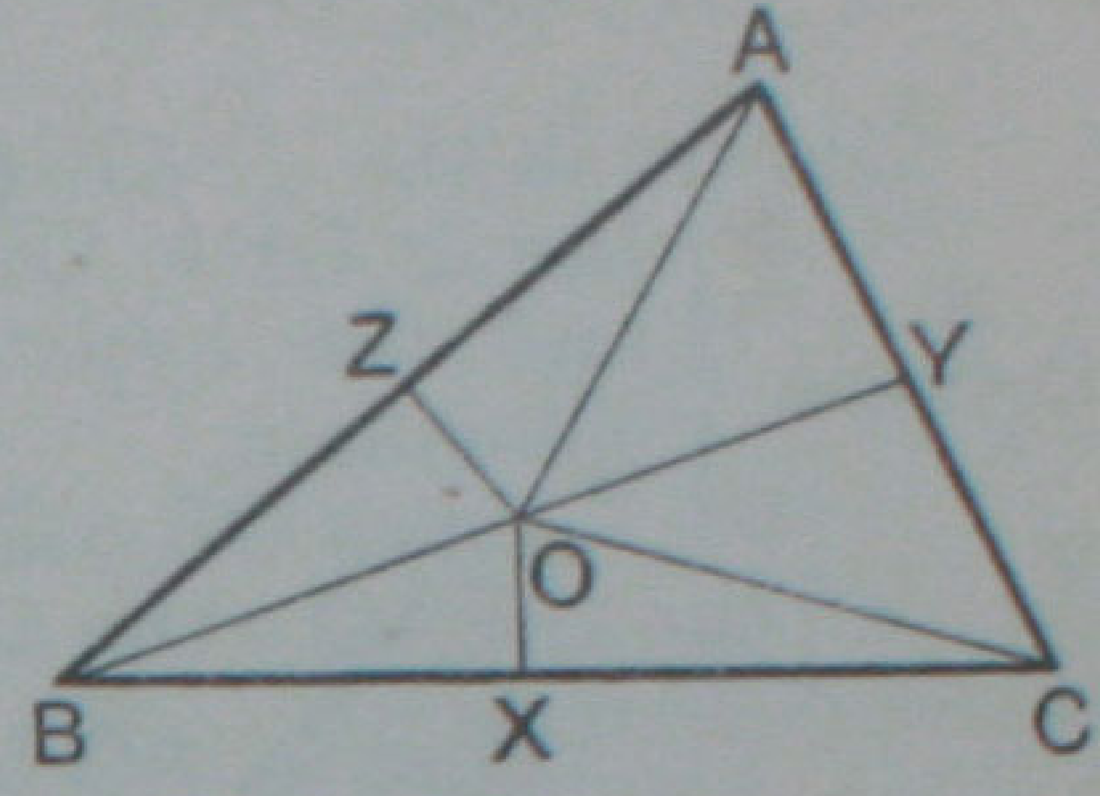
1. *The perpendiculars drawn to the sides of a triangle from their middle points are concurrent.*

Let ABC be a △, and X, Y, Z the middle points of its sides.

Then shall the perp^s drawn to the sides from X, Y, Z *be concurrent.*

From Z and Y draw perp^s to AB, AC; these perp^s, since they cannot be parallel, will meet at some point O. *Ax.* 12.

Join OX.



It is required to prove that OX *is perp. to* BC.

Join OA, OB, OC.

In the △^s OYA, OYC,

Because { YA=YC, *Hyp.*
and OY is common to both;
also the ∠ OYA = the ∠ OYC, being rt. ∠^s;

∴ OA=OC. I. 4.

Similarly, from the △^s OZA, OZB,
it may be proved that OA=OB.
Hence OA, OB, OC are all equal.

Again, in the △^s OXB, OXC,

Because { BX=CX, *Hyp.*
and XO is common to both;
also OB=OC: *Proved.*

∴ the ∠ OXB = the ∠ OXC; I. 8.

but these are adjacent ∠^s;
∴ they are rt. ∠^s; *Def.* 10.
that is, OX is perp. to BC.

Hence the three perp^s OX, OY, OZ meet at the point O.

Q.E.D.

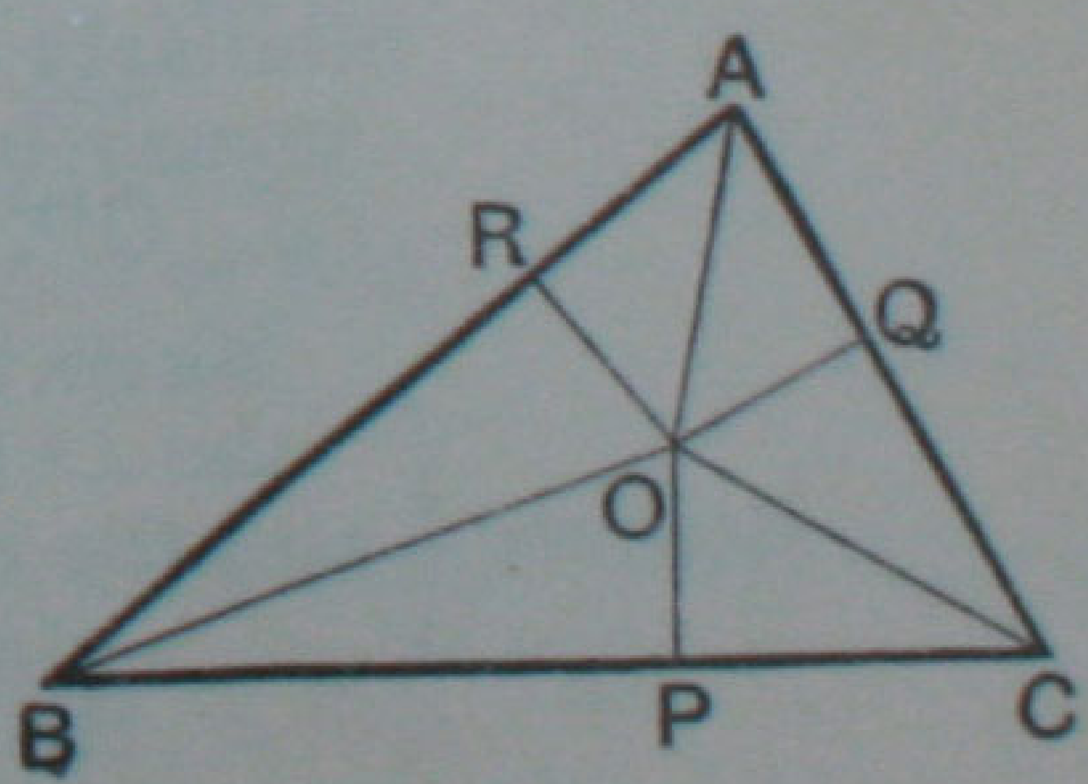
2. *The bisectors of the angles of a triangle are concurrent.*

Let ABC be a △. Bisect the ∠^s ABC, BCA, by straight lines which must meet at some point O. *Ax.* 12.

Join AO.

It is required to prove that AO *bisects the* ∠ BAC.

From O draw OP, OQ, OR perp. to the sides of the △.

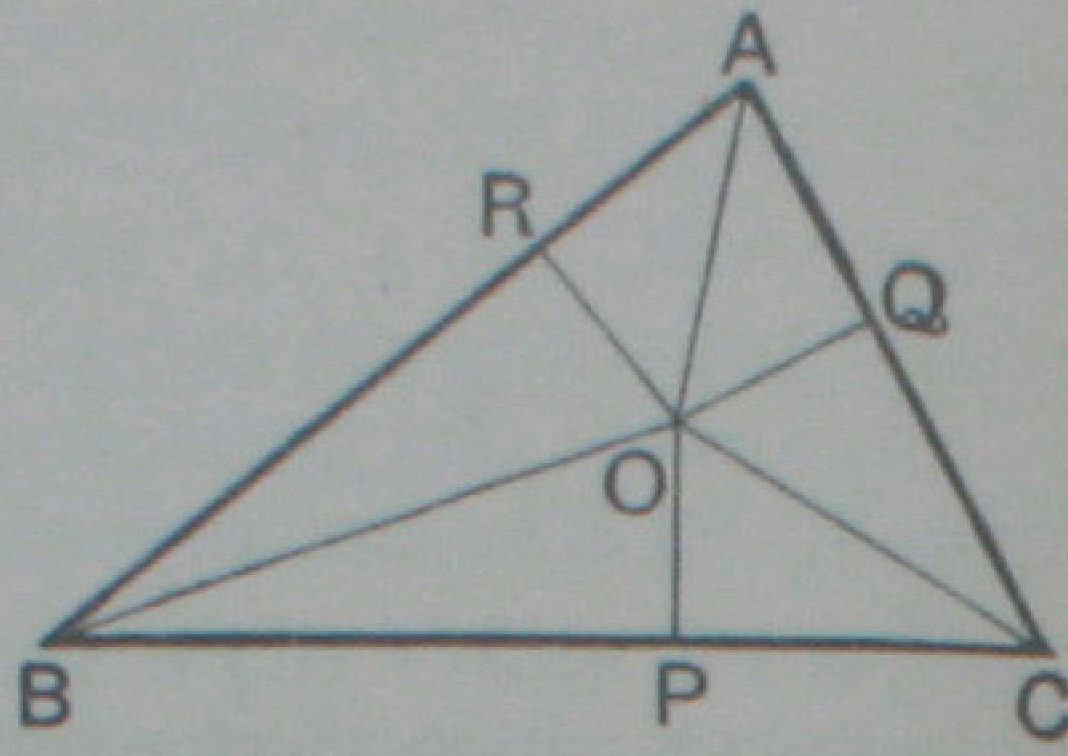


Then in the △^s OBP, OBR,

Because { the ∠ OBP = the ∠ OBR, *Constr.*
and the ∠ OPB = the ∠ ORB, being rt. ∠^s,
and OB is common;

∴ OP=OR. I. 26.

Similarly from the △ˢ OCP, OCQ,
it may be shewn that OP=OQ,
∴ OP, OQ, OR are all equal.



Again in the △ˢ ORA, OQA,

Because { the ∠ˢ ORA, OQA are rt. ∠ˢ, and the hypotenuse OA is common, also OR=OQ; *Proved.*

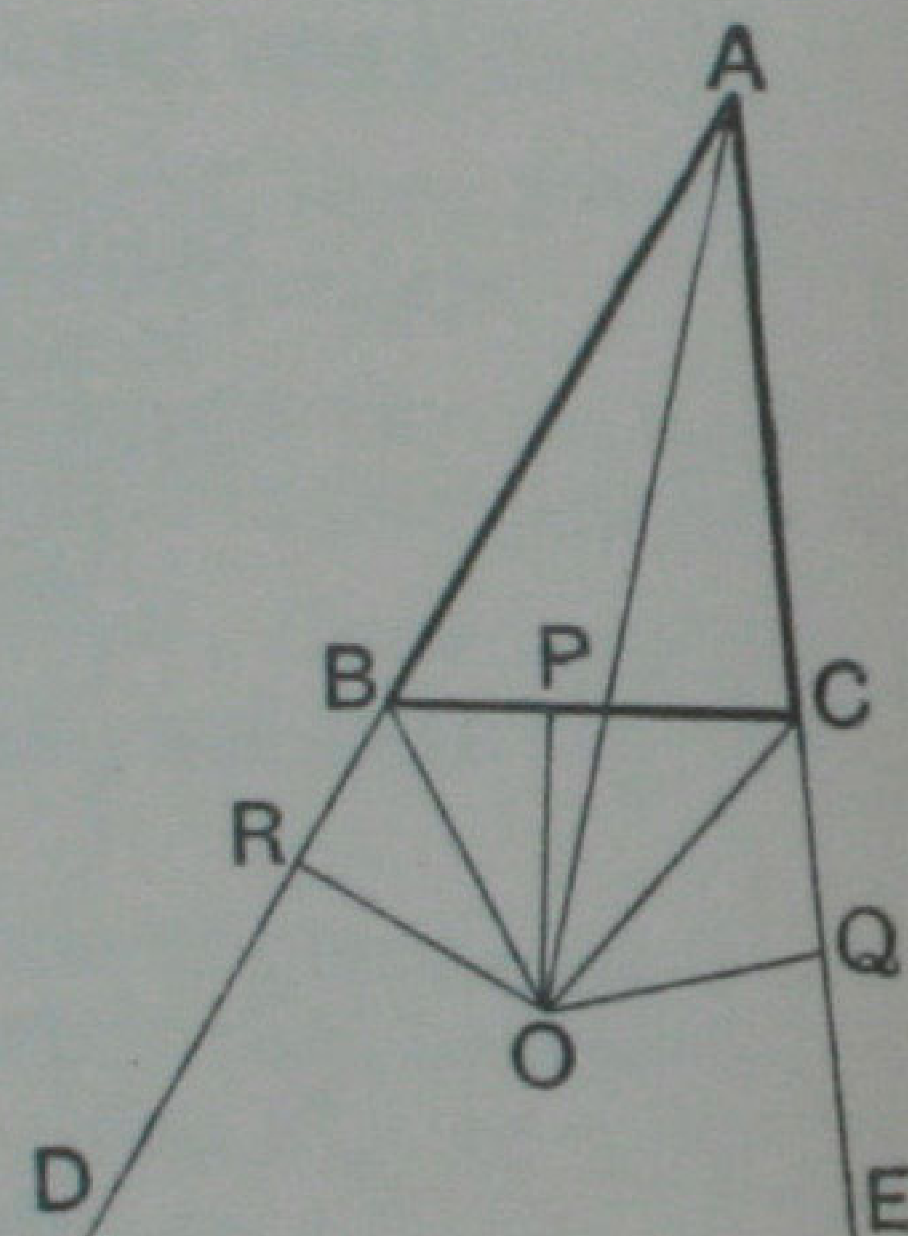
∴ the ∠ RAO=the ∠ QAO. Ex. 12, p. 99.

That is, AO is the bisector of the ∠ BAC.

Hence the bisectors of the three ∠ˢ meet at the point O.

Q.E.D.

3. *The bisectors of two exterior angles of a triangle and the bisector of the third angle are concurrent.*



Let ABC be a △, of which the sides AB, AC are produced to any points D and E.

Then shall the bisectors of the ∠ˢ DBC, ECB, BAC *be concurrent.*

Bisect the ∠ˢ DBC, ECB by straight lines which must meet at some point O. *Ax.* 12.

Join AO.

It is required to prove that AO *bisects the angle* BAC.

From O draw OP, OQ, OR perp. to the sides of the △.

Then in the △ˢ OBP, OBR,

Because { the ∠ OBP=the ∠ OBR, *Constr.* also the ∠ OPB=the ∠ ORB, being rt. ∠ˢ, and OB is common;

∴ OP=OR. I. 26.

Similarly from the △ˢ OCP, OCQ,
it may be shewn that OP=OQ:
∴ OP, OQ, OR are all equal.

Again in the △ˢ ORA, OQA,

Because { the ∠ˢ ORA, OQA are rt. ∠ˢ, and the hypotenuse OA is common, also OR=OQ; *Proved.*

∴ the ∠ RAO=the ∠ QAO. Ex. 12, p. 99.

That is, AO is the bisector of the ∠ BAC.
∴ the bisectors of the two exterior ∠ˢ DBC, ECB, and of the interior ∠ BAC meet at the point O.

Q.E.D.

4. *The medians of a triangle are concurrent.*

Let ABC be a △.

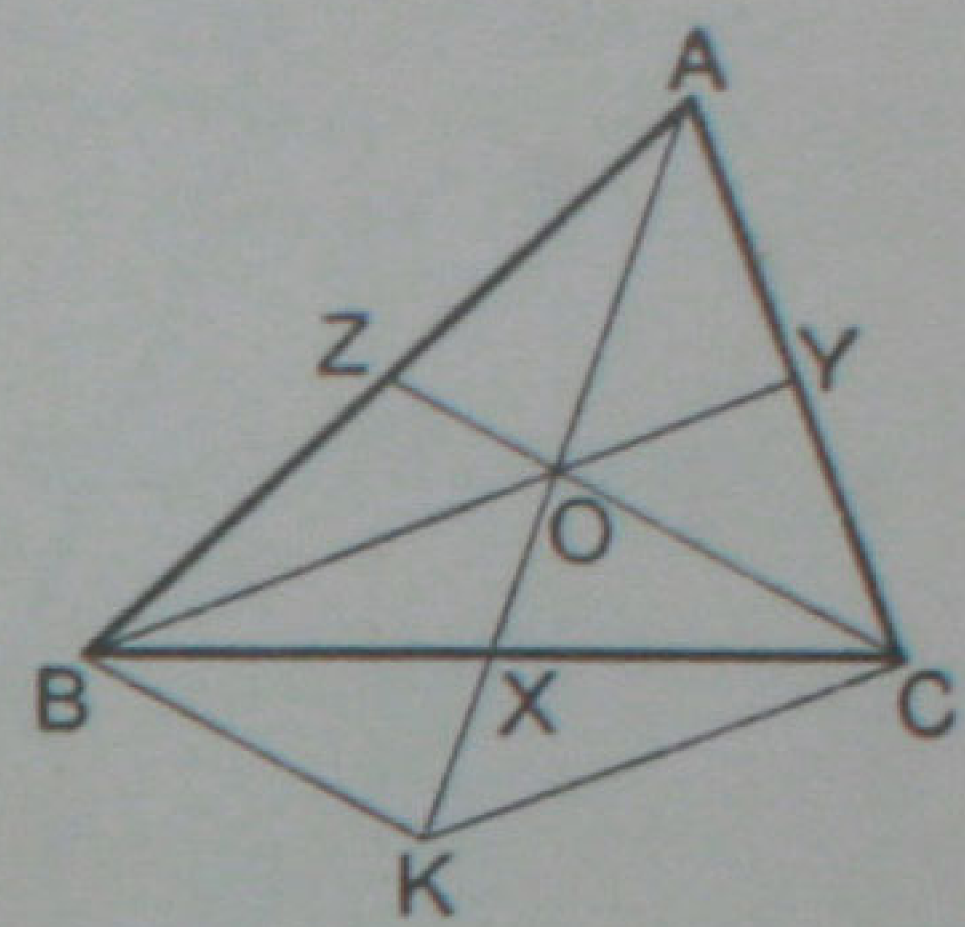
Then shall its three medians be concurrent.

Let BY and CZ be two of its medians, and let them intersect at O.

Join AO,

and produce it to meet BC in X.

It is required to shew that AX *is the remaining median of the* △.



Through C draw CK parallel to BY:

produce AX to meet CK at K.

Join BK.

In the △ AKC,

because Y is the middle point of AC, and YO is parallel to CK,

∴ O is the middle point of AK. Ex. 1, p. 104.

