Mathematics

F. A. N. Pessôa

F. A. N. Pessôa,

Durban High School,

February 1904.

Form VI Mathematics.

F. A. N. Pessôa

A TEXT-BOOK OF EUCLID'S ELEMENTS

Para [illegible]

A TEXT-BOOK

OF

EUCLID'S ELEMENTS

FOR THE USE OF SCHOOLS

BOOKS I.–VI. AND XI.

BY

H. S. HALL, M.A.

FORMERLY SCHOLAR OF CHRIST'S COLLEGE, CAMBRIDGE;

AND

F. H. STEVENS, M.A.

FORMERLY SCHOLAR OF QUEEN'S COLLEGE, OXFORD.

NEW EDITION

London

MACMILLAN AND CO., LIMITED

NEW YORK: THE MACMILLAN COMPANY

1902

First Edition, 1888. Second Edition (Book XI. added), 1889.
Reprinted 1890, 1891, 1892, 1893, 1894, 1895 (twice), 1896, 1897, 1898, 1899.
New Edition, 1900. Reprinted 1901, 1902.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS
BY ROBERT MACLEHOSE AND CO.

1144

EXTRACT FROM THE PREFACE TO THE FIRST EDITION.

This volume contains the first Six Books and part of the Eleventh Book of Euclid's Elements, together with additional Theorems and Examples, giving the most important elementary developments of Euclidean Geometry.

The text has been carefully revised, and special attention given to those points which experience has shewn to present difficulties to beginners.

In the course of this revision the enunciations have been altered as little as possible: and very few departures have been made from Euclid's proofs; in each case changes have been adopted only where the old text has been generally found a cause of difficulty; and such changes are for the most part in favour of well-recognised alternatives.

In Book I., for example, the ambiguity has been removed from the Enunciations of Propositions 18 and 19, and the fact that Propositions 8 and 26 establish the complete equality of the two triangles considered has been strongly urged: thus the redundant step has been removed from Proposition 34.

In Book II. Simson's arrangement of Proposition 13 has been abandoned for a well-known equivalent.

In Book III. Propositions 35 and 36 have been treated generally, and it has not been thought necessary to do more than call attention in a note to the special cases.

These are the chief deviations from the ordinary text as regards method and arrangement of proof; they are points familiar as difficulties to most teachers, and to name them indicates sufficiently, without further enumeration, the general principles which have guided our revision.

A few alternative proofs of difficult propositions are given for the convenience of those teachers who care to use them.

One purpose of the book is *gradually* to familiarise the student with the use of legitimate symbols and abbreviations; for a geometrical argument may thus be thrown into a form which is not only more readily seized by an advanced reader, but is useful as a guide to the way in which Euclid's propositions may be handled in written work. On the other hand, we think it very desirable to defer the introduction of symbols until the beginner has learnt that they can only be properly used in Pure Geometry as abbreviations for verbal argument: and we hope thus to prevent the slovenly and inaccurate habits which are very apt to arise from their employment before this principle is fully recognised.

Accordingly in Book I. we have used no contractions or symbols of any kind, though we have introduced verbal alterations into the text wherever it appeared that conciseness or clearness would be gained.

In Book II. abbreviated forms of constantly recurring words are used, and the phrases *therefore* and *is equal to* are replaced by the usual symbols.

In the Third and following Books, and in additional matter throughout the whole, we have employed all such signs and abbreviations as we believe to add to the clearness of the reasoning, care being taken that the symbols chosen are compatible with a rigorous geometrical method, and are recognised by the majority of teachers.

If this arrangement should be thought fanciful or wanting in uniformity, we may plead that it is the outcome of long experience in the use of various text-books. For some years, for example, we were accustomed to teach from a symbolical text, but in consequence of the frequent misconceptions and inaccuracies which too great brevity was found to generate among beginners, we were compelled to return to one of the older and unabbreviated editions. The gain to our younger boys was immediate and unmistakeable; but the change has not been unattended with disadvantage to more advanced students, who on reaching the Third or Fourth Book may not only be safely trusted with a carefully chosen system of abbreviations, but are certainly retarded by the monotonous and lengthy formalities of the old text.

It must be understood that our use of symbols, and the removal of unnecessary verbiage and repetition, by no means implies a desire to secure brevity at all hazards. On the contrary, nothing appears to us more mischievous than an abridgement which is attained by omitting steps, or condensing two or more steps into one. Such uses spring from the pressure of examinations; but an examination is not, or ought not to be, a mere race; and while we wish to indicate generally in the later books how a geometrical argument may be abbreviated for the purposes of written work, we have not attempted to reduce the propositions to the barest skeleton which a lenient Examiner may be supposed to accept. Indeed it does not follow that the form most suitable for the page of a text-book is also best adapted to examination purposes; for the object to be attained in each case is entirely different. The text-book should present the argument in the clearest possible manner to the mind of a reader to whom it is new: the written proposition need only convey to the Examiner the assurance that the proposition has been thoroughly grasped and remembered by the pupil.

From first to last we have kept in mind the undoubted fact that a very small proportion of those who study Elementary Geometry, and study it with profit, are destined to become mathematicians in any special sense; and that, to a large majority of students, Euclid is intended to serve not so much as a first lesson in *mathematical* reasoning, as the first, and sometimes the only, model of formal argument presented in an elementary education.

This consideration has determined not only the full treatment of the earlier Books, but the retention of the formal, if somewhat cumbrous, methods of Euclid in many places where proofs of greater brevity and mathematical elegance are available.

We hope that the additional matter introduced into the book will provide sufficient exercise for pupils whose study of Euclid is preliminary to a mathematical education.

The questions distributed through the text follow very easily from the propositions to which they are attached, and we think that teachers are likely to find in them all that is needed for an average pupil reading the subject for the first time.

The Theorems and Examples at the end of each Book contain questions of a slightly more difficult type: they have been very carefully classified and arranged, and brought into close connection with typical examples worked out either partially or in full; and it is hoped that this section of the book, on which much thought has been expended, will do something towards removing that extreme want of freedom in solving deductions that is so commonly found even among students who have a good knowledge of the text of Euclid.

To Volumes containing only Books I.-III., or Books I.-IV. an Appendix is added, giving an elementary account of the properties of Pole and Polar, and Radical Axis. In the complete book these subjects, together with a short account of Harmonic Section, Centres of Similitude, and Transversals, appear as *Theorems and Examples on Book VI.*

. . . .

Throughout the book we have italicised those deductions on which we desired to lay special stress as being in themselves important geometrical results: this arrangement we think will be useful to teachers who have little time to devote to riders, or who wish to sketch out a suitable course for revision.

. . . .

H. S. HALL.
F. H. STEVENS.

CLIFTON, *December*, 1886.

PREFATORY NOTE TO THE NEW EDITION.

In the present edition the text has received further revision, and the notes have been for the most part re-written, with a view to greater clearness and simplicity.

References to the *Definitions* being frequent in the text of Book I., the convenience of a *standard order* has been pointed out to us by many elementary teachers. We have therefore thought it advisable to re-number the Definitions in accordance with Simson's edition. This has involved the insertion of certain definitions hitherto omitted as of slight importance: such insertions have now been printed in subordinate type.

A few typographical improvements have been introduced: notably the italicising of *Particular Enunciations.* Some changes in pagination have also been effected for the purpose of presenting the whole of a proposition at one view, or of bringing notes and exercises into closer connection with the text to which they refer. Further, the symbols "$\therefore$" for *therefore*, and "$=$" for *is equal to* are now introduced from the 35th Proposition of Book I.

Groups of Test Questions for Revision have been inserted at various stages. These may be useful to beginners, and suggestive to teachers in framing examination papers, which so often consist of mere monotonous lists of propositions and examples.

One important change has been made. The algebraical treatment of the subject-matter of Book V. has been entirely separated from the stricter general treatment, so as to present in the simplest form such Definitions and Theorems of Proportion as are necessary before entering upon Book VI. This Introduction will be found immediately preceding Book VI. in a chapter called *The Elementary Principles of Proportion.*

H. S. H.
F. H. S.

February, 1900.

CONTENTS

BOOK I.

Theorems and Examples on Book I.

BOOK II.

BOOK III.

BOOK IV.

BOOK V.

BOOK VI.

BOOK XI.

BOOK XII.

*** *The First Proposition of Book XII. will be found at p. 364, worked out as an Example on Proposition 20 of Book VI., of which it is a development. Prop. 3 of Book XII. is briefly treated as a Corollary to Prop. 1.*

Suppose that AB and AC lie on opposite sides of a sphere (or circle). If the sphere (or circle) be considered as having circumference = ∞, then lines will be ~~straight~~ & parallel.

But the sphere with circumference ∞ has also radius = ∞; its radius = . Thus if the radius = circumference, circumference.

EUCLID'S ELEMENTS.

BOOK I.

DEFINITIONS.

1. A **point** is that which has position, but no magnitude.

2. A **line** is that which has length without breadth.

3. The extremities of a line are points, and the intersection of two lines is a point.

4. A **straight line** is that which lies evenly between its extreme points.

Any portion cut off from a straight line is called a **segment** of it.

5. A **surface** (or superficies) is that which has length and breadth, but no thickness.

6. The boundaries of a surface are lines.

7. A **plane surface** is one in which any two points being taken, the straight line between them lies wholly in that surface.

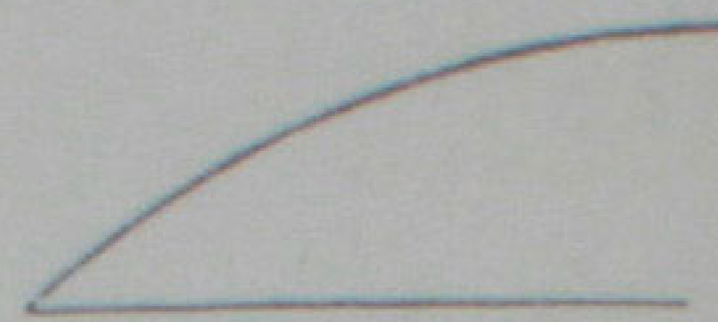
A plane surface is frequently referred to simply as a *plane*.

NOTE. Euclid regards a point merely as a *mark of position*, and he therefore attaches to it no idea of size and shape.

Similarly he considers that the properties of a line arise only from its *length* and *position*, without reference to that minute breadth which every line must really have *if actually drawn*, even though the most perfect instruments are used.

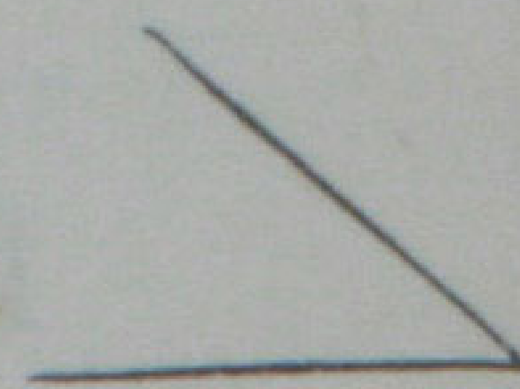
The definition of a surface is to be understood in a similar way.

8. A **plane angle** is the inclination of two lines to one another, which meet together, but are not in the same direction.

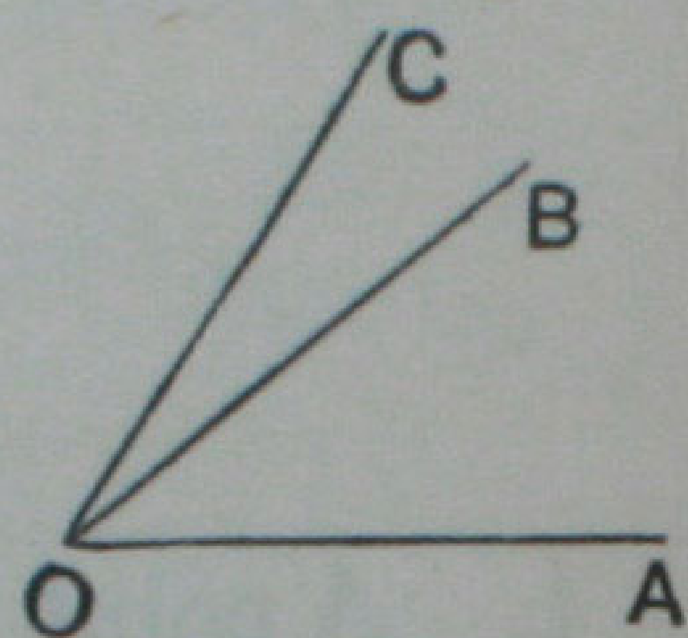


[Definition 8 is not required in Euclid's Geometry, the only angles employed by him being those formed by *straight* lines. See Def. 9.]

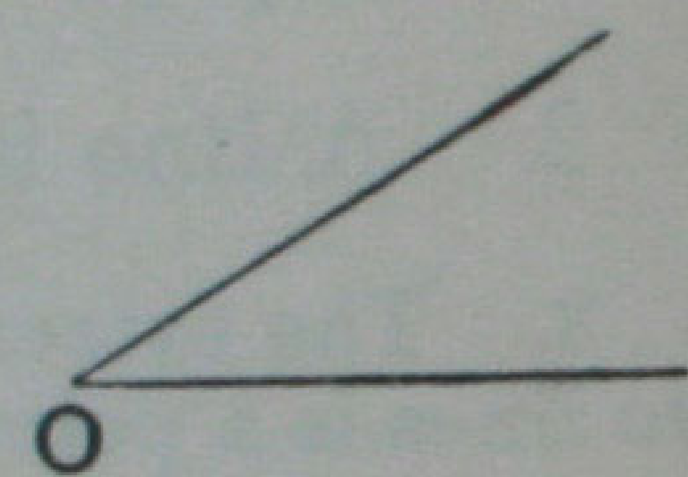
9. A **plane rectilineal angle** is the inclination of two *straight* lines to one another, which meet together, but are not in the same straight line.



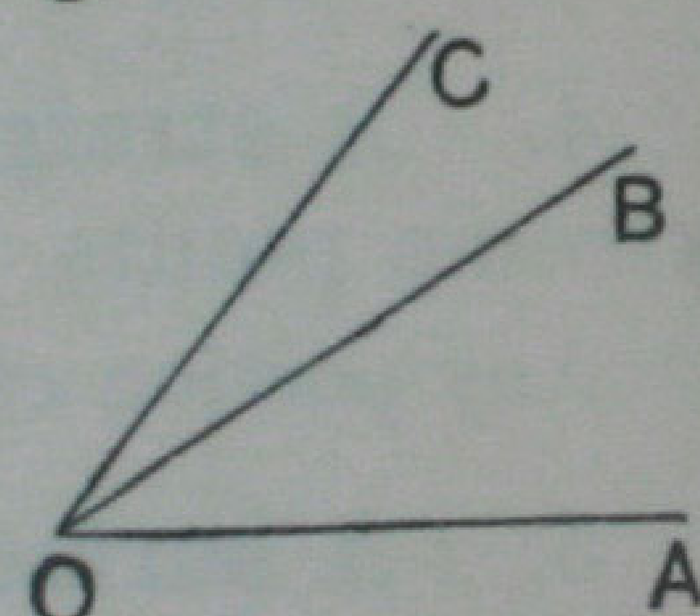
The point at which the straight lines meet is called the **vertex** of the angle, and the straight lines themselves the **arms** of the angle.



NOTE. When there are several angles at one point, each is expressed by three letters, of which the letter that refers to the vertex is put between the other two. Thus the angle contained by the straight lines OA, OB is named the angle AOB or BOA; and the angle contained by OA, OC is named the angle AOC or COA. But if there is only one angle at a point, it may be expressed by a single letter, as *the angle at* O.



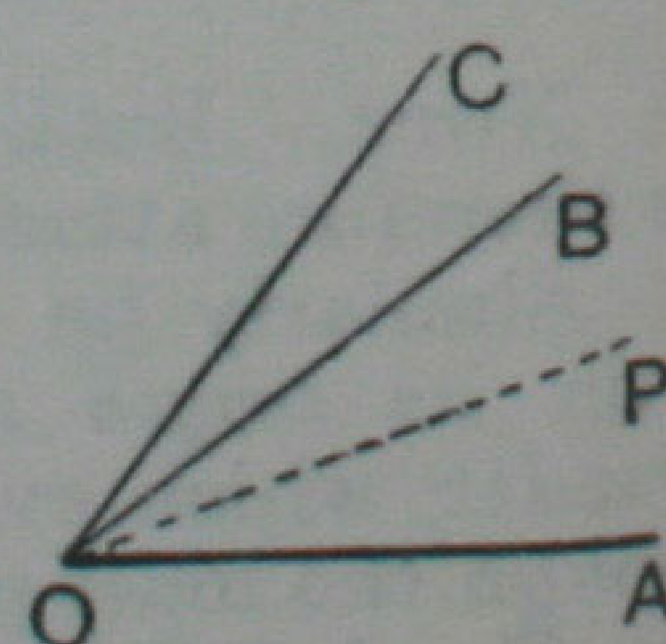
Of the two straight lines OB, OC shewn in the adjoining diagram, we recognize that OC is *more inclined* than OB to the straight line OA: this we express by saying that the angle AOC is greater than the angle AOB. Thus an angle must be regarded as having *magnitude*.



It must be carefully observed that the size of an angle in no way depends on the length of its arms, but only on their *inclination* to one another.

The angle AOC is the *sum* of the angles AOB and BOC; and AOB is the *difference* of the angles AOC and BOC.

[Another view of an angle is recognized in many branches of mathematics; and though not employed by Euclid, it is here given because it furnishes more clearly than any other a conception of what is meant by the *magnitude* of an angle.

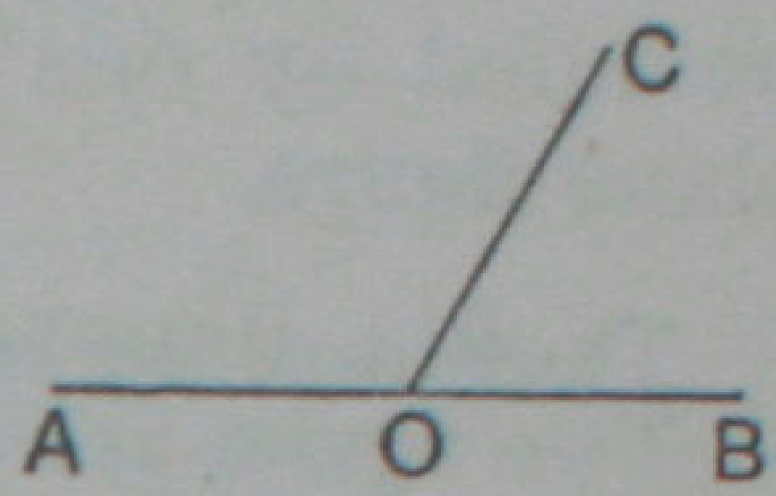


Suppose that the straight line OP in the diagram is capable of revolution about the point O, like the hand of a watch, but in the opposite direction; and suppose that in this way it has passed successively from the position OA to the positions occupied by OB and OC. Such a line must have undergone *more turning* in passing from OA to

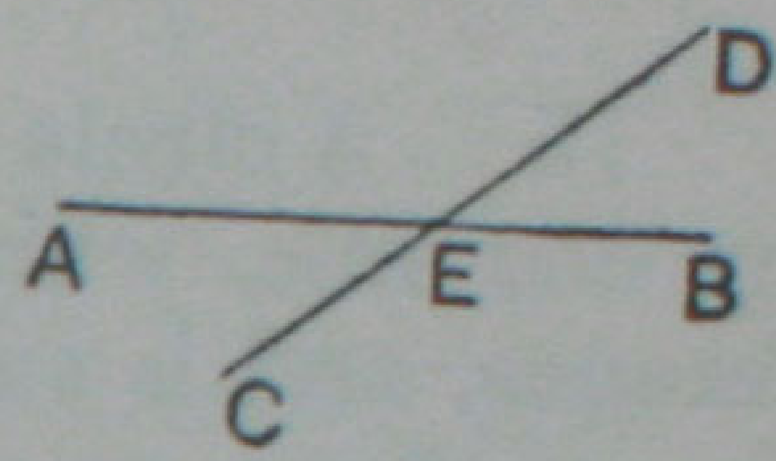
OC, than in passing from OA to OB; and consequently the angle AOC is said to be greater than the angle AOB.]

Angles which lie on either side of a common arm are called **adjacent** angles.

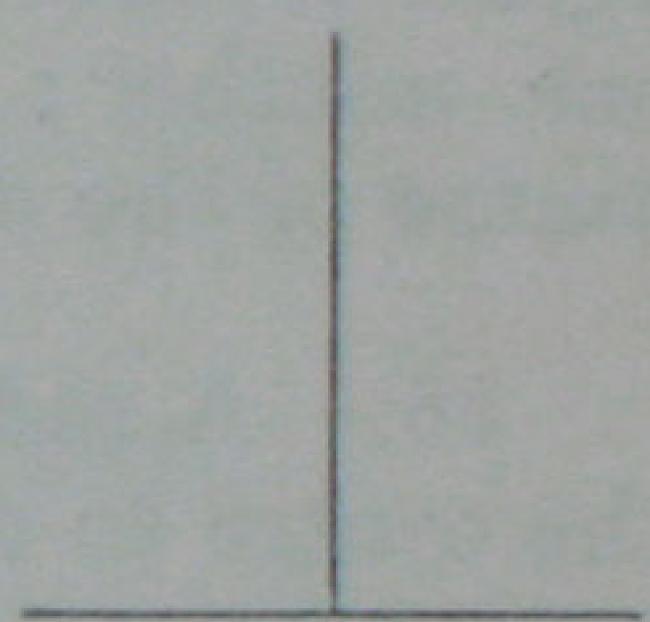
For example, when one straight line OC is drawn from a point in another straight line AB, the angles COA, COB are *adjacent*.



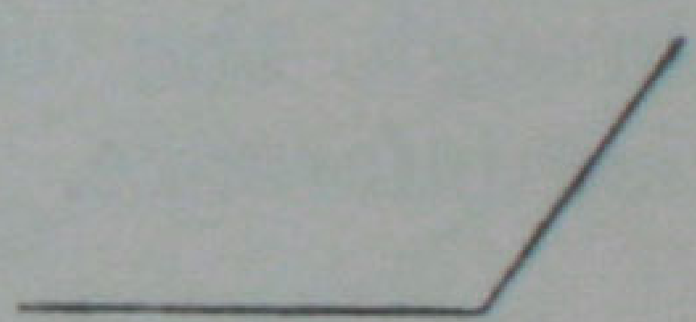
When two straight lines, such as AB, CD, cross one another at E, the two angles CEA, BED are said to be **vertically opposite**. The two angles CEB, AED are also vertically opposite to one another.



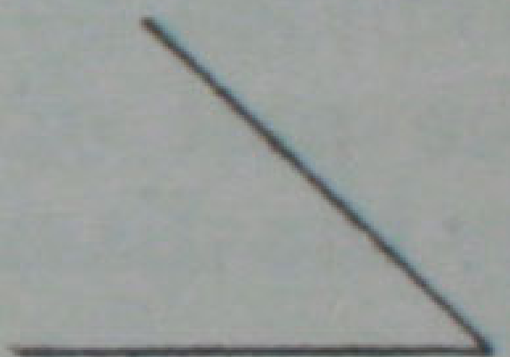
10. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a **right angle**; and the straight line which stands on the other is called a **perpendicular** to it.



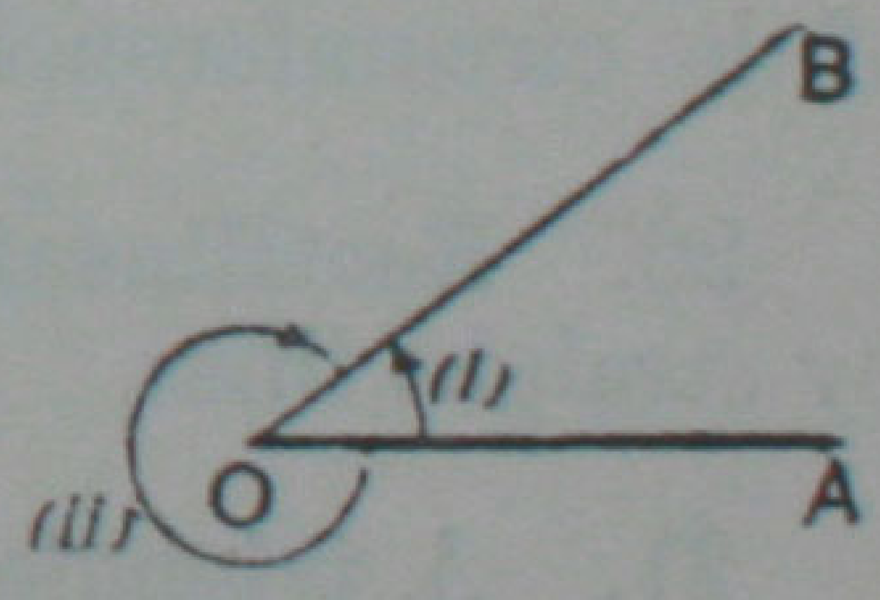
11. An **obtuse angle** is an angle which is greater than a right angle.



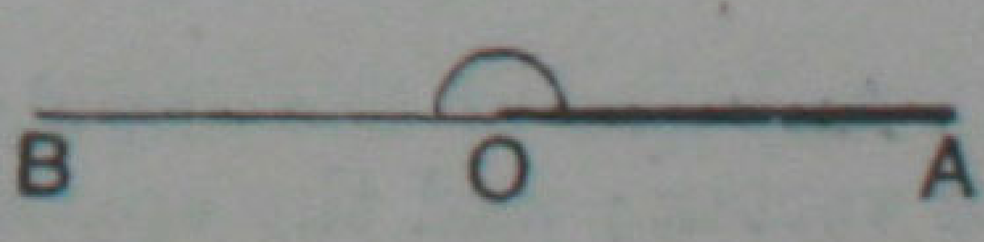
12. An **acute angle** is an angle which is less than a right angle.



[In the adjoining figure the straight line OB may be supposed to have arrived at its present position, from the position occupied by OA, by revolution about the point O in *either* of the two directions indicated by the arrows: thus two straight lines drawn from a point may be considered as forming *two* angles (marked (i) and (ii) in the figure), of which the greater (ii) is said to be **reflex**.

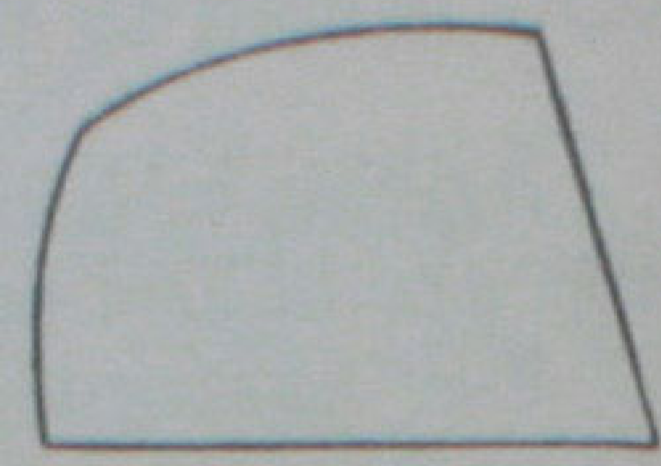


If the arms OA, OB are in the same straight line, the angle formed by them on either side is called a **straight angle**.]



13. A term or boundary is the extremity of anything.

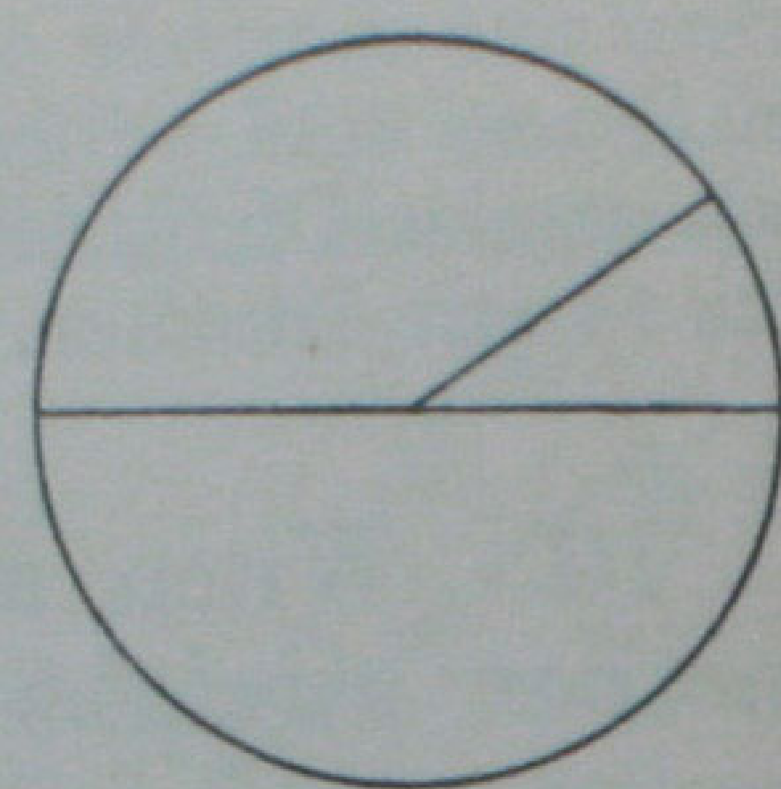
14. Any portion of a plane surface bounded by one or more lines is called a **plane figure.**



The sum of the bounding lines is called the **perimeter** of the figure.

Two figures are said to be equal in **area** when they enclose equal portions of a plane surface.

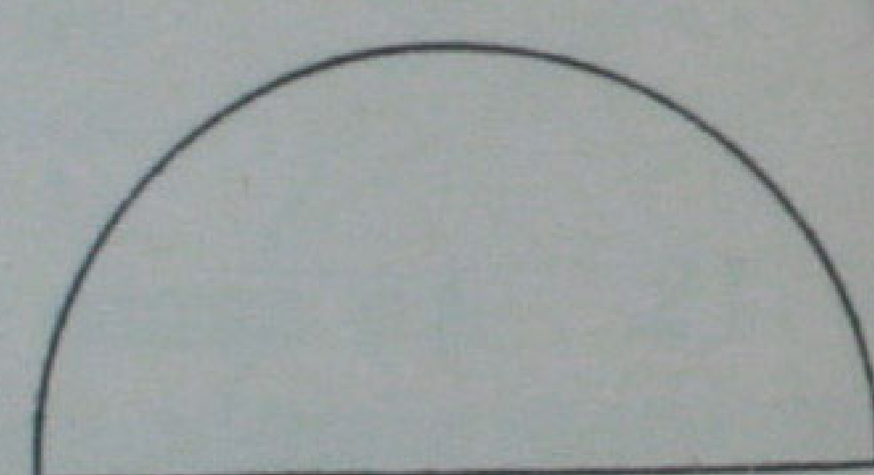
15. **A circle** is a plane figure contained by one line, which is called the **circumference**, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another; this point is called **the centre** of the circle.



16. A **radius** of a circle is a straight line drawn from the centre to the circumference.

17. A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

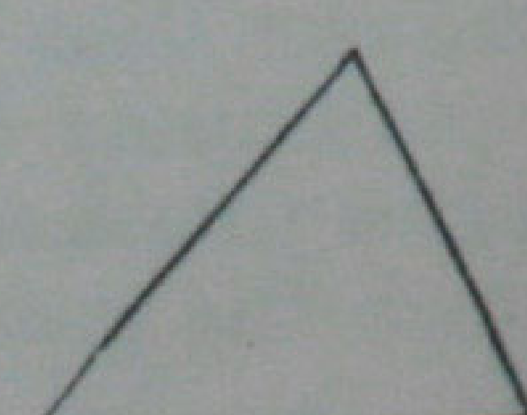
18. A **semicircle** is the figure bounded by a diameter of a circle and the part of the circumference cut off by the diameter.



19. **A segment of a circle** is the figure bounded by a straight line and the part of the circumference which it cuts off.

