18. If $\frac{\tan(\theta+a)}{\tan(\theta-a)}=\frac{a+b}{a-b}$, and $a\cos 2a+b\cos 2\theta=c$,

shew that $$a^2+c^2-2ac\cos 2a=b^2.$$

19. If $x=a(\sin 3\theta-\sin\theta)$, and $y=a(\cos\theta-\cos 3\theta)$,

shew that $$(x^2+y^2)(2a^2-x^2-y^2)^2=4a^4x^2.$$

Eliminate θ from the equations:

20. $x\cos\theta-y\sin\theta=a\cos 2\theta,\quad x\sin\theta+y\cos\theta=2a\sin 2\theta.$

21. $x\sin\theta-y\cos\theta=\sqrt{x^2+y^2},\quad \frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}=\frac{1}{x^2+y^2}.$

22. $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1,\quad x\sin\theta-y\cos\theta=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}.$

23. If $\cos(a-3\theta)=m\cos^3\theta$, and $\sin(a-3\theta)=m\sin^3\theta$,

shew that $$m^2+m\cos a=2.$$

Eliminate θ and ϕ from the equations:

24. $\tan\theta+\tan\phi=x,\quad \cot\theta+\cot\phi=y,\quad \theta+\phi=a.$

25. $\sin\theta+\sin\phi=a,\quad \cos\theta+\cos\phi=b,\quad \theta-\phi=a.$

26. $a\sin^2\theta+b\cos^2\theta=a\cos^2\phi+b\sin^2\phi=1,\quad a\tan\theta=b\tan\phi.$

27. If $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$, and $\theta-\phi=a$,

shew that $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\sec^2\frac{a}{2}.$$

28. If $\tan\theta+\tan\phi=a,\quad \cot\theta+\cot\phi=b,\quad \theta-\phi=a$,

shew that $$ab(ab-4)=(a+b)^2\tan^2 a.$$

Eliminate θ and ϕ between the equations:

29. $a\cos^2\theta+b\sin^2\theta=m\cos^2\phi,\quad a\sin^2\theta+b\cos^2\theta=n\sin^2\phi,$
$$m\tan^2\theta-n\tan^2\phi=0.$$

30. $x\cos\theta+y\sin\theta=2a\sqrt{3},\quad x\cos(\theta+\phi)+y\sin(\theta+\phi)=4a,$
$$x\cos(\theta-\phi)+y\sin(\theta-\phi)=2a.$$

31. $c\sin\theta=a\sin(\theta+\phi),\quad a\sin\phi=b\sin\theta,\quad \cos\theta-\cos\phi=2m.$

Application of Trigonometry to the Theory of Equations.

326. In the Theory of Equations it is shewn that the solution of *any* cubic equation may be made to depend on the solution of a cubic equation of the form $x^3+ax+b=0$. In certain cases the solution is very conveniently obtained by Trigonometry.

327. Consider the equation

$$x^3-qx-r=0 \quad\text{.........................}(1),$$

in which each of the letters q and r represents a positive quantity.

From the identity $\cos 3\theta=4\cos^3\theta-3\cos\theta$,

we have $$\cos^3\theta-\frac{3}{4}\cos\theta-\frac{\cos 3\theta}{4}=0 \quad\text{.................}(2).$$

Let $x=y\cos\theta$, where y is a positive quantity; then from (1),

$$\cos^3\theta-\frac{q}{y^2}\cos\theta-\frac{r}{y^3}=0 \quad\text{......................}(3).$$

If the equations (2) and (3) are identical, we have $\frac{q}{y^2}=\frac{3}{4}$, so that $y=+\sqrt{\frac{4q}{3}}$, since y is positive; and

$$\frac{\cos 3\theta}{4}=\frac{r}{y^3}=\sqrt{\frac{27r^2}{64q^3}};$$

whence $$\cos 3\theta=\sqrt{\frac{27r^2}{4q^3}}.$$

Hence the values of θ are real if $27r^2<4q^3$;

that is, if $$\left(\frac{r}{2}\right)^2<\left(\frac{q}{3}\right)^3.$$

Let a be the smallest angle whose cosine is equal to $\sqrt{\frac{27r^2}{4q^3}}$; then $\cos 3\theta=\cos a$; whence $3\theta=2n\pi\pm a$.