Again in the △ ABK,

since Z and O are the middle points of AB, AK,

∴ ZO is parallel to BK, Ex. 2, p. 104.

that is, OC is parallel to BK:

∴ the figure BKCO is a par^{m}.

But the diagonals of a par^{m} bisect one another, Ex. 5, p. 70.

∴ X is the middle point of BC.

That is, AX is a median of the △.

Hence the three medians meet at the point O. Q.E.D.

Corollary. *The three medians of a triangle cut one another at a point of trisection, the greater segment in each being towards the angular point.*

For in the above figure it has been proved that

AO = OK,

also that OX is half of OK;

∴ OX is half of OA:

that is, OX is one third of AX.

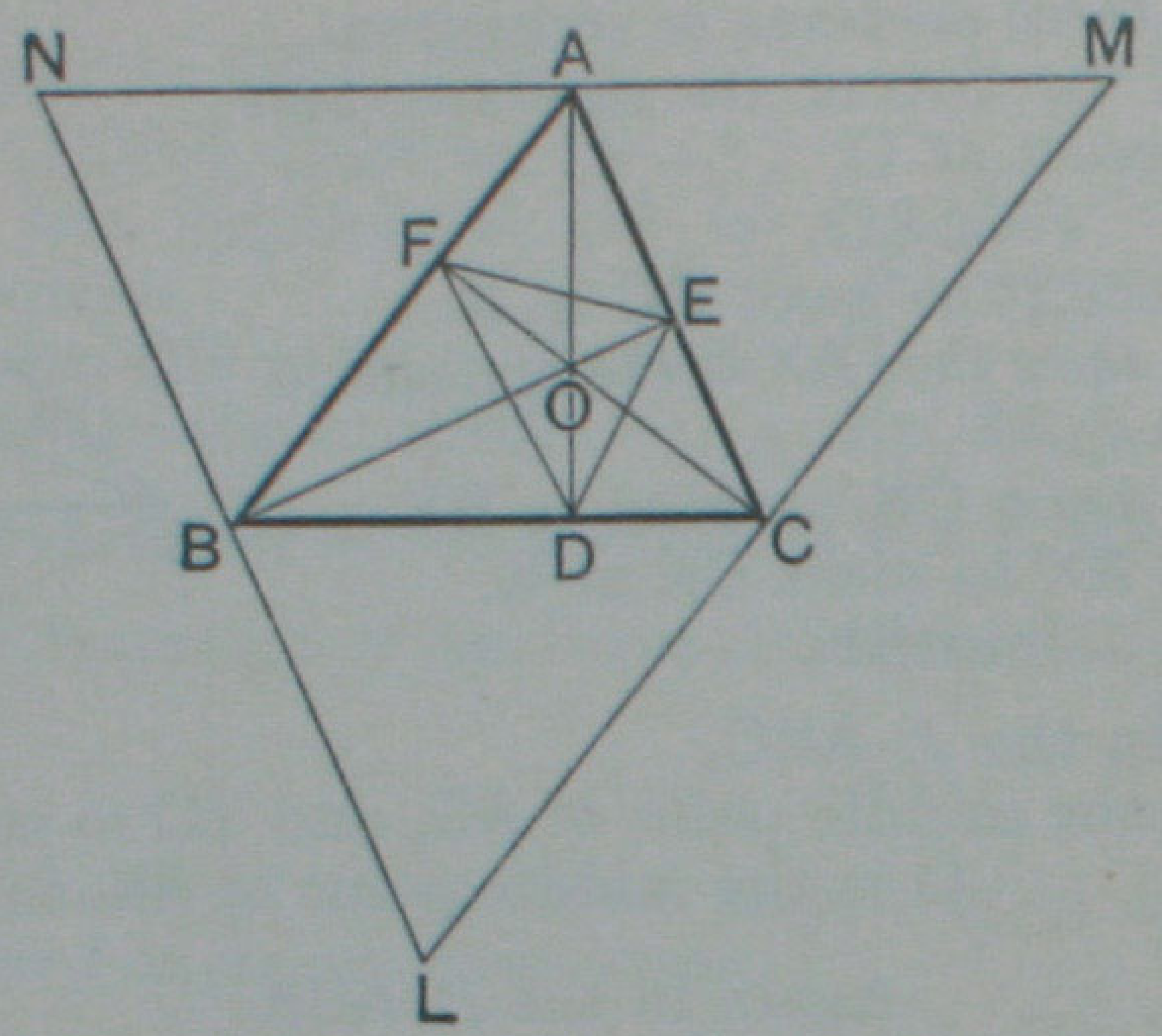
Similarly OY is one third of BY,

and OZ is one third of CZ. Q.E.D.

By means of this Corollary it may be shewn that in any triangle the shorter median bisects the greater side.

[The point of intersection of the three medians of a triangle is called the **centroid**. It is shewn in Mechanics that a thin triangular plate will balance in any position about this point: therefore the centroid of a triangle is also its centre of gravity.]

5. *The perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*



Let ABC be a △, and AD, BE, CF the three perp^s drawn from the vertices to the opposite sides.

Then shall the perp^s AD, BE, CF *be concurrent.*

Through A, B, and C draw straight lines MN, NL, LM parallel to the opposite sides of the △.

Then the figure BAMC is a par^m. *Def.* 36.
∴ AB = MC. I. 34.
Also the figure BACL is a par^m.
∴ AB = LC,
∴ LC = CM :
that is, C is the middle point of LM.
So also A and B are the middle points of MN and NL.
Hence AD, BE, CF are the perp^s to the sides of the △ LMN from their middle points. Ex. 3, p. 60.

But these perp^s meet in a point : Ex. 1, p. 111.
that is, the perp^s drawn from the vertices of the △ ABC to the opposite sides meet in a point. Q.E.D.

[For another proof see Theorems and Examples on Book III.]

DEFINITIONS.

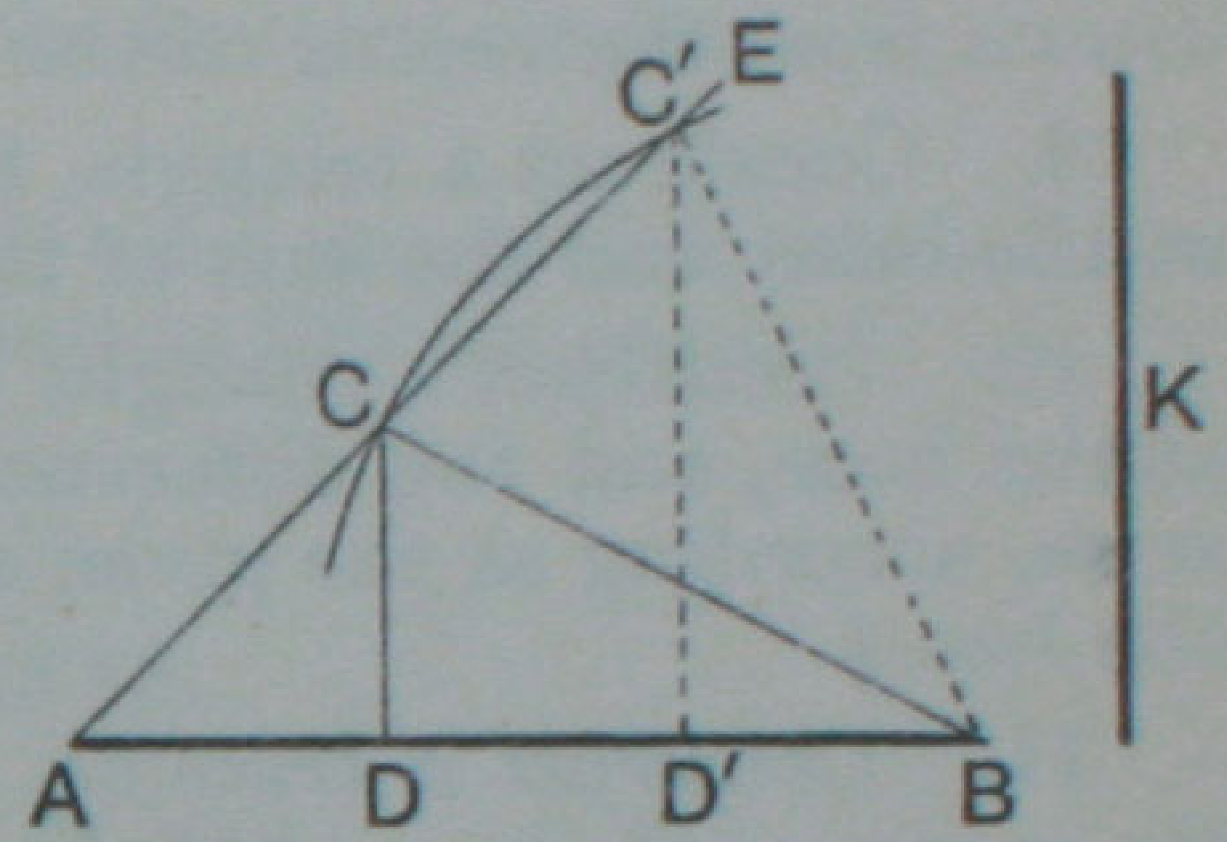
(i) The intersection of the perpendiculars drawn from the vertices of a triangle to the opposite sides is called its **orthocentre**.

(ii) The triangle formed by joining the feet of the perpendiculars is called the **pedal triangle**.

VII. ON THE CONSTRUCTION OF TRIANGLES WITH GIVEN PARTS.

Obs. No general rules can be laid down for the solution of problems in this section; but in a few typical cases we give constructions, which the student will find little difficulty in adapting to other questions of the same class.

1. *Construct a right-angled triangle, having given the hypotenuse and the sum of the remaining sides.*



[It is required to construct a rt.-angled △, having its hypotenuse equal to the given straight line K, and the sum of its remaining sides equal to AB.

From A draw AE making with BA an ∠ equal to half a rt. ∠. From centre B, with radius equal to K, describe a circle cutting AE in the points C, C′.

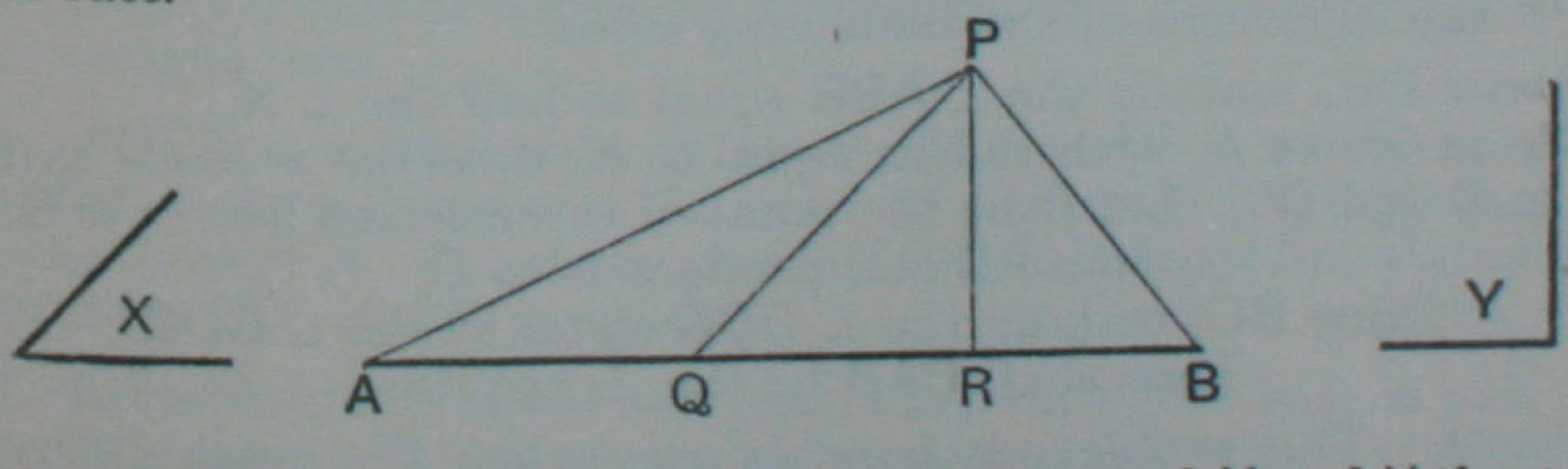
From C and C′ draw perp^s CD, C′D′ to AB; and join CB, C′B. Then either of the △^s CDB, C′D′B will satisfy the given conditions.

Note. If the given hypotenuse K be greater than the perpendicular drawn from B to AE, there will be *two* solutions. If the line K be equal to this perpendicular, there will be *one* solution; but if less, the problem is *impossible*.]

2. Construct a right-angled triangle, having given the hypotenuse and the difference of the remaining sides.

3. Construct an isosceles right-angled triangle, having given the sum of the hypotenuse and one side.

4. *Construct a triangle, having given the perimeter and the angles at the base.*



[Let AB be the perimeter of the required △, and X and Y the ∠^s at the base.

From A draw AP, making the ∠ BAP equal to half the ∠ X.
From B draw BP, making the ∠ ABP equal to half the ∠ Y.
From P draw PQ, making the ∠ APQ equal to the ∠ BAP.
From P draw PR, making the ∠ BPR equal to the ∠ ABP.

Then shall PQR be the required △.]

5. Construct a right-angled triangle, having given the perimeter and one acute angle.

6. Construct an isosceles triangle of given altitude, so that its base may be in a given straight line, and its two equal sides may pass through two fixed points. [See Ex. 7, p. 55.]

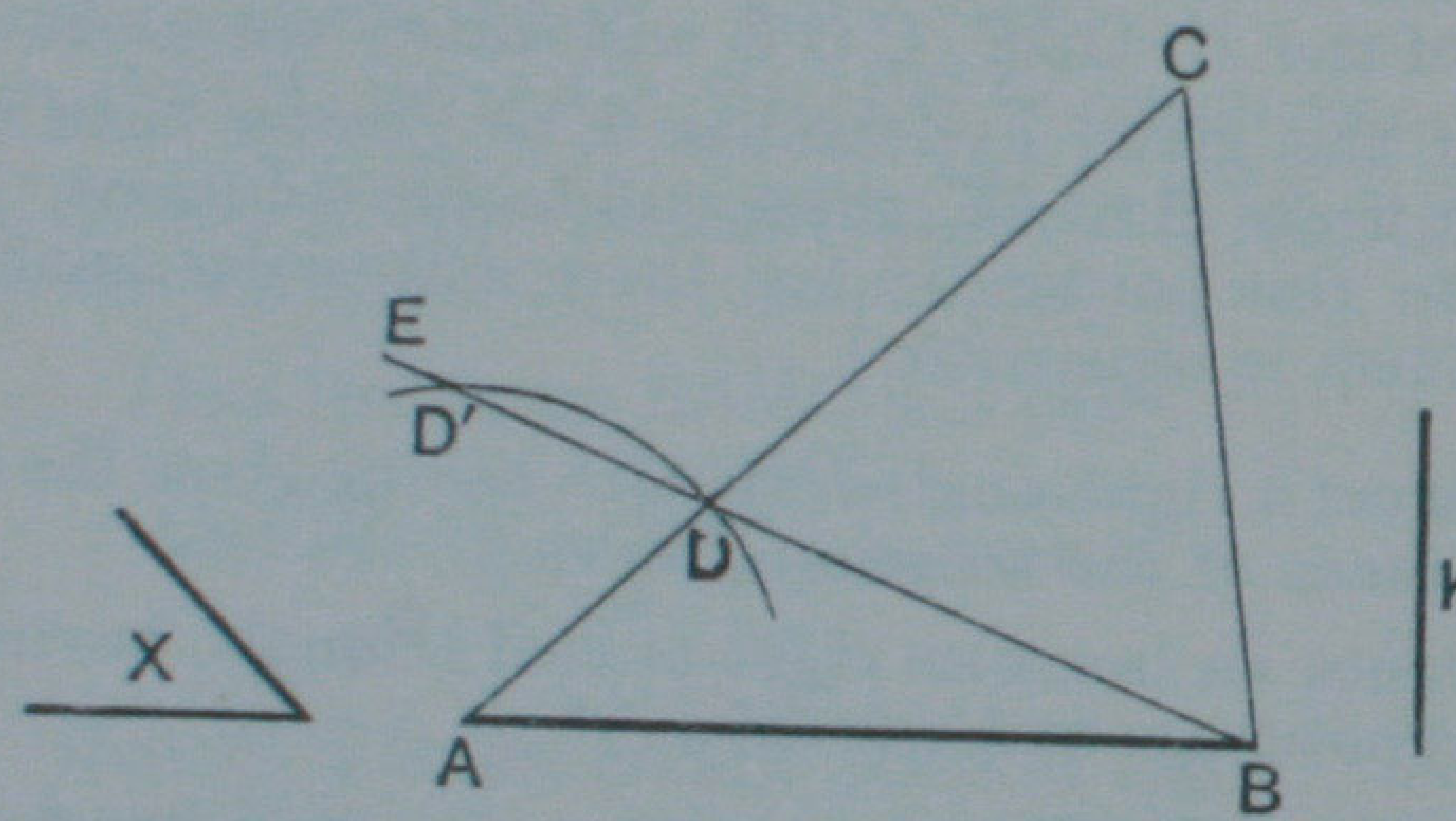
7. Construct an equilateral triangle, having given the length of the perpendicular drawn from one of the vertices to the opposite side.

8. Construct an isosceles triangle, having given the base, and the difference of one of the remaining sides and the perpendicular drawn from the vertex to the base. [See Ex. 1, p. 96.]

9. Construct a triangle, having given the base, one of the angles at the base, and the sum of the remaining sides.

10. Construct a triangle, having given the base, one of the angles at the base, and the difference of the remaining sides. [Two cases arise, according as the given angle is adjacent to the greater side or the less.]

11. *Construct a triangle, having given the base, the difference of the angles at the base, and the difference of the remaining sides.*



[Let AB be the given base, X the difference of the ∠s at the base, and K the difference of the remaining sides.

Draw BE, making the ∠ ABE equal to half the ∠ X.

From centre A, with radius equal to K, describe a circle cutting BE in D and D′. Let D be the point of intersection *nearer to* B.

Join AD and produce it to C.

Draw BC, making the ∠ DBC equal to the ∠ BDC.

Then shall CAB be the △ required. Ex. 7, p. 109.

NOTE. This problem is possible only when the given difference K is greater than the perpendicular drawn from A to BE.]

12. Construct a triangle, having given the base, the difference of the angles at the base, and the sum of the remaining sides.

13. Construct a triangle, having given the perpendicular from the vertex on the base, and the difference between each side and the adjacent segment of the base.

14. Construct a triangle, having given two sides and the median which bisects the remaining side. [See Ex. 18, p. 110.]

15. Construct a triangle, having given one side, and the medians which bisect the two remaining sides.

[See Fig. to Ex. 4, p. 113.
Let BC be the given side. Take two-thirds of each of the given medians; hence construct the triangle BOC. The rest of the construction follows easily.]