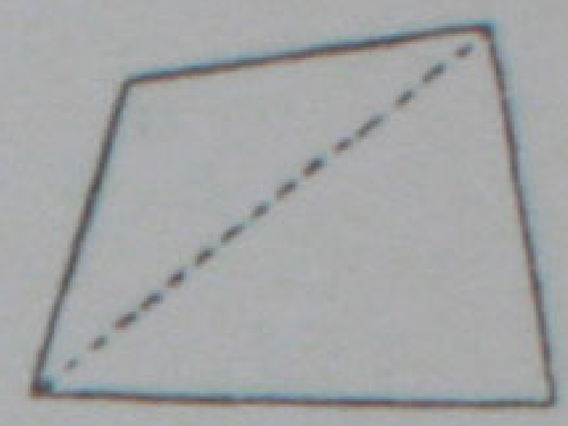
20. **Rectilineal figures** are those which are bounded by straight lines.

21. A **triangle** is a plane figure bounded by *three* straight lines.



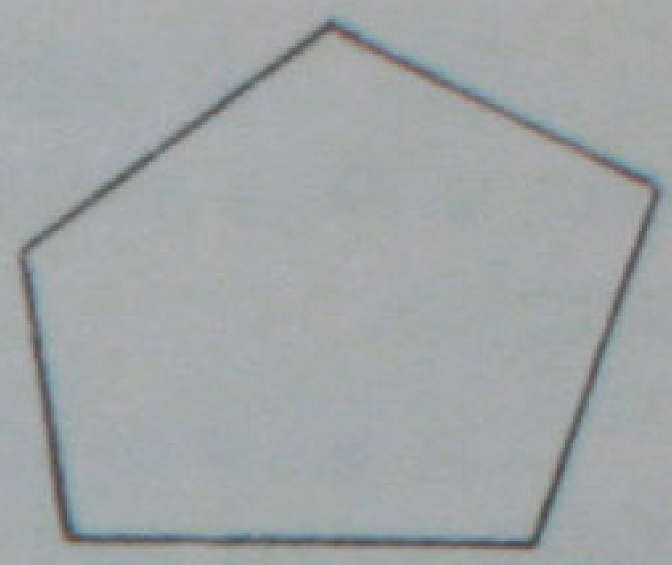
Any one of the angular points of a triangle may be regarded as its **vertex**; and the opposite side is then called the **base**.

22. A **quadrilateral** is a plane figure bounded by *four* straight lines.



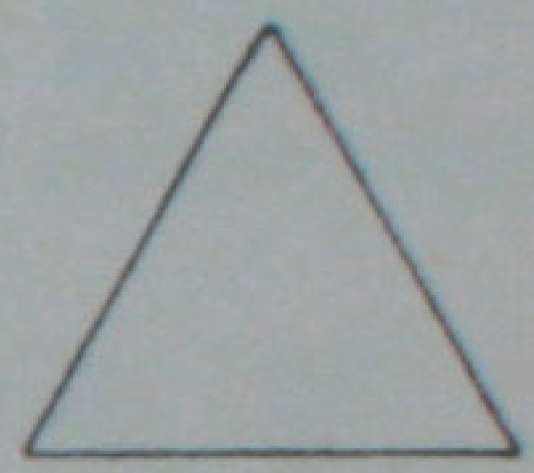
The straight line which joins opposite angular points in a quadrilateral is called a **diagonal**.

23. A **polygon** is a plane figure bounded by more than four straight lines.

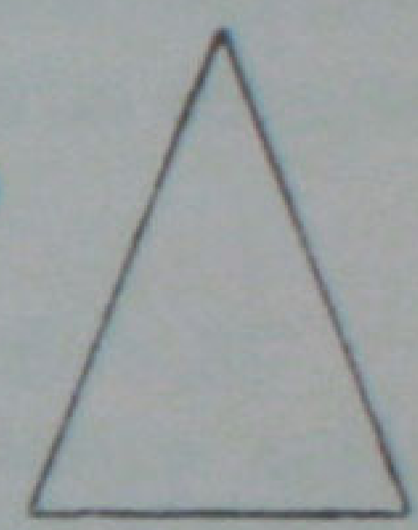


TRIANGLES.

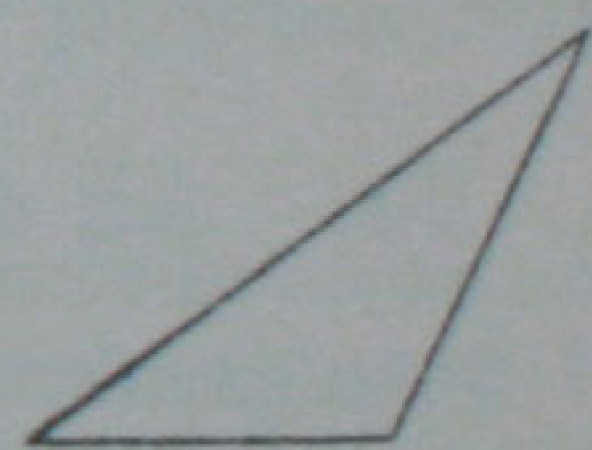
24. An **equilateral triangle** is a triangle whose three sides are equal.



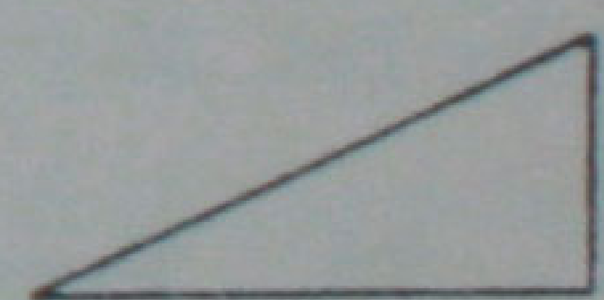
25. An **isosceles triangle** is a triangle two of whose sides are equal.



26. A **scalene triangle** is a triangle which has three unequal sides.

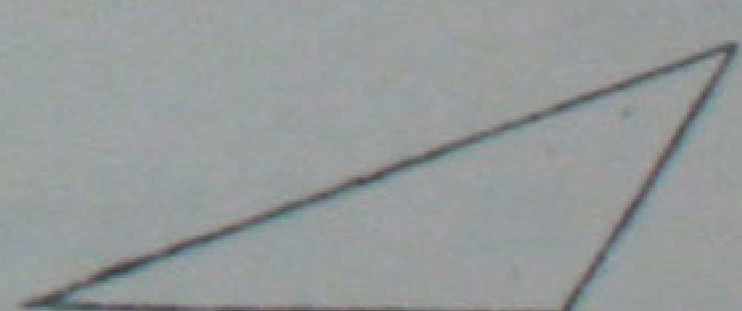


27. A **right-angled triangle** is a triangle which has a right angle.

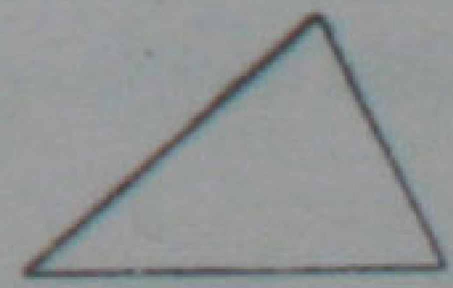


The side opposite to the right angle in a right-angled triangle is called the **hypotenuse**.

28. An **obtuse-angled triangle** is a triangle which has an obtuse angle.

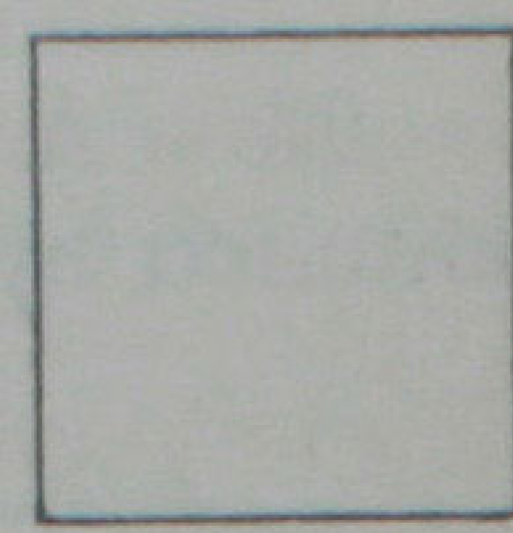


29. An **acute-angled triangle** is a triangle which has *three* acute angles.



[It will be seen hereafter (Book I. Proposition 17) *that every triangle must have at least two acute angles.*]

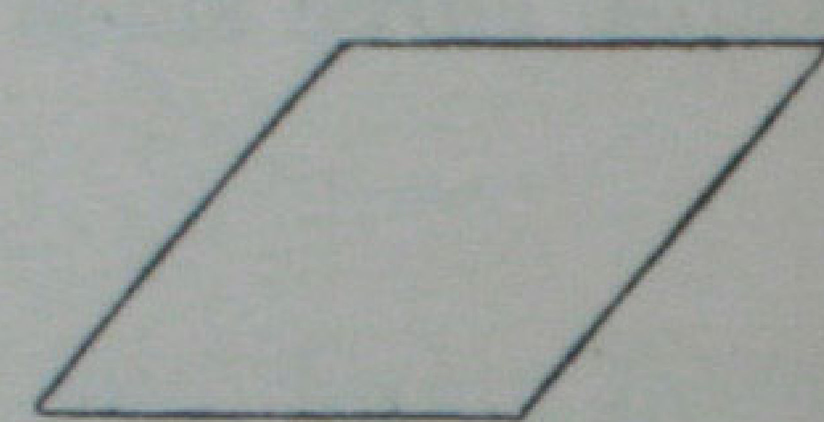
QUADRILATERALS.



30. A **square** is a four-sided figure which has all its sides equal and all its angles right angles.

[It may be shewn that if a quadrilateral has all its sides equal and *one* angle a right angle, then *all* its angles will be rignt angles.]

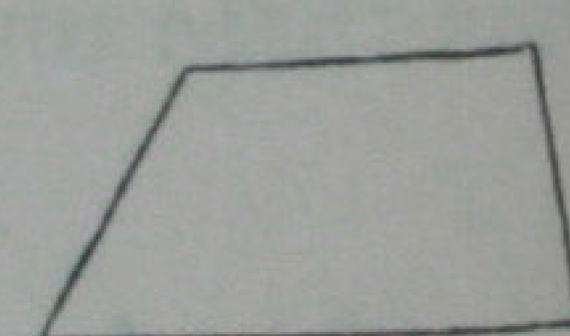
31. An **oblong** is a four-sided figure which has all its angles right angles, but not all its sides equal.



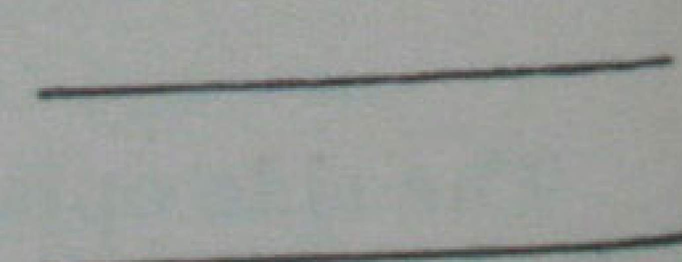
32. A **rhombus** is a four-sided figure which has all its sides equal, but its angles are not right angles.

33. A **rhomboid** is a four-sided figure which has its opposite sides equal to one another, but all its sides are not equal nor its angles right angles.

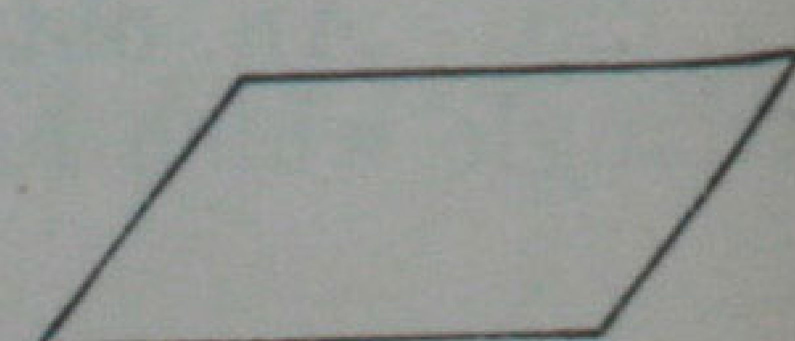
34. All other four-sided figures are called **trapeziums.**



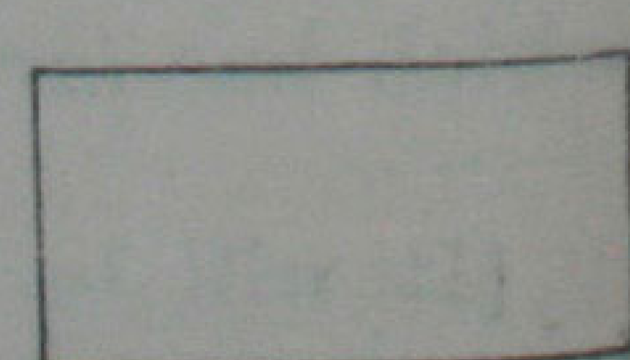
It is usual now to restrict the term *trapezium* to a quadrilateral which has two of its sides *parallel.* [See Def. 35.]



35. **Parallel straight lines** are such as, being in the same plane, do not meet, however far they are produced in either direction.



36. A **Parallelogram** is a four-sided figure which has its opposite sides parallel.



37. A **rectangle** is a parallelogram which has one of its angles a right angle.

The Postulates.

Let it be granted,

1. *That a straight line may be drawn from any one point to any other point.*

2. *That a* finite, *that is to say a terminated, straight line may be* produced *to any length in that straight line.*

3. *That a circle may be described from any centre, at any distance from that centre, that is, with a radius equal to any finite straight line drawn from the centre.*

Notes on the Postulates.

1. In order to draw the diagrams required in Euclid's Geometry certain instruments are necessary. These are

(i) A *ruler* with which to draw straight lines.

(ii) A *pair of compasses* with which to draw circles.

In the *Postulates*, or requests, Euclid claims the use of these instruments, and assumes that they suffice for the purposes mentioned above.

2. It is important to notice that the Postulates include no means of *direct measurement*: hence the straight ruler is not supposed to be *graduated*; and the compasses are not to be employed for *transferring distances* from one part of a diagram to another.

3. When we draw a straight line from the point A to the point B, we are said to *join* AB.

To *produce* a straight line means to *prolong* or *lengthen* it.

The expression *to describe* is used in Geometry in the sense of *to draw*.

On the Axioms.

The science of Geometry is based upon certain simple statements, the truth of which is so evident that they are accepted without proof.

These self-evident truths, called by Euclid *Common Notions*, are known as the **Axioms**.

General Axioms.

1. *Things which are equal to the same thing are equal to one another.*

2. *If equals be added to equals, the wholes are equal.*

3. *If equals be taken from equals, the remainders are equal.*

4. *If equals be added to unequals, the wholes are unequal, the greater sum being that which includes the greater of the unequals.*

5. *If equals be taken from unequals, the remainders are unequal, the greater remainder being that which is left from the greater of the unequals.*

6. *Things which are double of the same thing, or of equal things, are equal to one another.*

7. *Things which are halves of the same thing, or of equal things, are equal to one another.*

9.* *The whole is greater than its part.*

* To preserve the classification of general and geometrical axioms, we have placed Euclid's *ninth* axiom before the *eighth*.

Geometrical Axioms.

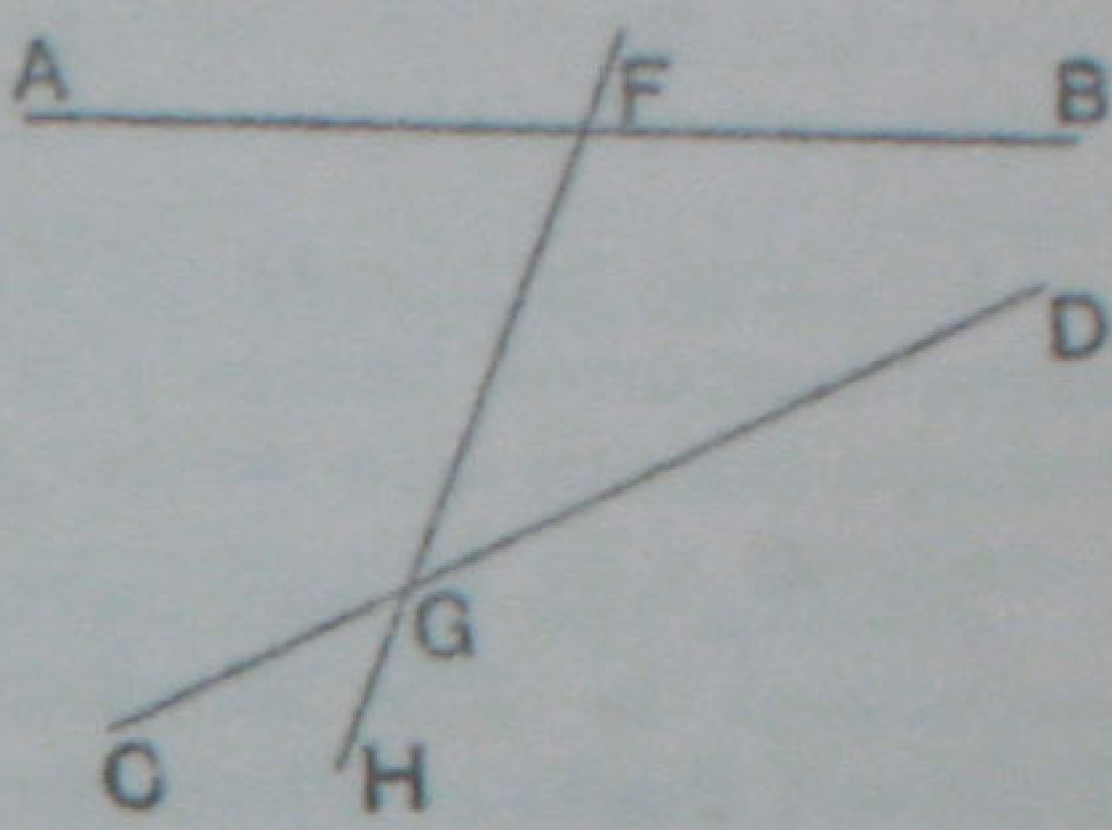
8. *Magnitudes which can be made to coincide with one another, are equal.*

10. *Two straight lines cannot enclose a space.*

11. *All right angles are equal.*

12. *If a straight line meet two straight lines so as to make the interior angles on one side of it together less than two right angles, these straight lines will meet if continually produced on the side on which are the angles which are together less than two right angles.*

That is to say, if the two straight lines AB and CD are met by the straight line EH at F and G, in such a way that the angles BFG, DGF are together less than two right angles, it is asserted that AB and CD will meet if continually produced in the direction of B and D.



Notes on the Axioms.

1. The necessary characteristics of an Axiom are

(i) That it should be *self-evident*; that is, that its truth should be immediately accepted without proof.

(ii) That it should be *fundamental*; that is, that its truth should not be derivable from any other truth more simple than itself.

(iii) That it should supply a basis for the establishment of further truths.

These characteristics may be summed up in the following definition.

Definition. An **Axiom** is a self-evident truth, which neither requires nor is capable of proof, but which serves as a foundation for future reasoning.

2. Euclid's Axioms may be classified as *general* and *geometrical*.

General Axioms apply to *magnitudes of all kinds*. Geometrical Axioms refer specially to *geometrical magnitudes*, as lines, angles, and figures.

3. Axiom 8 is Euclid's test of the equality of two geometrical magnitudes. It implies that any line, angle, or figure, may be taken up from its position, and without change in size or form, laid down upon a second line, angle, or figure, for the purpose of comparison, and it states that two such magnitudes are equal when one can be exactly placed over the other without overlapping.

This process is called **superposition**, and the first magnitude is said to be **applied** to the other.

4. Axiom 12 has been objected to on the double ground that it cannot be considered self-evident, and that its truth may be deduced from simpler principles. It is employed for the first time in the 29th Proposition of Book I., where a short discussion of the difficulty will be found.

Introductory.

1. Little is known of Euclid beyond the fact that he lived about three centuries before Christ (325-285) at Alexandria, where he became famous as a writer and teacher of Mathematics.

Among the works ascribed to him, the best known and most important is *The Elements*, written in Greek, and consisting of Thirteen Books. Of these it is now usual to read Books I.-IV. and VI. (which deal with Plane Geometry), together with parts of Books XI. and XII. (on the Geometry of Solids). The remaining Books deal with subjects which belong to the theory of Arithmetic.

2. Plane Geometry deals with the properties of all lines and figures that may be drawn upon a plane surface.

Euclid in his first Six Books confines himself to the properties of straight lines, rectilineal figures, and circles.

3. The subject is divided into a number of separate discussions, called **propositions**.

Propositions are of two kinds, **Problems** and **Theorems**.

A **Problem** proposes to perform some geometrical construction, such as to draw some particular line, or to construct some required figure.

A **Theorem** proposes to prove the truth of some geometrical statement.

4. A Proposition consists of the following parts:

The *General Enunciation*, the *Particular Enunciation*, the *Construction*, and the *Proof*.

(i) The **General Enunciation** is a preliminary statement, describing in general terms the purpose of the proposition.

(ii) The **Particular Enunciation** repeats in special terms the statement already made, and refers it to a diagram, which enables the reader to follow the reasoning more easily.

(iii) The **Construction** then directs the drawing of such straight lines and circles as may be required to effect the purpose of a problem, or to prove the truth of a theorem.

(iv) The **Proof** shews that the object proposed in a problem has been accomplished, or that the property stated in a theorem is true.

5. Euclid's reasoning is said to be **Deductive**, because by a connected chain of argument it *deduces* new truths from truths already proved or admitted. Thus each proposition, though in one sense complete in itself, is derived from the Postulates, Axioms, or former propositions, and itself leads up to subsequent propositions.

6. The initial letters Q.E.F., placed at the end of a problem, stand for **Quod erat Faciendum**, *which was to be done.*

The letters Q.E.D. are appended to a theorem, and stand for **Quod erat Demonstrandum**, *which was to be proved.*

7. A **Corollary** is a statement the truth of which follows readily from an established proposition; it is therefore appended to the proposition as an inference or deduction, which usually requires no further proof.

8. The attention of the beginner is drawn to the special use of the *future tense* in the Particular Enunciations of Euclid's propositions.

The future is only used in a statement of which the truth is *about to be proved.* Thus: "*The triangle* ABC SHALL BE *equilateral*" means that the triangle *has yet to be proved* equilateral. While, "*The triangle* ABC IS *equilateral*" means that the triangle *has already been proved* (or given) equilateral.

9. The following symbols and abbreviations may be employed in writing out the propositions of Book I., though their use is not recommended to beginners.

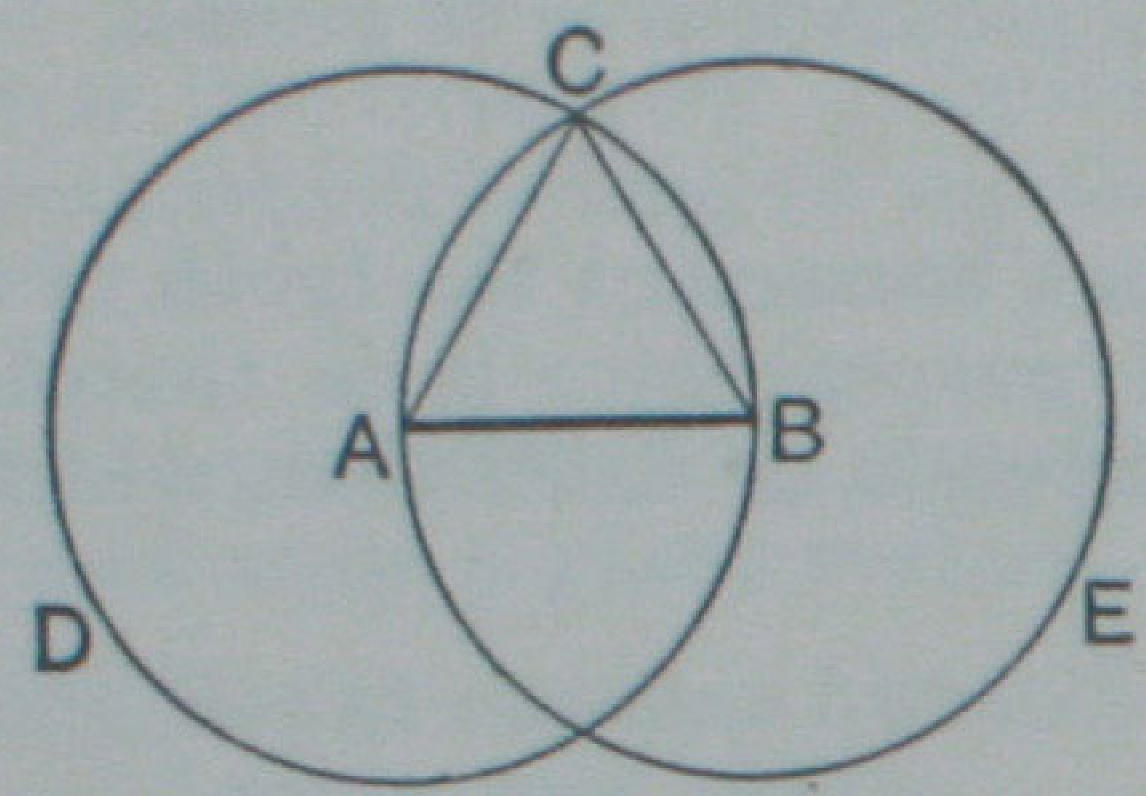
$\therefore$	*for*	therefore,	par[l] (or $\parallel$)	*for*	parallel,
$=$	,,	is, or are, equal to,	par[m]	,,	parallelogram,
$\angle$	,,	angle,	sq.	,,	square,
rt. $\angle$	,,	right angle,	rectil.	,,	rectilineal,
$\triangle$	,,	triangle,	st. line	,,	straight line,
perp.	,,	perpendicular,	pt.	,,	point;

and all obvious contractions of words, such as opp., adj., diag., etc., for opposite, adjacent, diagonal, etc.

SECTION I.

PROPOSITION 1. PROBLEM.

To describe an equilateral triangle on a given finite straight line.



Let AB be the given straight line.

It is required to describe an equilateral triangle on AB.

Construction. With centre A, and radius AB, describe the circle BCD. *Post.* 3.

With centre B, and radius BA, describe the circle ACE. *Post.* 3.

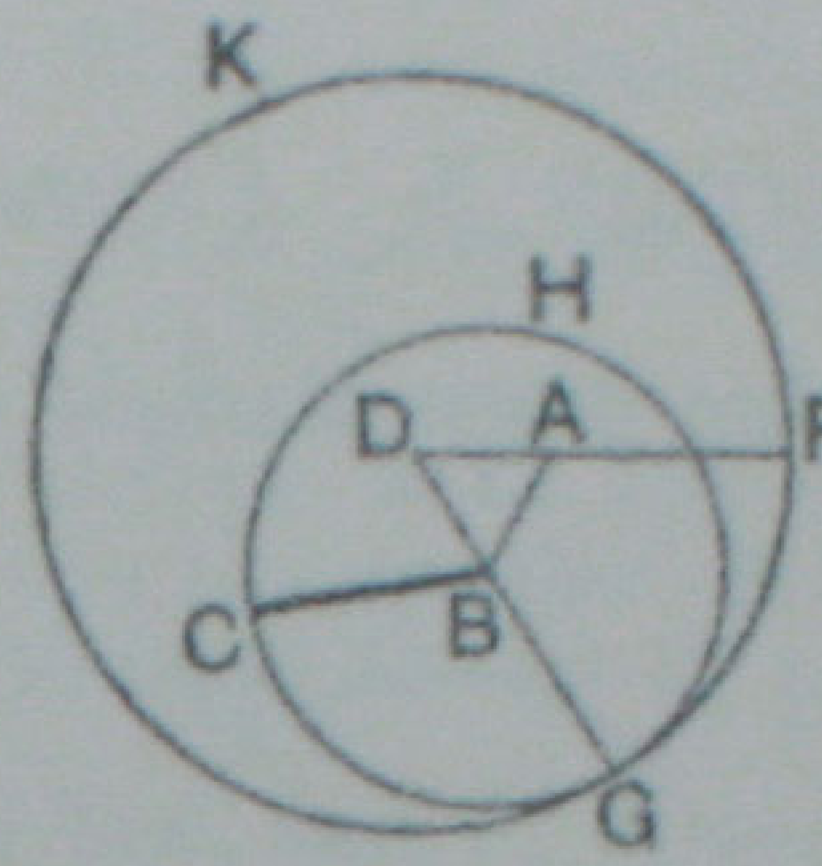
From the point C at which the circles cut one another, draw the straight lines CA and CB to the points A and B. *Post.* 1.

Then shall the triangle ABC *be equilateral.*

Proof. Because A is the centre of the circle BCD,
therefore AC is equal to AB. *Def.* 15.
And because B is the centre of the circle ACE,
therefore BC is equal to AB. *Def.* 15.
Therefore AC and BC are each equal to AB.
But things which are equal to the same thing are equal to one another. *Ax.* 1.
Therefore AC is equal to BC.
Therefore AC, AB, BC are equal to one another.
Therefore the triangle ABC is equilateral;
and it is described on the given straight line AB. Q.E.F.

Proposition 2. Problem.

From a given point to draw a straight line equal to a given straight line.



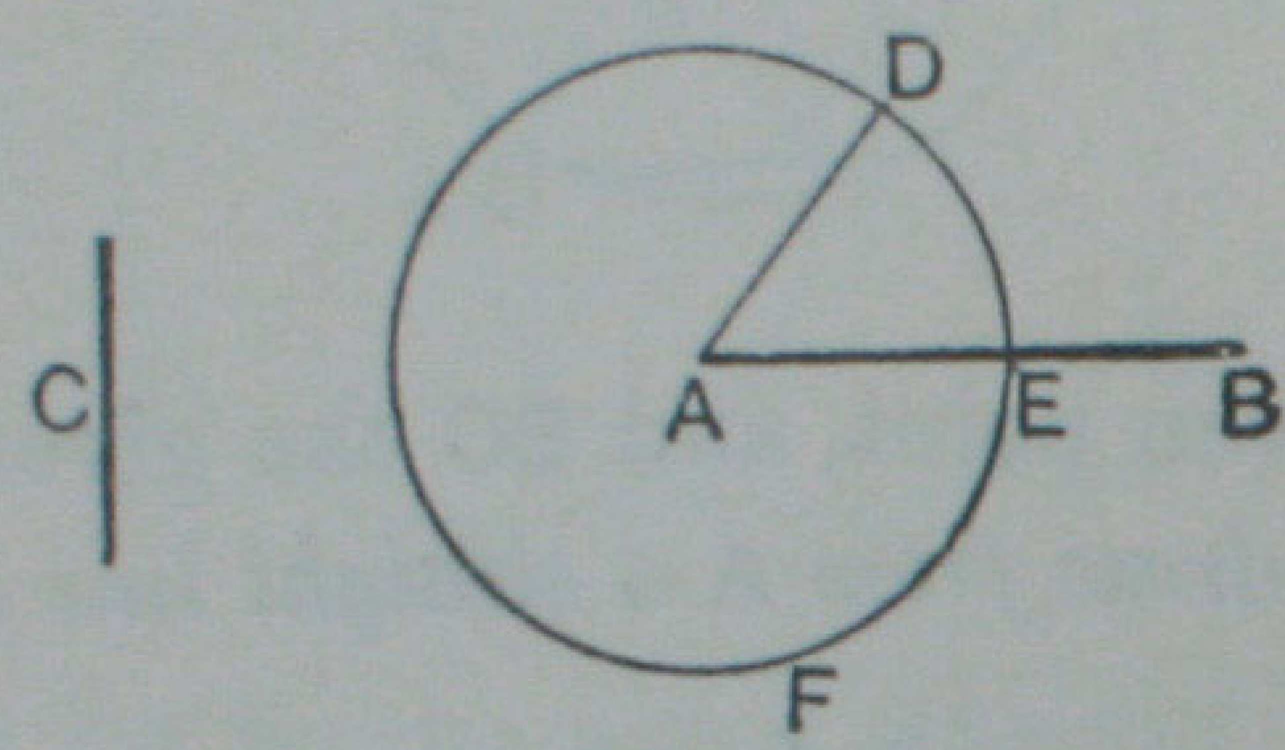
Let A be the given point, and BC the given straight line.
It is required to draw from A *a straight line equal to* BC.