Thus the values of $\cos\theta$ are

$$\cos\frac{a}{3},\quad \cos\frac{2\pi+a}{3},\quad \cos\frac{2\pi-a}{3}.$$ [See Art. 264.]

But $$x = y\cos\theta = \sqrt{\frac{4q}{3}}\cos\theta,$$

and therefore the roots of $x^3 - qx - r = 0$ are

$$\sqrt{\frac{4q}{3}}\cos a,\quad \sqrt{\frac{4q}{3}}\cos\frac{2\pi + a}{3},\quad \sqrt{\frac{4q}{3}}\cos\frac{2\pi - a}{3}.$$

328. Following the method explained in the preceding article, we may use the identity

$$\sin^3\theta - \frac{3}{4}\sin\theta + \frac{\sin 3\theta}{4} = 0$$

to obtain the solution of the equation

$$x^3 - qx + r = 0,$$

each of the quantities represented by q and r being positive.

Example. Solve the equation $x^3 - 12x + 8 = 0$.

We have $$\sin^3\theta - \frac{3}{4}\sin\theta + \frac{\sin 3\theta}{4} = 0.$$

In the given equation put $x = y\sin\theta$, where y is positive; then

$$\sin^3\theta - \frac{12}{y^2}\sin\theta + \frac{8}{y^3} = 0.$$

$$\therefore\ \frac{3}{4} = \frac{12}{y^2};\ \text{whence } y = 4;$$

and $$\frac{\sin 3\theta}{4} = \frac{8}{y^3} = \frac{1}{8};\ \text{whence } \sin 3\theta = \frac{1}{2}.$$

Suppose that θ is estimated in sexagesimal measure; then

$$3\theta = n \, . \, 180^\circ + (-1)^n \, 30^\circ.$$

By ascribing to n the values 0, 1, 2, 3, 4 we obtain

$$\theta = 10^\circ,\quad \theta = 50^\circ,\quad \theta = 130^\circ,\quad \theta = 170^\circ,\quad \theta = 250^\circ;$$

and by further ascribing to n the values 5, 6, 7,... it will easily be seen that the values of $\sin\theta$ are equal to some one of the three quantities

$$\sin 10^\circ,\quad \sin 50^\circ,\quad -\sin 70^\circ.$$

But $x = y\sin\theta = 4\sin\theta$, and therefore the roots are

$$4\sin 10^\circ,\quad 4\sin 50^\circ,\quad -4\sin 70^\circ.$$

Application of the Theory of Equations to Trigonometry.

329. In the Theory of Equations it is shewn that the equation whose roots are $a_1, a_2, a_3, \ldots\ldots, a_n$ is

$$(x-a_1)(x-a_2)(x-a_3)\ldots\ldots\ldots\ldots(x-a_n)=0,$$

or $$x^n - S_1x^{n-1} + S_2x^{n-2} - S_3x^{n-3} + \ldots\ldots + (-1)^nS_n = 0,$$

where S_1 = sum of the roots;

S_2 = sum of the products of the roots taken two at a time;

S_3 = sum of the products of the roots taken three at a time;

....

S_n = product of the roots.

[See Hall and Knight's *Higher Algebra*, Art. 538 and Art. 539.]

Example 1. If α, β, γ are the values of θ which satisfy the equation

$$a\tan^3\theta + (2a-x)\tan\theta + y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

shew that (i) if $\tan\alpha + \tan\beta = h$, then $ah^3 + (2a-x)h = y$;

(ii) if $\tan\alpha\tan\beta = k$, then $y^2 + (2a-x)ak^2 = a^2k^3$.