16. *Construct a triangle, having given its three medians.*

[See Fig. to Ex. 4, p. 113.
Take two-thirds of each of the given medians, and construct the triangle OKC. The rest of the construction follows easily.]

VIII. ON AREAS.

See Propositions 35—48.

Obs. It must be understood that throughout this section the word *equal* as applied to rectilineal figures will be used as denoting *equality of area* unless otherwise stated.

1. *Shew that a parallelogram is bisected by any straight line which passes through the middle point of one of its diagonals.* [I. 29, 26.]

2. Bisect a parallelogram by a straight line drawn through a given point.

3. Bisect a parallelogram by a straight line drawn perpendicular to one of its sides.

4. Bisect a parallelogram by a straight line drawn parallel to a given straight line.

5. ABCD *is a trapezium in which the side* AB *is parallel to* DC. *Shew that its area is equal to the area of a parallelogram formed by drawing through* X, *the middle point of* BC, *a straight line parallel to* AD, *meeting* DC, *or* DC *produced.* [I. 29, 26.]

6. A trapezium is equal to a parallelogram whose base is half the sum of the parallel sides of the given figure, and whose altitude is equal to the perpendicular distance between them.

7. ABCD is a trapezium in which the side AB is parallel to DC; shew that it is double of the triangle formed by joining the extremities of AD to X, the middle point of BC.

8. Shew that a trapezium is bisected by the straight line which joins the middle points of its parallel sides. [I. 38.]

Obs. In the following group of Exercises the proofs depend chiefly on Propositions 37 and 38, and the two converse theorems.

9. If two straight lines AB, CD intersect at X, and if the straight lines AC and BD, which join their extremities are parallel, shew that the triangle AXD is equal to the triangle BXC.

10. If two straight lines AB, CD intersect at X, so that the triangle AXD is equal to the triangle XCB, then AC and BD are parallel.

11. ABCD is a parallelogram, and X any point in the diagonal AC produced; shew that the triangles XBC, XDC are equal. [See Ex. 13, p. 70.]

12. ABC is a triangle, and R, Q the middle points of the sides AB, AC; shew that if BQ and CR intersect in X, the triangle BXC is equal to the quadrilateral AQXR. [See Ex. 5, p. 79.]

13. If the middle points of the sides of a quadrilateral be joined in order, the *parallelogram* so formed [see Ex. 9, p. 105] is equal to half the given figure.

14. Two triangles of equal area stand on the same base but on opposite sides of it: shew that the straight line joining their vertices is bisected by the base, or by the base produced.

15. The straight line which joins the middle points of the diagonals of a trapezium is parallel to each of the two parallel sides.

16. (i) *A triangle is equal to the sum or difference of two triangles on the same base (or on equal bases), if the altitude of the first is equal to the sum or difference of the altitudes of the others.*

(ii) *A triangle is equal to the sum or difference of two triangles of the same altitude, if the base of the first is equal to the sum or difference of the bases of the others.*

Similar statements hold good of parallelograms.

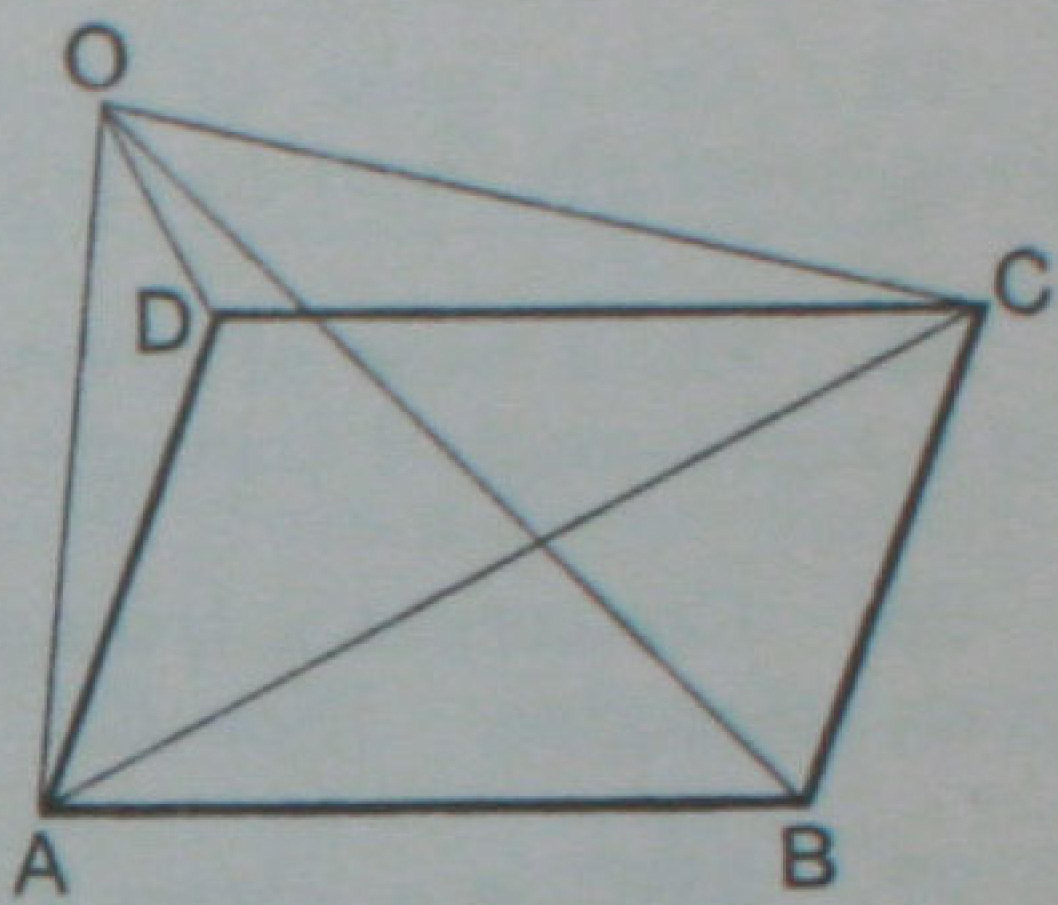
17. ABCD is a parallelogram, and O is any point outside it; shew that the sum or difference of the triangles OAB, OCD is equal to half the parallelogram. Distinguish between the two cases.

Obs. On the following proposition depends an important theorem in Mechanics: we give a proof of the first case, leaving the second case to be deduced by a similar method.

18. (i) ABCD *is a parallelogram, and* O *is any point* without the angle BAD and its opposite vertical angle; *shew that the triangle* OAC *is equal to the* sum *of the triangles* OAD, OAB.

(ii) *If* O *is* within the angle BAD or its opposite vertical angle, *the triangle* OAC *is equal to the difference of the triangles* OAD, OAB.

CASE I. If O is without the ∠ DAB and its opp. vert. ∠, then OA is without the par^{m} ABCD: therefore the perp. drawn from C to OA is equal to the sum of the perp^{s} drawn from B and D to OA. [See Ex. 20, p. 107.]



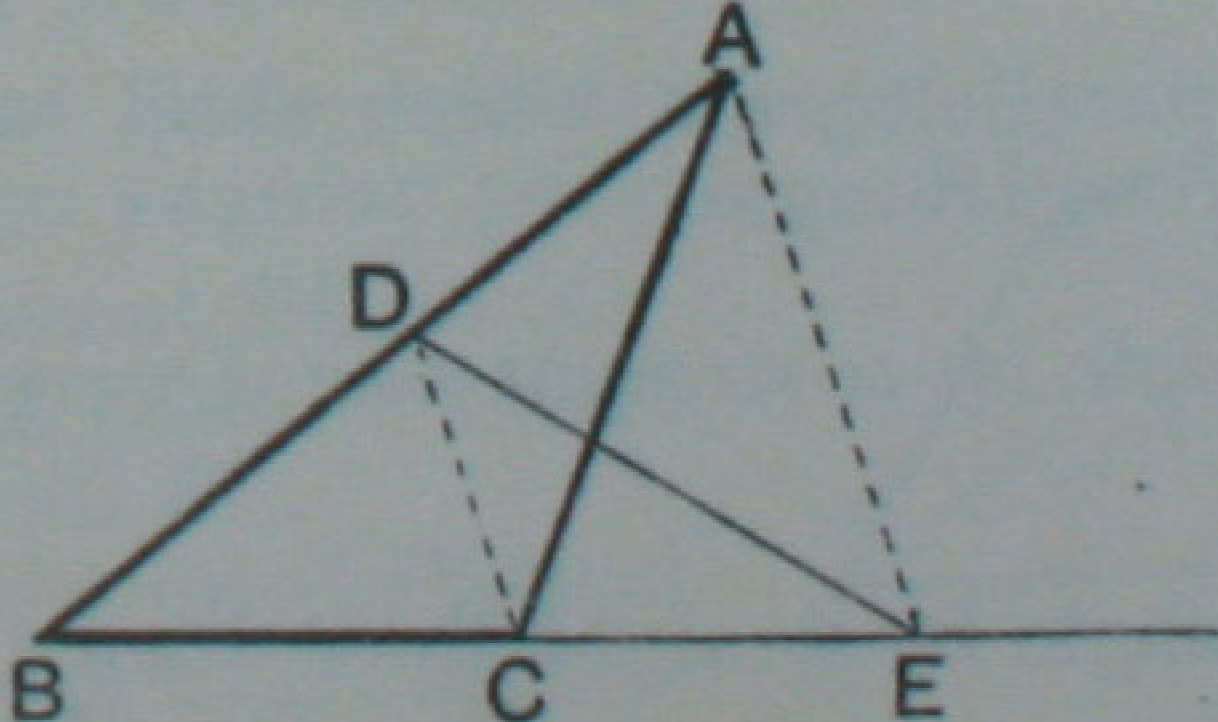
Now the △^{s} OAC, OAD, OAB are upon the same base OA;
and the altitude of the △ OAC with respect to this base has been shewn to be equal to the sum of the altitudes of the △^{s} OAD, OAB.

Therefore the △ OAC is equal to the sum of the △^{s} OAD, OAB. [See Ex. 16, p. 118.] Q.E.D.

19. ABCD is a parallelogram, and through O, any point within it, straight lines are drawn parallel to the sides of the parallelogram; shew that the difference of the parallelograms DO, BO is double of the triangle AOC. [See preceding theorem (ii).]

20. The area of a quadrilateral is equal to the area of a triangle having two of its sides equal to the diagonals of the given figure, and the included angle equal to either of the angles between the diagonals.

21. ABC *is a triangle, and* D *is any point in* AB; *it is required to draw through* D *a straight line* DE *to meet* BC *produced in* E, *so that the triangle* DBE *may be equal to the triangle* ABC.



[Join DC. Through A draw AE parallel to DC. I. 31.
Join DE.
The △ EBD shall be equal to the △ ABC.] I. 37.

22. On a base of given length describe a triangle equal to a given triangle and having an angle equal to an angle of the given triangle.

23. Construct a triangle equal in area to a given triangle, and having a given altitude.

24. On a base of given length construct a triangle equal to a given triangle, and having its vertex on a given straight line.

25. On a base of given length describe (i) an isosceles triangle; (ii) a right-angled triangle, equal to a given triangle.

26. Construct a triangle equal to the sum or difference of two given triangles. [See Ex. 16, p. 118.]

27. ABC is a given triangle, and X a given point: describe a triangle equal to ABC, having its vertex at X, and its base in the same straight line as BC.

28. ABCD *is a quadrilateral. On the base* AB *construct a triangle equal in area to* ABCD, *and having the angle at* A *common with the quadrilateral.*

[Join BD. Through C draw CX parallel to BD, meeting AD produced in X; join BX.]

29. *Construct a rectilineal figure equal to a given rectilineal figure, and having fewer sides by one than the given figure.*

Hence shew how to construct a triangle equal to a given rectilineal figure.

30. ABCD is a quadrilateral: it is required to construct a triangle equal in area to ABCD, having its vertex at a given point X in DC, and its base in the same straight line as AB.

31. Construct a rhombus equal to a given parallelogram.

32. Construct a parallelogram which shall have the same area and perimeter as a given triangle.

33. *Bisect a triangle by a straight line drawn through one of its angular points.*

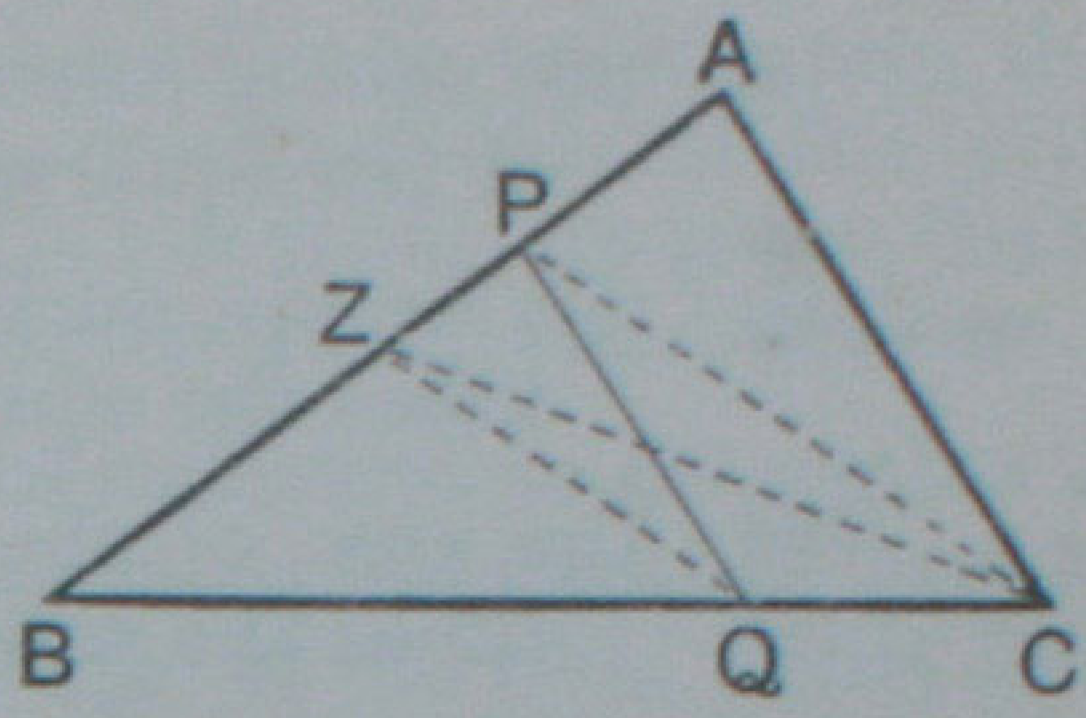
34. Trisect a triangle by straight lines drawn through one of its angular points. [See Ex. 19, p. 110, and I. 38.]

35. Divide a triangle into any number of equal parts by straight lines drawn through one of its angular points. [See Ex. 19, p. 107, and I. 38.]

36. *Bisect a triangle by a straight line drawn through a given point in one of its sides.*

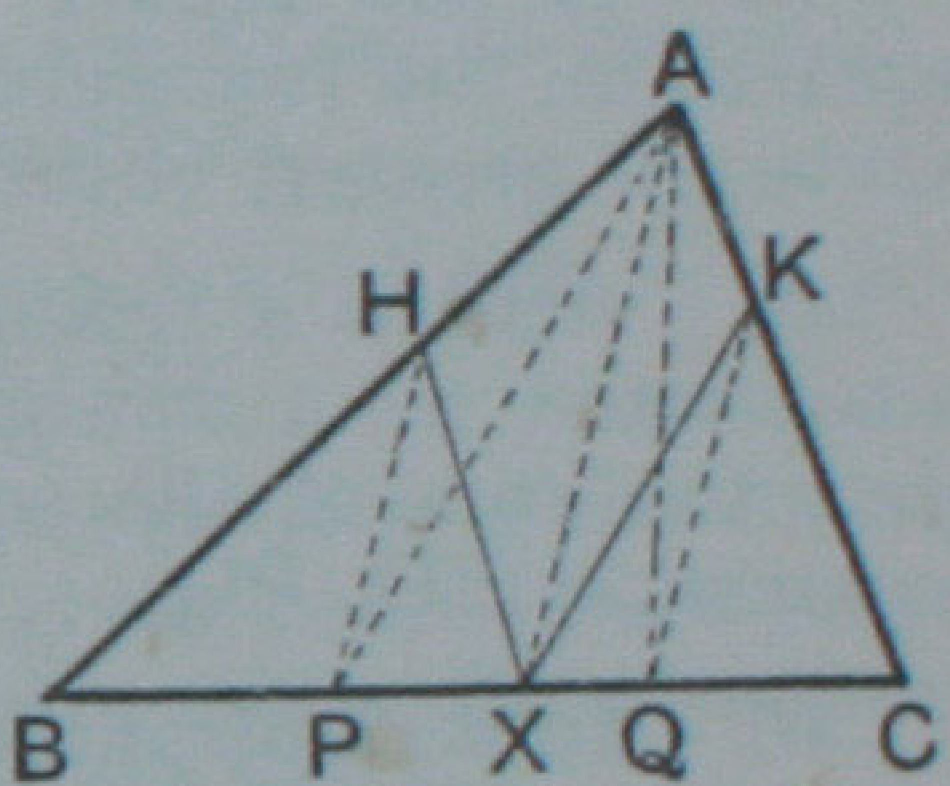
[Let ABC be the given △, and P the given point in the side AB.

Bisect AB at Z; and join CZ, CP.
Through Z draw ZQ parallel to CP.
Join PQ.
Then shall PQ bisect the △.
See Ex. 21, p. 119.]



37. *Trisect a triangle by straight lines drawn from a given point in one of its sides.*

[Let ABC be the given △, and X the given point in the side BC.
Trisect BC at the points P, Q. Ex. 19, p. 107.
Join AX, and through P and Q draw PH and QK parallel to AX.
Join XH, XK.
These straight lines shall trisect the △; as may be shewn by joining AP, AQ.
See Ex. 21, p. 119.]