Construction. Join AB; *Post.* 1.
and on AB describe an equilateral triangle DAB. I. 1.
With centre B, and radius BC, describe the circle CGH. *Post.* 3.
Produce DB to meet the circle CGH at G. *Post.* 2.
With centre D, and radius DG, describe the circle GKF.
Produce DA to meet the circle GKF at F. *Post.* 2.
Then AF *shall be equal to* BC.

Proof. Because B is the centre of the circle CGH,
therefore BC is equal to BG. *Def.* 15.
And because D is the centre of the circle GKF,
therefore DF is equal to DG. *Def.* 15.
And DA, a part of DF, is equal to DB, a part of DG; *Def.* 24.
therefore the remainder AF is equal to the remainder BG. *Ax.* 3.
But BC has been proved equal to BG;
therefore AF and BC are each equal to BG.
And things which are equal to the same thing are equal to one another. *Ax.* 1.
Therefore AF is equal to BC;
and it has been drawn from the given point A. Q.E.F.

Proposition 3. Problem.

From the greater of two given straight lines to cut off a part equal to the less.



Let AB and C be the two given straight lines, of which AB is the greater.

It is required to cut off from AB *a part equal to* C.

Construction. From the point A draw the straight line AD equal to C; I. 2.
and with centre A and radius AD, describe the circle DEF, cutting AB at E. *Post* 3.

Then AE *shall be equal to* C.

Proof. Because A is the centre of the circle DEF,
therefore AE is equal to AD. *Def.* 15.
But C is equal to AD. *Constr.*
Therefore AE and C are each equal to AD.
Therefore AE is equal to C; *Ax.* 1.
and it has been cut off from the given straight line AB.

Q.E.F.

EXERCISES ON PROPOSITIONS 1 TO 3.

1. If the two circles in Proposition 1 cut one another again at F, prove that AFB is an equilateral triangle.

2. If the two circles in Proposition 1 cut one another at C and F, prove that the figure ACBF is a rhombus.

3. AB is a straight line of given length: shew how to draw from A a line double the length of AB.

4. Two circles are drawn with the same centre O, and two radii OA, OB are drawn in the smaller circle. If OA, OB are produced to cut the outer circle at D and E, prove that AD = BE.

5. AB is a straight line, and P, Q are two points, one on each side of AB. Shew how to find points in AB, whose distance from P is equal to PQ. How many such points will there be?

6. In the figure of Proposition 2, if AB is equal to BC, shew that D, the vertex of the equilateral triangle, will fall on the circumference of the circle CGH.

7. In Proposition 2 the point A may be joined to either extremity of BC. Draw the figure, and prove the proposition in the case when A is joined to C.

8. On a given straight line AB describe an isosceles triangle having each of its equal sides equal to a given straight line PQ.

9. On a given base describe an isosceles triangle having each of its equal sides double of the base.

10. In a given straight line the points A, M, N, B are taken in order. On AB describe a triangle ABC, such that the side AC may be equal to AN, and the side BC to BM.

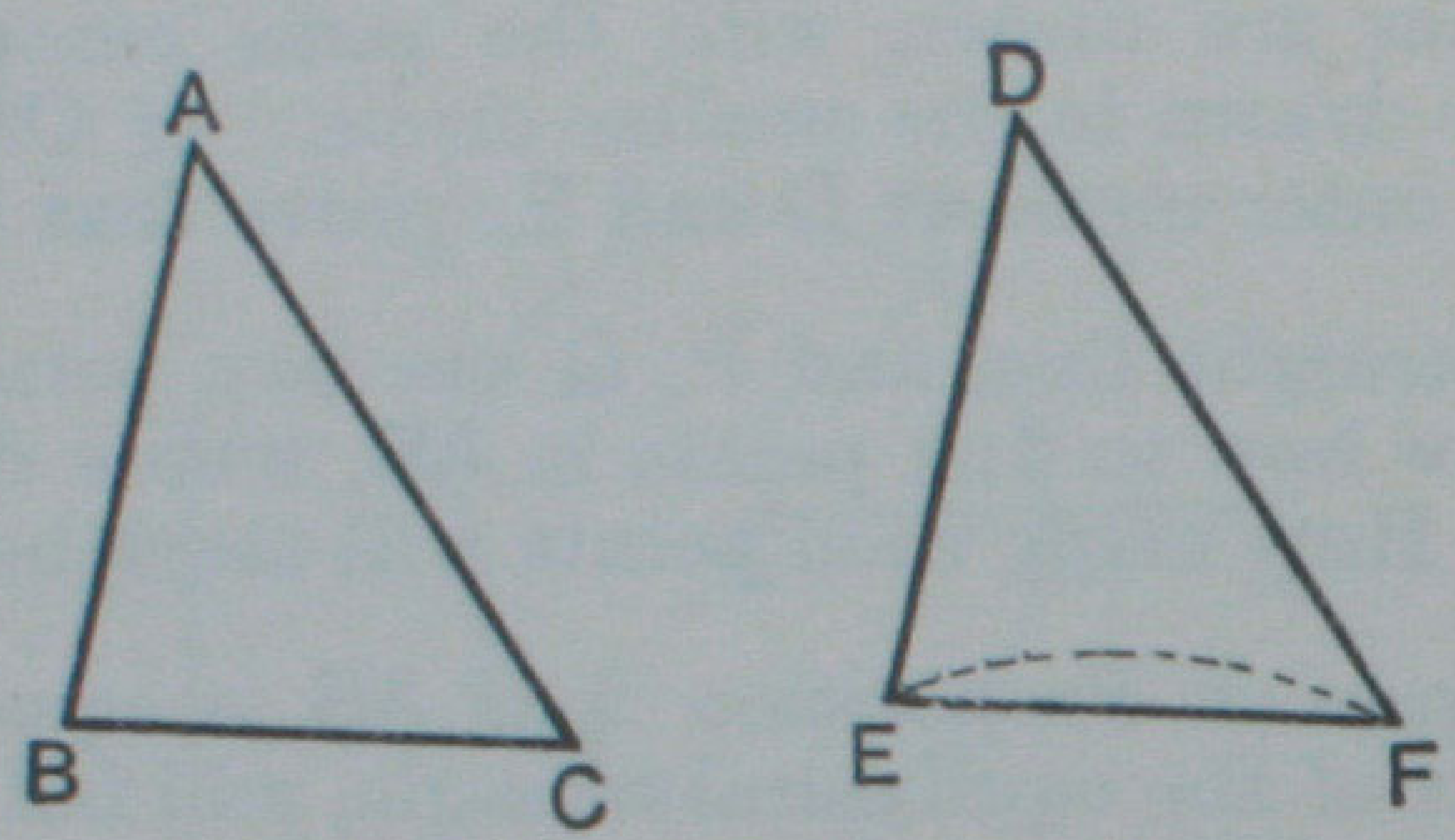
NOTE ON PROPOSITIONS 2 AND 3.

Propositions 2 and 3 are rendered necessary by the restriction tacitly imposed by Euclid, that compasses shall not be used to *transfer distances*. [See Notes on the Postulates.]

In carrying out the construction of Prop. 2 the point A may be joined to *either* extremity of the line BC; the equilateral triangle may be described on *either* side of the line so drawn; and the sides of the equilateral triangle may be produced in *either* direction. Thus there are in general $2 \times 2 \times 2$, or *eight*, possible constructions. The student should exercise himself in drawing the various figures that may arise.

Proposition 4. Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal, then the triangles shall be equal in all respects; that is to say, their bases or third sides shall be equal, and their remaining angles shall be equal, each to each, namely those to which the equal sides are opposite; and the triangles shall be equal in area.



Let ABC, DEF be two triangles, in which
the side AB is equal to the side DE,
the side AC is equal to the side DF, and
the contained angle BAC is equal to the contained angle EDF.

Then (i) *the base* BC *shall be equal to the base* EF;
(ii) *the angle* ABC *shall be equal to the angle* DEF;
(iii) *the angle* ACB *shall be equal to the angle* DFE;
(iv) *the triangle* ABC *shall be equal to the triangle* DEF *in area.*

Proof. If the triangle ABC be applied to the triangle DEF,
so that the point A may lie on the point D,
and the straight line AB along the straight line DE;
then because AB is equal to DE, *Hyp.*
therefore the point B must coincide with the point E.
And because AB falls along DE,
and the angle BAC is equal to the angle EDF, *Hyp.*
therefore AC must fall along DF.
And because AC is equal to DF, *Hyp.*
therefore the point C must coincide with the point F.
Then since B coincides with E, and C with F,
therefore the base BC must coincide with the base EF;

for if not, two straight lines would enclose a space; which is impossible. *Ax.* 10.

Thus the base BC coincides with the base EF, and is therefore equal to it. *Ax.* 8.

And the remaining angles of the triangle ABC coincide with the remaining angles of the triangle DEF, and are therefore equal to them;

namely, the angle ABC is equal to the angle DEF,
and the angle ACB is equal to the angle DFE.

And the triangle ABC coincides with the triangle DEF, and is therefore equal to it in area. *Ax.* 8.

That is, the triangles are equal in all respects. Q.E.D.

NOTE. The sides and angles of a triangle are known as its *six parts*. A triangle may also be considered in regard to its *area*.

Two triangles are said to be **equal in all respects**, or **identically equal**, when the sides and angles of one are respectively equal to the sides and angles of the other. We have seen that such triangles may be made to *coincide* with one another by *superposition*, so that they are also equal in *area*. [See Note on Axiom 8.]

[It will be shewn later that triangles can be equal in *area* without being equal in their several parts; that is to say, triangles can have the same *area* without having the same *shape*.]

EXERCISES ON PROPOSITION 4.

1. ABCD is a square: prove that the diagonals AC, BD are equal to one another.

2. ABCD is a square, and L, M, and N are the middle points of AB, BC, and CD: prove that

(i) LM = MN. (ii) AM = DM.
(iii) AN = AM. (iv) BN = DM.

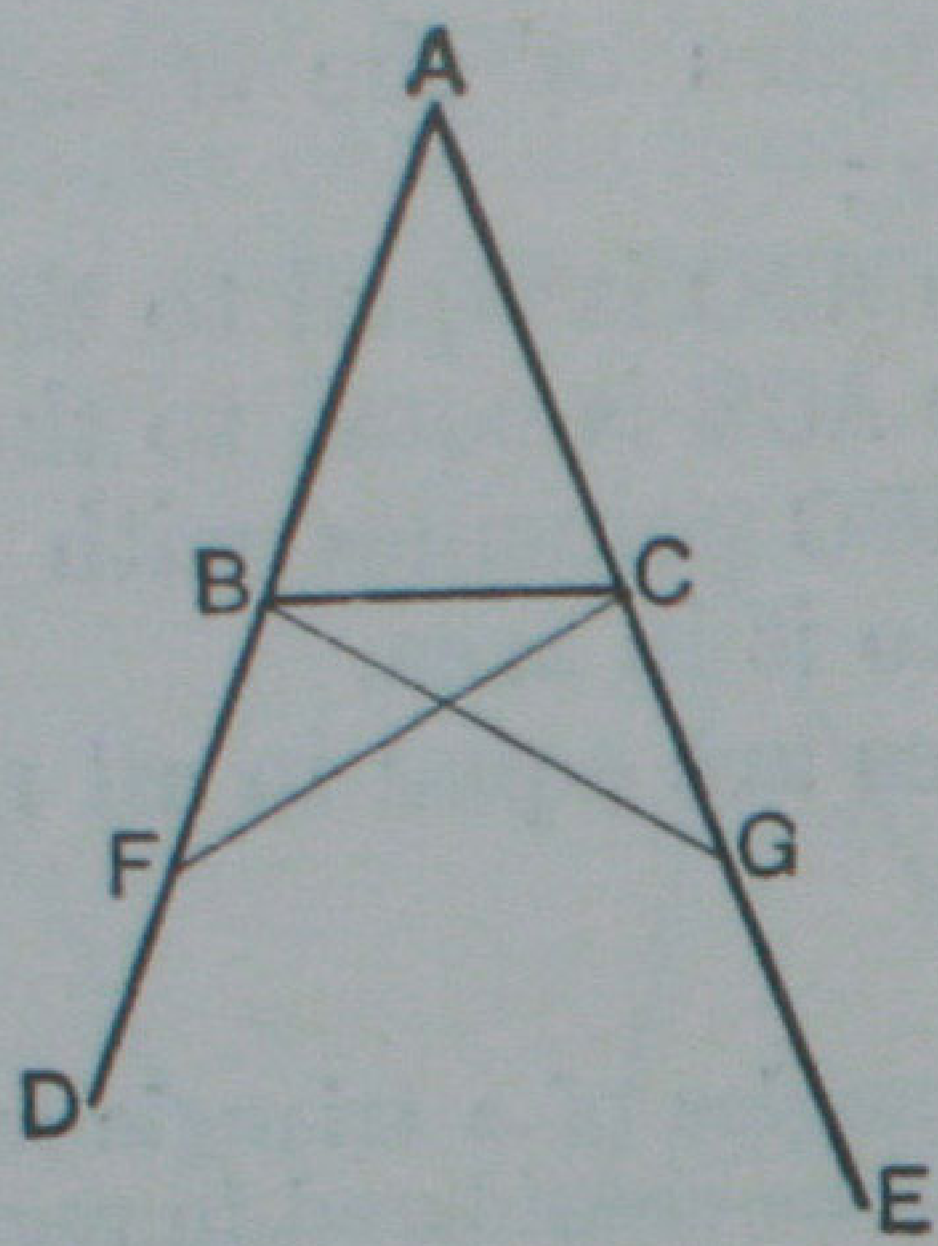
[Draw a separate figure in each case.]

3. ABC is an isosceles triangle: from the equal sides AB, AC two equal parts AX, AY are cut off, and BY and CX are joined. Prove that BY = CX.

4. ABCD is a quadrilateral having the opposite sides BC, AD equal, and also the angle BCD equal to the angle ADC: prove that BD is equal to AC.

PROPOSITION 5. THEOREM.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles on the other side of the base shall also be equal to one another.



Let ABC be an isosceles triangle, in which
the side AB is equal to the side AC,
and let the straight lines AB, AC be produced to D and E.

Then (i) *the angle* ABC *shall be equal to the angle* ACB;
(ii) *the angle* CBD *shall be equal to the angle* BCE.

Construction. In BD take any point F;
and from AE cut off a part AG equal to AF. I. 3.
Join FC, GB.

Proof. Then in the triangles FAC, GAB,

Because { FA is equal to GA, *Constr.*
and AC is equal to AB, *Hyp.*
also the contained angle at A is common to the two triangles:

therefore the triangle FAC is equal to the triangle GAB in all respects; I. 4.
that is, the base FC is equal to the base GB,
and the angle ACF is equal to the angle ABG,
also the angle AFC is equal to the angle AGB.

Again, because AF is equal to AG,
and AB, a part of AF, is equal to AC, a part of AG; *Hyp.*
therefore the remainder BF is equal to the remainder CG.

Then in the two triangles BFC, CGB,

Because { BF is equal to CG, *Proved.*
and FC is equal to GB, *Proved.*
also the contained angle BFC is equal to the contained angle CGB, *Proved.*

therefore the triangle BFC is equal to the triangle CGB in all respects ; I. 4.

so that the angle FBC is equal to the angle GCB,
and the angle BCF to the angle CBG.

Now it has been shewn that the angle ABG is equal to the angle ACF,

and that the angle CBG, a part of ABG, is equal to the angle BCF, a part of ACF ;

therefore the remaining angle ABC is equal to the remaining angle ACB ; *Ax.* 3.

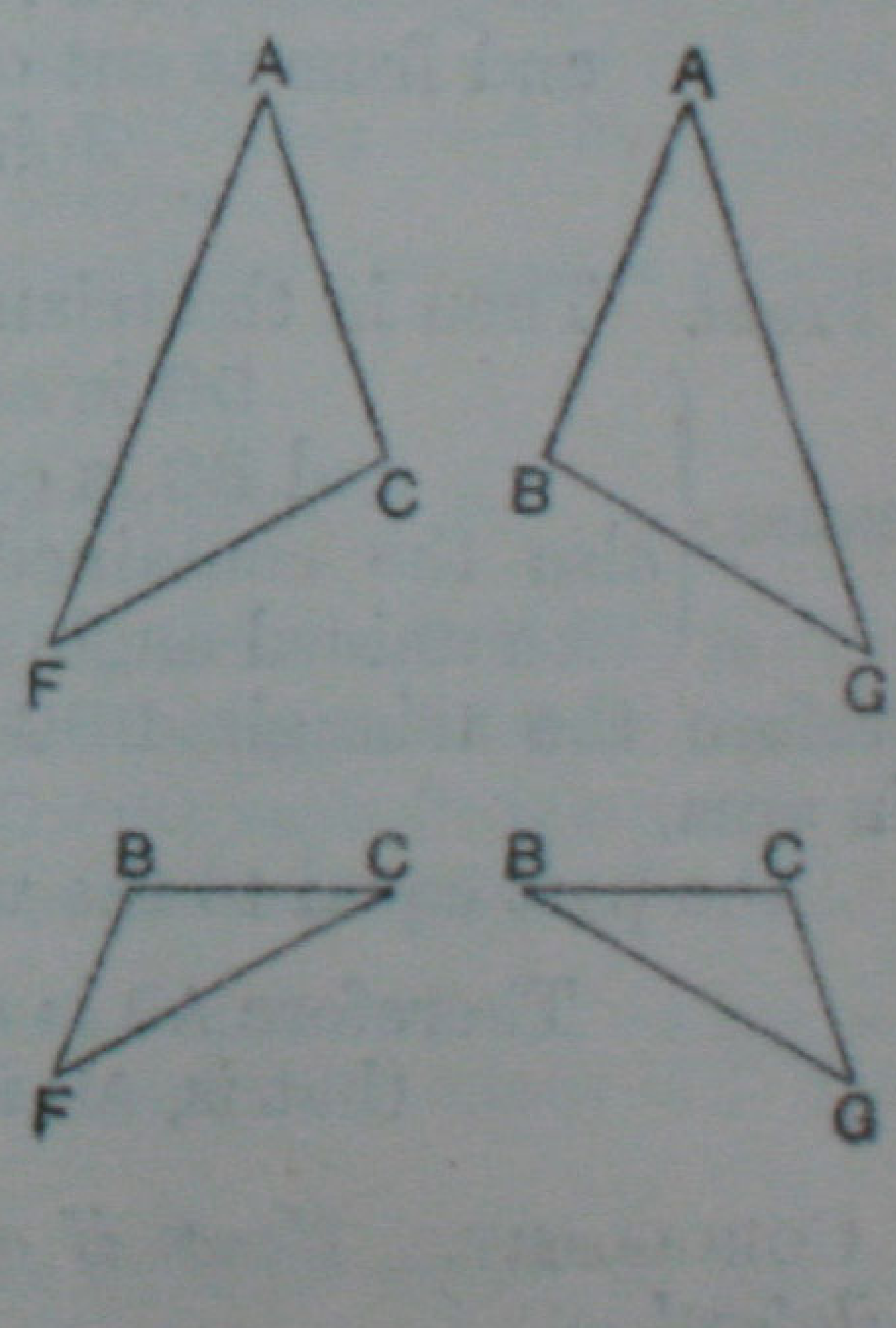
and these are the angles at the base of the triangle ABC.

Also it has been shewn that the angle FBC is equal to the angle GCB ;

and these are the angles on the other side of the base. Q.E.D.

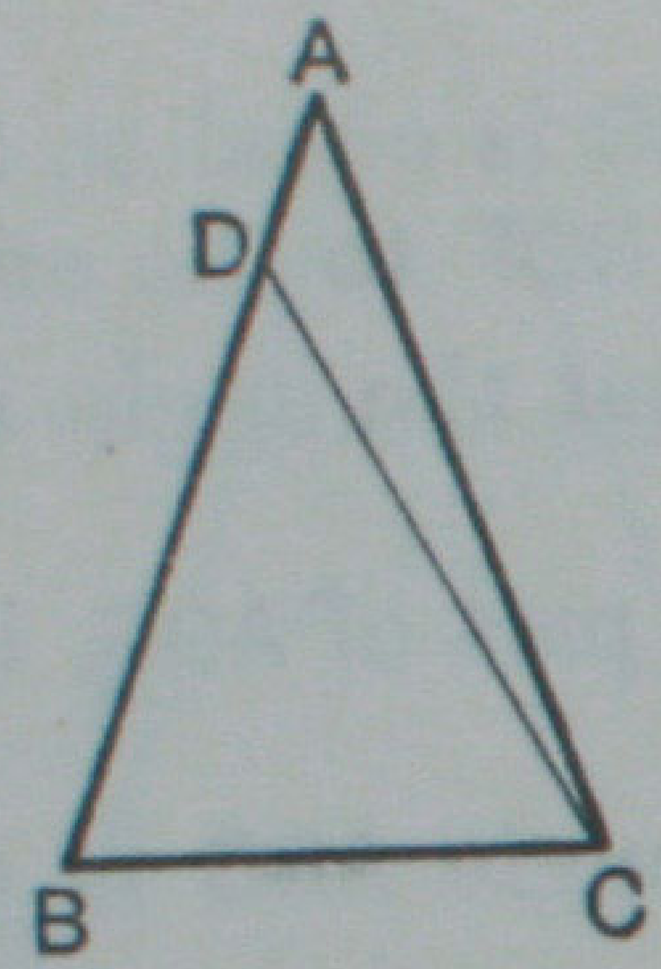
COROLLARY. *Hence if a triangle is equilateral it is also equiangular.*

NOTE. The difficulty which beginners find with this proposition arises from the fact that the triangles to be compared overlap one another in the diagram. This difficulty may be diminished by detaching each pair of triangles from the rest of the figure, as shewn in the margin.



PROPOSITION 6. THEOREM.

If two angles of a triangle be equal to one another, then the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.



Let ABC be a triangle, in which
the angle ABC is equal to the angle ACB.
Then shall the side AC *be equal to the side* AB.

Construction. For if AC be not equal to AB,
one of them must be greater than the other.
If possible, let AB be the greater;
and from it cut off BD equal to AC. I. 3.
Join DC.

Proof. Then in the triangles DBC, ACB,

Because { DB is equal to AC, *Constr.*
and BC is common to both,
also the contained angle DBC is equal to the contained angle ACB; *Hyp.* }

therefore the triangle DBC is equal to the triangle ACB in area, I. 4.
the part equal to the whole; which is absurd. *Ax.* 9.

Therefore AB is not unequal to AC;
that is, AB is equal to AC. Q.E.D.

COROLLARY. *Hence if a triangle is equiangular it is also equilateral.*

NOTE ON PROPOSITIONS 5 AND 6.

The enunciation of a theorem consists of two clauses. The first clause tells us what we are to *assume*, and is called the **hypothesis**; the second tells us what *it is required to prove*, and is called the **conclusion**.

For example, the enunciation of Proposition 5 assumes that in a certain triangle ABC *the side* AB = *the side* AC: this is the *hypothesis*. From this it is required to prove that *the angle* ABC = *the angle* ACB: this is the *conclusion*.

If we interchange the hypothesis and conclusion of a theorem, we enunciate a new theorem which is called the **converse** of the first.

For example, in Prop. 5

it is *assumed* that AB = AC;
it is *required to prove* that the angle ABC = the angle ACB.

Now in Prop. 6

it is *assumed* that the angle ABC = the angle ACB;
it is *required to prove* that AB = AC.

Thus we see that Prop. 6 is the converse of Prop. 5; for *the hypothesis of each is the conclusion of the other.*

In Proposition 6 Euclid employs for the first time an *indirect method of proof* frequently used in geometry. It consists in shewing that the theorem *cannot be untrue*; since, if it were, we should be led to some *impossible conclusion.* This form of proof is known as **Reductio ad Absurdum**, and is most commonly used in demonstrating the converse of some foregoing theorem.

The converse of *all* true theorems are not themselves necessarily true. [See Note on Prop 8.]

EXERCISES ON PROPOSITION 5.

1. ABCD is a rhombus, in which the diagonal BD is drawn: shew that

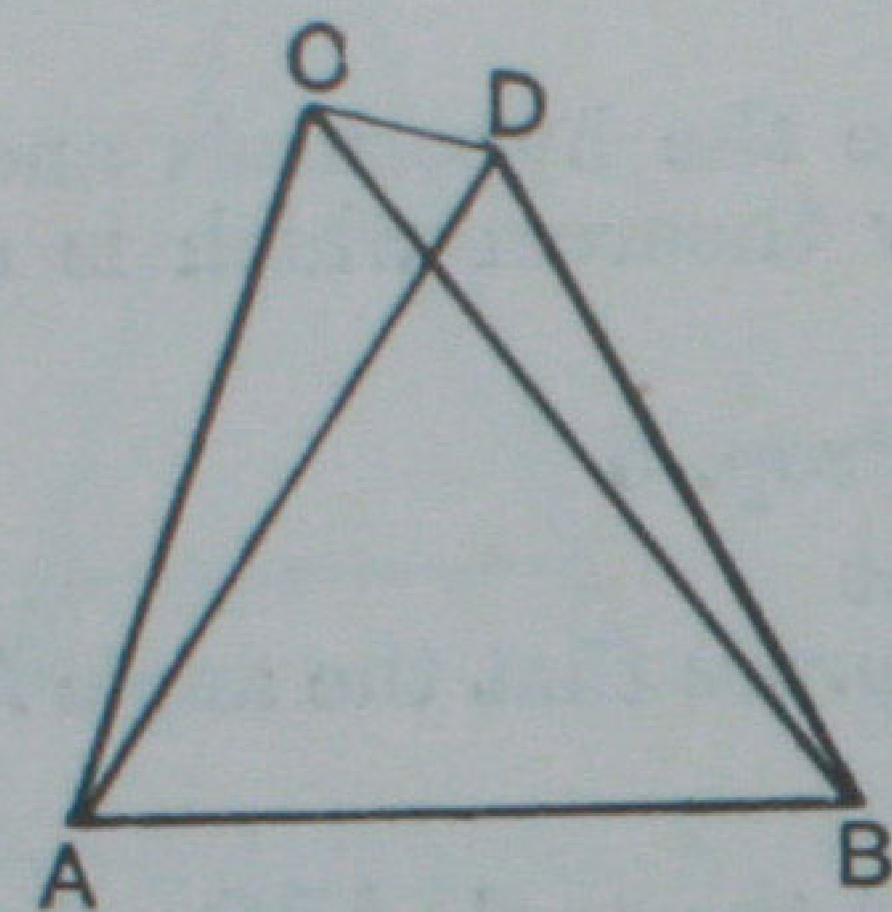
(i) the angle ABD = the angle ADB;
(ii) the angle CBD = the angle CDB;
(iii) the angle ABC = the angle ADC.

2. ABC, DBC are two isosceles triangles drawn on the same base BC, but on *opposite* sides of it: prove (by means of I. 5) that the angle ABD = the angle ACD.

3. ABC, DBC are two isosceles triangles drawn on the same base BC and on *the same* side of it: employ I. 5 to prove that the angle ABD = the angle ACD.

Proposition 7. Theorem.

On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.



If it be possible, on the same base AB, and on the same side of it, let there be two triangles ACB, ADB in which
the side AC is equal to the side AD,
and also the side BC is equal to the side BD.

Case I. When the vertex of each triangle is without the other triangle.

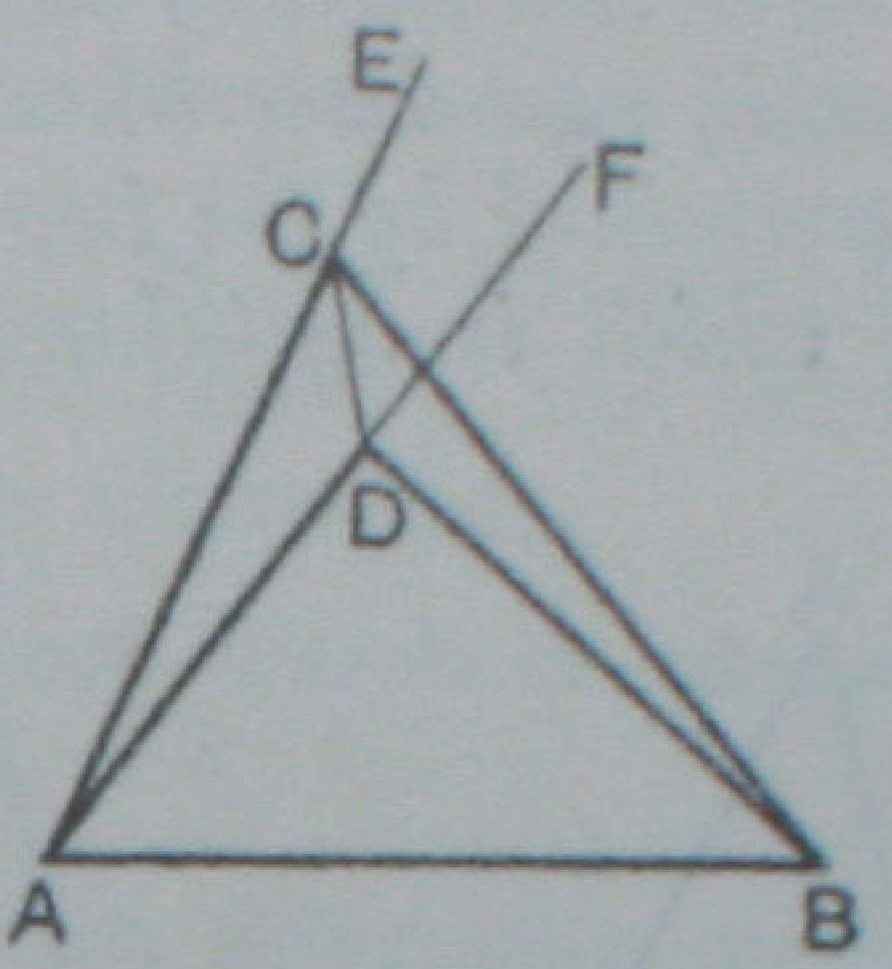
Construction. Join CD.

Proof. Then in the triangle ACD,
because AC is equal to AD, *Hyp.*
therefore the angle ACD is equal to the angle ADC. I. 5.

But the whole angle ACD is greater than its part, the angle BCD;
therefore also the angle ADC is greater than the angle BCD;
still more then is the angle BDC greater than the angle BCD.

Again, in the triangle BCD,
because BC is equal to BD, *Hyp.*
therefore the angle BDC is equal to the angle BCD: I. 5.
but it was shewn to be greater; which is impossible.

CASE II. When one of the vertices, as D, is within the other triangle ACB.



Construction. As before, join CD;
and produce AC, AD to E and F.

Proof. Then in the triangle ACD,
because AC is equal to AD, *Hyp.*
therefore the angle ECD is equal to the angle FDC,
these being the angles on the other side of the base. I. 5.
But the angle ECD is greater than its part, the angle BCD; therefore the angle FDC is also greater than the angle BCD:
still more then is the angle BDC greater than the angle BCD.

Again, in the triangle BCD,
because BC is equal to BD, *Hyp.*
therefore the angle BDC is equal to the angle BCD: I. 5.
but it has been shewn to be greater; which is impossible.