(i) From the theory of equations, we have from (1),

$$\tan\alpha + \tan\beta + \tan\gamma = 0;$$

$$\therefore\ h + \tan\gamma = 0, \text{ or } \tan\gamma = -h.$$

But $$a\tan^3\gamma + (2a-x)\tan\gamma + y = 0;$$

$$\therefore\ ah^3 + (2a-x)h - y = 0.$$

(ii) From the theory of equations, we have from (1),

$$\tan\alpha\tan\beta\tan\gamma = -\frac{y}{a};$$

$$\therefore\ k\tan\gamma = -\frac{y}{a}, \text{ or } \tan\gamma = -\frac{y}{ak}.$$

Substituting in $a\tan^3\gamma + (2a-x)\tan\gamma + y = 0$, we have

$$-\frac{ay^3}{a^3k^3} - (2a-x)\frac{y}{ak} + y = 0;$$

$$\therefore\ y^2 + (2a-x)ak^2 - a^2k^3 = 0.$$

Example 2. Shew that

$$\cos^2 \alpha + \cos^2\left(\frac{2\pi}{3}+\alpha\right) + \cos^2\left(\frac{2\pi}{3}-\alpha\right) = \frac{3}{2}.$$

Suppose that $\cos 3\theta = k$;

then $4\cos^3\theta - 3\cos\theta = \cos 3\theta = k$;

$$\therefore \cos^3\theta - \frac{3}{4}\cos\theta - \frac{k}{4} = 0.$$

The roots of this cubic in $\cos\theta$ are

$$\cos\alpha,\ \cos\left(\frac{2\pi}{3}+\alpha\right),\ \text{and}\ \cos\left(\frac{2\pi}{3}-\alpha\right),$$

where α is any angle which satisfies the equation $\cos 3\alpha = k$. For shortness, denote the roots by a, b, c; then

$$a^2+b^2+c^2 = (a+b+c)^2 - 2(bc+ca+ab)$$

$$= 0 - 2\left(-\frac{3}{4}\right);$$

$$\therefore \cos^2 \alpha + \cos^2\left(\frac{2\pi}{3}+\alpha\right) + \cos^2\left(\frac{2\pi}{3}-\alpha\right) = \frac{3}{2}.$$

330. If $5\theta = 2n\pi$, where n is any integer, we have

$$3\theta = 2n\pi - 2\theta;$$

$$\therefore \sin 3\theta = -\sin 2\theta.$$

The values of $\sin\theta$ found from this equation are

$$0,\ \sin\frac{2\pi}{5},\ \sin\frac{4\pi}{5},\ \sin\frac{6\pi}{5},\ \sin\frac{8\pi}{5},$$

being obtained by giving to n the values 0, 1, 2, 3, 4. It will easily be seen that no new values of $\sin\theta$ are obtained by ascribing to n the values 5, 6, 7, ...

But $$\sin\frac{6\pi}{5} = -\sin\frac{4\pi}{5} = -\sin\frac{\pi}{5},$$

and $$\sin\frac{8\pi}{5} = -\sin\frac{2\pi}{5};$$

hence rejecting the zero solution, the values of $\sin\theta$ found from the equation $\sin 3\theta = -\sin 2\theta$ are