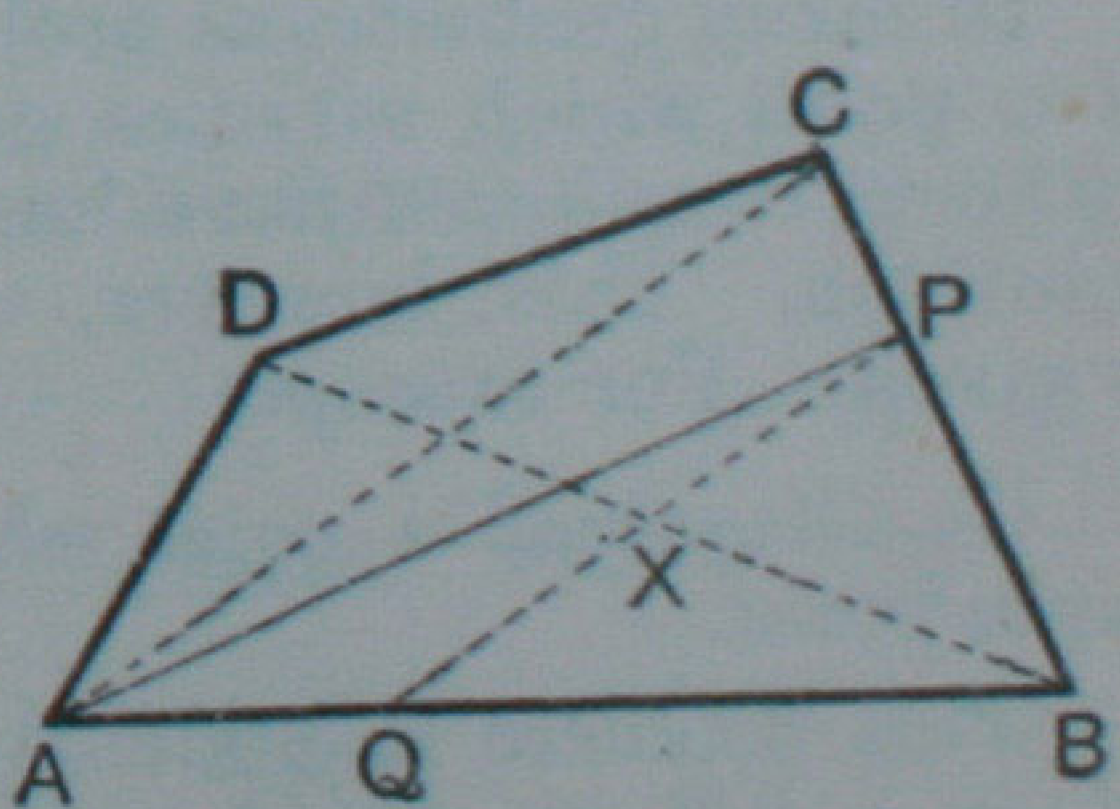
38. Cut off from a given triangle a fourth, fifth, sixth, or any part required by a straight line drawn from a given point in one of its sides. [See Ex. 19, p. 107, and Ex. 21, p. 119.]

39. *Bisect a quadrilateral by a straight line drawn through an angular point.*

[Two constructions may be given for this problem: the first will be suggested by Exercises 28 and 33, p. 120.

The second method proceeds thus.

Let ABCD be the given quadrilateral, and A the given angular point.
Join AC, BD, and bisect BD in X.
Through X draw PXQ parallel to AC, meeting BC in P; join AP.
Then shall AP bisect the quadrilateral.
Join AX, CX, and use I. 37, 38.]



40. Cut off from a given quadrilateral a third, a fourth, a fifth, or any part required, by a straight line drawn through a given angular point. [See Exercises 28 and 35, p. 120.]

Obs. The following Theorems depend on I. 47.

41. In the figure of I. 47, shew that

(i) the sum of the squares on AB and AE is equal to the sum of the squares on AC and AD.

(ii) the square on EK is equal to the square on AB with four times the square on AC.

(iii) the sum of the squares on EK and FD is equal to five times the square on BC.

42. If a straight line is divided into any two parts, the square on the straight line is greater than the sum of the squares on the two parts.

43. If the square on one side of a triangle is less than the squares on the remaining sides, the angle contained by these sides is acute; if greater, obtuse.

44. ABC is a triangle, right-angled at A; the sides AB, AC are intersected by a straight line PQ, and BQ, PC are joined: shew that the sum of the squares on BQ, PC is equal to the sum of the squares on BC, PQ.

45. In a right-angled triangle four times the sum of the squares on the medians which bisect the sides containing the right angle is equal to five times the square on the hypotenuse.

46. Describe a square whose area shall be three times that of a given square.

47. Divide a straight line into two parts such that the sum of their squares shall be equal to a given square.

IX. ON LOCI.

In many geometrical problems we are required to find the position of a point which satisfies given conditions; and all such problems hitherto considered have been found to admit of a *limited number* of solutions. This, however, will not be the case if *only one* condition is given. For example:

(i) *Required a point which shall be at a given distance from a given point.*

This problem is evidently *indeterminate*, that is to say, it admits of an indefinite number of solutions; for the condition stated is satisfied by *any* point on the circumference of the circle described from the given point as centre, with a radius equal to the given distance. Moreover this condition is satisfied by no other point within or without the circle.

(ii) *Required a point which shall be at a given distance from a given straight line.*

Here again there are an infinite number of such points, and they lie on two parallel straight lines drawn on either side of the given straight line at the given distance from it: further, no point that is not on one or other of these parallels satisfies the given condition.

Hence we see that *one* condition is not sufficient to determine the position of a point absolutely, but it may have the effect of restricting it to some definite line or lines, straight or curved. This leads us to the following definition.

DEFINITION. The **Locus** of a point satisfying an assigned condition consists of the line, lines, or part of a line, to which the point is thereby restricted; provided that the condition is satisfied by every point on such line or lines, and by no other.

A locus is sometimes defined as the path traced out by a point which moves in accordance with an assigned law.

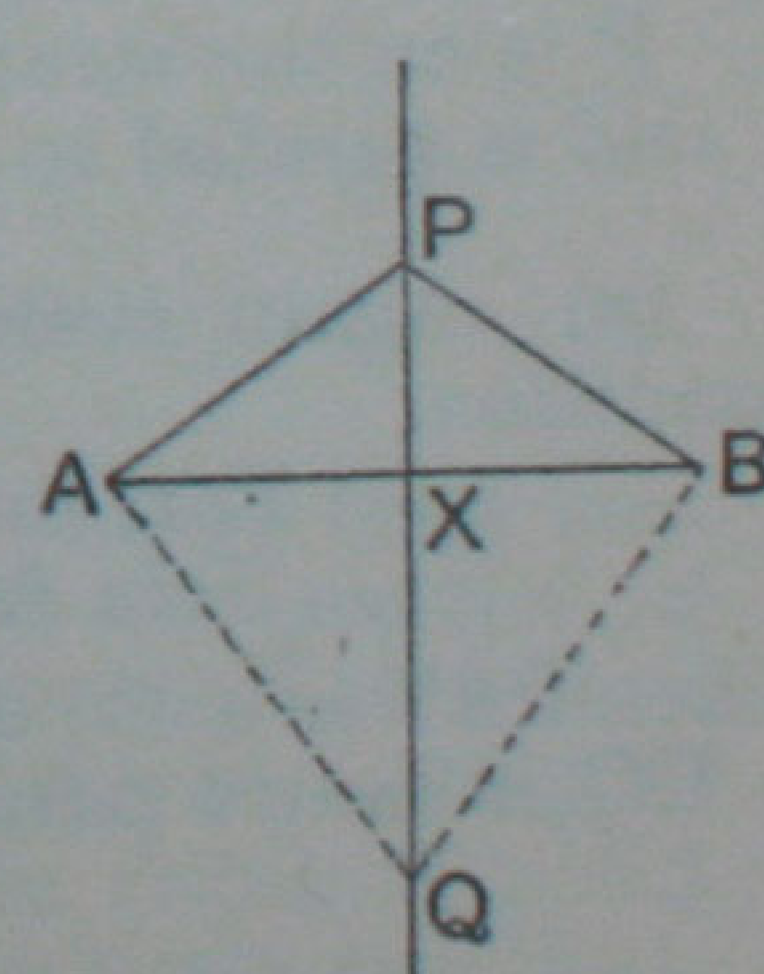
Thus the locus of a point, which is always at a given distance from a given point, is a circle of which the given point is the centre: and the locus of a point, which is always at a given distance from a given straight line, is a pair of parallel straight lines.

We now see that in order to infer that a certain line, or system of lines, is the locus of a point under a given condition, it is necessary to prove

(i) that any point which fulfils the given condition is on the supposed locus;

(ii) that every point on the supposed locus satisfies the given condition.

1. *Find the locus of a point which is always equidistant from two given points.*



Let A, B be the two given points.

(*a*) Let P be any point equidistant from A and B, so that AP = BP.

Bisect AB at X, and join PX.

Then in the △s AXP, BXP,

Because { AX = BX, *Constr.*
and PX is common to both,
also AP = BP, *Hyp.*

∴ the ∠ PXA = the ∠ PXB; I. 8.

and they are adjacent ∠s;

∴ PX is perp. to AB. *Def.* 10.

∴ any point which is equidistant from A and B is on the straight line which bisects AB at right angles.

(β) Also every point in this line is equidistant from A and B.

For let Q be any point in this line.

Join AQ, BQ.

Then in the $\triangle^s$ AXQ, BXQ,

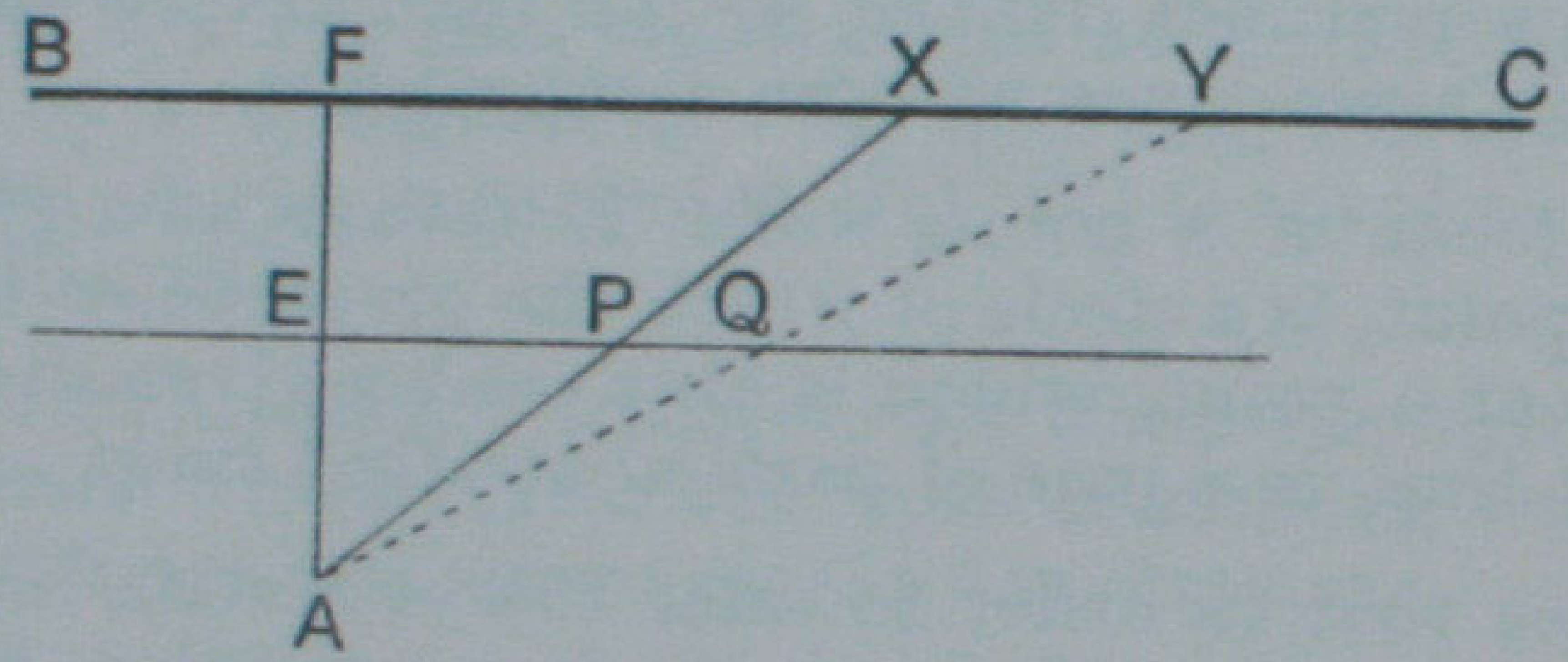
Because $\begin{cases} AX = BX, \\ \text{and } XQ \text{ is common to both;} \\ \text{also the } \angle AXQ = \text{the } \angle BXQ, \text{ being rt. } \angle^s; \end{cases}$

$\therefore$ AQ = BQ. I. 4.

That is, Q is equidistant from A and B.

Hence we conclude that the locus of the point equidistant from two given points A, B is the straight line which bisects AB at right angles.

2. *To find the locus of the middle point of a straight line drawn from a given point to meet a given straight line of unlimited length.*



Let A be the given point, and BC the given straight line of unlimited length.

(α) Let AX be any straight line drawn through A to meet BC, and let P be its middle point.

Draw AF perp. to BC, and bisect AF at E.

Join EP, and produce it indefinitely.

Since AFX is a $\triangle$, and E, P the middle points of the two sides AF, AX,

$\therefore$ EP is parallel to the remaining side FX. Ex. 2, p. 104.

$\therefore$ P is on the straight line which passes through the *fixed point* E, and is parallel to BC.

(β) Again, every point in EP, or EP produced, fulfils the required condition.

For, in this straight line take any point Q.

Join AQ, and produce it to meet BC in Y.

Then FAY is a $\triangle$, and through E, the middle point of the side AF, EQ is drawn parallel to the side FY;

$\therefore$ Q is the middle point of AY. Ex. 1, p. 104.

Hence the required locus is the straight line drawn parallel to BC, and passing through E, the middle point of the perp. from A to BC.

3. *Find the locus of a point equidistant from two given intersecting straight lines.* [See Ex. 3, p. 55.]

4. Find the locus of a point at a given radial distance from the circumference of a given circle.

5. Find the locus of a point which moves so that the sum of its distances from two given intersecting straight lines of unlimited length is constant.

6. Find the locus of a point when the differences of its distances from two given intersecting straight lines of unlimited length is constant.

7. A straight rod of given length slides between two straight rulers placed at right angles to one another: find the locus of its middle point. [See Ex. 2, p. 108.]

8. On a given base as hypotenuse right-angled triangles are described: find the locus of their vertices. [See Ex. 2, p. 108.]

9. AB is a given straight line, and AX is the perpendicular drawn from A to any straight line passing through B: find the locus of the middle point of AX.

10. Find the locus of the vertex of a triangle, when the base and area are given.

11. Find the locus of the intersection of the diagonals of a parallelogram, of which the base and area are given.

12. Find the locus of the intersection of the medians of triangles described on a given base and of given area.

X. ON THE INTERSECTION OF LOCI.

It appears from various problems which have already been considered, that we are often required to find a point, the position of which is subject to two given conditions. The method of loci is very useful in solving problems of this kind; for corresponding to each condition there will be a locus on which the required point must lie. Hence all points which are common to these two loci, that is, all the points of intersection of the loci, will satisfy *both* the given conditions.

EXAMPLE 1. *To construct a triangle, having given the base, the altitude, and the length of the median which bisects the base.*

Let AB be the given base, and P and Q the lengths of the altitude and median respectively:

then the triangle is known if its *vertex* is known.

(i) Draw a straight line CD parallel to AB, and at a distance from it equal to P:

then the required vertex must lie on CD.

(ii) Again, from the middle point of AB as centre, with radius equal to Q, describe a circle:

then the required vertex must lie on this circle.

Hence any points which are common to CD and the circle, satisfy both the given conditions: that is to say, if CD intersect the circle in E, F each of the points of intersection might be the vertex of the required triangle. This supposes the length of the median Q to be greater than the altitude.

EXAMPLE 2. *To find a point equidistant from three given points* A, B, C, *which are not in the same straight line.*

(i) The locus of points equidistant from A and B is the straight line PQ, which bisects AB at right angles. Ex. 1, p. 123.

(ii) Similarly the locus of points equidistant from B and C is the straight line RS which bisects BC at right angles.

Hence the point common to PQ and RS must satisfy both conditions: that is to say, the point of intersection of PQ and RS will be equidistant from A, B, and C.

Obs. These principles may also be used to prove the theorems relating to concurrency already given on page 111.

EXAMPLE. *To prove that the bisectors of the angles of a triangle are concurrent.*

Let ABC be a triangle.

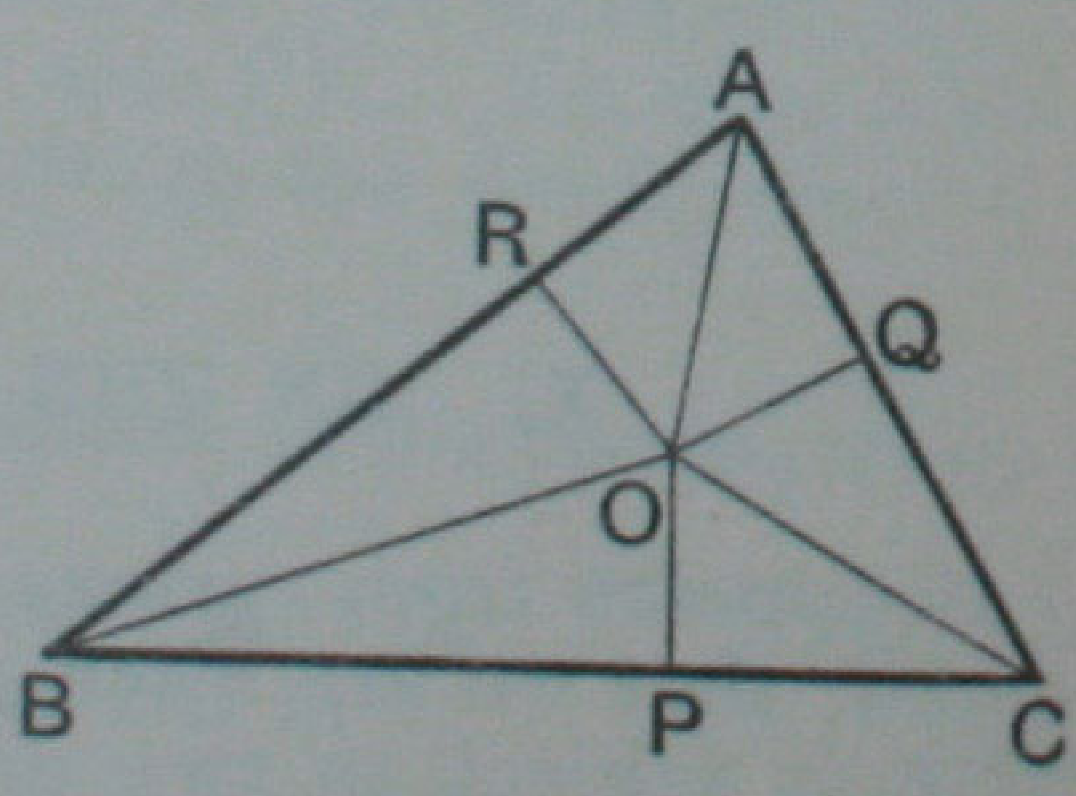
Bisect the ∠ˢ ABC, BCA by straight lines BO, CO: these must meet at some point O. *Ax.* 12.