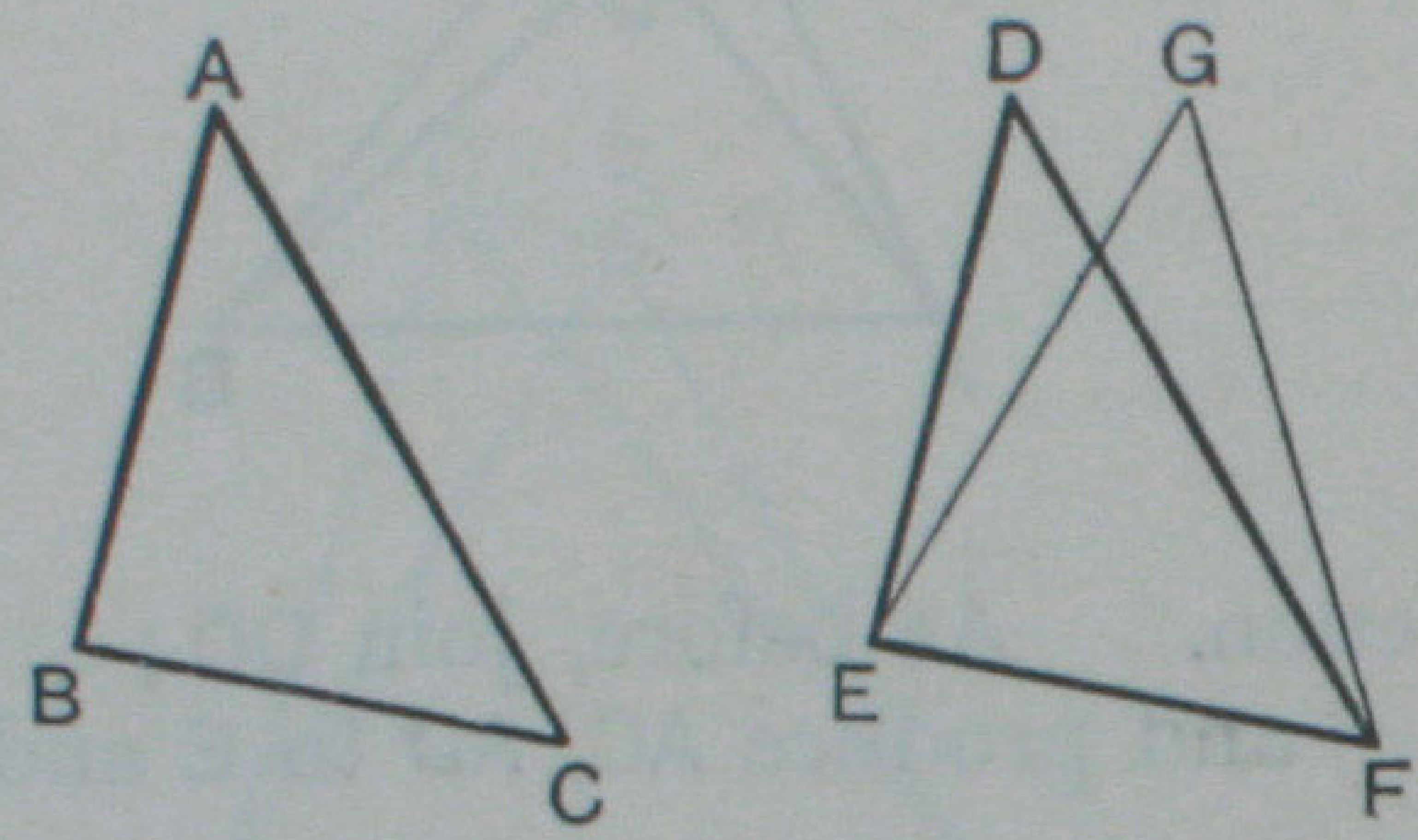
The case in which the vertex of one triangle is on a side of the other needs no demonstration.

Therefore AC cannot be equal to AD, and *at the same time*, BC equal to BD. Q.E.D.

NOTE. The sides AC, AD are called **conterminous sides**; similarly the sides BC, BD are conterminous.

PROPOSITION 8. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, then the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides of the other.



Let ABC, DEF be two triangles, in which
the side AB is equal to the side DE,
the side AC is equal to the side DF,
and the base BC is equal to the base EF.

Then shall the angle BAC *be equal to the angle* EDF.

Proof. If the triangle ABC be applied to the triangle DEF, so that the point B falls on the point E, and the base BC along the base EF;
then because BC is equal to EF, *Hyp.*
therefore the point C must coincide with the point F.

Then since BC coincides with EF,
it follows that BA and AC must coincide with ED and DF: for if they did not, but took some other position, as EG, GF, then on the same base EF, and on the same side of it, there would be two triangles EDF, EGF, having their *conterminous* sides equal: namely ED equal to EG, and FD equal to FG.
But this is impossible. I. 7.

Therefore the sides BA, AC coincide with the sides ED, DF.
That is, the angle BAC coincides with the angle EDF, and is therefore equal to it. *Ax.* 8.

Q.E.D.

NOTE 1. In this Proposition the three sides of one triangle are given equal respectively to the three sides of the other; and from this it is shewn that the two triangles may be made *to coincide with one another*.

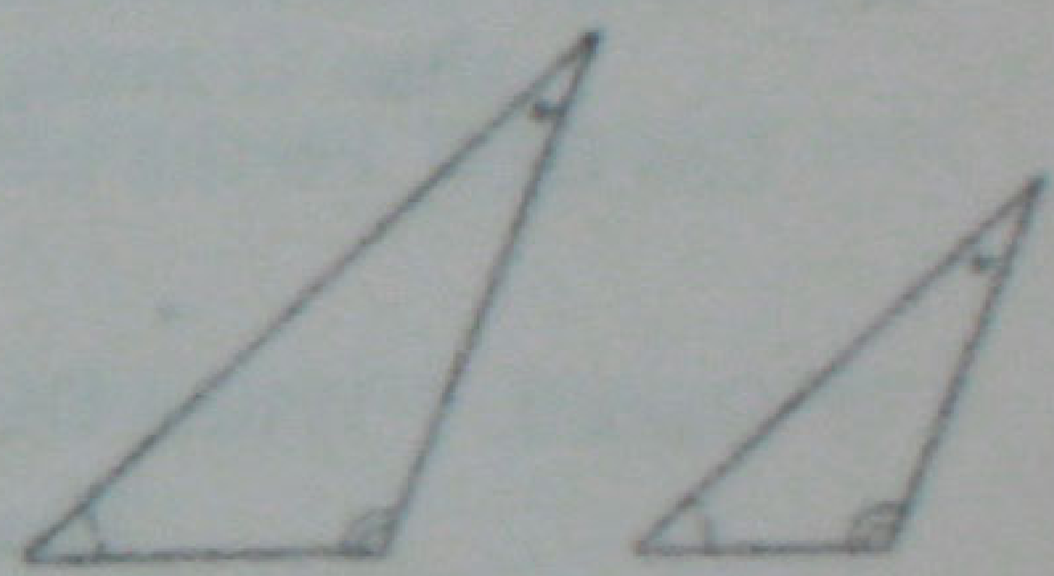
Hence we are led to the following important Corollary.

COROLLARY. *If in two triangles the three sides of the one are equal to the three sides of the other, each to each, then the triangles are equal in all respects.*

[An alternative proof, which is independent of Prop. 7, will be found on page 26.]

NOTE 2. Proposition 8 furnishes an instance of a true theorem of which the *converse* is not necessarily true.

It is proved above that *if the sides of one triangle are severally equal to the sides of another, then the angles of the first triangle are severally equal to the angles of the second.*



The *converse* of this enunciation would be as follows: *If the angles of one triangle are severally equal to the angles of another, then the sides of the first triangle are equal to the sides of the second.*

But this, as the diagram in the margin shews, is by no means necessarily true.

EXERCISES ON PROPOSITION 8.

1. Shew (by drawing a diagonal) that the opposite angles of a rhombus are equal.

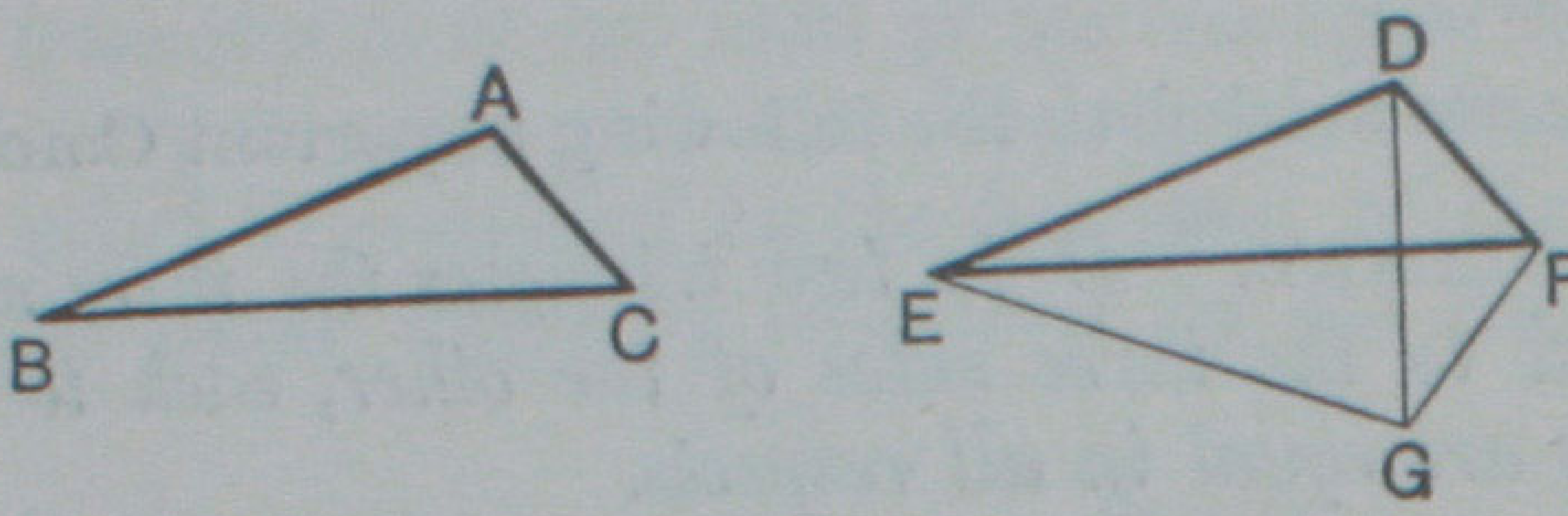
2. If ABCD is a quadrilateral, in which AB = CD and AD = CB, prove that the angle ADC = the angle ABC.

3. If ABC and DBC are two isosceles triangles drawn on the same base BC, prove (by means of I. 8) that the angle ABD = the angle ACD, taking (i) the case where the triangles are on the *same* side of BC, (ii) the case where they are on *opposite* sides of BC.

4. If ABC, DBC are two isosceles triangles drawn on opposite sides of the same base BC, and if AD be joined, prove that each of the angles BAC, BDC will be divided into two equal parts.

5. If in the figure of Ex. 4 the line AD meets BC in E, prove that BE = EC.

PROPOSITION 8. ALTERNATIVE PROOF.



Let ABC and DEF be two triangles, which have the sides BA, AC equal respectively to the sides ED, DF, and the base BC equal to the base EF.

Then shall the angle BAC *be equal to the angle* EDF.

For apply the triangle ABC to the triangle DEF, so that B may fall on E, and BC along EF, and so that the point A may be on the side of EF remote from D;

then C must fall on F, since BC is equal to EF.

Let GEF be the new position of the triangle ABC.

Join DG.

CASE I. When DG intersects EF.

Then because ED = EG,

∴ the angle EDG = the angle EGD. I. 5.

Again because FD = FG,

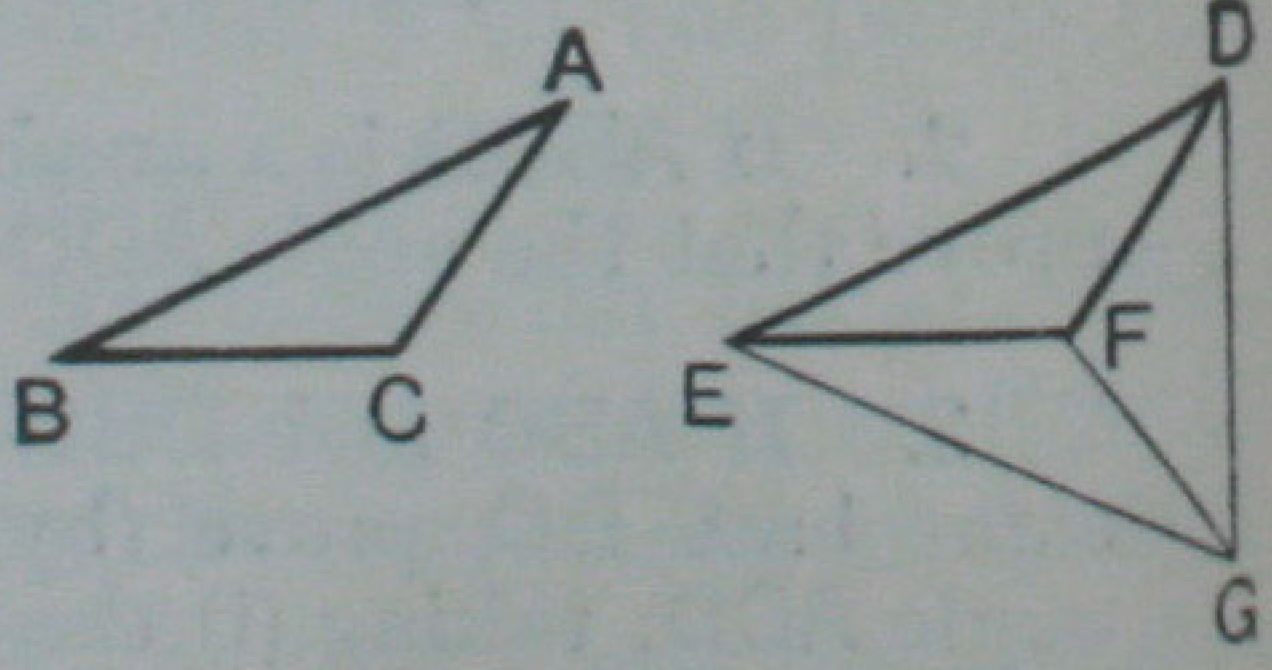
∴ the angle FDG = the angle FGD. I. 5.

Hence the whole angle EDF = the whole angle EGF; *Ax.* 2.

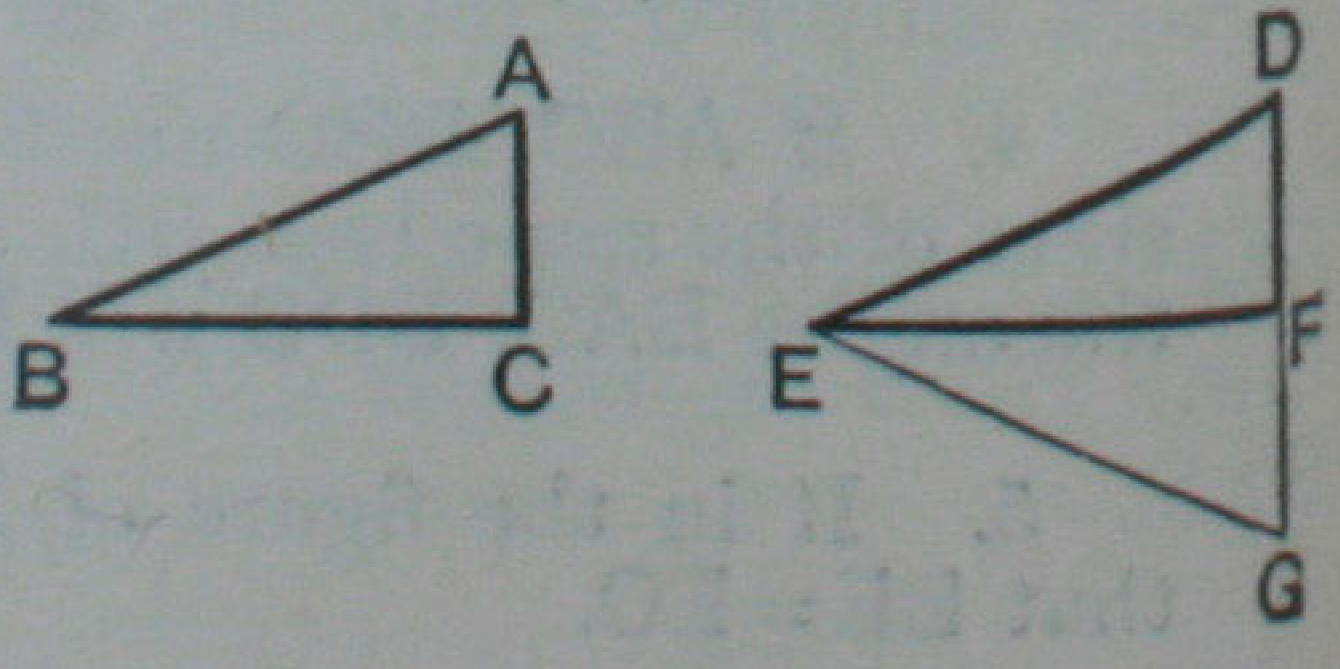
that is, the angle EDF = the angle BAC.

Two cases remain which may be dealt with in a similar manner: namely,

CASE II. When DG meets EF produced.



CASE III. When one pair of sides, as DF, FG are in one straight line.



QUESTIONS AND EXERCISES FOR REVISION.

1. Define *adjacent angles, a right angle, vertically opposite angles.*

2. Explain the words *enunciation, hypothesis, conclusion.*

3. Distinguish between the meanings of the following statements:

(i) then AB *is* equal to PQ;
(ii) then AB *shall be* equal to PQ.

4. When are two theorems said to be *converse* to one another. Give an example.

5. Shew by an example that the converse of a true theorem is not itself necessarily true.

6. What is a *corollary*? Quote the corollary to Proposition 5; and shew how its truth follows from that proposition.

7. Name the six *parts* of a triangle. When are triangles said to be *equal in all respects*?

8. What do you understand by the expression *geometrical magnitudes*? Give examples?

9. What is meant by *superposition*? Explain the test by which Euclid determines if two geometrical magnitudes are equal to one another. Illustrate by an example.

10. Quote and explain the *third postulate.* What restrictions does Euclid impose on the use of compasses, and what problems are thereby made necessary?

11. Define an *axiom.* Quote the axioms referred to (i) in Proposition 2; (ii) in Proposition 7.

12. Prove by the method of *superposition* that two squares are equal in area, if a side of one is equal to a side of the other.

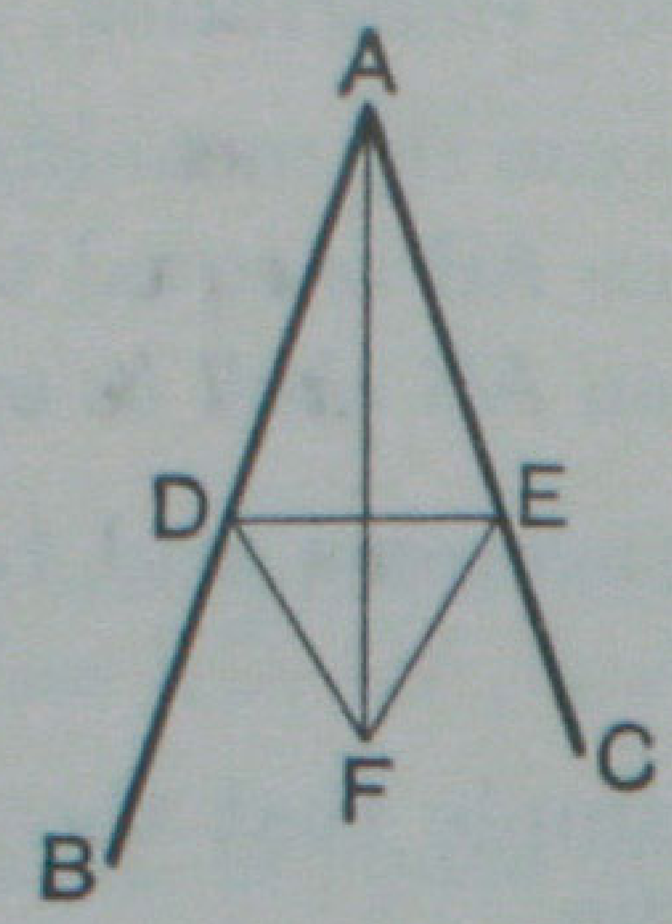
13. Two quadrilaterals ABCD, EFGH have the sides AB, BC, CD, DA equal respectively to the sides EF, FG, GH, HE, and have also the angle BAD equal to the angle FEH. Shew that the figures may be made to coincide with one another.

14. AB, AC are the equal sides of an isosceles triangle ABC; and L, M, N are the middle points of AB, BC, and CA respectively: prove that

(i) LM = MN. (ii) BN = CL.
(iii) the angle ALM = the angle ANM.

Proposition 9. Problem.

To bisect a given rectilineal angle, that is, to divide it into two equal parts.



Let BAC be the given angle.
It is required to bisect the angle BAC.

Construction. In AB take any point D;
and from AC cut off AE equal to AD. I. 3.
Join DE;
and on DE, on the side remote from A, describe an equilateral triangle DEF. I. 1.
Join AF.
Then shall the straight line AF *bisect the angle* BAC.

Proof. For in the two triangles DAF, EAF,

Because { DA is equal to EA, *Constr.*
and AF is common to both;
and the third side DF is equal to the third side EF; } *Def.* 24.

therefore the angle DAF is equal to the angle EAF. I. 8.
Therefore the given angle BAC is bisected by the straight line AF.
Q.E.F.

EXERCISES.

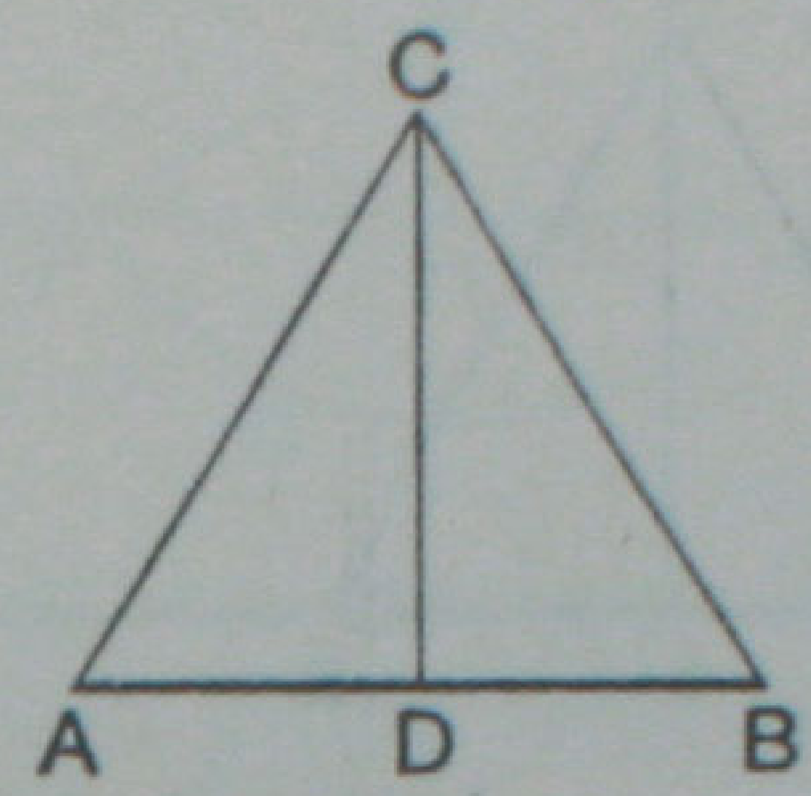
1. If in the above figure the equilateral triangle DFE were described on the same side of DE as A, what different cases would arise? And under what circumstances would the construction fail?

2. In the same figure, shew that AF also bisects the angle DFE.

3. Divide an angle into four equal parts.

PROPOSITION 10. PROBLEM.

To bisect a given finite straight line, that is, to divide it into two equal parts.



Let AB be the given straight line.
It is required to divide AB *into two equal parts.*

Constr. On AB describe an equilateral triangle ABC; I. 1.
and bisect the angle ACB by the straight line CD, meeting AB at D. I. 9.

Then shall AB *be bisected at the point* D.

Proof. For in the triangles ACD, BCD,

Because { AC is equal to BC, *Def.* 24.
and CD is common to both;
also the contained angle ACD is equal to the contained angle BCD; *Constr.* }

therefore the triangle ACD is equal to the triangle BCD in all respects: I. 4.

so that the base AD is equal to the base BD.

Therefore the straight line AB is bisected at the point D

Q.E.F.

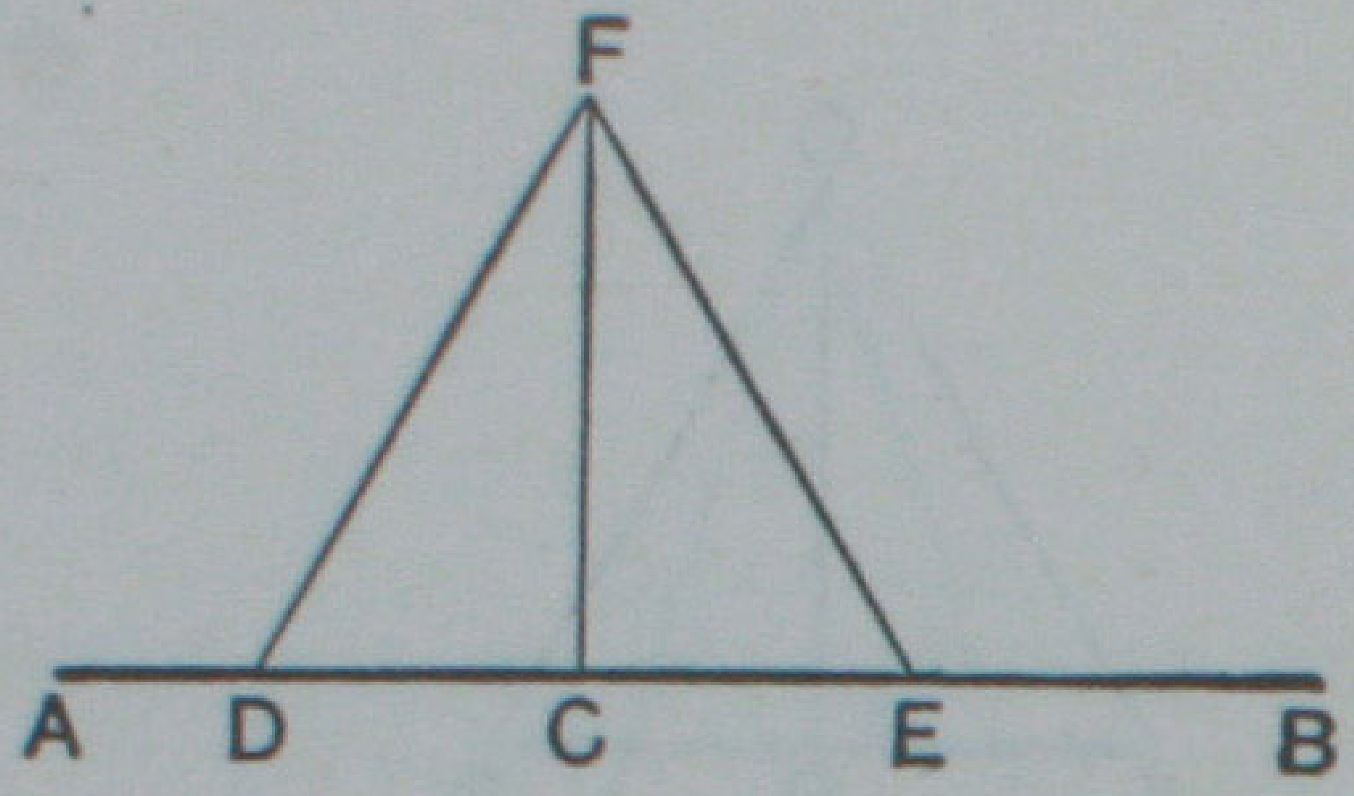
EXERCISES.

1. Shew that the straight line which bisects the vertical angle of an isosceles triangle, also bisects the base.

2. On a given base describe an isosceles triangle such that the sum of its equal sides may be equal to a given straight line.

Proposition 11. Problem.

To draw a straight line at right angles to a given straight line, from a given point in the same.



Let AB be the given straight line, and C the given point in it.

It is required to draw from C *a straight line at right angles to* AB.

Construction. In AC take any point D,
and from CB cut off CE equal to CD. I. 3.
On DE describe the equilateral triangle DFE. I. 1.
Join CF.
Then shall CF *be at right angles to* AB.

Proof. For in the triangles DCF, ECF,
Because { DC is equal to EC, *Constr.*
and CF is common to both;
and the third side DF is equal to the third side EF: *Def.* 24. }
therefore the angle DCF is equal to the angle ECF: I. 8.
and these are adjacent angles.

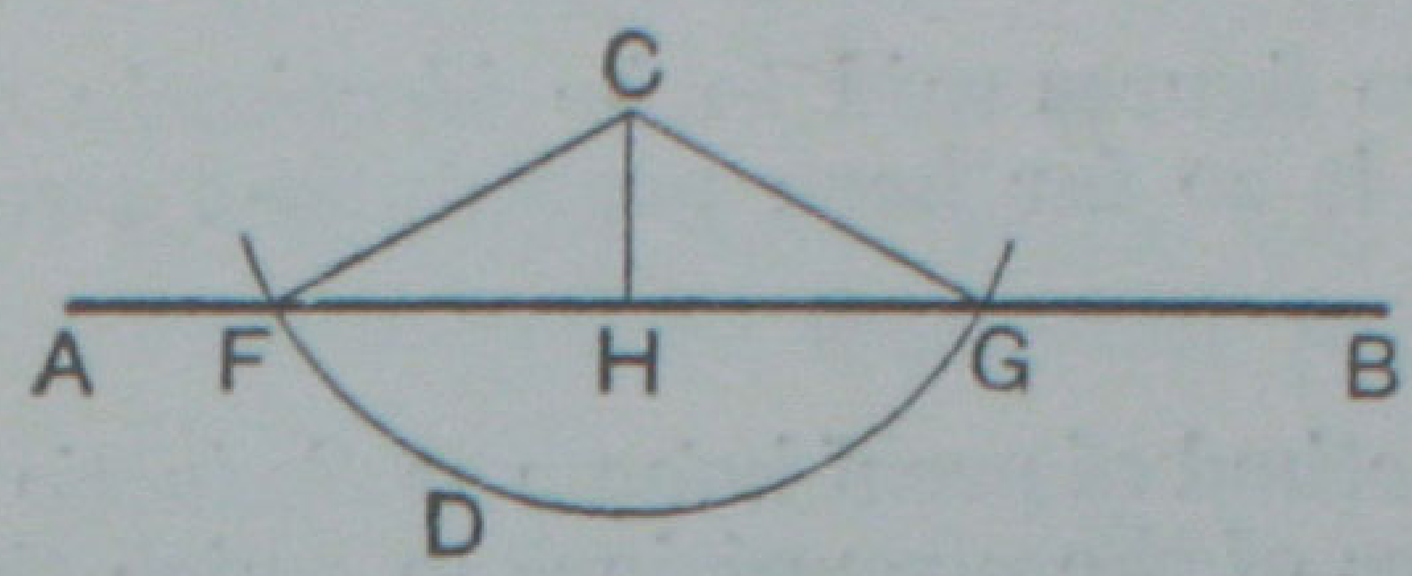
But when one straight line, standing on another, makes the adjacent angles equal, each of these angles is called a right angle; *Def.* 10.
therefore each of the angles DCF, ECF is a right angle.
Therefore CF is at right angles to AB,
and has been drawn from a point C in it. Q.E.F.

EXERCISE.

In the figure of the above proposition, shew that *any* point in FC, or FC produced, is equidistant from D and E.

PROPOSITION 12. PROBLEM.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.



Let AB be the given straight line of unlimited length, and let C be the given point without it.

It is required to draw from C *a straight line perpendicular to* AB.

Construction. On the side of AB remote from C take any point D;
and with centre C, and radius CD, describe the circle FDG, cutting AB at F and G

Bisect FG at H; I. 10.
and join CH.

Then shall CH *be perpendicular to* AB.

Join CF and CG.

Proof. Then in the triangles FHC, GHC,

Because { FH is equal to GH, *Constr.*
and HC is common to both;
and the third side CF is equal to the third side CG, being radii of the circle FDG; *Def.* 15.

therefore the angle CHF is equal to the angle CHG; I. 8.
and these are adjacent angles.

But when one straight line, standing on another, makes the adjacent angles equal, each of these angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it. *Def.* 10.

Therefore CH is perpendicular to AB,
and has been drawn from the point C without it. Q.E.F.

NOTE. The line AB must be of unlimited length, that is, capable of production to an indefinite length in either direction, to ensure its being intersected in two points by the circle FDG.

QUESTIONS AND EXERCISES FOR REVISION.

1. Distinguish between a *problem* and a *theorem*.

2. When are two figures said to be *identically equal*? Under what conditions has it so far been proved that two *triangles* are identically equal?

3. Explain the method of proof known as *Reductio ad Absurdum*. Quote the enunciations of the propositions in which this method has so far been used.

4. Quote the corollaries of Propositions 5 and 6, and shew that each is the converse of the other.

5. What is meant by saying that Euclid's reasoning is *deductive*? Shew, for instance, that the proof of Proposition 5 is a deductive argument.