$$\pm\sin\frac{\pi}{5},\ \text{and}\ \pm\sin\frac{2\pi}{5}.$$

If we put $\sin\theta = x$, the equation $\sin 3\theta = -\sin 2\theta$ becomes

$$3x - 4x^3 = -2x\sqrt{1-x^2}.$$

Dividing by x, and thus removing the solution $x=0$, we have

$$(3-4x^2)^2 = 4(1-x^2),$$

or

$$16x^4 - 20x^2 + 5 = 0.$$

This is a quadratic in x^2, and as we have just seen the values of x^2 are

$$\sin^2\frac{\pi}{5} \text{ and } \sin^2\frac{2\pi}{5}.$$

From the theory of quadratic equations, we have

$$\sin^2\frac{\pi}{5} + \sin^2\frac{2\pi}{5} = \frac{20}{16} = \frac{5}{4};$$

$$\sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5} = \frac{5}{16}.$$

Example. Shew that

$$\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \frac{1}{2}\sqrt{7}.$$

If $7\theta = 2n\pi$, where n is any integer, we have

$$\sin 4\theta = -\sin 3\theta.$$

The values of $\sin\theta$ found from this equation are

$$0,\ \pm\sin\frac{2\pi}{7},\ \pm\sin\frac{4\pi}{7},\ \pm\sin\frac{8\pi}{7},$$

since

$$\sin\frac{6\pi}{7} = -\sin\frac{8\pi}{7}.$$

If $\sin\theta = x$, the equation $\sin 4\theta = -\sin 3\theta$ becomes

$$4x(1-2x^2)\sqrt{1-x^2} = 4x^3 - 3x;$$

whence rejecting the solution $x=0$, we obtain

$$16(1-4x^2+4x^4)(1-x^2) = 16x^4 - 24x^2 + 9,$$

or

$$64x^6 - 112x^4 + 56x^2 - 7 = 0 \quad \text{.....................(1).}$$

The values of x^2 found from this equation are

$$\sin^2\frac{2\pi}{7},\ \sin^2\frac{4\pi}{7},\ \sin^2\frac{8\pi}{7};$$

hence $$\sin^2\frac{2\pi}{7}+\sin^2\frac{4\pi}{7}+\sin^2\frac{8\pi}{7}=\frac{112}{64}=\frac{7}{4}.$$

But $$\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}+\sin\frac{2\pi}{7}\sin\frac{8\pi}{7}+\sin\frac{4\pi}{7}\sin\frac{8\pi}{7}$$

$$=\frac{1}{2}\left\{\left(\cos\frac{2\pi}{7}-\cos\frac{6\pi}{7}\right)+\left(\cos\frac{6\pi}{7}-\cos\frac{10\pi}{7}\right)+\left(\cos\frac{4\pi}{7}-\cos\frac{12\pi}{7}\right)\right\}$$

$=0.$

$$\therefore\ \left(\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}\right)^2=\sin^2\frac{2\pi}{7}+\sin^2\frac{4\pi}{7}+\sin^2\frac{8\pi}{7}=\frac{7}{4};$$

$$\therefore\ \sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}=\frac{1}{2}\sqrt{7}.$$

331. If $7\theta=2n\pi$, where n is any integer, we have

$$4\theta=2n\pi-3\theta;$$

$$\therefore\ \cos 4\theta=\cos 3\theta.$$

By giving to n the values 0, 1, 2, 3, the values of $\cos\theta$ obtained from this equation are

$$1,\ \cos\frac{2\pi}{7},\ \cos\frac{4\pi}{7},\ \cos\frac{6\pi}{7}.$$

It will easily be seen that no new values of $\cos\theta$ are found by ascribing to n the values 4, 5, 6, 7, ...; for

$$\cos\frac{8\pi}{7}=\cos\frac{6\pi}{7},\ \cos\frac{10\pi}{7}=\cos\frac{4\pi}{7},\ \ldots\ldots\ldots\ldots$$

Now $$\cos 4\theta=8\cos^4\theta-8\cos^2\theta+1,$$

and therefore if $x=\cos\theta$, the equation $\cos 4\theta=\cos 3\theta$ becomes

$$8x^4-8x^2+1=4x^3-3x,$$

or $$8x^4-4x^3-8x^2+3x+1=0.$$

Removing the factor $x-1$, which corresponds to the root $\cos\theta=1$, we obtain

$$8x^3+4x^2-4x-1=0,$$

the roots of which equation are

$$\cos\frac{2\pi}{7},\ \cos\frac{4\pi}{7},\ \cos\frac{6\pi}{7}.$$

Example 1. Find the values of

$$\tan^2\frac{\pi}{7}+\tan^2\frac{2\pi}{7}+\tan^2\frac{3\pi}{7} \text{ and } \tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{3\pi}{7}.$$