Join OA.

Then shall OA *bisect the* ∠ BAC.

Now BO is the locus of points equidistant from BC, BA; Ex. 3, p. 55.

∴ OP=OR.



Similarly CO is the locus of points equidistant from BC, CA.

∴ OP=OQ; hence OR=OQ.

∴ O is on the locus of points equidistant from AB and AC: that is, OA is the bisector of the ∠ BAC.

Hence the bisectors of the three ∠ˢ meet at the point O.

It may happen that the data of the problem are so related to one another that the resulting loci do not intersect. In this case the problem is impossible.

For example, if in Ex. 1, page 126, the length of the given median *is less than* the given altitude, the straight line CD will not be intersected by the circle, and no triangle can fulfil the conditions of the problem. If the length of the median *is equal* to the given altitude, *one* point is common to the two loci; and consequently only one solution of the problem exists: and we have seen that there are two solutions, if the median is greater than the altitude.

In examples of this kind the student should make a point of investigating the relations which must exist among the data, in order that the problem may be possible; and he must observe that if under certain relations *two* solutions are possible, and under other relations no solution exists, there will always be some *intermediate* relation under which *one* and *only one* solution is possible.

EXAMPLES.

1. Find a point in a given straight line which is equidistant from two given points.

2. Find a point which is at given distances from each of two given straight lines. How many solutions are possible?

3. *On a given base construct a triangle, having given one angle at the base and the length of the opposite side. Examine the relations which must exist among the data in order that there may be two solutions, one solution, or that the problem may be impossible.*

4. On the base of a given triangle construct a second triangle equal in area to the first, and having its vertex in a given straight line.

5. Construct an isosceles triangle equal in area to a given triangle, and standing on the same base.

6. Find a point which is at a given distance from a given point, and is equidistant from two given parallel straight lines.

When does this problem admit of two solutions, when of one only, and when is it impossible?

BOOK II.

Book II. deals with the areas of rectangles and squares.

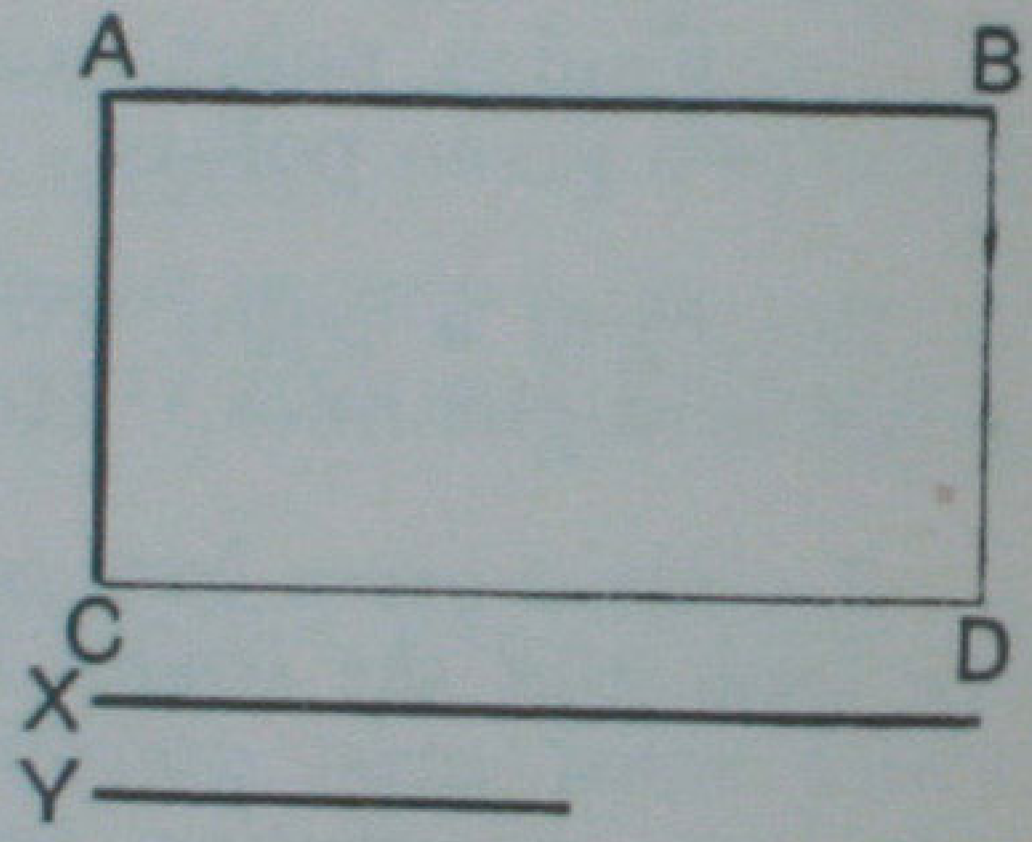
A **Rectangle** has been defined (Book I., Def. 37) as a parallelogram which has one of its angles a right angle.

It should be remembered that if a parallelogram has *one* right angle, *all* its angles are right angles. [I. 46, *Cor.*]

Definitions.

1. A rectangle is said to be **contained** by any two of its sides which form a right angle: for it is clear that both the form and magnitude of a rectangle are fully determined when the lengths of two such sides are given.

Thus the rectangle ACDB is said *to be contained* by AB, AC; or by CD, DB: and if X and Y are two straight lines equal respectively to AB and AC, then the rectangle contained by X and Y is equal to the rectangle contained by AB, AC.



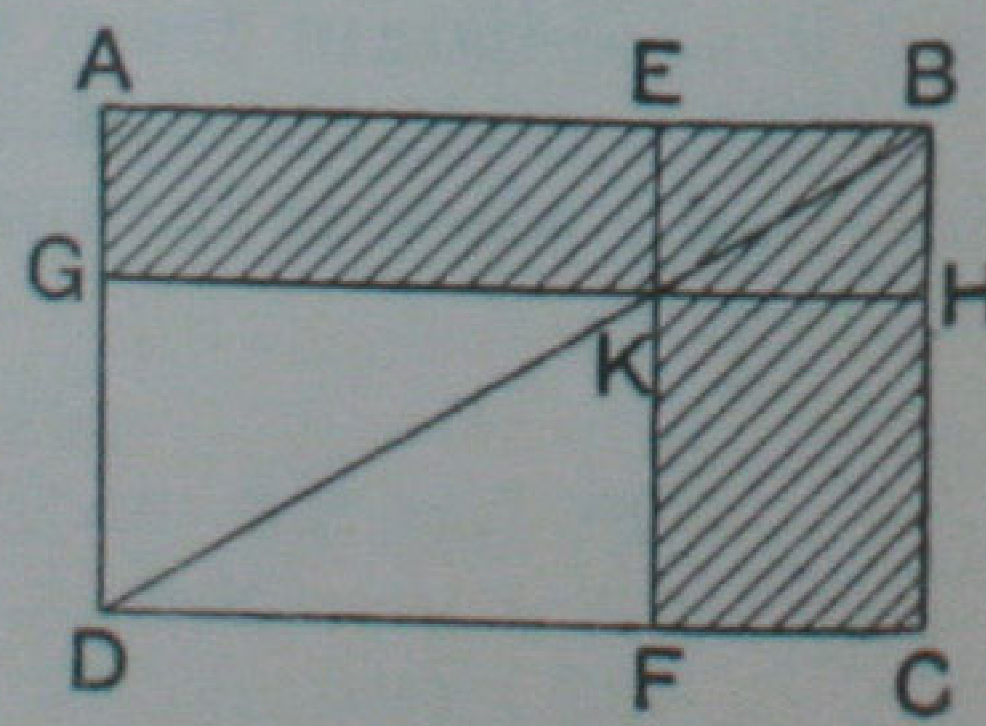
[See Ex. 12, p. 70.]

After Proposition 3, we shall use the abbreviation *rect.* AB, AC to denote *the rectangle contained by* AB *and* AC.

2. In any parallelogram the figure formed by either of the parallelograms about a diagonal together with the two complements is called a **gnomon**.

Thus the shaded portion of the annexed diagram, consisting of the parallelogram EH together with the complements AK, KC is the *gnomon* AHF.

The other gnomon in the diagram is that which is made up of the figures AK, GF and FH, namely the gnomon AFH.

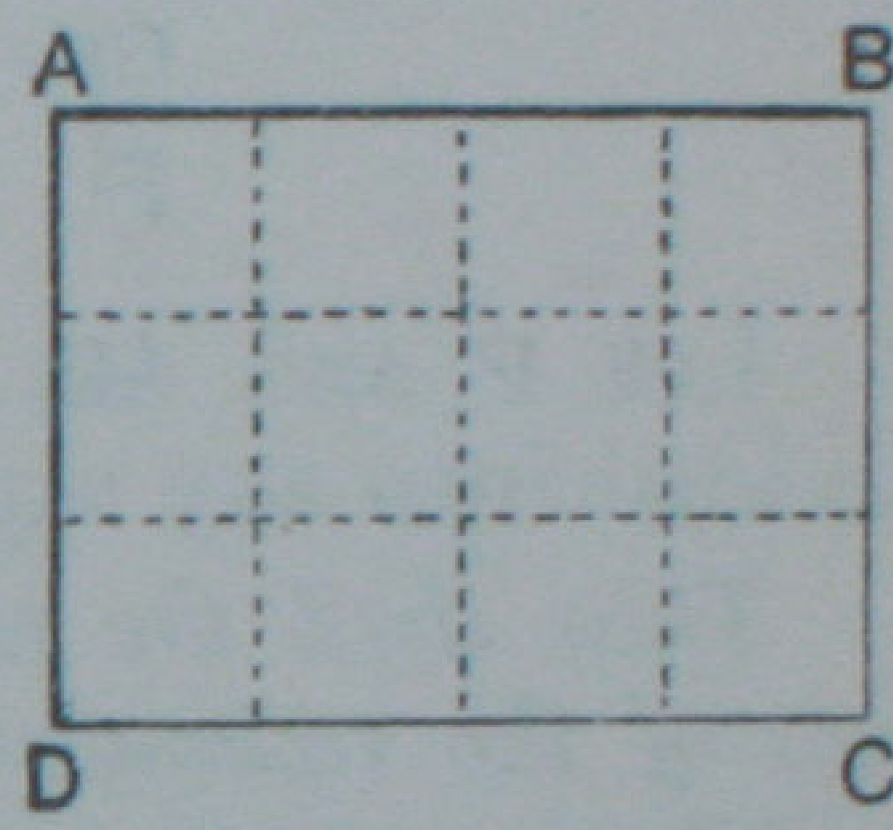


INTRODUCTORY.

Before entering upon Book II. the student is reminded of the following arithmetical rule:

RULE. *To find the area of a rectangle, multiply the number of units in the* **length** *by the number of units in the* **breadth**; *the product will be the number of* **square units** *in the* **area**.

For example, if the two sides AB, AD of the rectangle ABCD are respectively *four* and *three* inches long, and if through the points of division parallels are drawn as in the annexed figure, it is seen that the rectangle is divided into *three rows*, each containing *four* square inches, or into *four columns*, each containing *three* square inches.



Hence the whole rectangle contains 3×4, or 12, square inches.

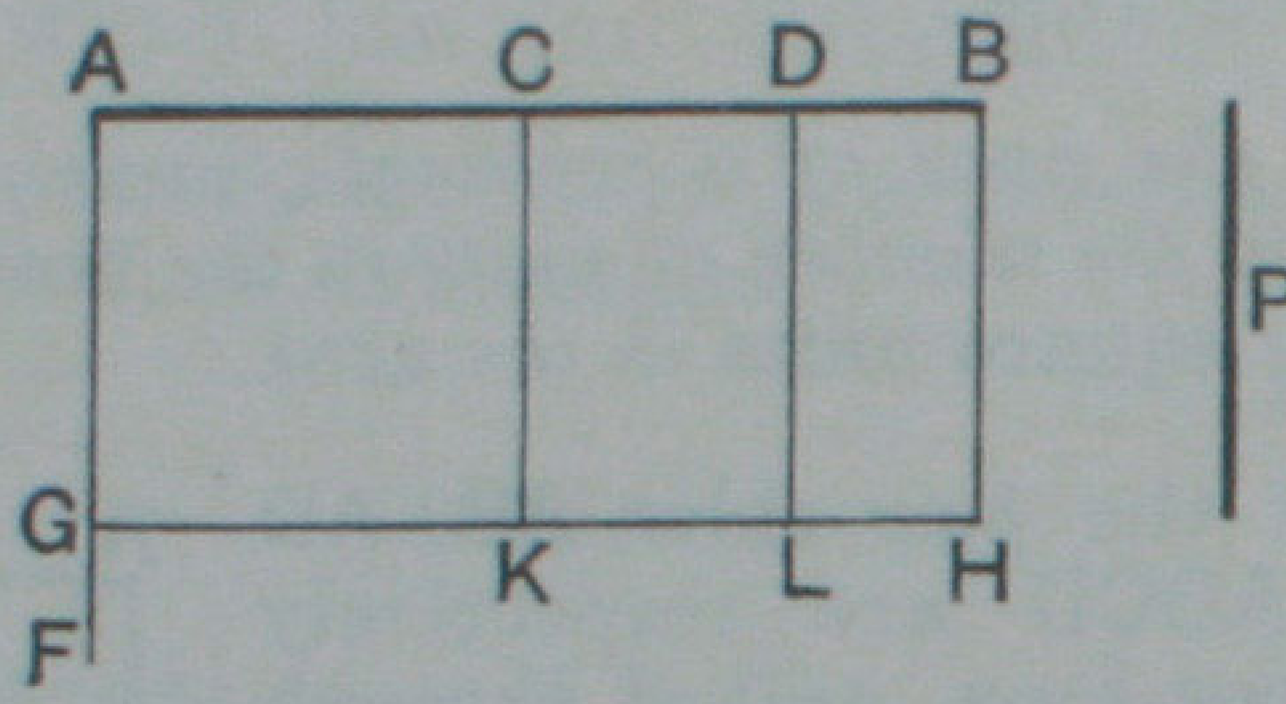
Similarly if AB and AD contain m and n units of length respectively, it follows that the rectangle ABCD will contain $m \times n$ units of area: further, if AB and AD are equal, each containing m units of length, the rectangle becomes a square, and contains m^2 units of area.

From this we conclude that *the rectangle contained by two straight lines* in Geometry corresponds to *the product of two numbers* in Arithmetic or Algebra; and that *the square described on a straight line* corresponds to *the square of a number*. Accordingly it will be found in the course of Book II. that several theorems relating to the areas of rectangles and squares are analogous to well-known algebraical formulæ.

In view of these principles the rectangle contained by two straight lines AB, BC is sometimes expressed in the form of a product, as AB . BC, and the square described on AB as AB^2. This notation, together with the signs + and −, will be employed in the additional matter appended to this book; *but it is not admitted into Euclid's text* because it is desirable in the first instance to emphasize the distinction between *geometrical magnitudes themselves* and the *numerical equivalents* by which they may be expressed arithmetically.

PROPOSITION 1. THEOREM.

If there are two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided straight line and the several parts of the divided line.



Let P and AB be two straight lines, and let AB be divided into any number of parts AC, CD, DB.

Then shall the rectangle contained by P, AB *be equal to the sum of the rectangles contained by* P, AC, *by* P, CD, *and by* P, DB.

Construction. From A draw AF perp. to AB; I. 11.
and make AG equal to P. I. 3.
Through G draw GH par[l] to AB; I. 31.
and through C, D, B draw CK, DL, BH par[l] to AG.

Proof. Now the fig. AH is made up of the figs. AK, CL, DH, and is therefore equal to their sum;
and of these,
the fig. AH is the rectangle contained by P, AB;
for it is contained by AG, AB; and AG = P:
and the fig. AK is the rectangle contained by P, AC;
for it is contained by AG, AC; and AG = P:
also the fig. CL is the rectangle contained by P, CD;
for it is contained by CK, CD;
and CK = the opp. side AG, and AG = P. I. 34.
Similarly the fig. DH is the rectangle contained by P, DB.

∴ the rectangle contained by P, AB is equal to the sum of the rectangles contained by P, AC, by P, CD, and by P, DB.

Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

In accordance with the principles explained on page 129, the result of this proposition may be written thus :

$$P . AB = P . AC + P . CD + P . DB.$$

Now if the line P contains p units of length, and if AC, CD, DB contain a, b, c units respectively,

then $AB = a + b + c$;

hence the statement

$$P . AB = P . AC + P . CD + P . DB$$

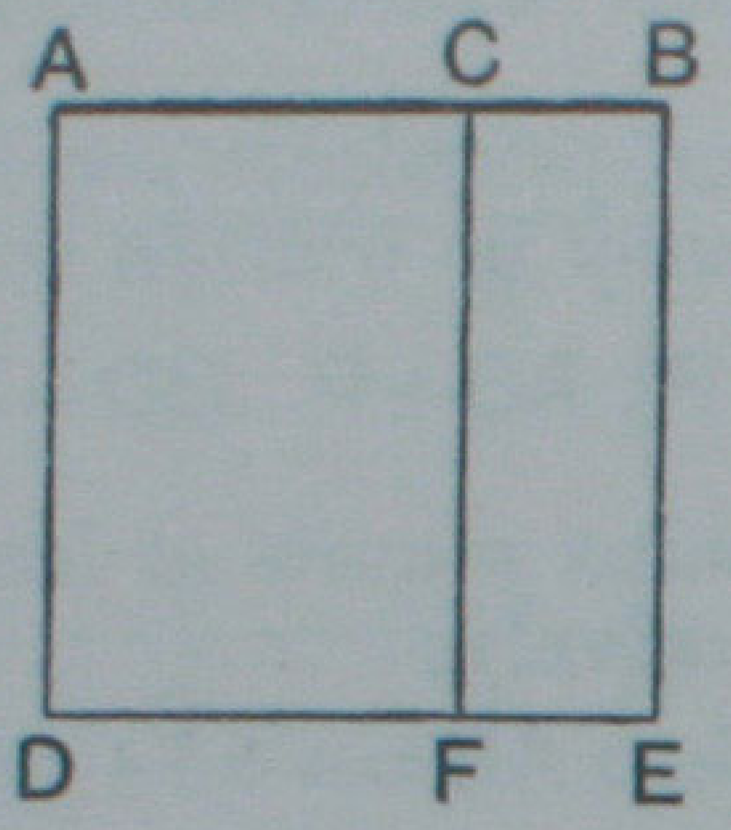
becomes $$p(a+b+c) = pa + pb + pc.$$

[NOTE. It must be understood that the rule given on page 129, for expressing the area of a rectangle as the product of the lengths of two adjacent sides, implies that those sides are **commensurable**, that is, that they can be expressed *exactly* in terms of some common unit.