6. Two forts defend the mouth of a river, one on each side; the forts are 4000 yards apart, and their guns have a range of 3000 yards. Taking *one inch* to represent a length of 1000 yards, draw a diagram shewing what part of the river is exposed to the fire of both forts.

7. Define *the perimeter* of a rectilineal figure. A square and an equilateral triangle each have a perimeter of 3 feet: compare the lengths of their sides.

8. Shew how to draw a rhombus each of whose sides is equal to a given straight line PQ, which is also to be one diagonal of the figure.

9. A and B are two given points. Shew how to draw a rhombus having A and B as opposite vertices, and having each side equal to a given line PQ. Is this always possible?

10. Two circles are described with the same centre O; and two radii OA, OB are drawn to the inner circle, and produced to cut the outer circle at D and E: prove that

(i) DB = EA;

(ii) the angle BAD = the angle ABE;

(iii) the angle ODB = the angle OEA.

EXERCISES ON PROPOSITIONS 1 TO 12.

1. Shew that the straight line which joins the vertex of an isosceles triangle to the middle point of the base is perpendicular to the base.

2. Shew that the straight lines which join the extremities of the base of an isosceles triangle to the middle points of the opposite sides, are equal to one another.

3. Two given points in the base of an isosceles triangle are equidistant from the extremities of the base: shew that they are also equidistant from the vertex.

4. If the opposite sides of a quadrilateral are equal, shew that the opposite angles are also equal.

5. Any two isosceles triangles XAB, YAB stand on the same base AB: shew that the angle XAY is equal to the angle XBY; and if XY be joined, that the angle AXY is equal to the angle BXY.

6. Shew that the opposite angles of a rhombus are bisected by the diagonal which joins them.

7. Shew that the straight lines which bisect the base angles of an isosceles triangle form with the base a triangle which is also isosceles.

8. ABC is an isosceles triangle having AB equal to AC; and the angles at B and C are bisected by straight lines which meet at O: shew that OA bisects the angle BAC.

9. Shew that the triangle formed by joining the middle points of the sides of an equilateral triangle is also equilateral.

10. The equal sides BA, CA of an isosceles triangle BAC are produced beyond the vertex A to the points E and F, so that AE is equal to AF; and FB, EC are joined: shew that FB is equal to EC.

11. Shew that the diagonals of a rhombus bisect one another at right angles.

12. In the equal sides AB, AC of an isosceles triangle ABC two points X and Y are taken, so that AX is equal to AY; and CX and BY are drawn intersecting in O: shew that

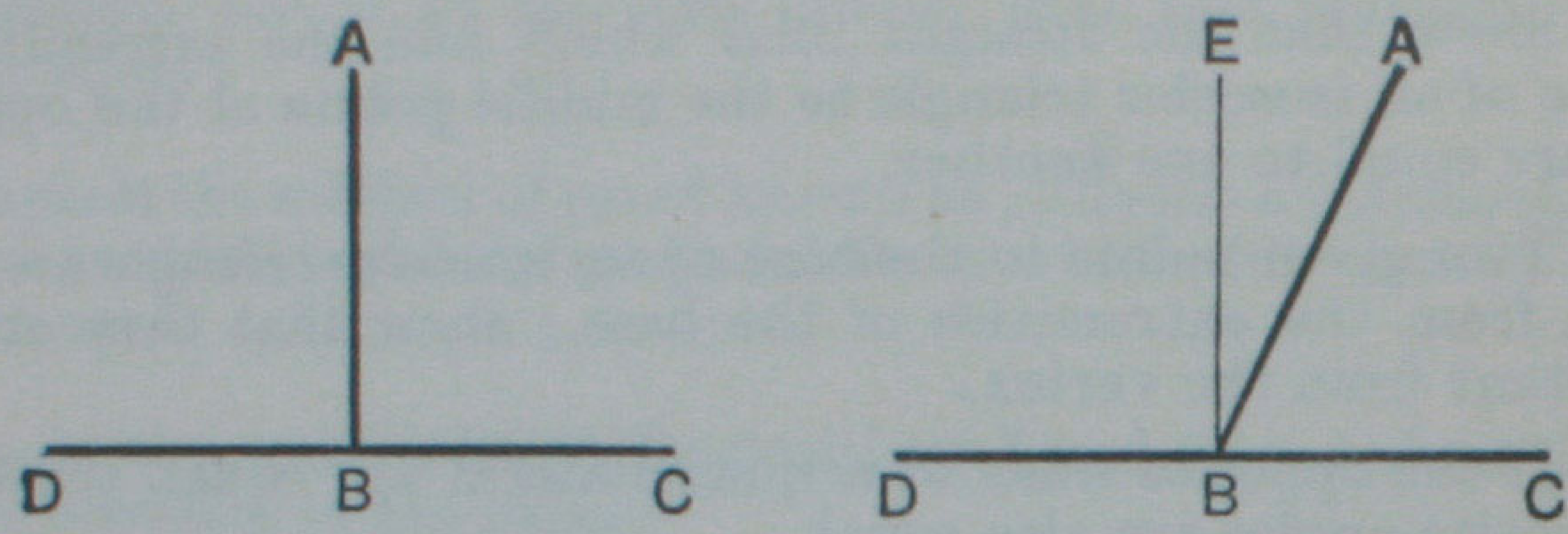
(i) the triangle BOC is isosceles;
(ii) AO bisects the vertical angle BAC;
(iii) AO, if produced, bisects BC at right angles.

13. Describe an isosceles triangle, having given the base and the length of the perpendicular drawn from the vertex to the base.

14. In a given straight line find a point that is equidistant from two given points. In what case is this impossible?

Proposition 13. Theorem.

The adjacent angles which one straight line makes with another straight line, on one side of it, are either two right angles or are together equal to two right angles.



Let the straight line AB meet the straight line DC.

Then the adjacent angles DBA, ABC *shall be either two right angles, or together equal to two right angles.*

Case I. For if the angle DBA is equal to the angle ABC, each of them is a right angle. *Def.* 10.

Case II. But if the angle DBA is not equal to the angle ABC,

from B draw BE at right angles to CD. I. 11.

Proof. Now the angle DBA is made up of the two angles DBE, EBA;

to each of these equals add the angle ABC;

then the two angles DBA, ABC are together equal to the three angles DBE, EBA, ABC. *Ax.* 2.

Again, the angle EBC is made up of the two angles EBA, ABC;

to each of these equals add the angle DBE;

then the two angles DBE, EBC are together equal to the three angles DBE, EBA, ABC. *Ax.* 2.

But the two angles DBA, ABC have been shewn to be equal to the same three angles;

therefore the angles DBA, ABC are together equal to the angles DBE, EBC. *Ax.* 1.

But the angles DBE, EBC are two right angles; *Constr.*

therefore the angles DBA, ABC are together equal to two right angles.

Q.E.D.

DEFINITIONS.

(i) The **complement** of an acute angle is its *defect from* a right angle, that is, the angle by which it falls short of a right angle.

Thus two angles are **complementary**, when their sum is a right angle.

(ii) The **supplement** of an angle is its *defect from* two right angles, that is, the angle by which it falls short of two right angles.

Thus two angles are **supplementary**, when their sum is two right angles.

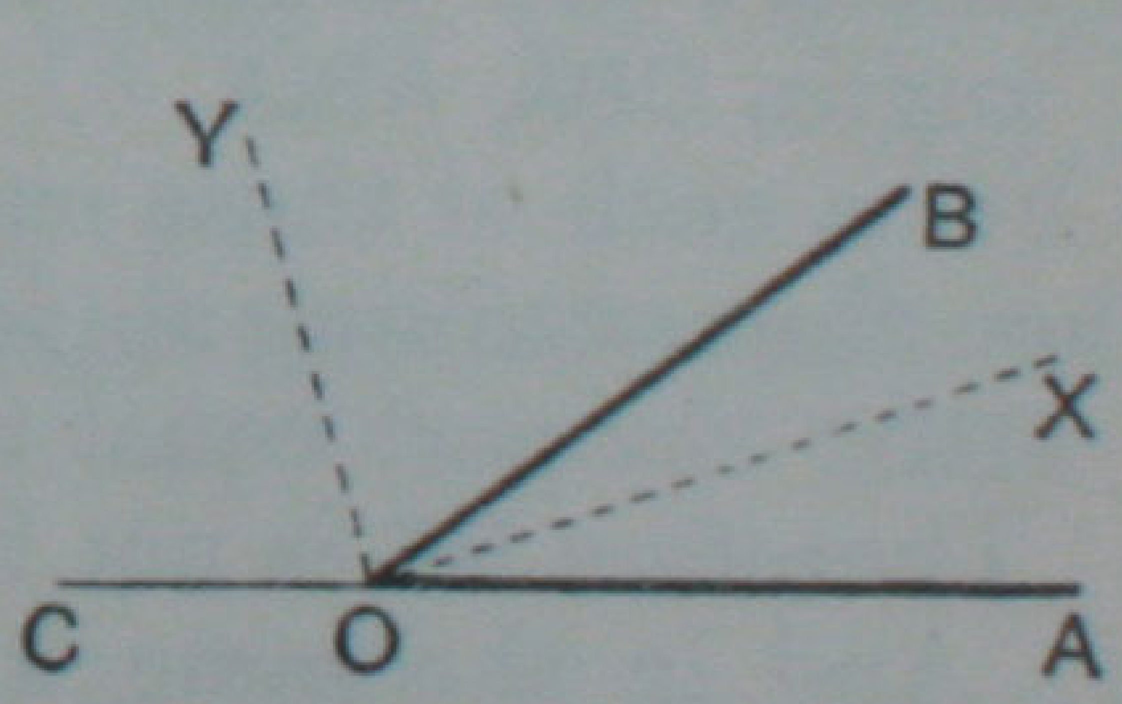
Corollary. *Angles which are complementary or supplementary to the same angle are equal to one another.*

EXERCISES.

1. If the two exterior angles formed by producing a side of a triangle both ways are equal, shew that the triangle is isosceles.

2. *The bisectors of the adjacent angles which one straight line makes with another contain a right angle.*

Note In the adjoining diagram AOB is a given angle; and one of its arms AO is produced to C: the adjacent angles AOB, BOC are bisected by OX, OY.



Then OX and OY are called respectively the **internal** and **external bisectors** of the angle AOB.

Hence Exercise 2 may be thus enunciated:

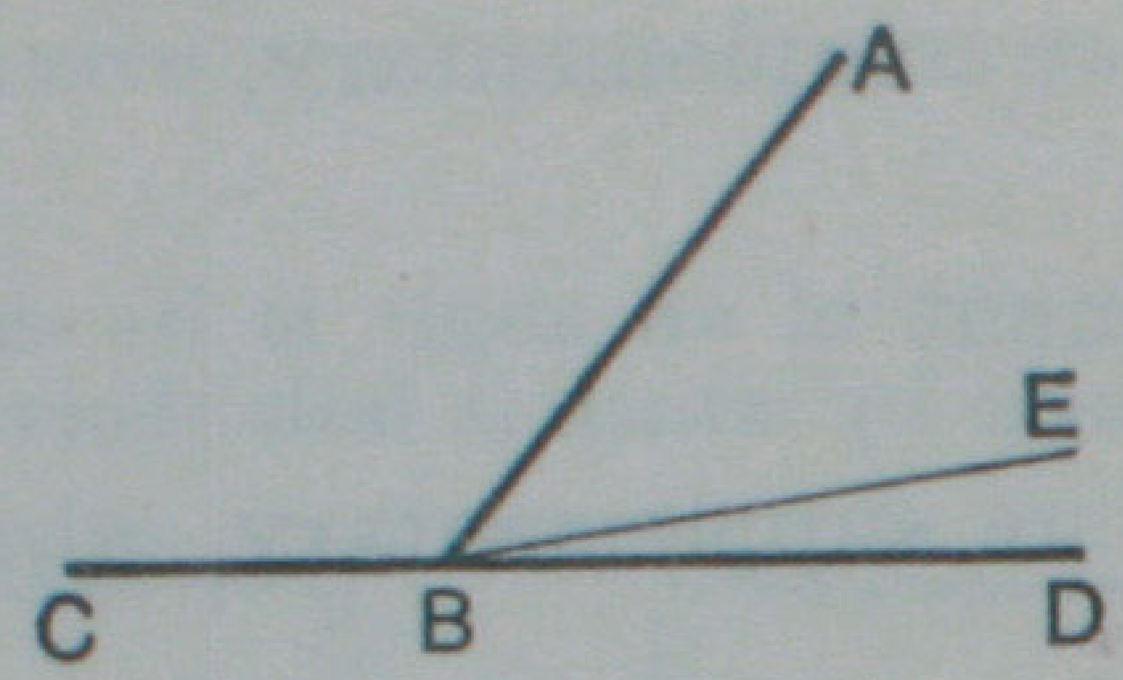
The internal and external bisectors of an angle are at right angles to one another.

3. Shew that the angles AOX and COY are complementary.

4. Shew that the angles BOX and COX are supplementary; and also that the angles AOY and BOY are supplementary.

PROPOSITION 14. THEOREM.

If, at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles, then these two straight lines shall be in one and the same straight line.



At the point B in the straight line AB, let the two straight lines BC, BD, on the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles.

Then BD *shall be in the same straight line with* BC.

Proof. For if BD be not in the same straight line with BC, if possible, let BE be in the same straight line with BC.

Then because AB meets the straight line CBE,
therefore the adjacent angles CBA, ABE are together equal to two right angles. I. 13.
But the angles CBA, ABD are also together equal to two right angles. *Hyp.*
Therefore the angles CBA, ABE are together equal to the angles CBA, ABD. *Ax.* 11.
From each of these equals take the common angle CBA;
then the remaining angle ABE is equal to the remaining angle ABD; the part equal to the whole; which is impossible.
Therefore BE is not in the same straight line with BC.

And in the same way it may be shewn that no other line but BD can be in the same straight line with BC.

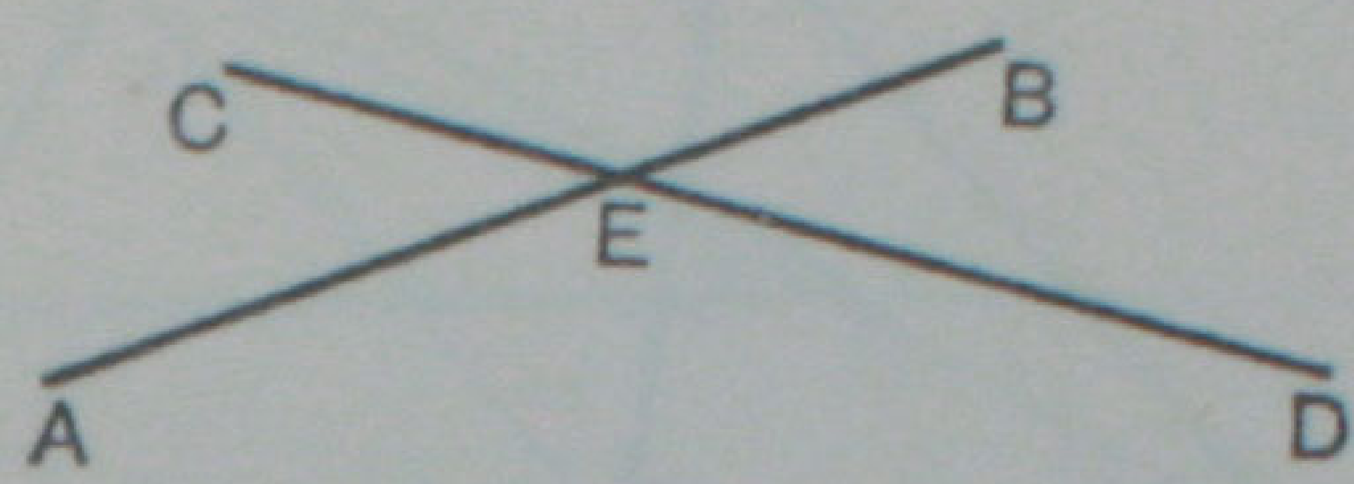
Therefore BD is in the same straight line with BC. Q.E.D.

EXERCISE.

ABCD is a rhombus; and the diagonal AC is bisected at O. If O is joined to the angular points B and D; shew that OB and OD are in one straight line.

PROPOSITION 15. THEOREM.

If two straight lines intersect one another, then the vertically opposite angles shall be equal.



Let the two straight lines AB, CD cut one another at the point E.

Then (i) *the angle* AEC *shall be equal to the angle* DEB;
(ii) *the angle* CEB *shall be equal to the angle* AED.

Proof. Because AE meets the straight line CD,
therefore the adjacent angles CEA, AED are together equal to two right angles. I. 13.

Again, because DE meets the straight line AB,
therefore the adjacent angles AED, DEB are together equal to two right angles. I. 13.

Therefore the angles CEA, AED are together equal to the angles AED, DEB.

From each of these equals take the common angle AED;
then the remaining angle CEA is equal to the remaining angle DEB. *Ax.* 3.

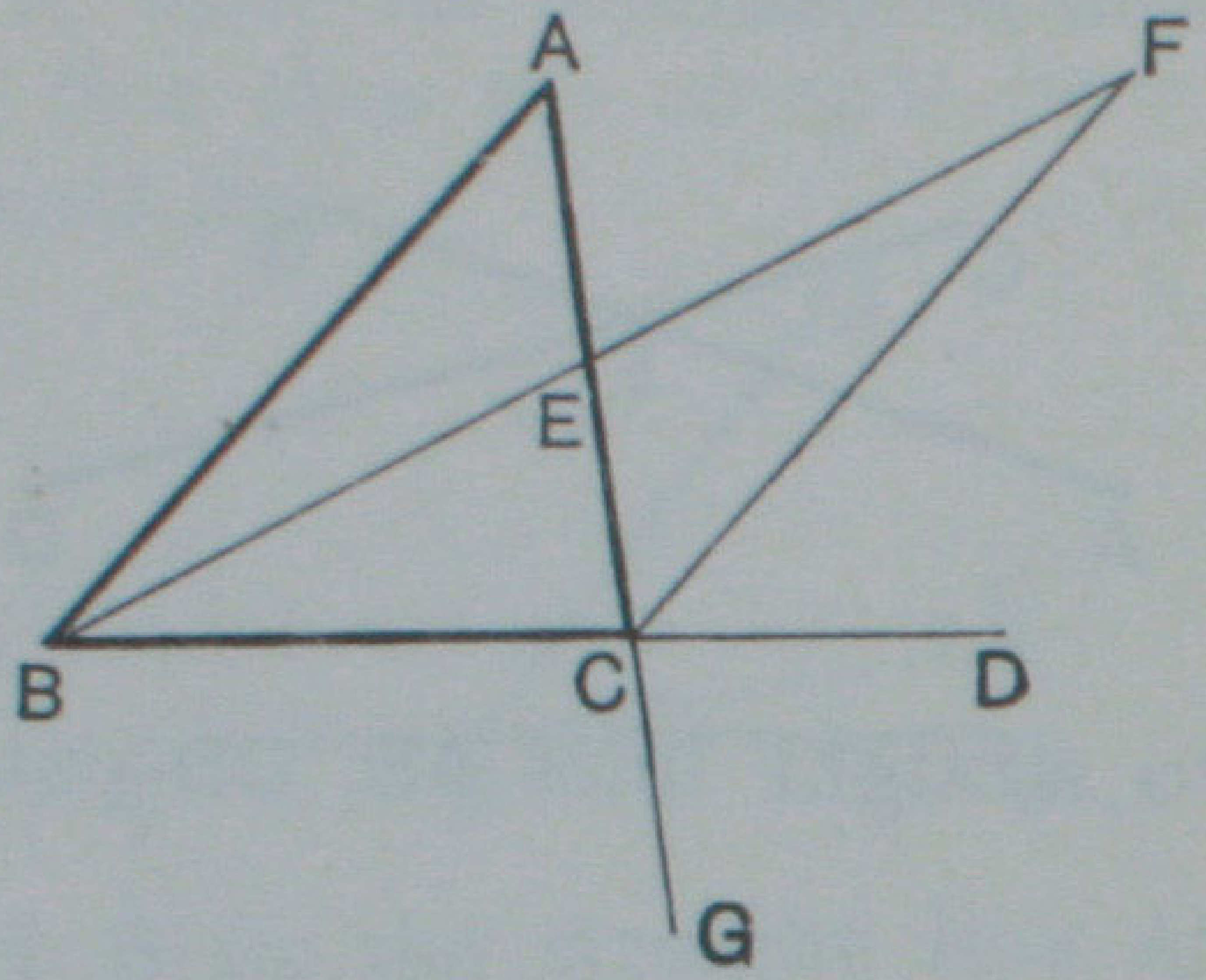
In the same way it may be proved that the angle CEB is equal to the angle AED. Q.E.D.

COROLLARY 1. *From this it follows that, if two straight lines cut one another, the four angles so formed are together equal to four right angles.*

COROLLARY 2. *Consequently, when any number of straight lines meet at a point, the sum of the angles made by consecutive lines is equal to four right angles.*

PROPOSITION 16. THEOREM.

If one side of a triangle be produced, then the exterior angle shall be greater than either of the interior opposite angles.



Let ABC be a triangle, and let BC be produced to D.

Then shall the exterior angle ACD *be greater than either of the interior opposite angles* ABC, BAC.

Construction. Bisect AC at E; I. 10.

Join BE; and produce it to F, making EF equal to BE. I. 3.

Join FC.

Proof. Then in the triangles AEB, CEF,

Because { AE is equal to CE, *Constr.*
and EB is equal to EF; *Constr.*
also the angle AEB is equal to the vertically opposite angle CEF; I. 15. }

therefore the triangle AEB is equal to the triangle CEF in all respects: I. 4.

so that the angle BAE is equal to the angle ECF.

But the angle ECD is greater than its part, the angle ECF;

therefore the angle ECD is greater than the angle BAE;

that is, the angle ACD is greater than the angle BAC.

In the same way, if BC be bisected, and the side AC produced to G, it may be proved that the angle BCG is greater than the angle ABC.

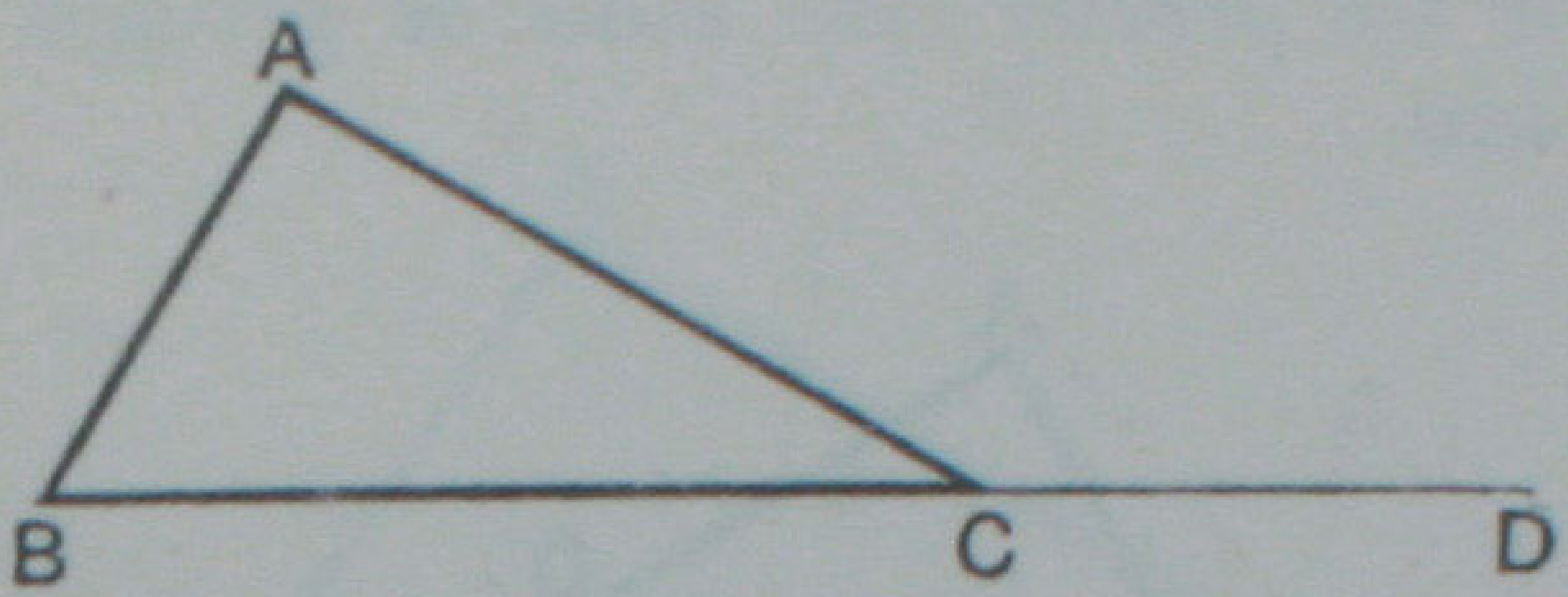
But the angle BCG is equal to the angle ACD: I. 15

therefore also the angle ACD is greater than the angle ABC.

Q.E.D.

PROPOSITION 17. THEOREM.

Any two angles of a triangle are together less than two right angles.



Let ABC be a triangle.

Then shall any two of the angles of the triangle ABC *be together less than two right angles.*

Construction. Produce the side BC to D.

Proof. Then because BC, a side of the triangle ABC, is produced to D;

therefore the exterior angle ACD is greater than the interior opposite angle ABC. I. 16.

To each of these add the angle ACB:

then the angles ACD, ACB are together greater than the angles ABC, ACB. *Ax.* 4.

But the adjacent angles ACD, ACB are together equal to two right angles. I. 13.

Therefore the angles ABC, ACB are together less than two right angles.

Similarly it may be shewn that the angles BAC, ACB, as also the angles CAB, ABC, are together less than two right angles. Q.E.D.

NOTE. It follows from this Proposition that *every triangle must have at least two acute angles:* for if one angle is obtuse, or a right angle, each of the other angles must be less than a right angle.

EXERCISES.

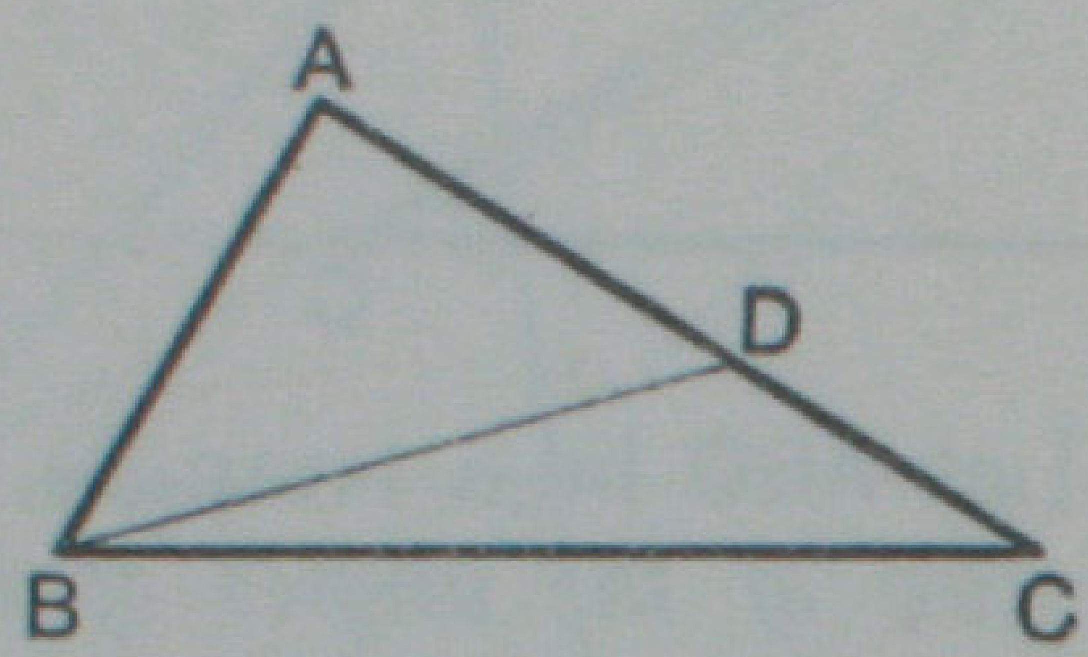
1. Enunciate this Proposition so as to shew that it is the converse of Axiom 12.

2. If any side of a triangle is produced both ways, the exterior angles so formed are together greater than two right angles.

3. Shew how a proof of Proposition 17 may be obtained by joining each vertex in turn to any point in the opposite side.

Proposition 18. Theorem.

If one side of a triangle be greater than another, then the angle opposite to the greater side shall be greater than the angle opposite to the less.



Let ABC be a triangle, in which the side AC is greater than the side AB.

Then shall the angle ABC *be greater than the angle* ACB.

Construction. From AC cut off a part AD equal to AB. I. 3.
Join BD.

Proof. Then in the triangle ABD,
because AB is equal to AD,
therefore the angle ABD is equal to the angle ADB. I. 5.

But the exterior angle ADB of the triangle DCB is greater than the interior opposite angle DCB, that is, greater than the angle ACB. I. 16.

Therefore also the angle ABD is greater than the angle ACB; still more then is the angle ABC greater than the angle ACB.

Q.E.D.

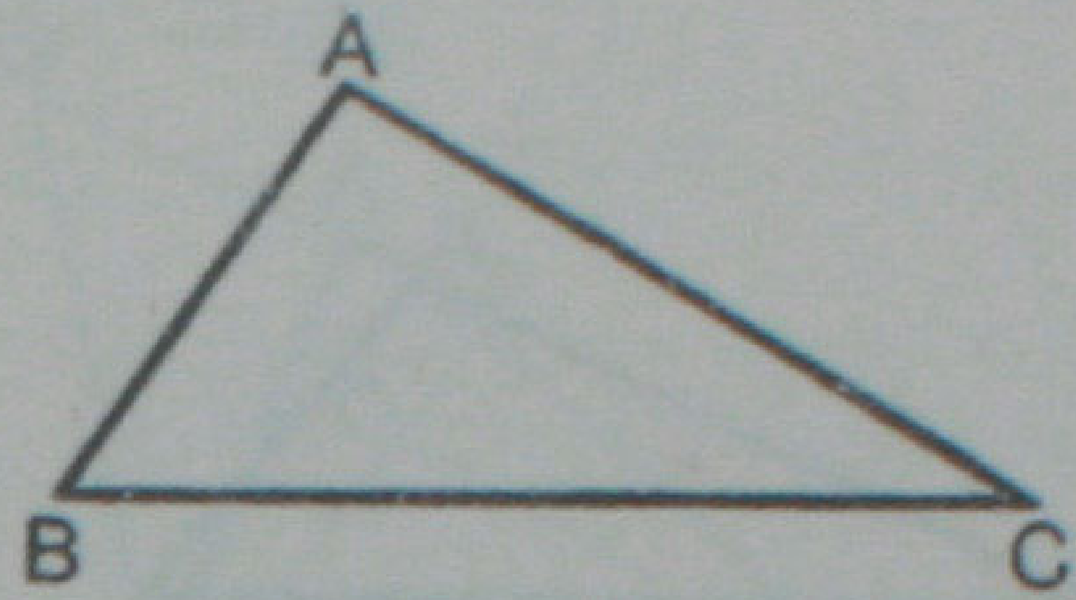
Euclid enunciated Proposition 18 as follows:

The greater side of every triangle has the greater angle opposite to it.

[This form of enunciation is found to be a common source of difficulty with beginners, who fail to distinguish what is *assumed* in it and what is *to be proved*. If Euclid's enunciations of Props. 18 and 19 are adopted, it is important to remember that in each case the part of the triangle *first named* points out the hypothesis.]

Proposition 19. Theorem.

If one angle of a triangle be greater than another, then the side opposite to the greater angle shall be greater than the side opposite to the less.



Let ABC be a triangle in which the angle ABC is greater than the angle ACB.

Then shall the side AC *be greater than the side* AB.

Proof. For if AC be not greater than AB,
it must be either equal to, or less than AB.

But AC is not equal to AB,
for then the angle ABC would be equal to the angle ACB; I. 5.
but it is not. *Hyp.*

Neither is AC less than AB;
for then the angle ABC would be less than the angle ACB; I. 18.
but it is not. *Hyp.*

That is, AC is neither equal to, nor less than AB.
Therefore AC is greater than AB. Q.E.D.