If $7\theta=n\pi$, where n is any integer, we have

$$\tan 4\theta=-\tan 3\theta.$$

By writing $\tan\theta=t$, this equation becomes

$$\frac{4t-4t^3}{1-6t^2+t^4}=-\frac{3t-t^3}{1-3t^2},$$

or

$$t^6-21t^4+35t^2-7=0.$$

The roots of this cubic in t^2 are

$$\tan^2\frac{\pi}{7},\quad \tan^2\frac{2\pi}{7},\quad \tan^2\frac{3\pi}{7}.$$

$$\therefore\ \tan^2\frac{\pi}{7}+\tan^2\frac{2\pi}{7}+\tan^2\frac{3\pi}{7}=21,$$

and

$$\tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{3\pi}{7}=\sqrt{7},$$

the positive value of the square root being taken, since each of the factors on the left is positive.

Example 2. Shew that

$$\cos^4\frac{\pi}{7}+\cos^4\frac{2\pi}{7}+\cos^4\frac{3\pi}{7}=\frac{13}{16};$$

and

$$\sec^4\frac{\pi}{7}+\sec^4\frac{2\pi}{7}+\sec^4\frac{3\pi}{7}=416.$$

Let y denote any one of the quantities

$$\cos^2\frac{\pi}{7},\quad \cos^2\frac{2\pi}{7},\quad \cos^2\frac{3\pi}{7};$$

then $2y=1+x$, where x denotes one of the quantities

$$\cos\frac{2\pi}{7},\quad \cos\frac{4\pi}{7},\quad \cos\frac{6\pi}{7}.$$

From Art. 331, the equation whose roots are

$$\cos\frac{2\pi}{7},\quad \cos\frac{4\pi}{7},\quad \cos\frac{6\pi}{7}$$

is

$$8x^3+4x^2-4x-1=0;$$

whence by substituting $x=2y-1$, it follows that

$$\cos^2\frac{\pi}{7}, \quad \cos^2\frac{2\pi}{7}, \quad \cos^2\frac{3\pi}{7}$$

are the roots of the equation

$$8(2y-1)^3+4(2y-1)^2-4(2y-1)-1=0,$$

or

$$64y^3-80y^2+24y-1=0.$$

$$\therefore \cos^2\frac{\pi}{7}+\cos^2\frac{2\pi}{7}+\cos^2\frac{3\pi}{7}=\frac{80}{64}=\frac{5}{4};$$

and

$$\Sigma\cos^2\frac{\pi}{7}\cos^2\frac{2\pi}{7}=\frac{24}{64}=\frac{3}{8}.$$

By squaring the first of these equations and subtracting twice the second equation, we have

$$\cos^4\frac{\pi}{7}+\cos^4\frac{2\pi}{7}+\cos^4\frac{3\pi}{7}=\frac{13}{16}.$$

By putting $z=\frac{1}{y}$, we see that

$$\sec^2\frac{\pi}{7}, \quad \sec^2\frac{2\pi}{7}, \quad \sec^2\frac{3\pi}{7}$$

are the roots of the equation

$$z^3-24z^2+80z-64=0;$$

$$\therefore \sec^4\frac{\pi}{7}+\sec^4\frac{2\pi}{7}+\sec^4\frac{3\pi}{7}=(24)^2-(2\times 80)=416.$$