This however is not always the case. Two straight lines may be so related that it is impossible to divide either of them into equal parts, *of which the other contains an exact number.* Such lines are said to be **incommensurable**. Hence if the adjacent sides of a rectangle are incommensurable, we cannot choose any linear unit in terms of which these sides may be *exactly* expressed ; and thus it will be impossible to subdivide the rectangle into squares of unit area, as illustrated in the figure of page 129. We do not here propose to enter further into the subject of incommensurable quantities: it is sufficient to point out that further knowledge of them will convince the student that the area of a rectangle may be expressed *to any required degree of accuracy* by the product of the lengths of two adjacent sides, whether those lengths are commensurable or not.]

Proposition 2. Theorem.

If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the rectangles contained by the whole line and each of the parts.



Let the straight line AB be divided at C into the two parts AC, CB.

Then shall the square on AB *be equal to the sum of the rectangles contained by* AB, AC, *and by* AB, BC.

Construction. On AB describe the square ADEB. I. 46.
Through C draw CF par[l] to AD. I. 31.

Proof. Now the fig. AE is made up of the figs. AF, CE: and of these,

the fig. AE is the sq. on AB: *Constr.*

and the fig. AF is the rectangle contained by AB, AC;
for it is contained by AD, AC; and AD = AB:

also the fig. CE is the rectangle contained by AB, BC;
for it is contained by BE, BC; and BE = AB.

∴ the sq. on AB = the sum of the rectangles contained by AB, AC, and by AB, BC. Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

The result of this proposition may be written

$$AB^2 = AB \, . \, AC + AB \, . \, BC.$$

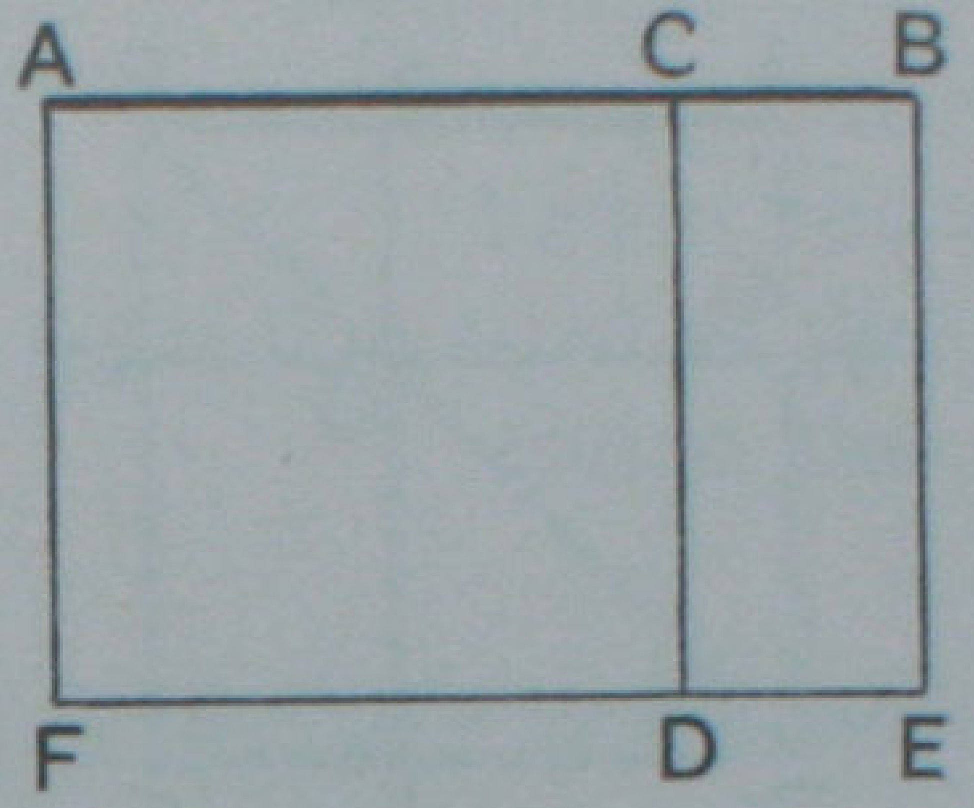
Let AC contain a units of length, and let CB contain b units,

then AB $= a + b$ units;

and we have $$(a+b)^2 = (a+b)a + (a+b)b.$$

PROPOSITION 3. THEOREM.

If a straight line is divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the square on that part together with the rectangle contained by the two parts.



Let the straight line AB be divided at C into the two parts AC, CB.

Then shall the rectangle contained by AB, AC *be equal to the square on* AC *together with the rectangle contained by* AC, CB.

Construction. On AC describe the square AFDC. I. 46.
Through B draw BE par[l] to AF, meeting FD produced in E. I. 31.

Proof. Now the fig. AE is made up of the figs. AD, CE; and of these,

the fig. AE is the rectangle contained by AB, AC;
for AF = AC;
and the fig. AD is the sq. on AC; *Constr.*
also the fig. CE is the rectangle contained by AC, CB;
for CD = AC.

∴ the rectangle contained by AB, AC is equal to the sq. on AC together with the rectangle contained by AC, CB.

Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

This result may be written AB . AC = AC^2 + AC . CB.

Let AC, CB contain a and b units of length respectively,

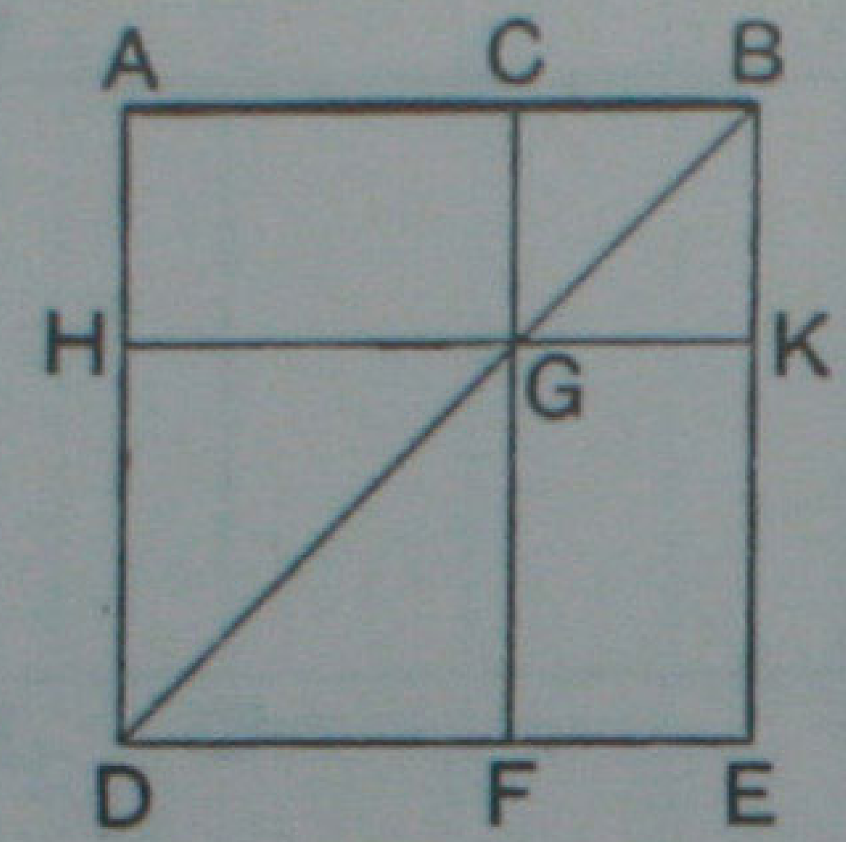
then AB = $a+b$ units;

and we have $$(a+b)a = a^2 + ab.$$

Note. It should be observed that Props. 2 and 3 are *special cases* of Prop. 1.

PROPOSITION 4. THEOREM.

If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the two parts.



Let the straight line AB be divided at C into the two parts AC, CB.

Then shall the sq. on AB *be equal to the sum of the sqq. on* AC, CB, *together with twice the rect.* AC, CB.

Construction. On AB describe the square ADEB. I. 46.
Join BD.
Through C draw CF par[l] to BE, meeting BD in G. I. 31.
Through G draw HGK par[l] to AB.

It is first required to shew that the fig. CK is the sq. on CB.

Proof. Because CF and AD are par[l], and BD meets them,
∴ the ext. angle CGB = the int. opp. angle ADB. I. 29.
And since AB = AD, being sides of a square;
∴ the angle ADB = the angle ABD; I. 5.
∴ the angle CGB = the angle CBG.
∴ CB = CG. I. 6.
And the opp. sides of the par[m] CK are equal; I. 34.
∴ the fig. CK is equilateral;
also the angle CBK is a right angle; *Def.* 30.
∴ CK is a square, and it is described on CB. I. 46, *Cor.*

Similarly, the fig. HF is the sq. on HG, that is, the sq. on AC;
for HG = the opp. side AC. I. 34.

Again, the complement AG = the complement GE ; I. 43.
and the fig. AG = the rect. AC, CB ; for CG = CB.
∴ the two figs. AG, GE = twice the rect. AC, CB.

* Now the sq. on AB = the fig. AE
= the figs. HF, CK, AG, GE
= the sqq. on AC, CB together with twice the rect. AC, CB.
∴ the sq. on AB = the sum of the sqq. on AC, CB with twice the rect. AC, CB. Q.E.D.

Corollary 1. *Parallelograms about the diagonals of a square are themselves squares.*

Corollary 2. *If a straight line is bisected, the square on the whole line is four times the square on half the line.*

* For the purpose of oral work, this step of the proof may conveniently be arranged as follows:

Now the sq. on AB is equal to the fig. AE,
that is, to the figs. HF, CK, AG, GE ;
that is, to the sqq. on AC, CB together with twice the rect. AC, CB.

CORRESPONDING ALGEBRAICAL FORMULA.

The result of this important Proposition may be written thus:

$$AB^2 = AC^2 + CB^2 + 2AC \cdot CB.$$

Let $AC = a$, and $CB = b$;

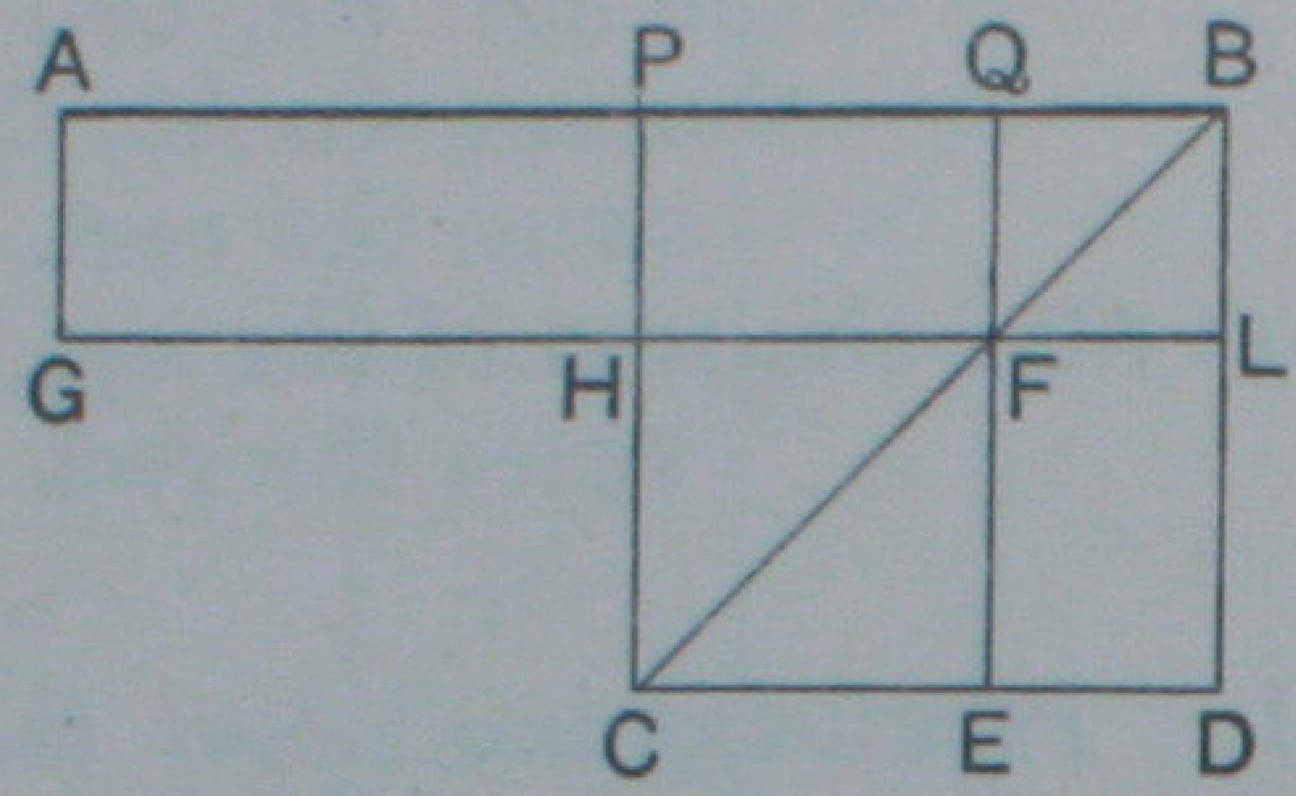
then $AB = a + b$;

hence the statement $AB^2 = AC^2 + CB^2 + 2AC \cdot CB$

becomes $(a+b)^2 = a^2 + b^2 + 2ab.$

Proposition 5. Theorem.

If a straight line is divided equally and also unequally, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.



Let the straight line AB be divided equally at P, and unequally at Q.

Then the rect. AQ, QB *together with the sq. on* PQ *shall be equal to the sq. on* PB.

Construction. On PB describe the square PCDB. I. 46.

Join BC.

Through Q draw QE par[l] to BD, cutting BC in F. I. 31.

Through F draw LFHG par[l] to AB.

Through A draw AG par[l] to BD.

Proof.

Now the complement PF = the complement FD: I. 43.

to each add the fig. QL;

then the fig. PL = the fig. QD.

But the fig. PL = the fig. AH, for they are par[ms] on equal bases and between the same par[ls]; I. 36.

∴ the fig. AH = the fig. QD.

To each add the fig. PF;

then the fig. AF = the gnomon PLE.

Now the fig. AF is the rect. AQ, QB; for QF = QB;

∴ the rect. AQ, QB = the gnomon PLE.

To each add the sq. on PQ, that is, the fig. HE; II. 4.

then the rect. AQ, QB with the sq. on PQ

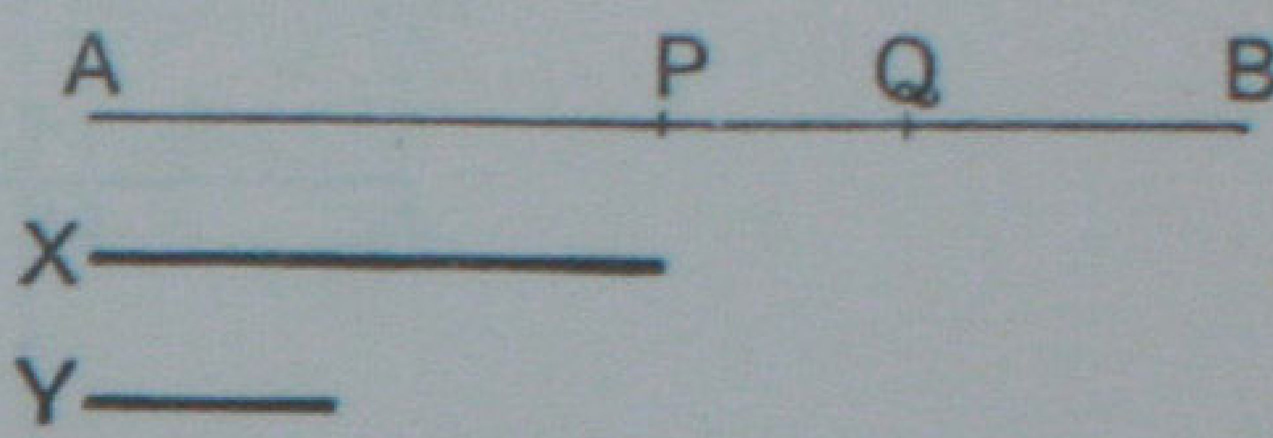
= the gnomon PLE with the fig. HE

= the whole fig. PD,

which is the sq. on PB.

That is, the rect. AQ, QB together with the square on PQ is equal to the sq. on PB. Q.E.D.

COROLLARY. From this Proposition it follows that *the difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference.*



For let X and Y be the given st. lines, of which X is the greater.

Draw AP equal to X, and produce it to B, making PB equal to AP, that is to X.

From PB cut off PQ equal to Y.

Then AQ is equal to the sum of X and Y,

and QB is equal to the difference of X and Y.

Now because AB is divided equally at P and unequally at Q,

$\therefore$ the rect. AQ, QB with sq. on PQ = the sq. on PB; II. 5.

that is, the difference of the sqq. on PB, PQ = the rect. AQ, QB.

or, the difference of the sqq. on X and Y = the rectangle contained by the sum and the difference of X and Y.

CORRESPONDING ALGEBRAICAL FORMULA.

This result may be written

$$AQ \,.\, QB + PQ^2 = PB^2.$$

Let $AB = 2a$; and let $PQ = b$;

then AP and PB each $= a$.

Also $AQ = a + b$; and $QB = a - b$.