NOTE. The mode of demonstration used in this Proposition is known as the **Proof by Exhaustion**. It is applicable to cases in which *one* of certain suppositions must necessarily be true; and it consists in shewing that each of these suppositions is false *with one exception*: hence the truth of the remaining supposition is inferred.

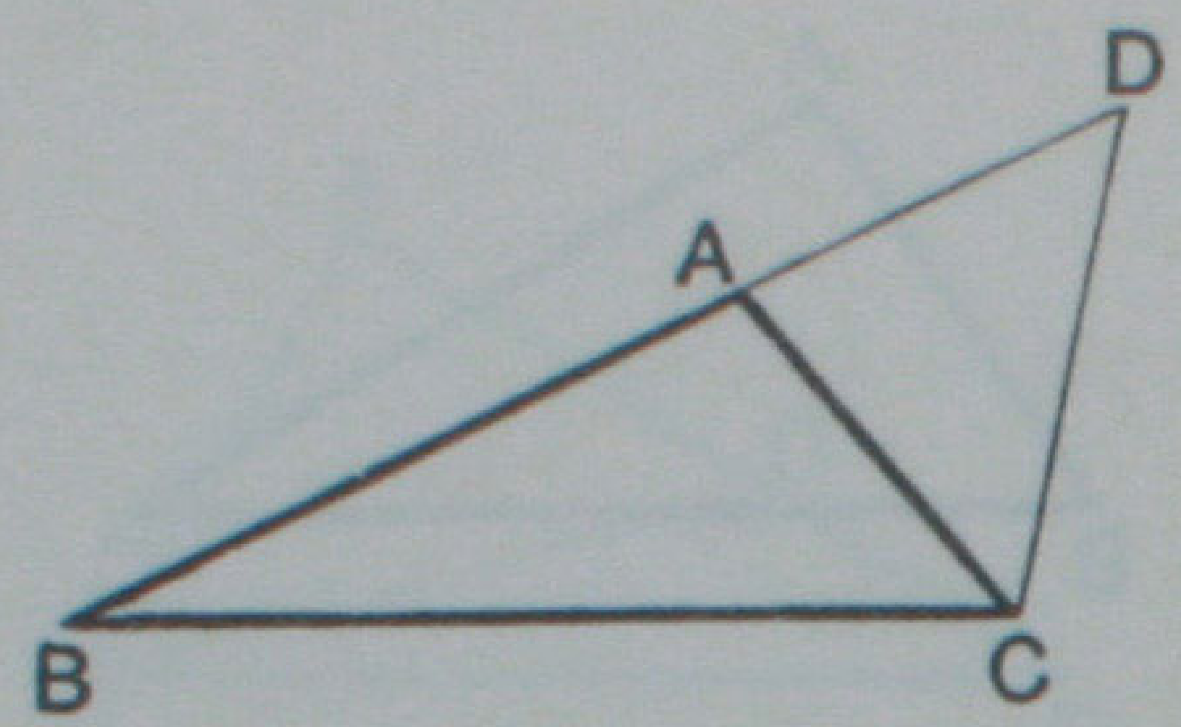
Euclid enunciated Proposition 19 as follows:

The greater angle of every triangle is subtended by the greater side, or, has the greater side opposite to it.

[For Exercises on Props. 18 and 19 see page 44.]

PROPOSITION 20. THEOREM.

Any two sides of a triangle are together greater than the third side.



Let ABC be a triangle.

Then shall any two of its sides be together greater than the third side:

namely, BA, AC, *shall be greater than* CB;
AC, CB *shall be greater than* BA;
and CB, BA *shall be greater than* AC.

Construction. Produce BA to D, making AD equal to AC. I. 3.
Join DC.

Proof. Then in the triangle ADC,
because AD is equal to AC, *Constr.*
therefore the angle ACD is equal to the angle ADC. I. 5.
But the angle BCD is greater than its part the angle ACD;
therefore also the angle BCD is greater than the angle ADC,
that is, than the angle BDC.

And in the triangle BCD,
because the angle BCD is greater than the angle BDC,
therefore the side BD is greater than the side CB. I. 19.

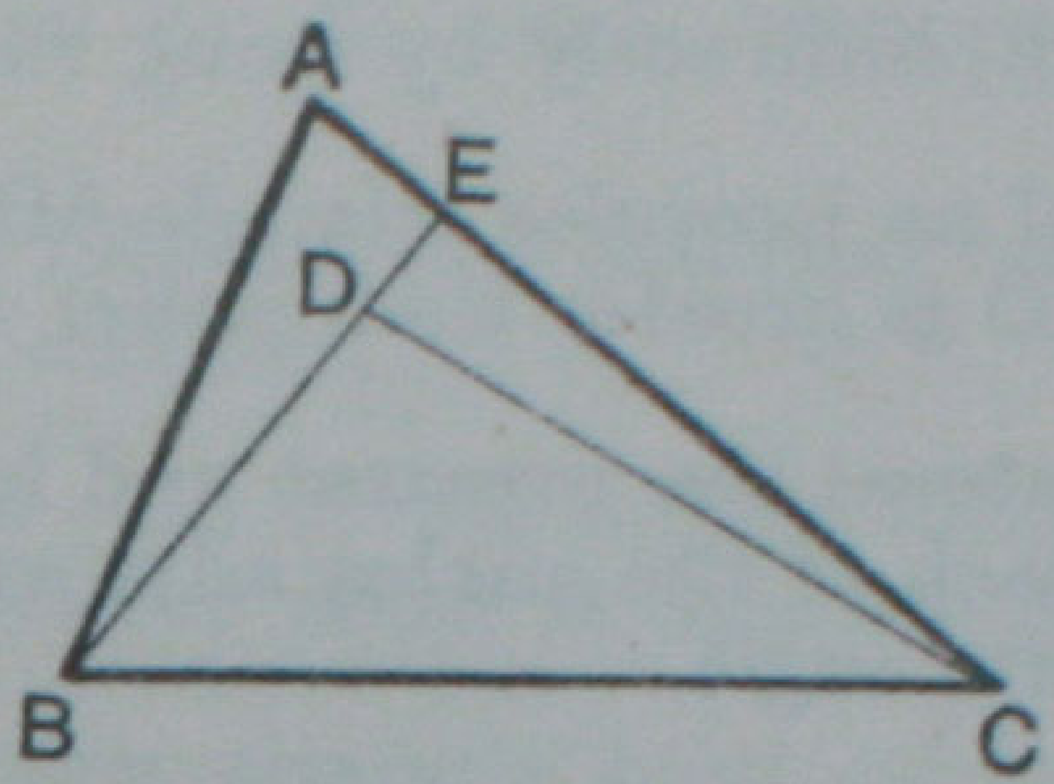
But BA and AC are together equal to BD;
therefore BA and AC are together greater than CB.

Similarly it may be shewn
that AC, CB are together greater than BA;
and CB, BA are together greater than AC. Q.E.D.

[For Exercises see page 44.]

PROPOSITION 21. THEOREM.

If from the ends of a side of a triangle, there be drawn two straight lines to a point within the triangle, then these straight lines shall be less than the other two sides of the triangle, but shall contain a greater angle.



Let ABC be a triangle, and from B, C, the ends of the side BC, let the straight lines BD, CD be drawn to a point D within the triangle

Then (i) BD *and* DC *shall be together less than* BA *and* AC ;

(ii) *the angle* BDC *shall be greater than the angle* BAC.

Construction. Produce BD to meet AC in E.

Proof. (i) In the triangle BAE, the two sides BA, AE are together greater than the third side BE ; I. 20.

to each of these add EC ;

then BA, AC are together greater than BE, EC. *Ax.* 4.

Again, in the triangle DEC, the two sides DE, EC are together greater than DC ; I. 20.

to each of these add BD ;

then BE, EC are together greater than BD, DC.

But it has been shewn that BA, AC are together greater than BE, EC :

still more then are BA, AC greater than BD, DC.

(ii) Again, the exterior angle BDC of the triangle DEC is greater than the interior opposite angle DEC ; I. 16.

and the exterior angle DEC of the triangle BAE is greater than the interior opposite angle BAE, that is, than the angle BAC ; I. 16.

still more then is the angle BDC greater than the angle BAC.

Q.E.D.

EXERCISES.

ON PROPOSITIONS 18 AND 19.

1. The hypotenuse is the greatest side of a right-angled triangle.

2. If two angles of a triangle are equal to one another, the sides also, which subtend the equal angles, are equal to one another. Prove this [*i.e.* Prop. 6] indirectly by using the result of Prop. 18.

3. BC, the base of an isosceles triangle ABC, is produced to any point D; shew that AD is greater than either of the equal sides.

4. If in a quadrilateral the greatest and least sides are opposite to one another, then each of the angles adjacent to the least side is greater than its opposite angle.

5. In a triangle ABC, if AC is not greater than AB, shew that any straight line drawn through the vertex A and terminated by the base BC, is less than AB.

6. ABC is a triangle, in which OB, OC bisect the angles ABC, ACB respectively: shew that, if AB is greater than AC, then OB is greater than OC.

ON PROPOSITION 20.

7. The difference of any two sides of a triangle is less than the third side.

8. In a quadrilateral, if two opposite sides which are not parallel are produced to meet one another; shew that the perimeter of the greater of the two triangles so formed is greater than the perimeter of the quadrilateral.

9. The sum of the distances of any point from the three angular points of a triangle is greater than half its perimeter.

10. The perimeter of a quadrilateral is greater than the sum of its diagonals.

11. Obtain a proof of Proposition 20 by bisecting an angle by a straight line which meets the opposite side.

ON PROPOSITION 21.

12. In Proposition 21 shew that the angle BDC is greater than the angle BAC by joining AD, and producing it towards the base.

13. The sum of the distances of any point within a triangle from its angular points is less than the perimeter of the triangle.

QUESTIONS FOR REVISION.

1. Define the *complement* of an angle. When are two angles said to be *supplementary*? Shew that two angles which are supplementary to the same angle are equal to one another.

2. What is meant by an angle being *bisected internally and externally*?

Prove that the internal and external bisectors of an angle are at right angles to one another.

3. Prove that the sum of the angles formed by any number of straight lines drawn from a point is equal to four right angles.

4. Why must every triangle have *at least two acute angles*? Quote the enunciation of the proposition from which this inference is drawn.

5. In the enunciation *The greater side of a triangle has the greater angle opposite to it*, point out what is assumed and what is to be proved.

6. What is meant by the *Proof by Exhaustion*? Illustrate the use of this method by naming the steps in the proof of Proposition 19.

7. What inference may be drawn respecting the triangles whose sides measure

(i) 4 inches, 5 inches, 4 inches;
(ii) 8 inches, 9 inches, 10 inches;
(iii) 6 inches, 10 inches, 4 inches?

8. Quote the enunciations of propositions which, from a hypothesis relating to the *sides* of triangle, establish a conclusion relating to the *angles*.

9. Quote the enunciations of propositions which, from a hypothesis relating to the *angles* of a triangle, establish a conclusion relating to the sides.

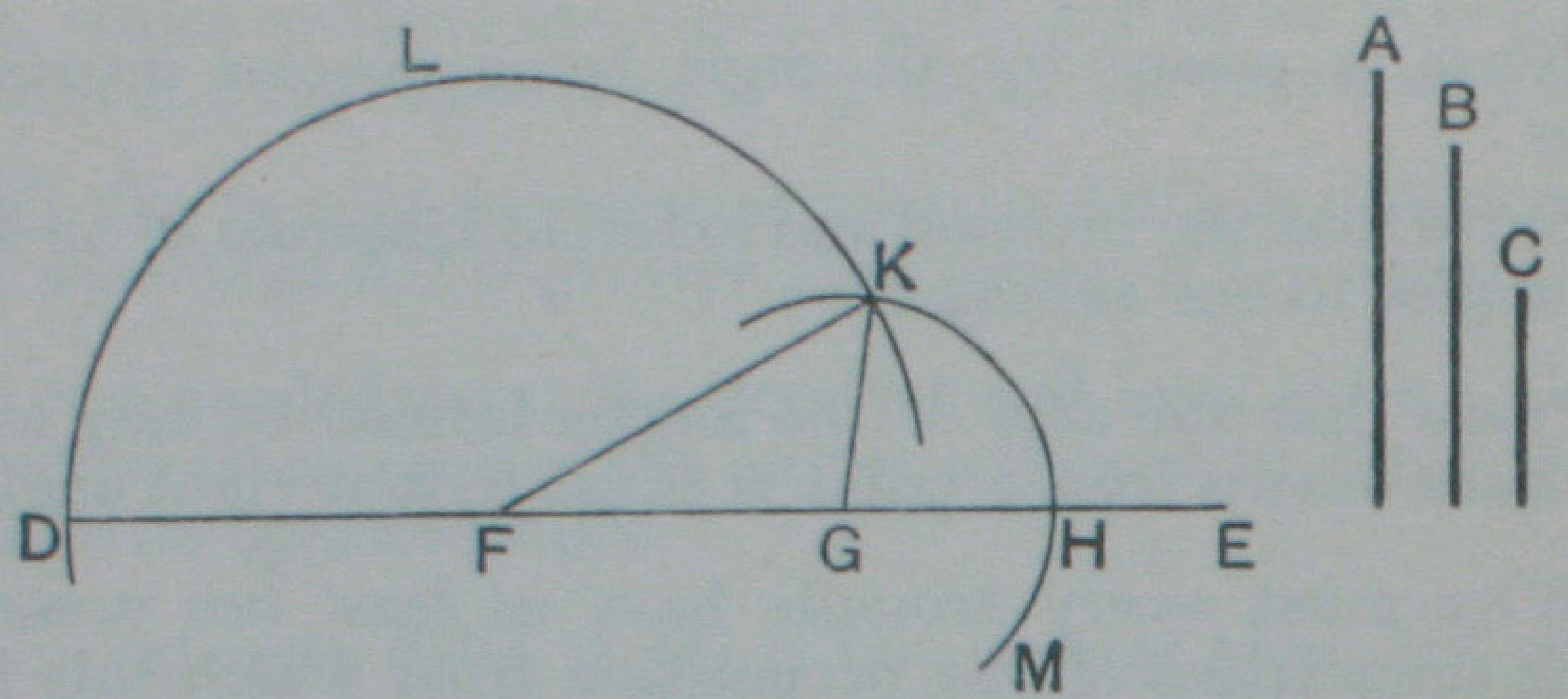
10. Explain why parallel straight lines must be *in the same plane*.

11. Prove by means of Prop. 7 that on a given base and on the same side of it only one equilateral triangle can be drawn.

12. In an isosceles triangle, if the equal sides are produced, shew that the angles on the other side of the base must be obtuse.

PROPOSITION 22. PROBLEM.

To describe a triangle having its sides equal to three given straight lines, any two of which are together greater than the third.



Let A, B, C be the three given straight lines, of which any two are together greater than the third.

It is required to describe a triangle of which the sides shall be equal to A, B, C.

Construction. Take a straight line DE terminated at the point D, but unlimited towards E.

Make DF equal to A, FG equal to B, and GH equal to C. I. 3.

With centre F and radius FD, describe the circle DLK.

With centre G and radius GH, describe the circle MHK cutting the former circle at K.

Join FK, GK.

Then shall the triangle KFG *have its sides equal to the three straight lines* A, B, C.

Proof. Because F is the centre of the circle DLK,

therefore FK is equal to FD: *Def.* 15.

but FD is equal to A; *Constr.*

therefore also FK is equal to A. *Ax.* 1.

Again, because G is the centre of the circle MHK,

therefore GK is equal to GH: *Def.* 15.

but GH is equal to C; *Constr.*

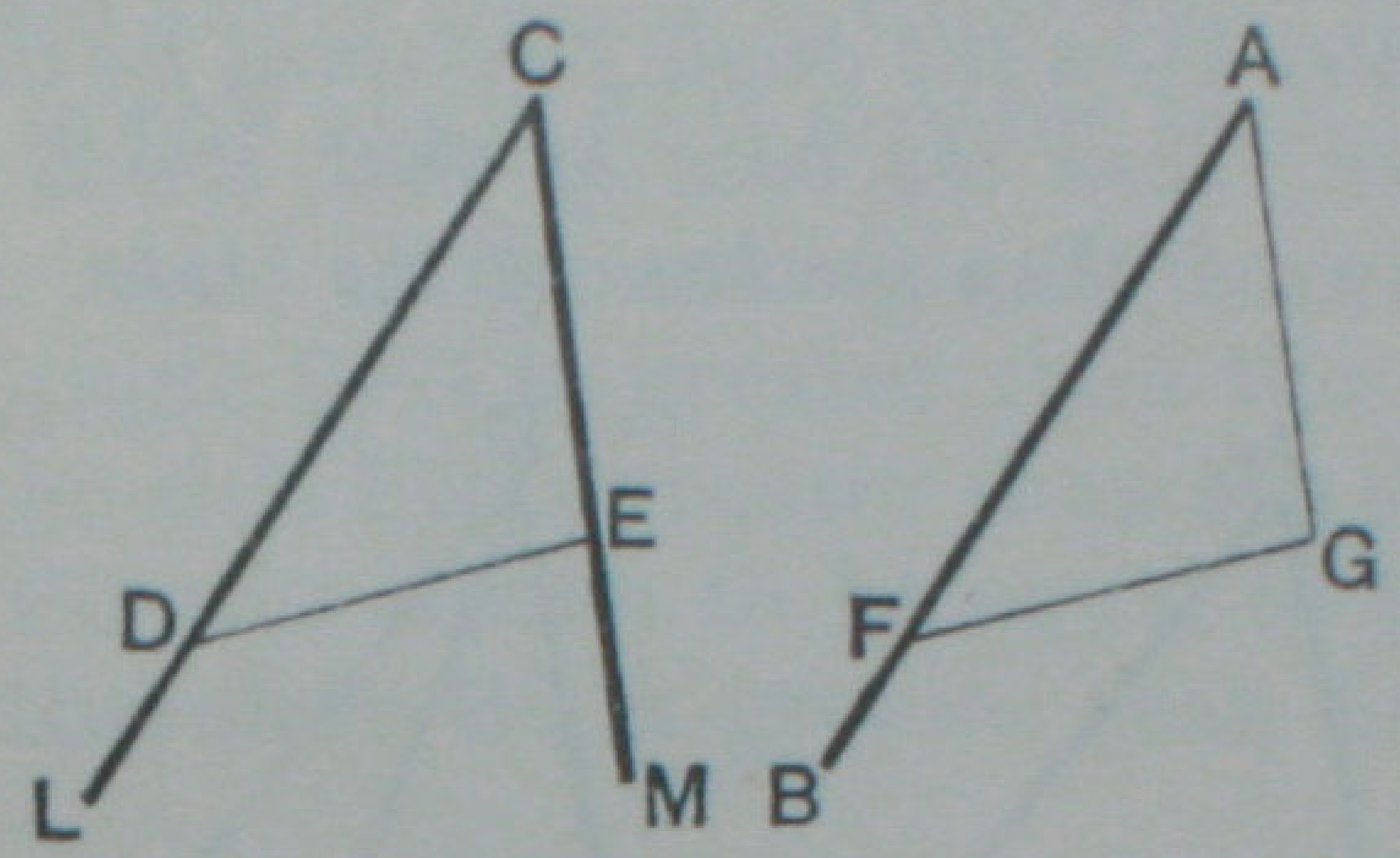
therefore also GK is equal to C. *Ax.* 1.

And FG is equal to B. *Constr.*

Therefore the triangle KFG has its sides KF, FG, GK equal respectively to the three given lines A, B, C. Q.E.F.

Proposition 23. Problem.

At a given point in a given straight line, to make an angle equal to a given rectilineal angle.



Let AB be the given straight line, and A the given point in it, and let LCM be the given angle.

It is required to draw from A *a straight line making with* AB *an angle equal to the given angle* DCE.

Construction. In CL, CM take any points D and E; and join DE.

From AB cut off AF equal to CD. I. 3.

On AF describe the triangle FAG, having the remaining sides AG, GF equal respectively to CE, ED. I. 22.

Then shall the angle FAG *be equal to the angle* DCE.

Proof. For in the triangles FAG, DCE,

Because { FA is equal to DC, *Constr.*
and AG is equal to CE; *Constr.*
and the base FG is equal to the base DE: *Constr.*

therefore the angle FAG is equal to the angle DCE. I. 8.

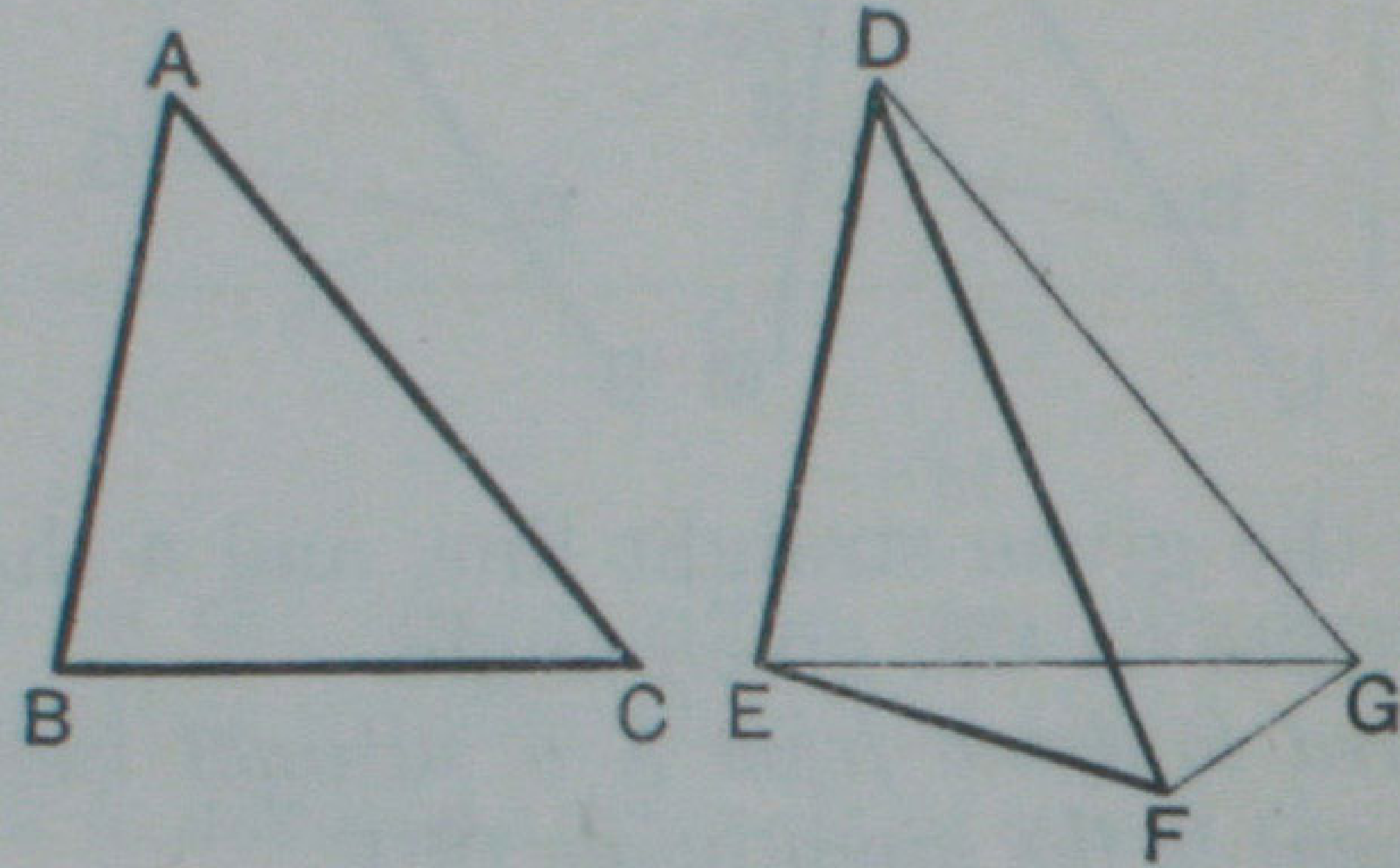
That is, AG makes with AB, at the given point A, an angle equal to the given angle DCE. Q.E.F.

EXERCISE.

On a given base describe a triangle, whose remaining sides shall be equal to two given straight lines. Point out how the construction fails, if any one of the three given lines is greater than the sum of the other two.

PROPOSITION 24. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one greater than the angle contained by the corresponding sides of the other; then the base of that which has the greater angle shall be greater than the base of the other.



Let ABC, DEF be two triangles, in which
the side BA is equal to the side ED,
and the side AC is equal to the side DF,
but the angle BAC is greater than the angle EDF.

Then shall the base BC *be greater than the base* EF.

Of the two sides DE, DF, let DE be that which is not greater than the other.*

Construction. At D in the straight line ED, and on the same side of it as DF, make the angle EDG equal to the angle BAC. I. 23.

Make DG equal to DF or AC; I. 3.
and join EG, GF.

Proof. Then in the triangles BAC, EDG,

Because { BA is equal to ED, *Hyp.*
and AC is equal to DG, *Constr.*
also the contained angle BAC is equal to the contained angle EDG; *Constr.* }

therefore the triangle BAC is equal to the triangle EDG in all respects; I. 4.
so that the base BC is equal to the base EG.

Again, in the triangle FDG,
because DG is equal to DF,
therefore the angle DFG is equal to the angle DGF. I. 5.

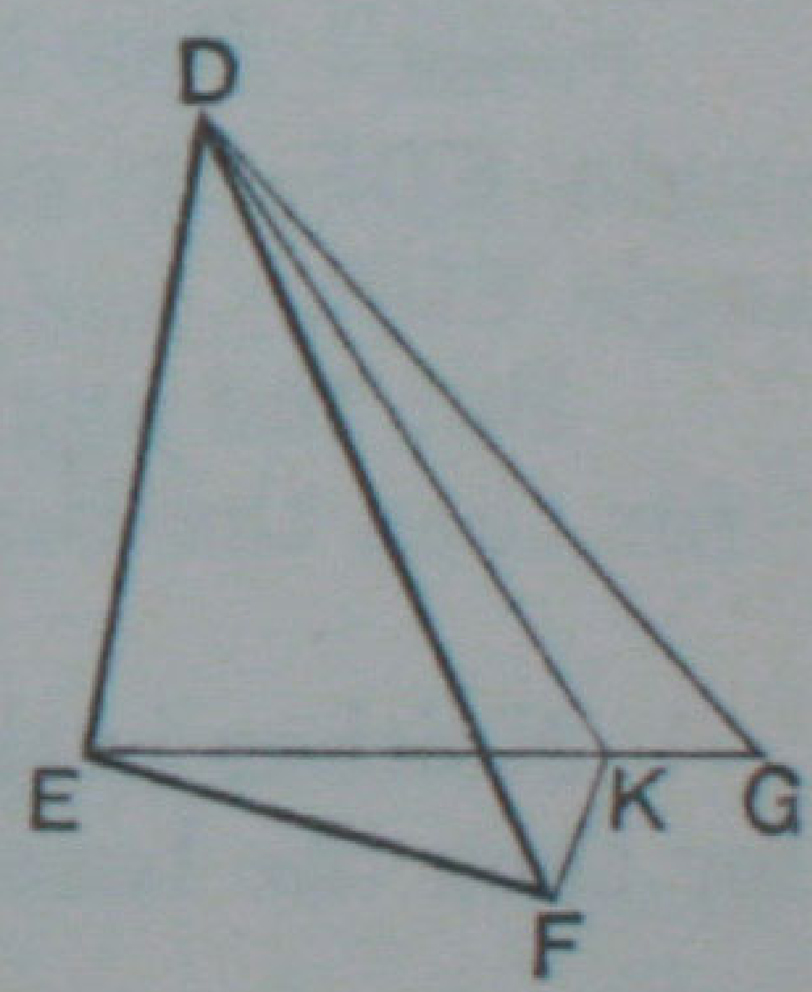
But the angle DGF is greater than its part the angle EGF;
therefore also the angle DFG is greater than the angle EGF;
still more then is the angle EFG greater than the angle EGF.

And in the triangle EFG,
because the angle EFG is greater than the angle EGF,
therefore the side EG is greater than the side EF; I. 19.
but EG was shewn to be equal to BC;
therefore BC is greater than EF. Q.E.D.

* The object of this step is to make the point F fall *below* EG. Otherwise F might fall *above*, *upon*, or *below* EG; and each case would require separate treatment. But as it is not *proved* that this condition fulfils its object, this demonstration of Prop. 24 must be considered defective.

An alternative construction and proof are given below.

Construction. At D in ED make the angle EDG equal to the angle BAC; and make DG equal to DF. Join EG.



Then, as before, it may be shewn that the triangle EDG = the triangle BAC in all respects.

Now if EG passes through F, then EG is greater than EF; that is, BC is greater than EF.

But if not, bisect the angle FDG by DK, meeting EG at K. Join FK.

Proof. Then in the triangles FDK, GDK,

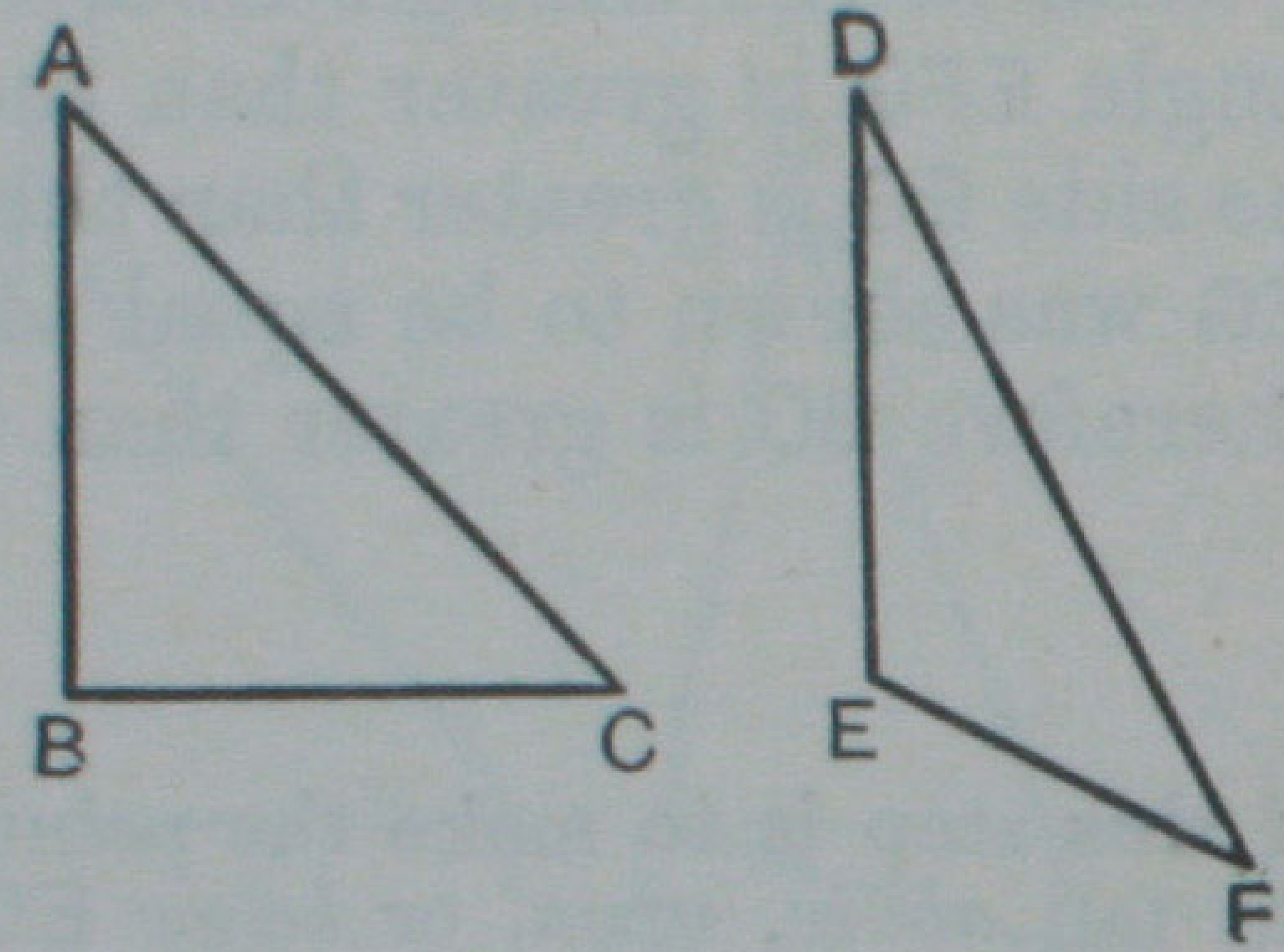
Because $\begin{cases} \text{FD} = \text{GD}, \\ \text{and DK is common to both,} \\ \text{and the angle FDK} = \text{the angle GDK;} \end{cases}$ *Constr.*

∴ FK = GK. I. 4.