332. To find $\cos 5\theta$ and $\sin 5\theta$, we may proceed as follows:

$$\cos 5\theta+\cos\theta=2\cos 3\theta\cos 2\theta$$
$$=(4\cos^3\theta-3\cos\theta)(4\cos^2\theta-2);$$
$$\therefore \cos 5\theta=16\cos^5\theta-20\cos^3\theta+5\cos\theta.$$

$$\sin 5\theta+\sin\theta=2\sin 3\theta\cos 2\theta$$
$$=(3\sin\theta-4\sin^3\theta)(2-4\sin^2\theta);$$
$$\therefore \sin 5\theta=16\sin^5\theta-20\sin^3\theta+5\sin\theta.$$

It is easy to prove that

$$\cos 6\theta=32\cos^6\theta-48\cos^4\theta+18\cos^2\theta-1,$$

and

$$\sin 6\theta=\cos\theta(32\sin^5\theta-32\sin^3\theta+6\sin\theta).$$

EXAMPLES. XXV. c.

Solve the following equations:

1. $x^3-3x-1=0$.
2. $x^3-3x+1=0$.
3. $x^3-3x-\sqrt{3}=0$.
4. $8x^3-6x+\sqrt{2}=0$.
5. $8a^3x^3-6ax+2\sin 3A=0$.
6. $x^3-3a^2x-2a^3\cos 3A=0$.

7. If $\sin\alpha$ and $\sin\beta$ are the roots of the equation

$$a\sin^2\theta+b\sin\theta+c=0,$$

shew that (1) if $\sin\alpha+2\sin\beta=1$, then $a^2+2b^2+3ab+ac=0$,

(2) if $c\sin\alpha=a\sin\beta$, then $a+c=\pm b$.

8. If $\tan\alpha$ and $\tan\beta$ are the roots of the equation

$a\tan^2\theta-b\tan\theta+c=0$, and if $a\tan\alpha+b\tan\beta=2b$,

shew that $$b^2(2a-b)+c(a-b)^2=0.$$

9. If $\tan\alpha$, $\tan\beta$, $\tan\gamma$ are the roots of the equation

$$a\tan^3\theta+(2a-x)\tan\theta+y=0,$$

and if $a(\tan^2\alpha+\tan^2\beta)=2x-5a$, shew that $x\pm y=3a$.

10. If $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the roots of the equation

$$\cos^3\theta+a\cos^2\theta+b\cos\theta+c=0,$$

and if $\cos\alpha(\cos\beta+\cos\gamma)=2b$, prove that $abc+2b^3+c^2=0$.

Prove the following identities:

11. $\sec\alpha+\sec\left(\frac{2\pi}{3}+\alpha\right)+\sec\left(\frac{2\pi}{3}-\alpha\right)=-3\sec 3\alpha$.

12. $\sin^2\alpha+\sin^2\left(\frac{2\pi}{3}+\alpha\right)+\sin^2\left(\frac{4\pi}{3}+\alpha\right)=\frac{3}{2}$.

13. $\operatorname{cosec}\alpha+\operatorname{cosec}\left(\frac{2\pi}{3}+\alpha\right)+\operatorname{cosec}\left(\frac{4\pi}{3}+\alpha\right)=3\operatorname{cosec}3\alpha$.

14. $\operatorname{cosec}^2\frac{\pi}{5}+\operatorname{cosec}^2\frac{2\pi}{5}=4$.

15. $\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=-\frac{1}{2}$, and $\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}=-\frac{1}{4}$.

16. Form the equation whose roots are

(1) $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$, $\cos\frac{5\pi}{7}$;

(2) $\sin^2\frac{\pi}{14}$, $\sin^2\frac{3\pi}{14}$, $\sin^2\frac{5\pi}{14}$.

17. Form the equation whose roots are

$$\sin^2\frac{\pi}{7},\ \sin^2\frac{2\pi}{7},\ \sin^2\frac{3\pi}{7};$$

and shew that $\sum_{n=1}^{n=3}\sin^4\frac{n\pi}{7}=\frac{21}{16}$ and $\sum_{n=1}^{n=3}\operatorname{cosec}^4\frac{n\pi}{7}=32$.