Hence the statement $AQ \,.\, QB + PQ^2 = PB^2$

becomes $$(a+b)(a-b) + b^2 = a^2,$$

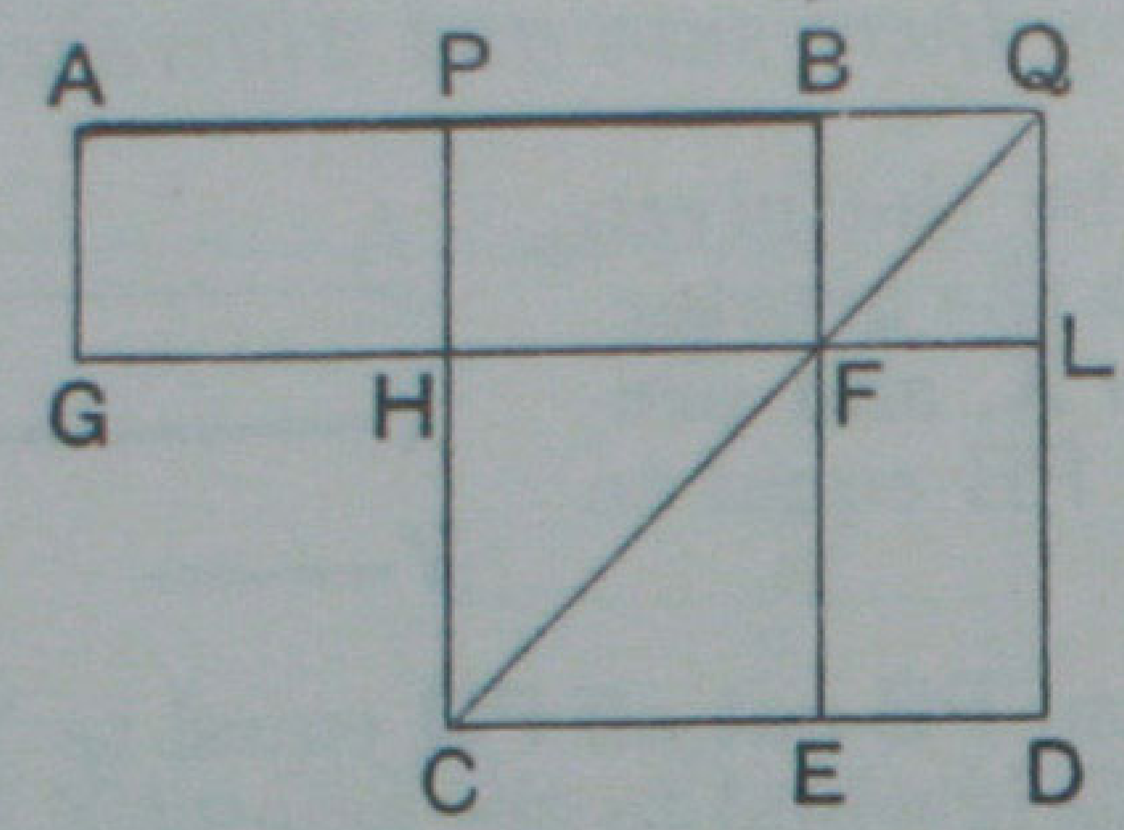
or $$(a+b)(a-b) = a^2 - b^2.$$

EXERCISE.

In the above figure shew that AP *is half the sum of* AQ *and* QB; *and that* PQ *is half their difference.*

Proposition 6. Theorem.

If a straight line is bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line made up of the half and the part produced.



Let the straight line AB be bisected at P, and produced to Q.

Then the rect. AQ, QB *together with the sq. on* PB *shall be equal to the sq. on* PQ.

Construction. On PQ describe the square PCDQ. I. 46.

Join QC.

Through B draw BE par^l to QD, meeting QC in F. I. 31.

Through F draw LFHG par^l to AQ.

Through A draw AG par^l to QD.

Proof. Now the complement PF = the complement FD. I. 43.

But the fig. PF = the fig. AH; for they are par^{ms} on equal bases and between the same par^{ls}. I. 36.

∴ the fig. AH = the fig. FD.

To each add the fig. PL;

then the fig. AL = the gnomon PLE.

Now the fig. AL is the rect. AQ, QB; for QL = QB;

∴ the rect. AQ, QB = the gnomon PLE.

To each add the sq. on PB, that is, the fig. HE;

then the rect. AQ, QB with the sq. on PB

= the gnomon PLE with the fig. HE

= the whole fig. PD,

which is the square on PQ.

That is, the rect. AQ, QB together with the sq. on PB is equal to the sq. on PQ. Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

This result may be written

$$AQ \,.\, QB + PB^2 = PQ^2.$$

Let $AB = 2a$; and let $PQ = b$;

then AP and PB each $= a$.

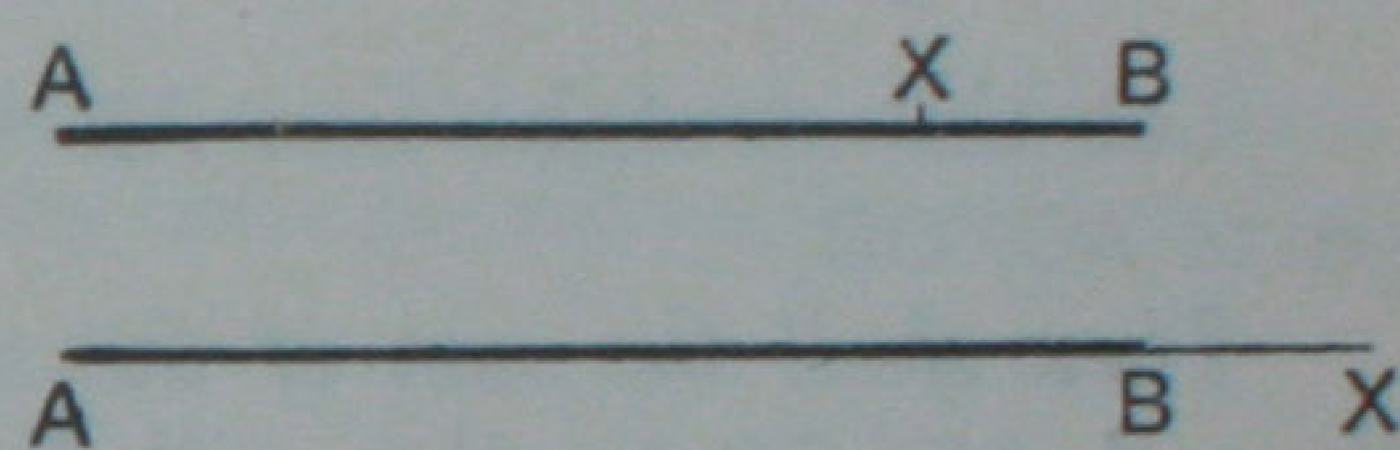
Also $AQ = a + b$; and $QB = b - a$.

Hence the statement $AQ \,.\, QB + PB^2 = PQ^2$

becomes $$(a+b)(b-a) + a^2 = b^2,$$

or $$(b+a)(b-a) = b^2 - a^2.$$

Definition. If a point X is taken in a straight line AB, or in AB produced, the distances of the point of section from the extremities of AB are said to be the **segments** into which AB is divided at X.



In the former case AB is divided **internally**, in the latter case **externally**.

Thus in *each* of the annexed figures, the segments into which AB is divided at X are the lines AX and XB.

This definition enables us to include Props. 5 and 6 in a single Enunciation.

If a straight line is bisected, and also divided (internally or externally) into two unequal segments, the rectangle contained by the unequal segments is equal to the difference of the squares on half the line, and on the line between the points of section.

EXERCISE.

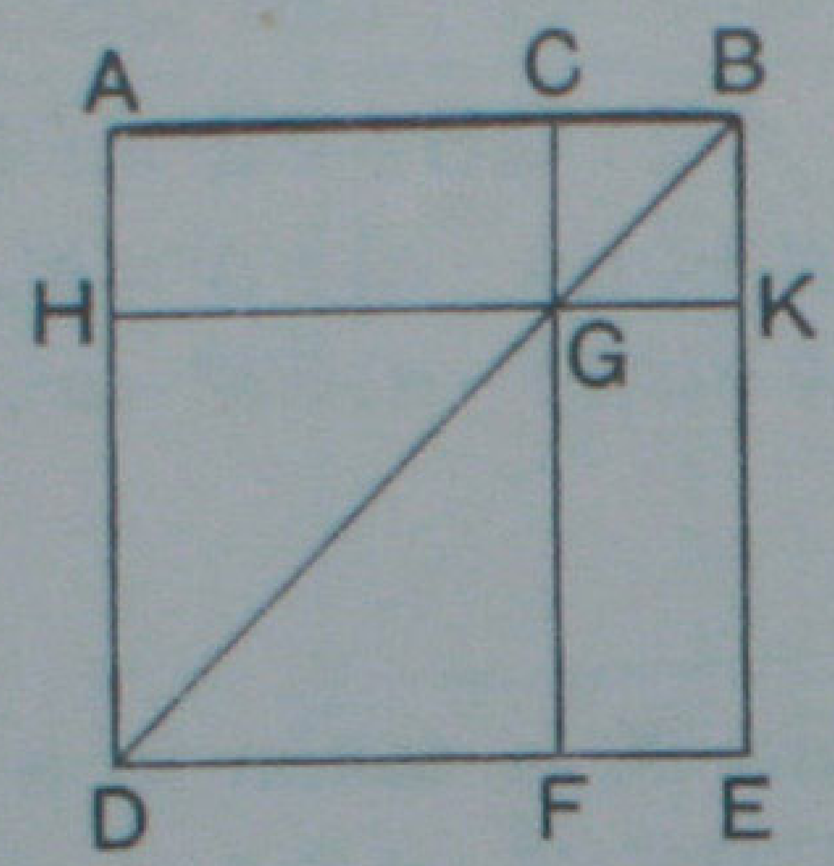
Shew that the Enunciations of Props. 5 and 6 may take the following form:

The rectangle contained by two straight lines is equal to the difference of the squares on half their sum and on half their difference.

[See Ex., p. 137.]

PROPOSITION 7. THEOREM.

If a straight line is divided into any two parts, the sum of the squares on the whole line and on one of the parts is equal to twice the rectangle contained by the whole and that part, together with the square on the other part.



Let the straight line AB be divided at C into the two parts AC, CB.

Then shall the sum of the sqq. on AB, BC *be equal to twice the rect.* AB, BC *together with the sq. on* AC.

Construction. On AB describe the square ADEB. I. 46.
Join BD.
Through C draw CF par[l] to BE, meeting BD in G. I. 31.
Through G draw HGK par[l] to AB.

Proof. Now the complement AG = the complement GE; I. 43.
to each add the fig. CK:
then the fig. AK = the fig. CE.
But the fig. AK is the rect. AB, BC; for BK = BC;
∴ the two figs AK, CE = twice the rect. AB, BC.
But the two figs. AK, CE make up the gnomon AKF and the fig. CK:
∴ the gnomon AKF with the fig. CK = twice the rect. AB, BC.
To each add the fig. HF, which is the sq. on AC:
then the gnomon AKF with the figs. CK, HF
= twice the rect. AB, BC with the sq. on AC.
But the gnomon AKF with the figs. CK, HF make up the figs. AE, CK, that is to say, the sqq. on AB, BC;
∴ the sqq. on AB, BC = twice the rect. AB, BC with the sq. on AC.

Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

The result of this proposition may be written

$$AB^2 + BC^2 = 2AB \cdot BC + AC^2.$$

Let $AB = a$, and $BC = b$; then $AC = a - b$.

Hence the statement

$$AB^2 + BC^2 = 2AB \cdot BC + AC^2$$

becomes $$a^2 + b^2 = 2ab + (a-b)^2,$$

or $$(a-b)^2 = a^2 - 2ab + b^2.$$

Comparing this result with that obtained from Prop. 4, we see that

(i) *The square on the* sum *of two straight lines is* greater *than the sum of the squares on those lines by twice the rectangle contained by them.* [Prop. 4.]

(ii) *The square on the* difference *of two straight lines is* less *than the sum of the squares on those lines by twice the rectangle contained by them.* [Prop. 7.]

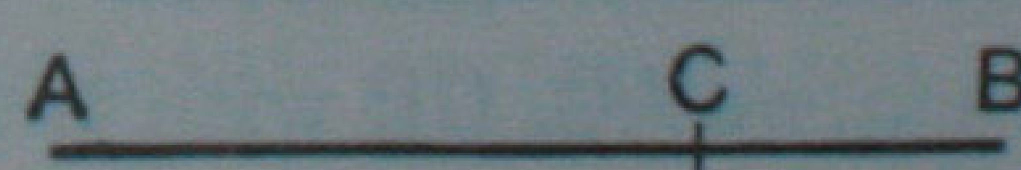
ALTERNATIVE PROOFS OF PROPOSITIONS 4, 5, 6, 7.

The following alternative proofs are recommended for purposes of revision, as affording useful exercise on the enunciations of preceding propositions, and illustrating the way in which many examples on Book II. may be solved. The beginner however should not adopt these proofs until he has thoroughly mastered those given in the text, where the rectangles and squares are actually represented in the diagrams.

PROPOSITION 4.

Let the straight line AB be divided at C into two parts AC, CB.

Then shall the sq. on AB *be equal to the sum of the sqq. on* AC, CB *with twice the rect.* AC, CB.



Now the sq. on AB = the rect. AB, AC with the rect. AB, CB. II. 2.

But the rect. AB, AC = the sq. on AC with the rect. AC, CB; II. 3.

and the rect. AB, CB = the sq. on CB with the rect. AC, CB. II. 3.

Hence the sq. on AB = the sum of the sqq. on AC, CB with twice the rect. AC, CB.

Proposition 5.

Let the straight line AB be divided equally at P, and unequally at Q.

Then shall the rect. AQ, QB *with the sq. on* PQ *be equal to the sq. on* PB.

Now the rect. AQ, QB = the rect. AP, QB with the rect. PQ, QB, II. 1.
= the rect. PB, QB with the rect. PQ, QB.
But the rect. PB, QB = the sq. on QB with the rect. PQ, QB; II. 3.
∴ the rect. AQ, QB = the sq. on QB with *twice* the rect. PQ, QB.
To each of these equals add the sq. on PQ.
Then the rect. AQ, QB with the sq. on PQ
= the sqq. on PQ, QB with twice the rect. PQ, QB
= the sq. on PB. II. 4.

Proposition 6.

Let the straight line AB be bisected at P, and produced to Q.

Then shall the rect. AQ, QB *with the sq. on* PB *be equal to the sq. on* PQ.

Now the rect. AQ, QB = the rect. AP, BQ with the rect. PQ, BQ II. 1.
= the rect. PB, BQ with the rect. PQ, BQ.
But the rect. PQ, BQ = the sq. on BQ with the rect. PB, BQ. II. 3.
∴ the rect. AQ, QB = the sq. on BQ with *twice* the rect. PB, BQ.
To each of these equals add the sq. on PB.
Then the rect. AQ, QB with the sq. on PB
= the sqq. on PB, BQ with twice the rect. PB, BQ
= the sq. on PQ. II. 4.

Proposition 7.

Let the straight line AB be divided at any point C.

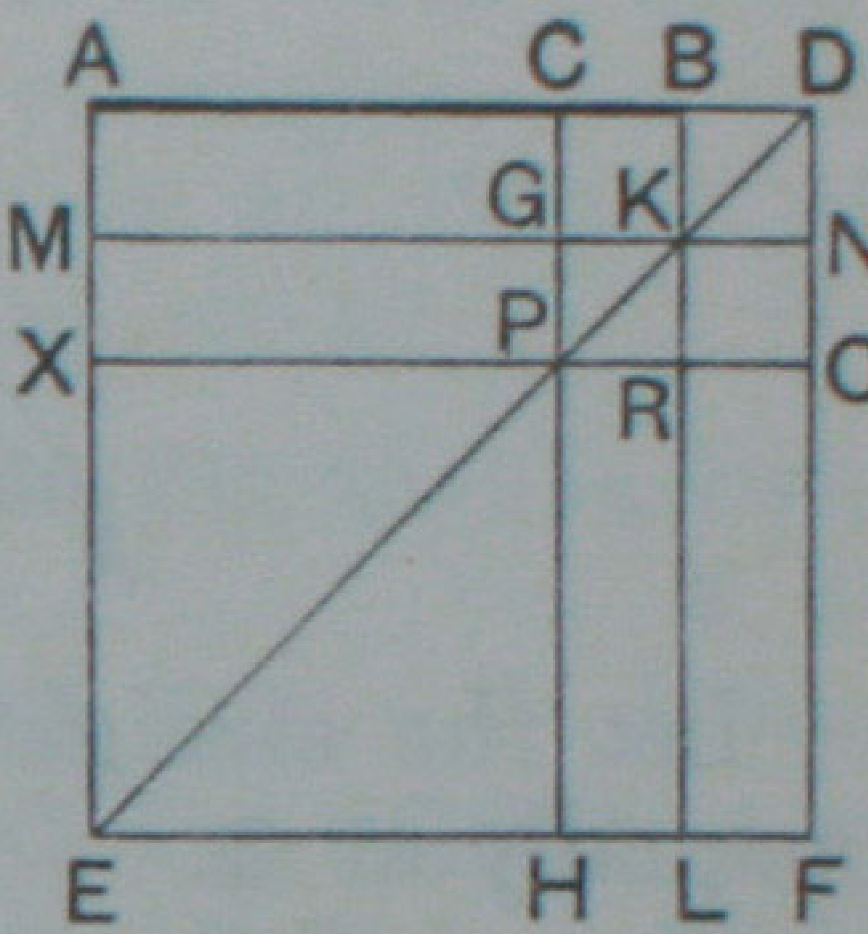
Then shall the sum of the sqq. on AB, BC *be equal to twice the rect.* AB, BC *with the sq. on* AC.

Now the sq. on AB = the sqq. on AC, CB with twice the rect. AC, CB. II. 4.
To each of these equals add the sq. on BC.
Then the sqq. on AB, BC = the sq. on AC with *twice* the sq. on BC and *twice* the rect. AC, CB.
But *twice* the sq. on BC with *twice* the rect. AC, CB
= *twice* the rect. AB, BC. II. 3.
∴ the sqq. on AB, BC = the sq. on AC with twice the rect. AB, BC.

Obs. The following proposition being little used, we merely give the figure and the leading points of Euclid's proof.

Proposition 8. Theorem.

If a straight line is divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and the first named part.



Let AB be divided at C.

Produce AB to D, making BD equal to BC.

Then shall four times the rect. AB, BC *with the sq. on* AC*=the sq. on* AD.

On AD describe the square AEFD; and complete the construction as indicated in the figure.

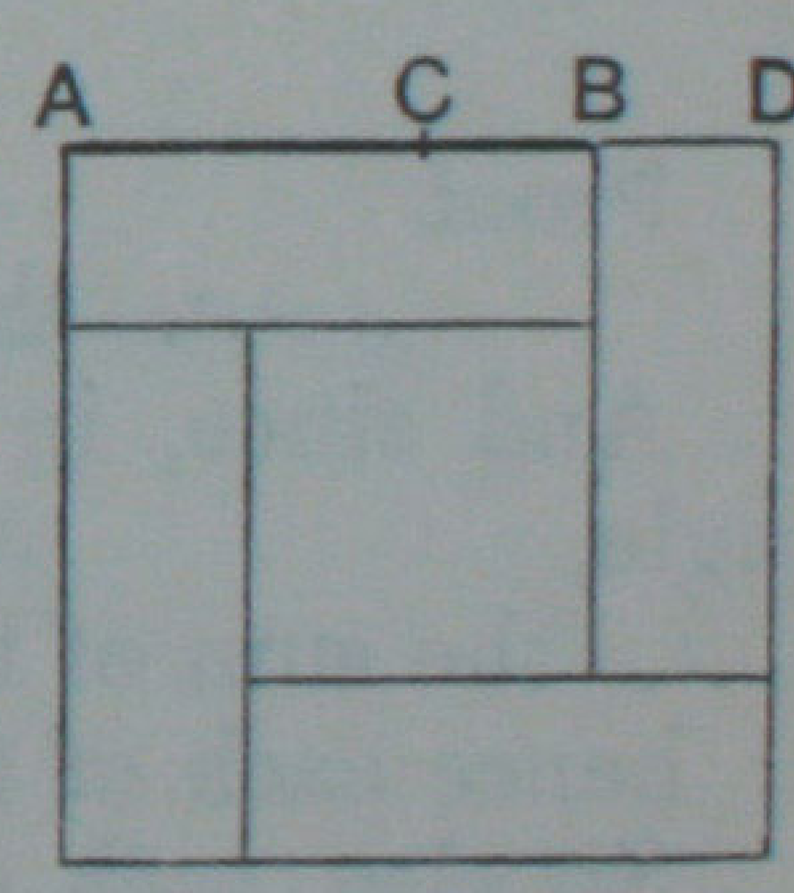
Euclid then proves (i) that the figs. CK, BN, GR, KO are all equal:

(ii) that the figs. AG, MP, PL, RF are all equal.