But in the triangle EKF, the two sides EK, KF are greater than EF;
that is, EK, KG are greater than EF.
Hence EG (or BC) is greater than EF.

PROPOSITION 25. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other; then the angle contained by the sides of that which has the greater base, shall be greater than the angle contained by the corresponding sides of the other.



Let ABC, DEF be two triangles in which
the side BA is equal to the side ED,
and the side AC is equal to the side DF,
but the base BC is greater than the base EF.
Then shall the angle BAC *be greater than the angle* EDF.

Proof. For if the angle BAC be not greater than the angle EDF, it must be either equal to, or less than the angle EDF.

But the angle BAC is not equal to the angle EDF,
for then the base BC would be equal to the base EF; I. 4.
but it is not. *Hyp.*

Neither is the angle BAC less than the angle EDF,
for then the base BC would be less than the base EF; I. 24.
but it is not. *Hyp.*

Therefore the angle BAC is neither equal to, nor less than the angle EDF;
that is, the angle BAC is greater than the angle EDF. Q.E.D.

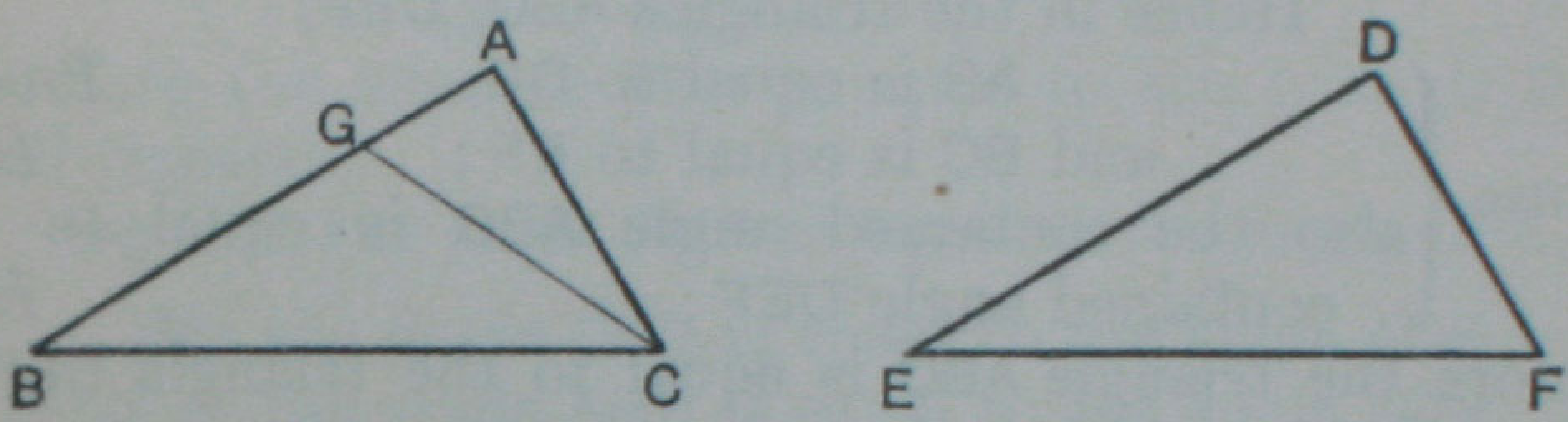
EXERCISE.

In a triangle ABC, the vertex A is joined to X, the middle point of the base BC; shew that the angle AXB is obtuse or acute, according as AB is greater or less than AC.

PROPOSITION 26. THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each, and a side of one equal to a side of the other, these sides being either adjacent to the equal angles, or opposite to equal angles in each; then shall the triangles be equal in all respects.

CASE I. When the equal sides are *adjacent* to the equal angles in the two triangles.



Let ABC, DEF be two triangles, in which
the angle ABC is equal to the angle DEF,
and the angle ACB is equal to the angle DFE,
and the side BC is equal to the side EF.

Then shall the triangle ABC *be equal to the triangle* DEF *in all respects; that is,* AB *shall be equal to* DE, *and* AC *to* DF, *and the angle* BAC *shall be equal to the angle* EDF.

For if AB be not equal to DE, one must be greater than the other. If possible, let AB be greater than DE.

Construction. From BA cut off BG equal to ED, I. 3.
and join GC.

Proof. Then in the two triangles GBC, DEF,

Because { GB is equal to DE, *Constr.*
and BC is equal to EF, *Hyp.*
also the contained angle GBC is equal to the contained angle DEF; *Hyp.* }

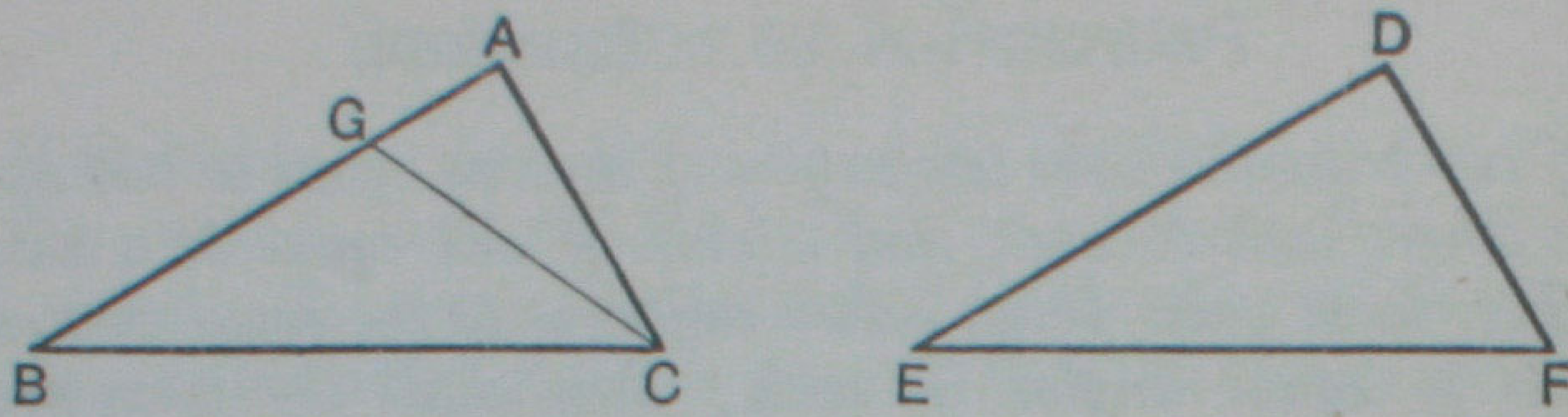
therefore the triangle GBC is equal to the triangle DEF in all respects; I. 4.

so that the angle GCB is equal to the angle DFE.

But the angle ACB is equal to the angle DFE; *Hyp.*

therefore also the angle GCB is equal to the angle ACB; *Ax.* 1.

the part equal to the whole, which is impossible.



Therefore AB is not unequal to DE;
that is, AB is equal to DE.

Hence in the triangles ABC, DEF,

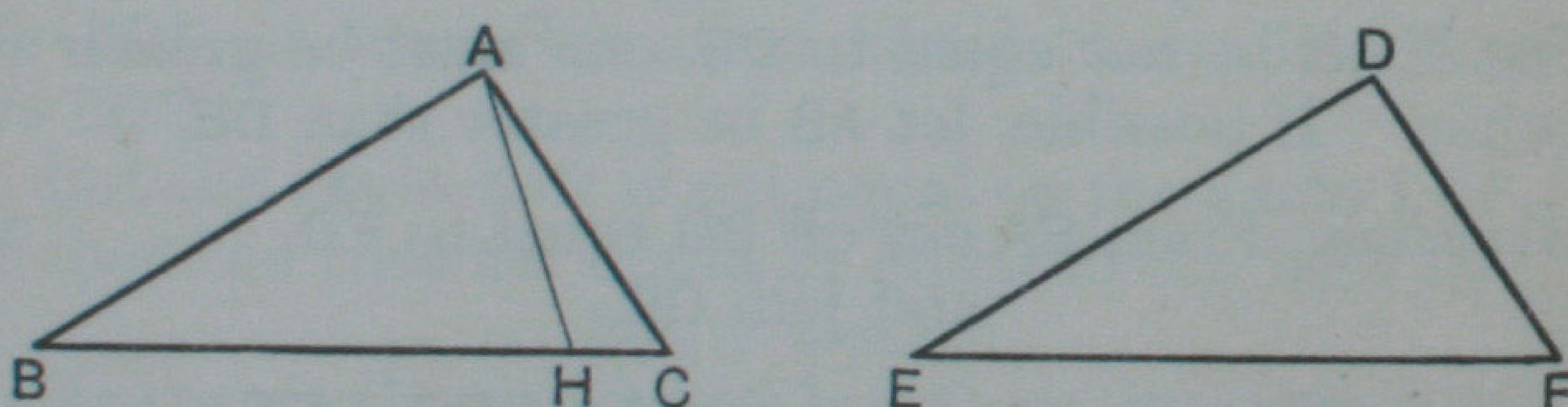
Because {
AB is equal to DE, *Proved.*
and BC is equal to EF; *Hyp.*
also the contained angle ABC is equal to the contained angle DEF; *Hyp.*
}

therefore the triangle ABC is equal to the triangle DEF in all respects: I. 4.

so that the side AC is equal to the side DF;
and the angle BAC is equal to the angle EDF.

Q.E.D.

CASE II. When the equal sides are *opposite* to equal angles in the two triangles.



Let ABC, DEF be two triangles, in which
the angle ABC is equal to the angle DEF,
and the angle ACB is equal to the angle DFE,
and the side AB is equal to the side DE.

Then the triangle ABC *shall be equal to the triangle* DEF *in all respects;*

namely, BC *shall be equal to* EF,
and AC *shall be equal to* DF,
and the angle BAC *shall be equal to the angle* EDF.

For if BC be not equal to EF, one must be greater than the other. If possible, let BC be greater than EF.

Construction. From BC cut off BH equal to EF, I. 3.
and join AH.

Proof. Then in the triangles ABH, DEF,

Because { AB is equal to DE, *Hyp.*
and BH is equal to EF, *Constr.*
also the contained angle ABH is equal to the contained angle DEF; *Hyp.* }

therefore the triangle ABH is equal to the triangle DEF in all respects; I. 4.

so that the angle AHB is equal to the angle DFE.

But the angle DFE is equal to the angle ACB; *Hyp.*
therefore the angle AHB is equal to the angle ACB; *Ax.* 1.
that is, an exterior angle of the triangle ACH is equal to an interior opposite angle; which is impossible. I. 16.

Therefore BC is not unequal to EF,
that is, BC is equal to EF.

Hence in the triangles ABC, DEF,

Because { AB is equal to DE, *Hyp.*
and BC is equal to EF; *Proved.*
also the contained angle ABC is equal to the contained angle DEF; *Hyp.* }

therefore the triangle ABC is equal to the triangle DEF in all respects; I. 4.

so that the side AC is equal to the side DF,
and the angle BAC is equal to the angle EDF.

Q.E.D.

Corollary. *In both cases of this Proposition it is seen that the triangles may be made* to coincide with one another; *and they are therefore equal* in area.

ON THE IDENTICAL EQUALITY OF TRIANGLES.

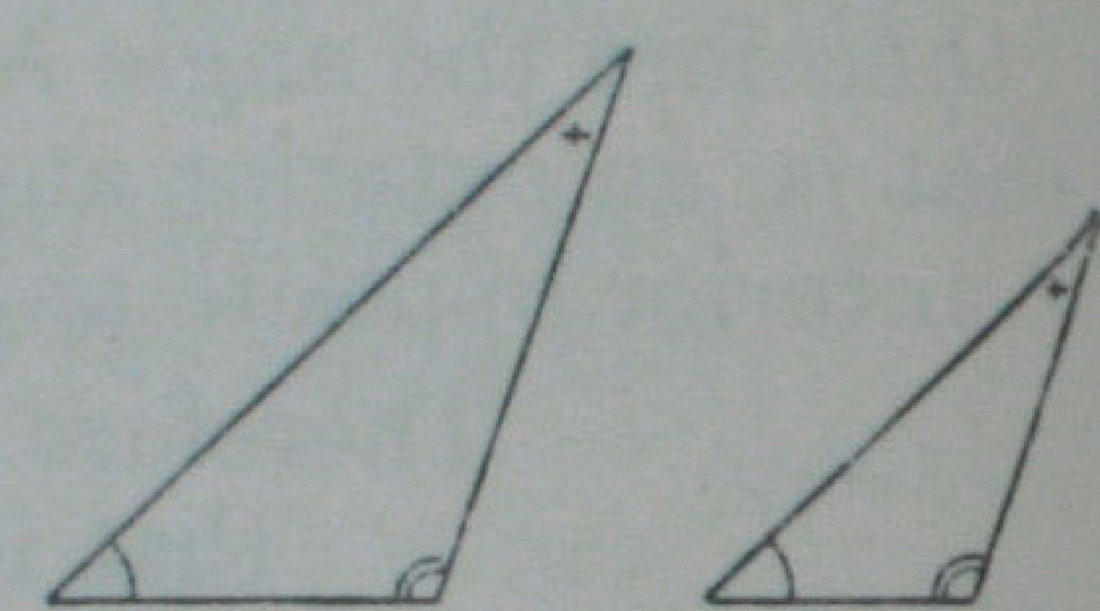
Three cases have been already dealt with in Propositions 4, 8, and 26, the results of which may be summarized as follows:

Two triangles are equal in all respects when the following three parts in each are severally equal:

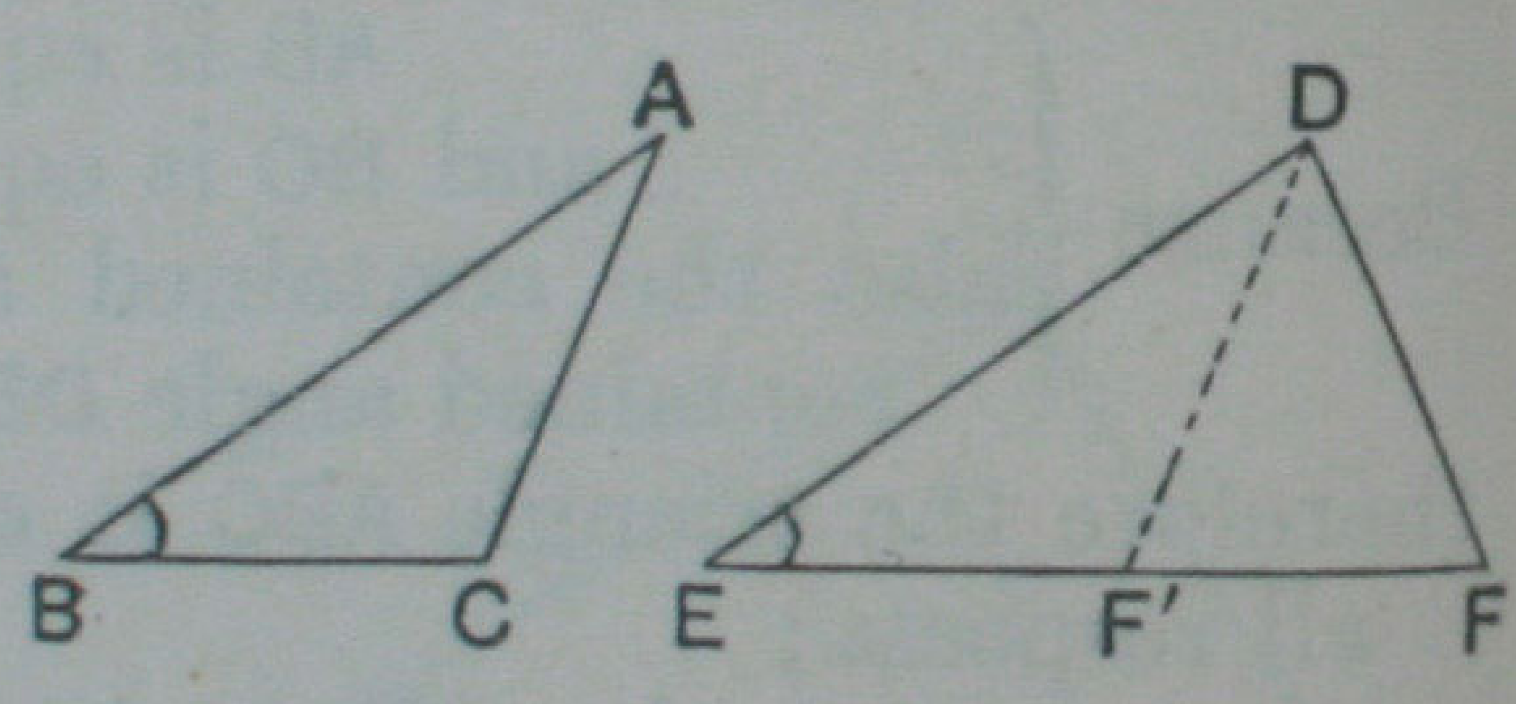
1. Two sides, and the included angle. *Prop.* 4.
2. The three sides. *Prop.* 8, *Cor.*
3. (*a*) Two angles, and the adjacent side;
(*b*) Two angles, and a side opposite one of them. } *Prop.* 26.

Two triangles are not, however, necessarily equal in all respects when *any three parts* of one are equal to the corresponding parts of the other. For example

(i) When the *three angles* of one are equal to the *three angles* of the other, each to each, the adjoining diagram shews that the triangles need not be equal in all respects.



(ii) When *two sides and one angle* in one are equal to *two sides and one angle* in the other, the given angles being *opposite* to equal sides, the diagram shews that the triangles need not be equal in all respects.



For it will be seen that if AB = DE, and AC = DF, and the angle ABC = the angle DEF, then the shorter of the given sides in the triangle DEF may lie in either of the positions DF or DF′.

In cases (i) and (ii) a further condition must be given before we can prove that the two triangles are identically equal.

We observe that in each of the three cases in which two triangles have been proved equal in all respects, namely in Propositions 4, 8, 26, it is shewn that the triangles may be made to *coincide with one another*; so that they are equal in *area*. Euclid however restricted himself to the use of Prop. 4, when he required to deduce the equality in *area* of two triangles from the equality of certain of their parts. This restriction is now generally abandoned.

EXERCISES ON PROPOSITIONS 12–26.

1. If BX and CY, the bisectors of the angles at the base BC of an isosceles triangle ABC, meet the opposite sides in X and Y, shew that the triangles YBC, XCB are equal in all respects.

2. Shew that the perpendiculars drawn from the extremities of the base of an isosceles triangle to the opposite sides are equal.

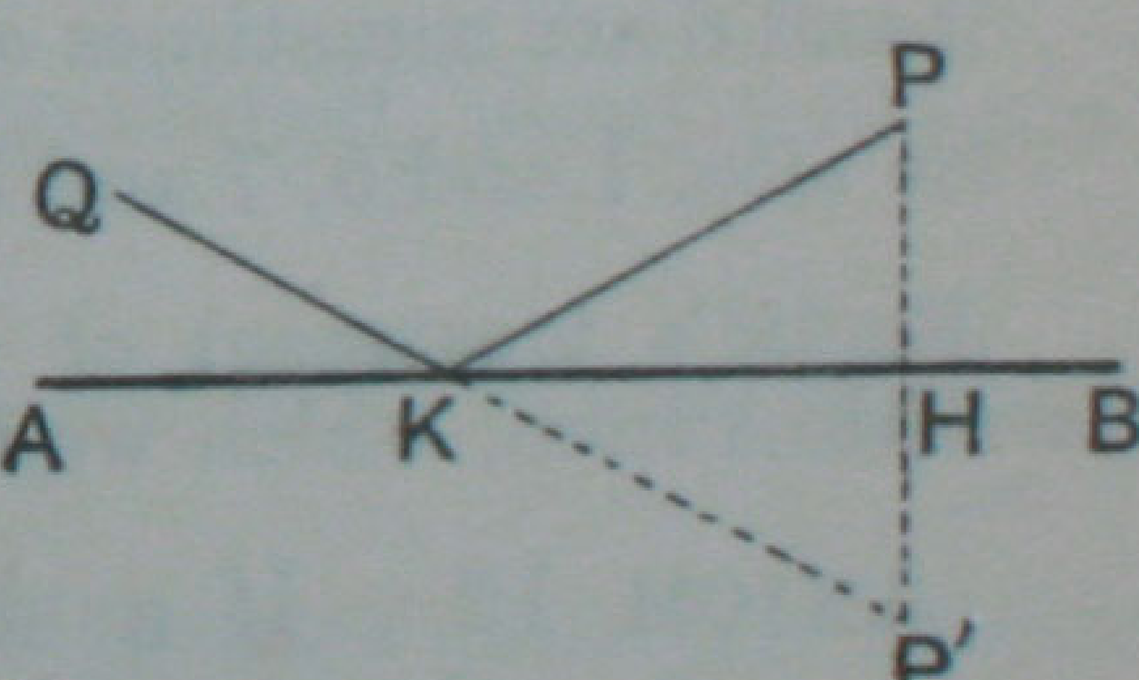
3. *Any point on the bisector of an angle is equidistant from the arms of the angle.*

4. Through O, the middle point of a straight line AB, any straight line is drawn, and perpendiculars AX and BY are dropped upon it from A and B: shew that AX is equal to BY.

5. If the bisector of the vertical angle of a triangle is at right angles to the base, the triangle is isosceles.

6. *The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line; and of others, that which is nearer to the perpendicular is less than the more remote; and two, and only two equal straight lines can be drawn from the given point to the given straight line, one on each side of the perpendicular.*

7. *From two given points on the same side of a given straight line, draw two straight lines, which shall meet in the given straight line, and make equal angles with it.*



Let AB be the given straight line, and P, Q the given points.

It is required to draw from P and Q to a point in AB, two straight lines that shall be equally inclined to AB.

Construction. From P draw PH perpendicular to AB: produce PH to P′, making HP′ equal to PH. Draw QP′, meeting AB in K. Join PK.

Then PK, QK shall be the required lines. [Supply the proof.]

8. In a given straight line find a point which is equidistant from two given intersecting straight lines. In what case is this impossible?

9. Through a given point draw a straight line such that the perpendiculars drawn to it from two given points may be equal.

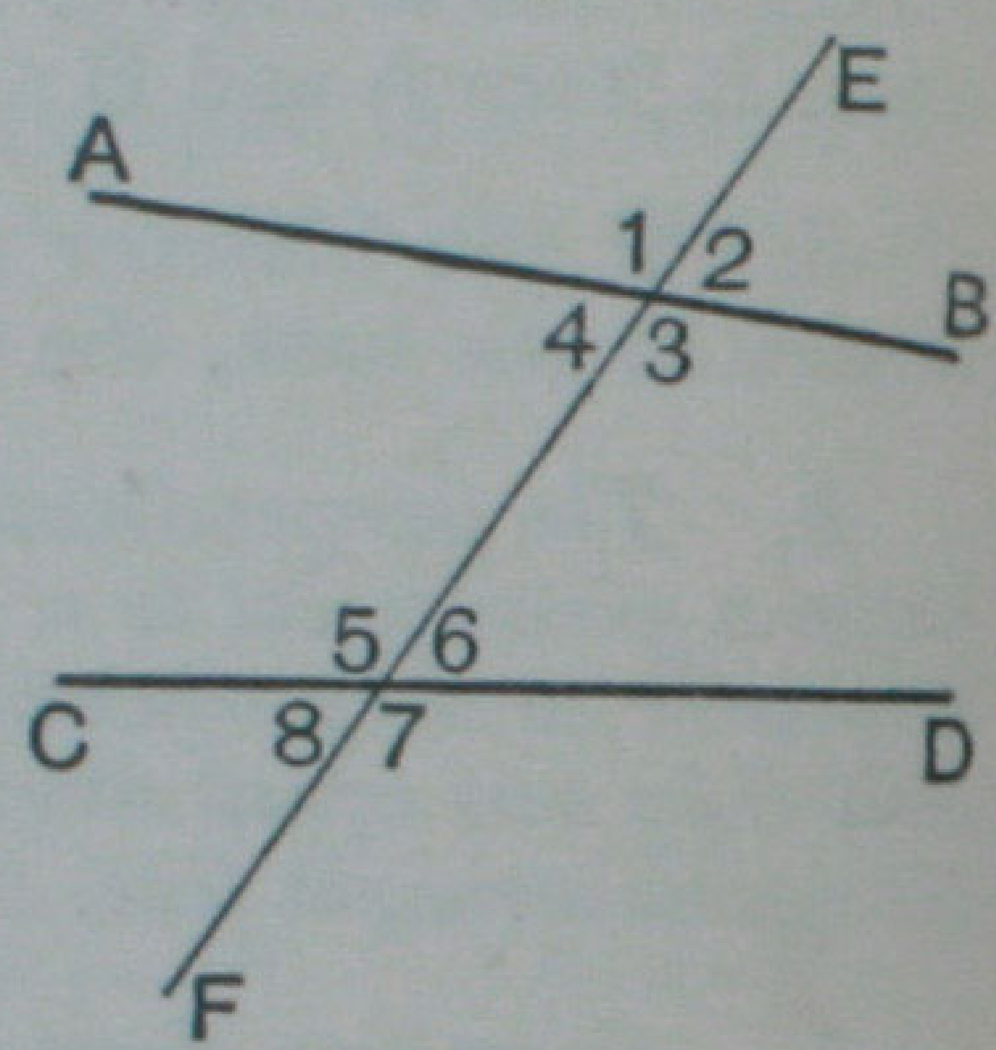
In what case is this impossible?

SECTION II.

PARALLEL STRAIGHT LINES AND PARALLELOGRAMS.

DEFINITION. Parallel straight lines are such as, being in the same plane, do not meet however far they are produced in both directions.

When two straight lines AB, CD are met by a third straight line EF, *eight* angles are formed, to which for the sake of distinction particular names are given.



Thus in the adjoining figure,

1, 2, 7, 8 are called **exterior** angles,

3, 4, 5, 6 are called **interior** angles,

4 and 6 are said to be **alternate** angles; so also the angles 3 and 5 are alternate to one another.

Of the angles 2 and 6, 2 is referred to as the exterior angle, and 6 as the **interior opposite** angle on the same side of EF.

2 and 6 are sometimes called **corresponding** angles.

So also, 1 and 5, 7 and 3, 8 and 4 are corresponding angles.

Euclid's treatment of parallel straight lines is based upon his twelfth Axiom, which we here repeat.

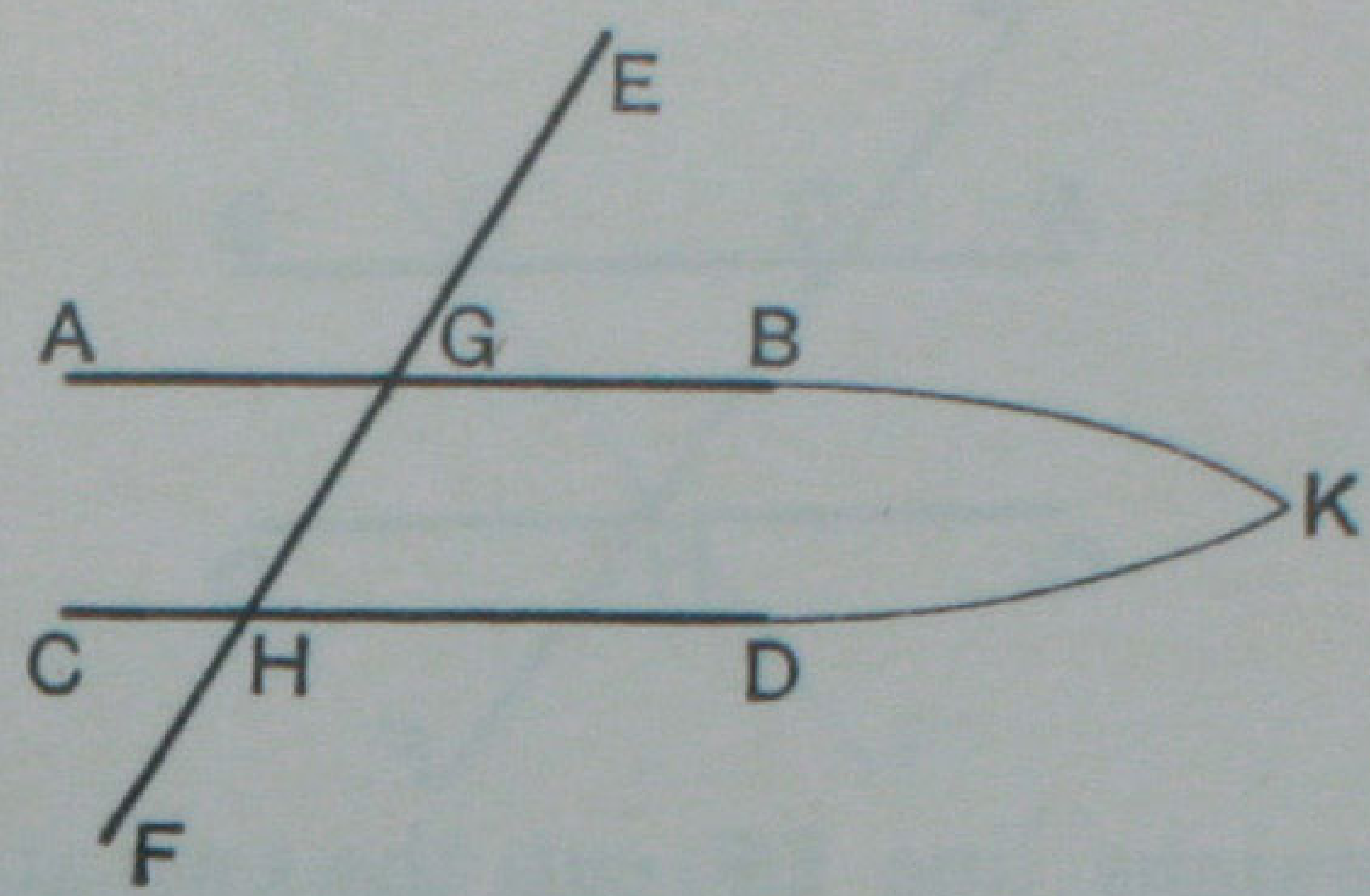
AXIOM 12. If a straight line cut two straight lines so as to make the two interior angles on the same side of it together less than two right angles, these straight lines, being continually produced, will at length meet on that side on which are the angles which are together less than two right angles.

Thus in the figure given above, if the two angles 3 and 6 are together less than two right angles, it is asserted that AB and CD will meet towards B and D.

This Axiom is used to establish I. 29: some remarks upon it will be found in a note on that Proposition.

PROPOSITION 27. THEOREM.

If a straight line, falling on two other straight lines, make the alternate angles equal to one another, then these two straight lines shall be parallel.



Let the straight line EF cut the two straight lines AB, CD at G and H, so as to make the alternate angles AGH, GHD equal to one another.

Then shall AB *and* CD *be parallel.*

Proof. For if AB and CD be not parallel,
they will meet, if produced, either towards B and D, or towards A and C.
If possible, let AB and CD, when produced, meet towards B and D, at the point K.

Then KGH is a triangle, of which one side KG is produced to A;
therefore the exterior angle AGH is greater than the interior opposite angle GHK. I. 16.

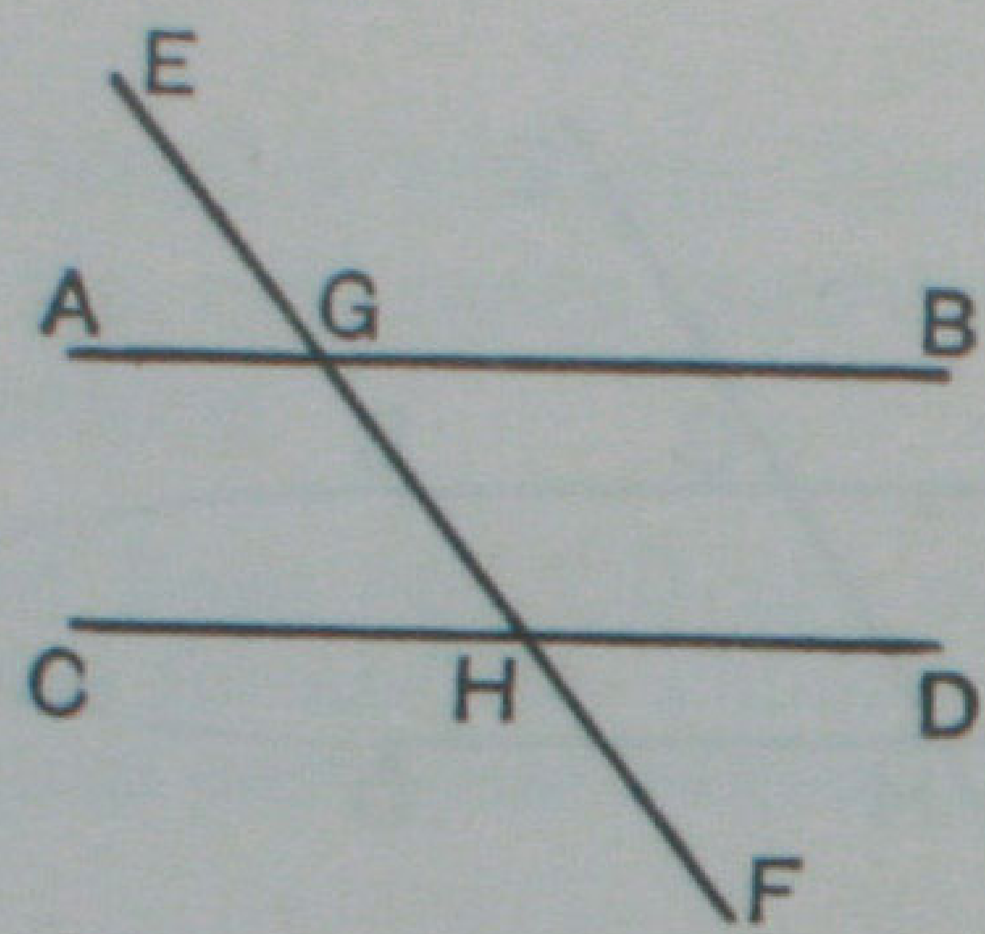
But the angle AGH was given equal to the angle GHK: *Hyp.*
hence the angles AGH and GHK are both equal and unequal;
which is impossible.

Therefore AB and CD cannot meet when produced towards B and D.
Similarly it may be shewn that they cannot meet towards A and C:

therefore AB and CD are parallel.

PROPOSITION 28. THEOREM.

If a straight line, falling on two other straight lines, make an exterior angle equal to the interior opposite angle on the same side of the line; or if it make the interior angles on the same side together equal to two right angles, then the two straight lines shall be parallel.



Let the straight line EF cut the two straight lines AB, CD in G and H: and

First, let the exterior angle EGB be equal to the interior opposite angle GHD.

Then shall AB *and* CD *be parallel.*

Proof. Because the angle EGB is equal to the angle GHD; and because the angle EGB is also equal to the vertically opposite angle AGH; I. 15.
therefore the angle AGH is equal to the angle GHD;
but these are alternate angles;
therefore AB and CD are parallel. I. 27.

Q.E.D.

Secondly, let the two interior angles BGH, GHD be together equal to two right angles.

Then shall AB *and* CD *be parallel.*

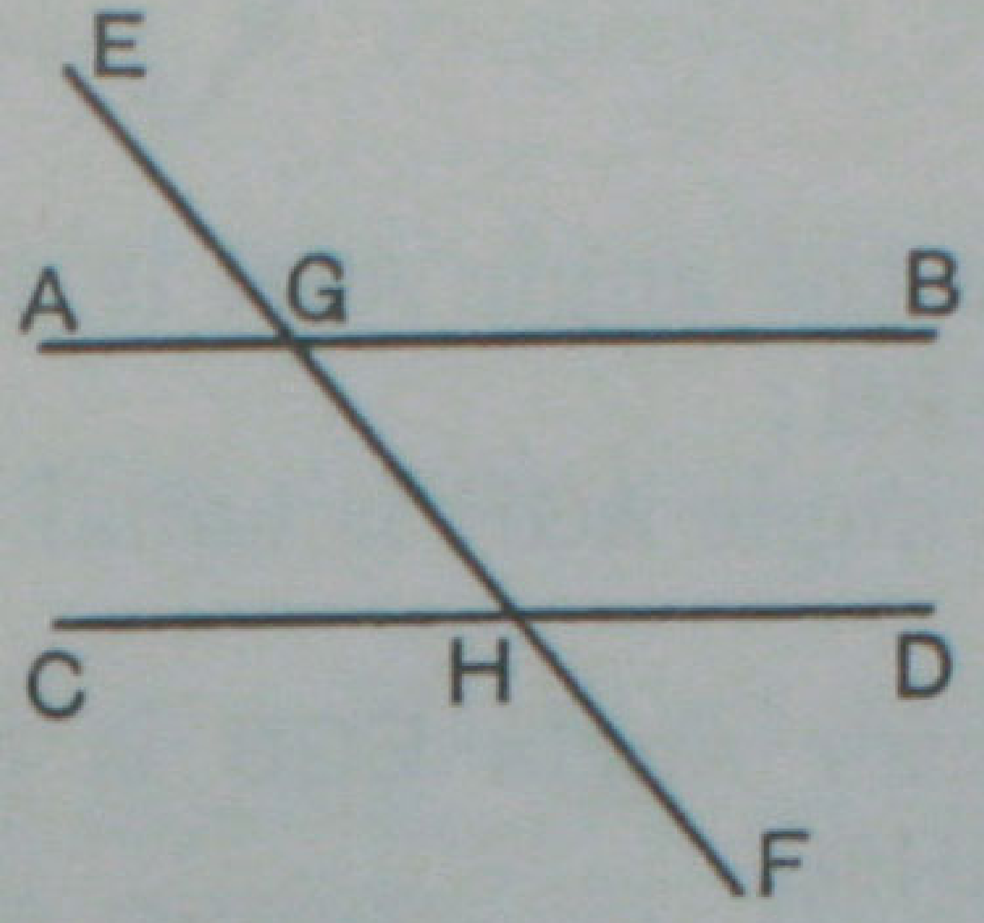
Proof. Because the angles BGH, GHD are together equal to two right angles; *Hyp.*
and because the adjacent angles BGH, AGH are also together equal to two right angles; I. 13.
therefore the angles BGH, AGH are together equal to the two angles BGH, GHD.

From these equals take the common angle BGH:
then the remaining angle AGH is equal to the remaining angle GHD: and these are alternate angles;
therefore AB and CD are parallel. I. 27.

Q.E.D.

PROPOSITION 29. THEOREM.

If a straight line fall on two parallel straight lines, then it shall make the alternate angles equal to one another, and the exterior angle equal to the interior opposite angle on the same side; and also the two interior angles on the same side equal to two right angles.



Let the straight line EF fall on the parallel straight lines AB, CD.

Then (i) *the angle* AGH *shall be equal to the alternate angle* GHD;

(ii) *the exterior angle* EGB *shall be equal to the interior opposite angle* GHD;

(iii) *the two interior angles* BGH, GHD *shall be together equal to two right angles.*

Proof. (i) For if the angle AGH be not equal to the angle GHD, one of them must be greater than the other.

If possible, let the angle AGH be greater than the angle GHD;

add to each the angle BGH:

then the angles AGH, BGH are together greater than the angles BGH, GHD.

But the adjacent angles AGH, BGH are together equal to two right angles; I. 13.

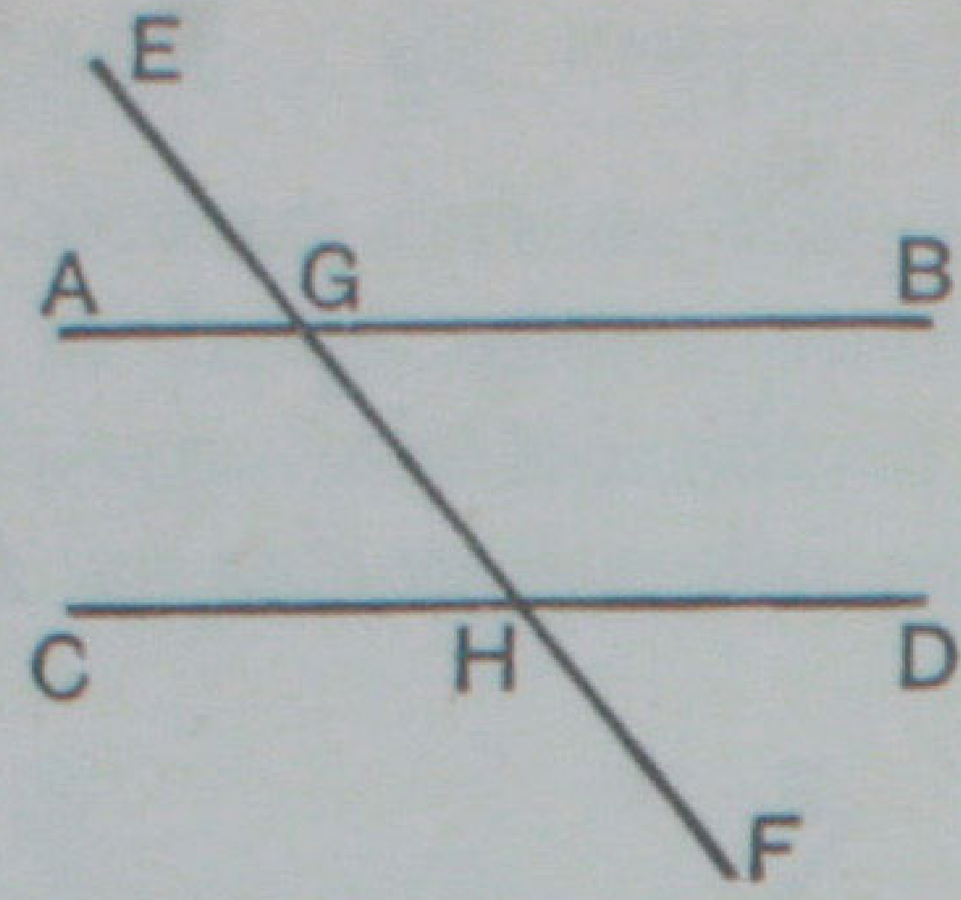
therefore the angles BGH, GHD are together less than two right angles;

therefore, *by Axiom 12*, AB and CD meet towards B and D.

But they never meet, since they are parallel. *Hyp.*

Therefore the angle AGH is not unequal to the angle GHD: that is, the angle AGH is equal to the alternate angle GHD.

(*Over*)



(ii) Again, because the angle AGH is equal to the vertically opposite angle EGB; I. 15.
and because the angle AGH is equal to the angle GHD; *Proved.*
therefore the exterior angle EGB is equal to the interior opposite angle GHD.

(iii) Lastly, the angle EGB is equal to the angle GHD; *Proved.*
add to each the angle BGH;
then the angles EGB, BGH are together equal to the angles BGH, GHD.
But the adjacent angles EGB, BGH are together equal to two right angles: I. 13.
therefore also the two interior angles BGH, GHD are together equal to two right angles. Q.E.D.

EXERCISES ON PROPOSITIONS 27, 28, 29.

1. Two straight lines AB, CD bisect one another at O: shew that the straight lines joining AC and BD are parallel. [I. 27.]

2. *Straight lines which are perpendicular to the same straight line are parallel to one another.* [I. 27 or I. 28.]

3. *If a straight line meet two or more parallel straight lines, and is perpendicular to one of them, it is also perpendicular to all the others.* [I. 29.]

4. *If two straight lines are parallel to two other straight lines, each to each, then the angles contained by the first pair are equal respectively to the angles contained by the second pair.* [I. 29.]

Note on the Twelfth Axiom.

Euclid's twelfth Axiom is unsatisfactory as the basis of a theory of parallel straight lines. It cannot be regarded as either simple or self-evident, and it therefore falls short of the essential characteristics of an axiom: nor is the difficulty entirely removed by considering it as a corollary to Proposition 17, of which it is the converse.

Of the many substitutes which have been proposed, we need only notice the following:

Axiom. *Two intersecting straight lines cannot be both parallel to a third straight line.*

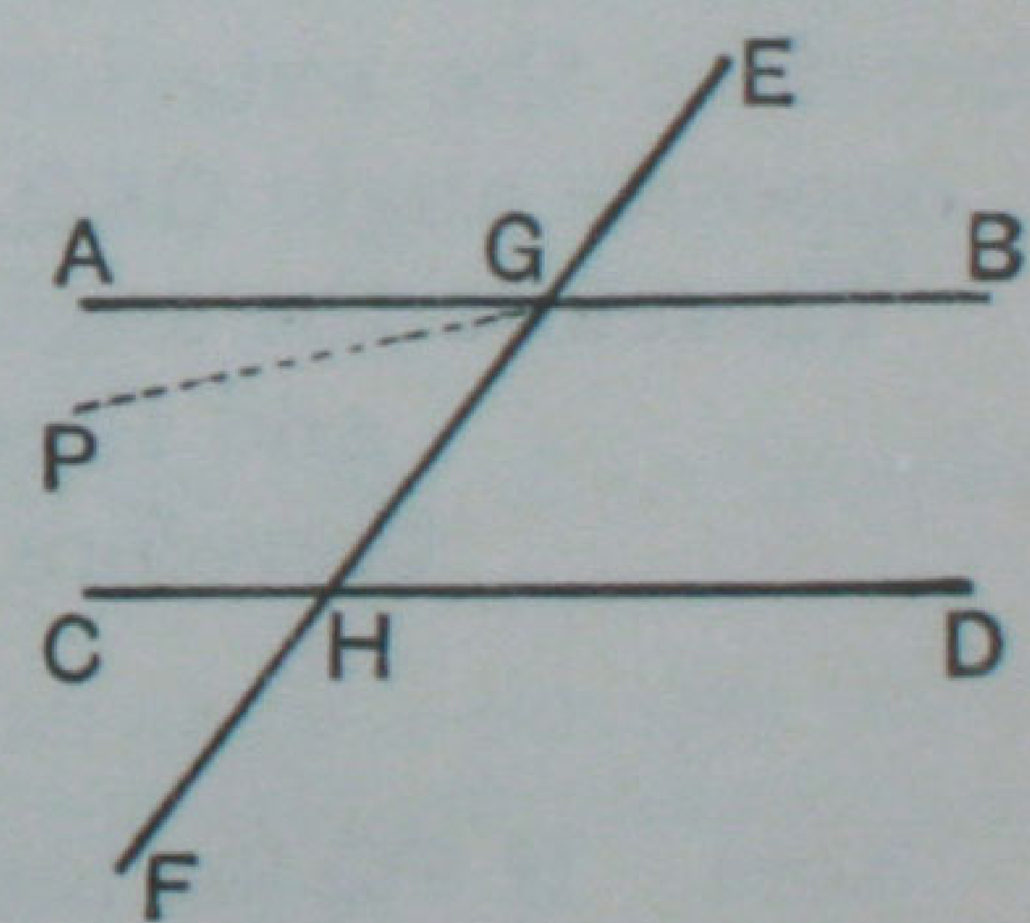
This statement is known as **Playfair's Axiom**; and though it is not altogether free from objection, it is no doubt simpler and more fundamental than that employed by Euclid, and more readily admitted without proof.

Propositions 27 and 28 having been proved in the usual way, the first part of Proposition 29 is then given thus.

Proposition 29. [Alternative Proof.]

If a straight line fall on two parallel straight lines, then it shall make the alternate angles equal.

Let the straight line EF meet the two parallel straight lines AB, CD at G and H.



Then shall the alternate angles AGH, GHD *be equal.*

For if the angle AGH is not equal to the angle GHD:

at G in the straight line HG make the angle HGP equal to the angle GHD, and alternate to it. I. 23.

Then PG and CD are parallel. I. 27.

But AB and CD are parallel: *Hyp.*

therefore the two intersecting straight lines AG, PG are both parallel to CD;

which is impossible. *Playfair's Axiom.*

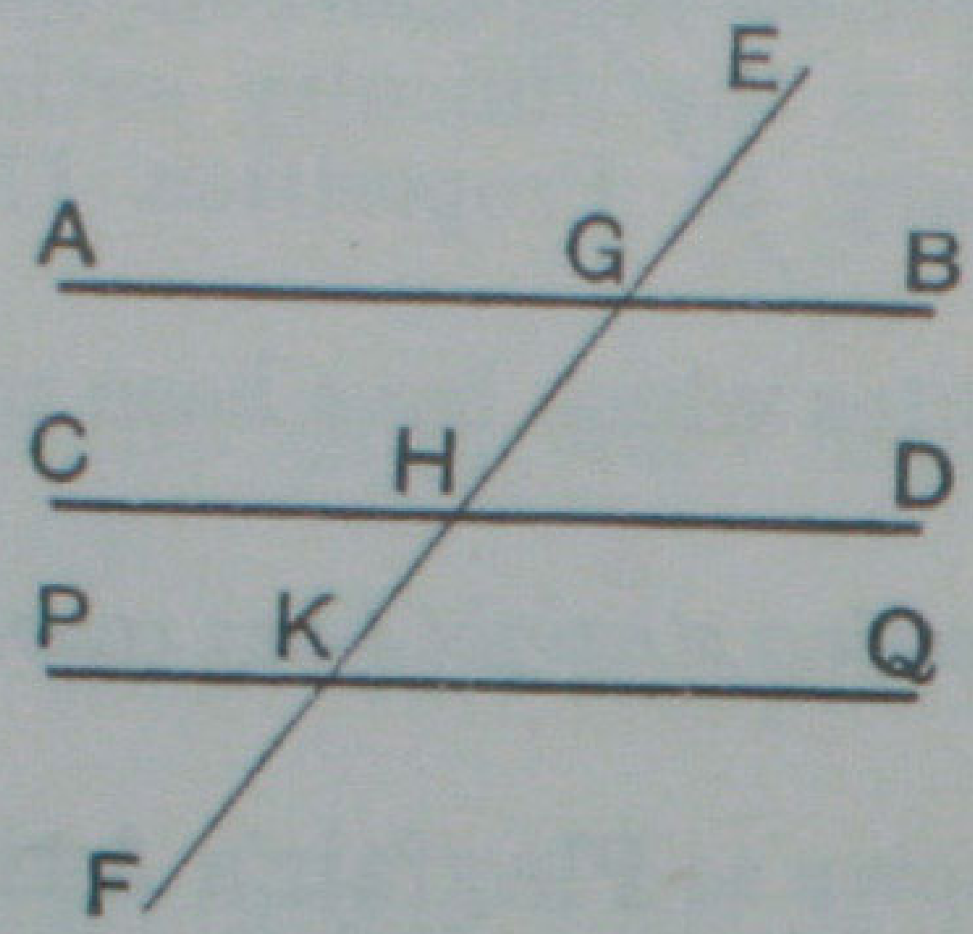
Therefore the angle AGH is not unequal to the angle GHD;

that is, the alternate angles AGH, GHD are equal. Q.E.D.

The second and third parts of the Proposition may then be deduced as in the text; and Euclid's Axiom 12 follows as a Corollary.

PROPOSITION 30. THEOREM.

Straight lines which are parallel to the same straight line are parallel to one another.



Let the straight lines AB, CD be each parallel to the straight line PQ.

Then shall AB *and* CD *be parallel to one another.*

Construction. Draw any straight line EF cutting AB, CD, and PQ in the points G, H, and K.

Proof. Then because AB and PQ are parallel, and EF meets them,
therefore the angle AGK is equal to the alternate angle GKQ
I. 29.
And because CD and PQ are parallel, and EF meets them,
therefore the exterior angle GHD is equal to the interior opposite angle GKQ. I. 29.
Therefore the angle AGH is equal to the angle GHD ;
and these are alternate angles ;
therefore AB and CD are parallel. I. 27.

Q.E.D.

NOTE. If PQ lies between AB and CD, the Proposition may be established in a similar manner, though in this case it scarcely needs proof ; for it is inconceivable that two straight lines, which do not meet an intermediate straight line, should meet one another.

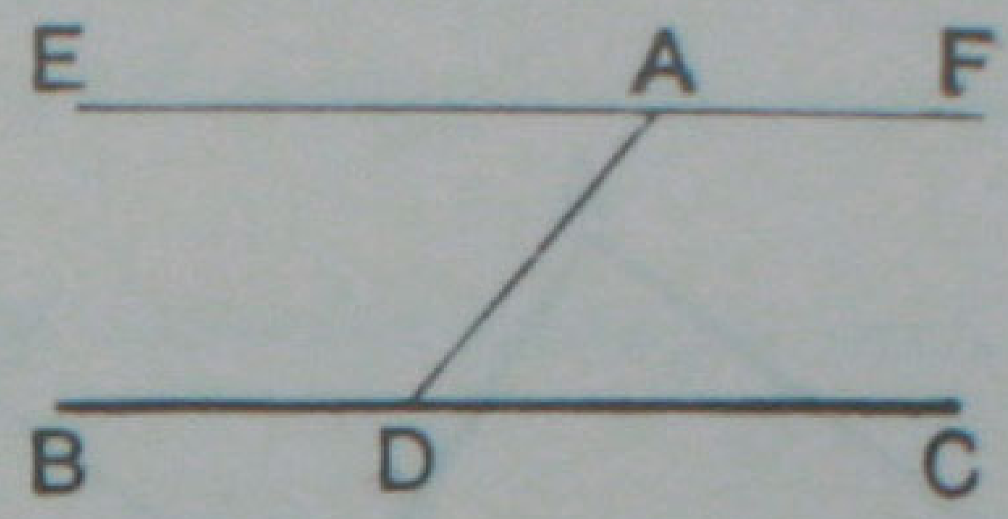
The truth of this Proposition may be readily deduced from Playfair's Axiom, of which it is the converse.

For if AB and CD were not parallel, they would meet when produced. Then there would be two intersecting straight lines both parallel to a third straight line : which is impossible.

Therefore AB and CD never meet ; that is, they are parallel.

PROPOSITION 31. PROBLEM.

To draw a straight line through a given point parallel to a given straight line.



Let A be the given point, and BC the given straight line.

It is required to draw through A *a straight line parallel to* BC.

Construction. In BC take any point D; and join AD.
At the point A in DA, make the angle DAE equal to the angle ADC, and alternate to it, I. 23.

and produce EA to F.

Then shall EF *be parallel to* BC.

Proof. Because the straight line AD, meeting the two straight lines EF, BC, makes the alternate angles EAD, ADC equal; *Constr.*

therefore EF is parallel to BC; I. 27.

and it has been drawn through the given point A.

Q.E.F.

EXERCISES.

1. Any straight line drawn parallel to the base of an isosceles triangle makes equal angles with the sides.

2. If from any point in the bisector of an angle a straight line is drawn parallel to either arm of the angle, the triangle thus formed is isosceles.

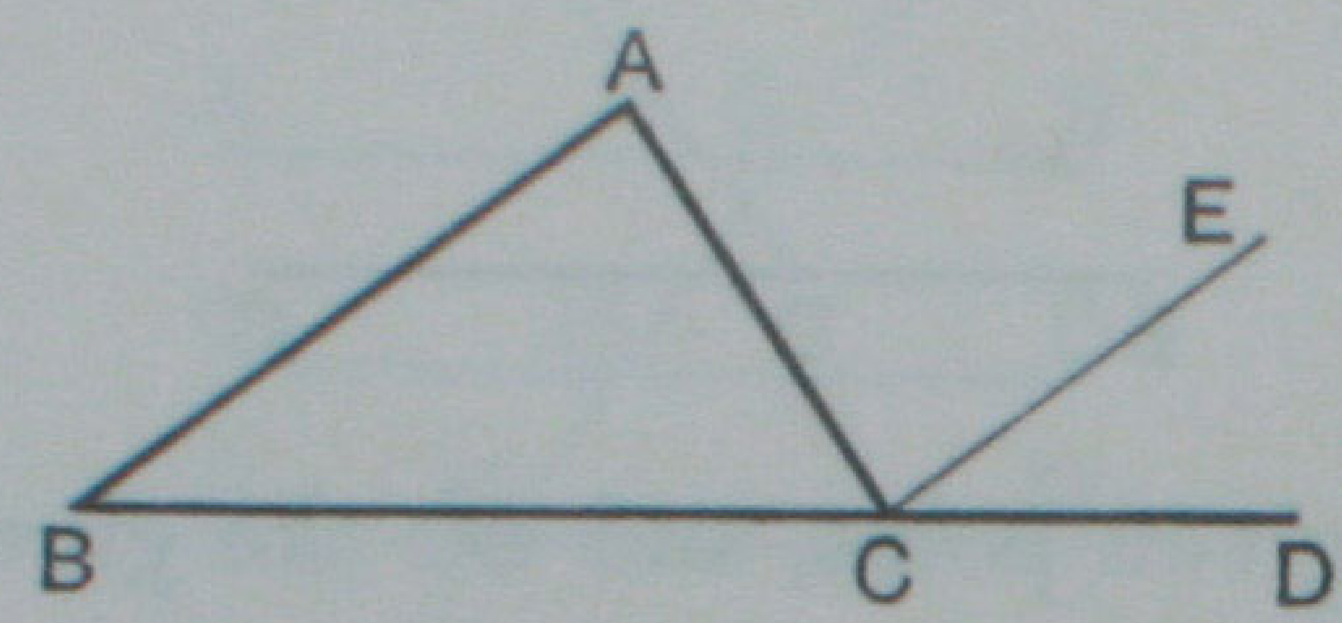
3. From a given point draw a straight line that shall make with a given straight line an angle equal to a given angle.

4. From X, a point in the base BC of an isosceles triangle ABC, a straight line is drawn at right angles to the base, cutting AB in Y, and CA produced in Z: shew the triangle AYZ is isosceles.

5. If the straight line which bisects an exterior angle of a triangle is parallel to the opposite side, shew that the triangle is isosceles.

PROPOSITION 32. THEOREM.

If a side of a triangle be produced, then the exterior angle shall be equal to the sum of the two interior opposite angles; also the three interior angles of a triangle are together equal to two right angles.



Let ABC be a triangle, and let one of its sides BC be produced to D.

Then (i) *the exterior angle* ACD *shall be equal to the sum of the two interior opposite angles* CAB, ABC;

(ii) *the three interior angles* ABC, BCA, CAB *shall be together equal to two right angles.*

Construction. Through C draw CE parallel to BA. I. 31.

Proof. (i) Then because BA and CE are parallel, and AC meets them,

therefore the angle ACE is equal to the alternate angle CAB. I. 29.

Again, because BA and CE are parallel, and BD meets them,

therefore the exterior angle ECD is equal to the interior opposite angle ABC. I. 29.

Therefore the whole exterior angle ACD is equal to the sum of the two interior opposite angles CAB, ABC.

(ii) Again, since the angle ACD is equal to the sum of the angles CAB, ABC; *Proved.*

to each of these equals add the angle BCA:

then the angles BCA, ACD are together equal to the three angles BCA, CAB, ABC.

But the adjacent angles BCA, ACD are together equal to two right angles. I. 13.

Therefore also the angles BCA, CAB, ABC are together equal to two right angles. Q.E.D.

From this Proposition we draw the following important inferences.

1. *If two triangles have two angles of the one equal to two angles of the other, each to each, then the third angle of the one is equal to the third angle of the other.*

2. *In any right-angled triangle the two acute angles are complementary.*

3. *In a right-angled isosceles triangle each of the equal angles is half a right angle.*

4. *If one angle of a triangle is equal to the sum of the other two, the triangle is right-angled.*

5. *The sum of the angles of any quadrilateral figure is equal to four right angles.*

6. *Each angle of an equilateral triangle is two-thirds of a right angle.*

EXERCISES ON PROPOSITION 32.

1. Prove that the three angles of a triangle are together equal to two right angles,

   (i) by drawing through the vertex a straight line parallel to the base;

   (ii) by joining the vertex to any point in the base.

2. If the base of any triangle is produced both ways, shew that the sum of the two exterior angles diminished by the vertical angle is equal to two right angles.

3. *If two straight lines are perpendicular to two other straight lines, each to each, the acute angle between the first pair is equal to the acute angle between the second pair.*

4. *Every right-angled triangle is divided into two isosceles triangles by a straight line drawn from the right angle to the middle point of the hypotenuse.*

Hence the joining line is equal to half the hypotenuse.

5. *Draw a straight line at right angles to a given finite straight line from one of its extremities, without producing the given straight line.*

[Let AB be the given straight line. On AB describe any isosceles triangle ACB. Produce BC to D, making CD equal to BC. Join AD. Then shall AD be perpendicular to AB.]

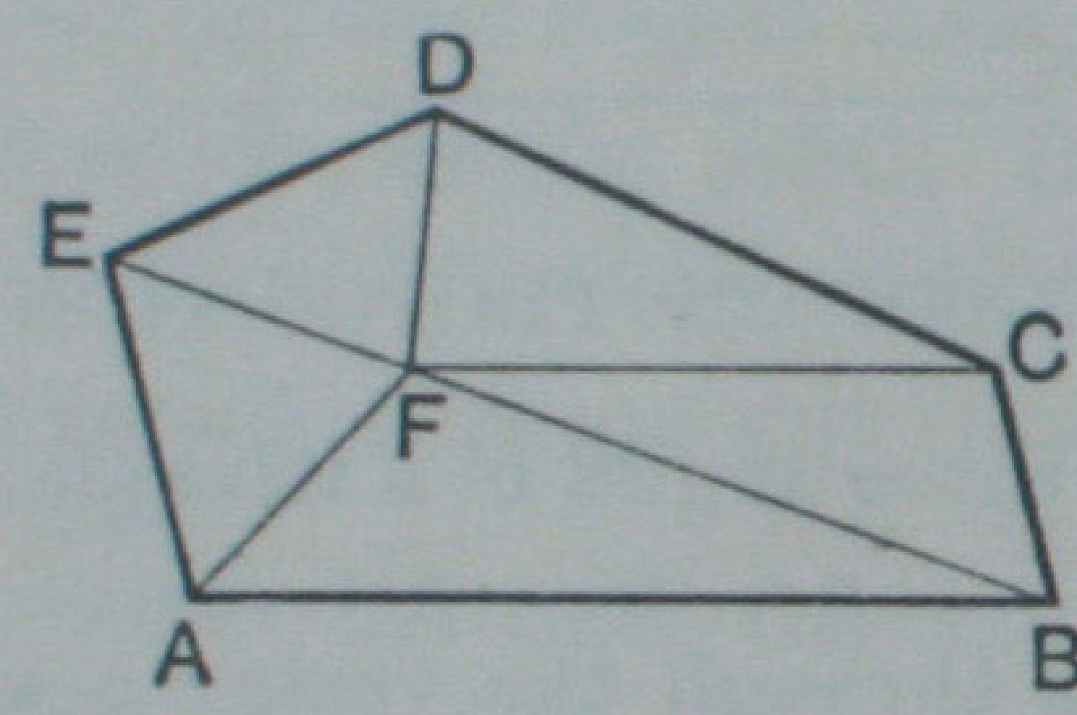
6. *Trisect a right angle.*

7. The angle contained by the bisectors of the angles at the base of an isosceles triangle is equal to an exterior angle formed by producing the base.

8. The angle contained by the bisectors of two adjacent angles of a quadrilateral is equal to half the sum of the remaining angles.

The following theorems were added as corollaries to Proposition 32 by Robert Simson, who edited Euclid's text in 1756.

COROLLARY 1. *All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*



Let ABCDE be any rectilineal figure.
Take F, any point within it,
and join F to each of the angular points of the figure.

Then the figure is divided into as many triangles as it has sides.

And the three angles of each triangle are together equal to two right angles. I. 32.

Hence *all* the angles of *all* the triangles are together equal to twice as many right angles as the figure has sides.

But all the angles of all the triangles make up all the interior angles of the figure, together with the angles at F, which are equal to four right angles. I. 15, *Cor.*

Therefore all the interior angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides. Q.E.D.