Hence the eight figures named above are together four times the sum of the figs. AG, CK; that is, four times the fig. AK; that is, four times the rect. AB, BC.

But the whole fig. AF, namely the sq. on AD, is made up of these eight figures, together with the fig. XH, which is the sq. on AC:

hence the sq. on AD=four times the rect. AB, BC, together with the sq. on AC. Q.E.D.



The accompanying figure will suggest a less cumbrous proof, which we leave as an Exercise to the student.

CORRESPONDING ALGEBRAICAL FORMULA.

The result of this proposition may be written

$$4AB \, . \, BC + AC^2 = AD^2.$$

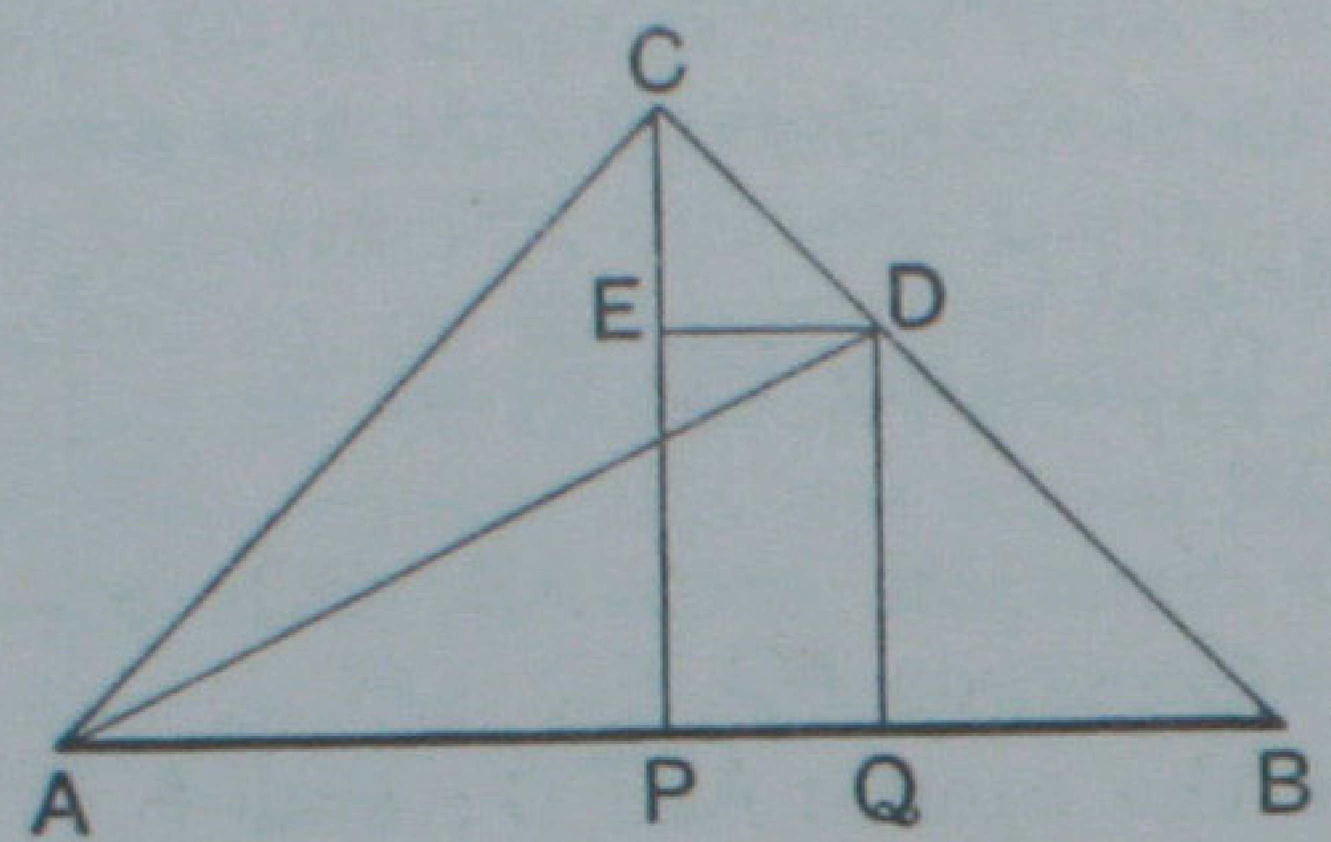
Let $AB = a$, and $BC = b$; then $AC = a - b$, and $AD = a + b$.

Hence we have $$4ab + (a-b)^2 = (a+b)^2;$$

or $$(a+b)^2 - (a-b)^2 = 4ab.$$

PROPOSITION 9. THEOREM. [EUCLID'S PROOF.]

If a straight line is divided equally and also unequally, the sum of the squares on the two unequal parts is twice the sum of the squares on half the line and on the line between the points of section.



Let the straight line AB be divided equally at P, and unequally at Q.

Then shall the sum of the sqq. on AQ, QB *be twice the sum of the sqq. on* AP, PQ.

Construction. At P draw PC at rt. angles to AB; I. 11.
and make PC equal to AP or PB. I. 3.
Join AC, BC.
Through Q draw QD par[l] to PC; I. 31.
and through D draw DE par[l] to AB.
Join AD.

Proof. Then since PA = PC, *Constr.*
∴ the angle PAC = the angle PCA. I. 5.

And since, in the triangle APC, the angle APC is a rt. angle, *Constr.*
∴ the sum of the angles PAC, PCA is a rt. angle: I. 32.
hence each of the angles PAC, PCA is half a rt. angle.

So also, each of the angles PBC, PCB is half a rt. angle.
∴ the whole angle ACB is a rt. angle.

Again, the ext. angle CED = the int. opp. angle CPB; I. 29.
∴ the angle CED is a rt. angle:
and the angle ECD is half a rt. angle. *Proved.*
∴ the remaining angle EDC is half a rt. angle; I. 32.
∴ the angle ECD = the angle EDC;
∴ EC = ED. I. 6

Again, the ext. angle DQB = the int. opp. angle CPB ; I. 29.
∴ the angle DQB is a rt. angle.
And the angle QBD is half a rt. angle ; *Proved.*
∴ the remaining angle QDB is half a rt. angle ; I. 32.
∴ the angle QBD = the angle QDB ;
∴ QD = QB.

Now the sq. on AP = the sq. on PC ; for AP = PC. *Constr.*
And since the angle APC is a rt. angle,
∴ the sq. on AC = the sum of the sqq. on AP, PC ; I. 47.
∴ the sq. on AC is twice the sq. on AP.

Similarly, the sq. on CD is twice the sq. on ED, that is, twice the sq. on the opp. side PQ. I. 34.
Now the sqq. on AQ, QB = the sqq. on AQ, QD *Proved.*
= the sq. on AD, for AQD is a rt. angle ; I. 47.
= the sum of the sqq. on AC, CD, for ACD is a rt. angle ; I. 47.
= twice the sq. on AP with twice the sq. on PQ. *Proved.*

That is,
the sum of the sqq. on AQ, QB = twice the sum of the sqq. on AP, PQ. Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

The result of this proposition may be written

$$AQ^2 + QB^2 = 2(AP^2 + PQ^2).$$

Let $AB = 2a$; and $PQ = b$;

then AP and PB each $= a$.

Also $AQ = a + b$; and $QB = a - b$.

Hence the statement

$$AQ^2 + QB^2 = 2(AP^2 + QP^2)$$

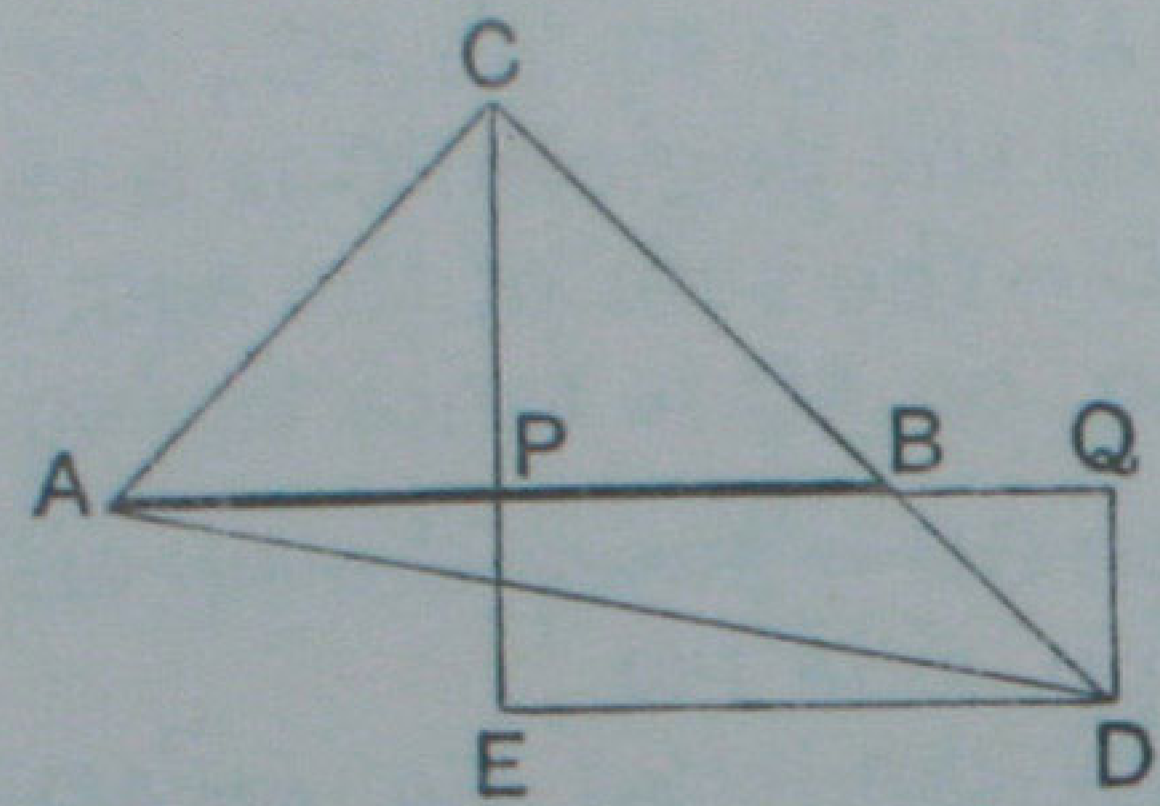
becomes

$$(a+b)^2 + (a-b)^2 = 2(a^2 + b^2).$$

[NOTE. For alternative proofs of this proposition, see page 148.]

PROPOSITION 10. THEOREM. [EUCLID'S PROOF.]

If a straight line is bisected and produced to any point, the sum of the squares on the whole line thus produced, and on the part produced, is twice the sum of the squares on half the line bisected and on the line made up of the half and the part produced.



Let the st. line AB be bisected at P, and produced to Q.

Then shall the sum of the sqq. on AQ, QB *be twice the sum of the sqq. on* AP, PQ.

Construction. At P draw PC at right angles to AB; I. 11.
and make PC equal to PA or PB. I. 3.
Join AC, BC.
Through Q draw QD par[l] to PC, to meet CB produced in D; I. 31.
and through D draw DE par[l] to AB, to meet CP produced in E.
Join AD.

Proof. Then since PA = PC, *Constr.*
∴ the angle PAC = the angle PCA. I. 5.
And since, in the triangle APC, the angle APC is a rt. angle,
∴ the sum of the angles PAC, PCA is a rt. angle. I. 32.
Hence each of the angles PAC, PCA is half a rt. angle.
So also, each of the angles PBC, PCB is half a rt. angle.
∴ the whole angle ACB is a rt. angle.

Again, the ext. angle CPB = the int. opp. angle CED: I. 29.
∴ the angle CED is a rt. angle:
and the angle ECD is half a rt. angle; *Proved.*
∴ the remaining angle EDC is half a rt. angle. I. 32.
∴ the angle ECD = the angle EDC;
∴ EC = ED. I. 6.

Again, the angle DQB = the alt. angle CPB ; I. 29.
∴ the angle DQB is a rt. angle.
Also the angle QBD = the vert. opp. angle CBP : I. 15.
that is, the angle QBD is half a rt. angle.
∴ the remaining angle QDB is half a rt. angle : I. 32.
∴ the angle QBD = the angle QDB ;
∴ QB = QD. I. 6.

Now the sq. on AP = the sq. on PC ; for AP = PC. *Constr.*
And since the angle APC is a rt. angle,
∴ the sq. on AC = the sum of the sqq. on AP, PC ; I. 47.
∴ the sq. on AC is twice the sq. on AP.

Similarly, the sq. on CD is twice the sq. on ED, that is, twice the sq. on the opp. side PQ. I. 34.

Now the sqq. on AQ, QB = the sqq. on AQ, QD *Proved.*
= the sq. on AD, for AQD is a rt. angle ; I. 47.
= the sum of the sqq. on AC, CD, for ACD is a rt. angle ; I. 47.
= twice the sq. on AP with twice the sq. on PQ. *Proved.*

That is,
the sum of the sqq. on AQ, QB is twice the sum of the sqq. on AP, PQ. Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

The result of this proposition may be written

$$AQ^2 + QB^2 = 2(AP^2 + PQ^2).$$

Let $AB = 2a$; and $PQ = b$;

then AP and PB each $= a$.

Also $AQ = a + b$; and $QB = b - a$.

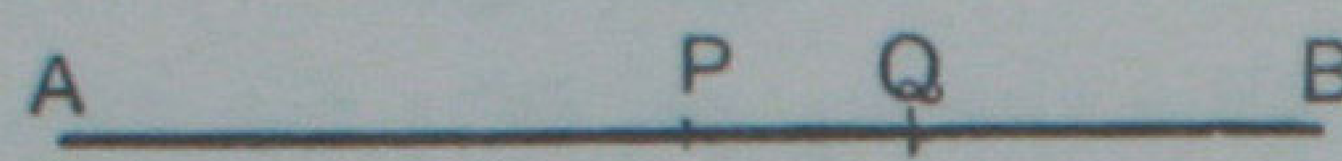
Hence we have

$$(a+b)^2 + (b-a)^2 = 2(a^2 + b^2).$$

[NOTE. For alternative proofs of this proposition, see page 149.]

Proposition 9. [Alternative Proof.]

If a straight line is divided equally and also unequally, the sum of the squares on the two unequal parts is twice the sum of the squares on half the line and on the line between the points of section.



Let the straight line AB be divided equally at P and unequally at Q.

Then shall the sum of the sqq. on AQ, QB *be twice the sum of the sqq. on* AP, PQ.

Proof.

The sq. on AQ = the sum of the sqq. on AP, PQ with twice the rect. AP, PQ   II. 4.
= the sum of the sqq. on AP, PQ with twice the rect. PB, PQ; for PB = AP.

To each of these equals add the sq. on QB.

Then the sqq. on AQ, QB = the sum of the sqq. on AP, PQ with twice the rect. PB, PQ and the sq. on QB.

But twice the rect. PB, PQ and the sq. on QB
= the sum of the sqq. on PB, PQ.   II. 7.

∴ the sqq. on AQ, QB = the sum of the sqq. on AP, PQ with the sum of the sqq. on PB, PQ
= *twice* the sum of the sqq. on AP, PQ.

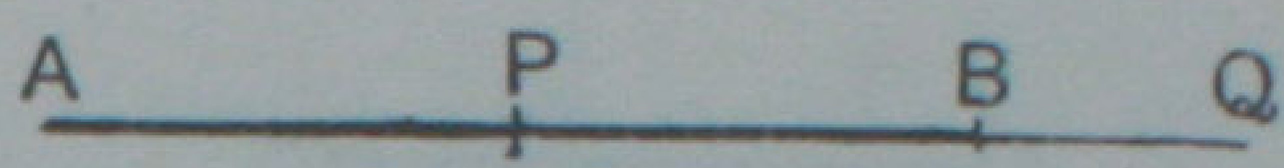
Q.E.D.

Note. The following concise proof, obtained from II. 4 and II. 5, is useful as an exercise, but it is hardly admissible as a formal demonstration owing to its algebraical use of the negative sign.

We have $AQ^2 + QB^2 = AB^2 - 2AQ \cdot QB$ II. 4.
$= 4PB^2 - 2AQ \cdot QB$ II. 4, *Cor.* 2.
$= 4PB^2 - 2(PB^2 - PQ^2)$ II. 5.
$= 2PB^2 + 2PQ^2$.

PROPOSITION 10. [ALTERNATIVE PROOF.]

If a straight line is bisected and produced to any point, the sum of the squares on the whole line thus produced and on the part produced, is twice the sum of the squares on half the line bisected and on the line made up of the half and the part produced.



Let the straight line AB be bisected at P, and produced to Q.

Then shall the sum of the sqq. on AQ, QB *be twice the sum of the sqq. on* AP, PQ.

Proof.

The sq. on AQ = the sum of the sqq. on AP, PQ with twice the rect. AP, PQ   II. 4.
= the sum of the sqq. on AP, PQ with twice the rect. PB, PQ ; for PB = AP.

To each of these equals add the sq. on QB.

Then the sqq. on AQ, QB = the sum of the sqq. on AP, PQ with twice the rect. PB, PQ and the sq. on QB.

But twice the rect. PB, PQ and the sq. on QB
= the sum of the sqq. on PB, PQ.   II. 7.

∴ the sqq. on AQ, QB = the sum of the sqq. on AP, PQ with the sum of the sqq. on PB, PQ
= *twice* the sum of the sqq. on AP, PQ.

Q.E.D.

NOTE. Another proof of this proposition, based on II. 7 and II. 6, is indicated by the following steps:

We have $AQ^2 + QB^2 = 2AQ \, . \, QB + AB^2$ II. 7.
$= 2AQ \, . \, QB + 4PB^2$ II. 4, *Cor.* 2.
$= 2(PQ^2 - PB^2) + 4PB^2$ II. 6.
$= 2PB^2 + 2PQ^2$.