18. Form the equation whose roots are

(1) $\cos\frac{2\pi}{9}$, $\cos\frac{4\pi}{9}$, $\cos\frac{6\pi}{9}$, $\cos\frac{8\pi}{9}$;

(2) $\cos\frac{\pi}{9}$, $\cos\frac{3\pi}{9}$, $\cos\frac{5\pi}{9}$, $\cos\frac{7\pi}{9}$.

19. Form the equation whose roots are

$$\cos^2\frac{\pi}{9},\ \cos^2\frac{2\pi}{9},\ \cos^2\frac{3\pi}{9},\ \cos^2\frac{4\pi}{9},$$

and shew that $\sum_{n=1}^{n=4}\cos^4\frac{n\pi}{9}=\frac{19}{16}$, and $\sum_{n=1}^{n=4}\sec^4\frac{n\pi}{9}=1120$.

20. Form the equation whose roots are

$$\tan^2\frac{\pi}{9},\ \tan^2\frac{2\pi}{9},\ \tan^2\frac{3\pi}{9},\ \tan^2\frac{4\pi}{9},$$

and shew that $\cot^2\frac{\pi}{9}+\cot^2\frac{2\pi}{9}+\cot^2\frac{4\pi}{9}=9$.

21. Prove that

(1) $\operatorname{cosec}^2\frac{\pi}{7}+\operatorname{cosec}^2\frac{2\pi}{7}+\operatorname{cosec}^2\frac{3\pi}{7}=8$;

(2) $\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11}=\frac{1}{32}$.

MISCELLANEOUS EXAMPLES. I.

1. If $$a \tan \alpha + b \tan \beta = (a+b) \tan \frac{\alpha+\beta}{2},$$
prove that $$a \cos \beta = b \cos \alpha.$$

2. If $$\frac{\sin^4 \alpha}{a} + \frac{\cos^4 \alpha}{b} = \frac{1}{a+b},$$
prove that $$\frac{\sin^8 \alpha}{a^3} + \frac{\cos^8 \alpha}{b^3} = \frac{1}{(a+b)^3}.$$

3. Shew that
$$2 \tan^{-1}\left\{\tan \frac{\alpha}{2} \tan\left(\frac{\pi}{4} - \frac{\beta}{2}\right)\right\} = \tan^{-1}\left(\frac{\sin \alpha \cos \beta}{\cos \alpha + \sin \beta}\right).$$

4. If the equation
$$\frac{\sin^{2n+2} \theta}{\sin^{2n} \alpha} + \frac{\cos^{2n+2} \theta}{\cos^{2n} \alpha} = 1$$
is true when $n=1$, prove that it will be true when n is any positive integer.

5. If $a \cos \theta + b \sin \theta = c$ and $a \cos^2 \theta + b \sin^2 \theta = c$,
prove that $$4a^2b^2 + (b-c)(a-c)(a-b)^2 = 0.$$

6. Prove the following identities:

(i) $\Sigma \sin(\beta-\gamma)\cos(\alpha-\beta)\cos(\alpha-\gamma) = -\Pi \sin(\beta-\gamma)$;

(ii) $\Sigma \sin \alpha \sin(\beta-\gamma)\cos(\beta+\gamma-\alpha) = 0$;

(iii) $\Sigma \sin \alpha \sin(\beta-\gamma)\sin(\beta+\gamma-\alpha) = 2\Pi \sin(\beta-\gamma)$.

7. If P be a point within a triangle ABC, such that
$$\angle PAB = \angle PBC = \angle PCA = \omega,$$
prove that (1) $\cot \omega = \cot A + \cot B + \cot C$;

(2) $\operatorname{cosec}^2 \omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$.

8. A hill of inclination 1 in 169 faces West. Shew that a railway on it which runs S.E. has an inclination of 1 in 239.

9. Two vertical walls of equal height a are inclined to one another at an angle α. At noon the breadth of their shadows are b and c: shew that the altitude θ of the sun is given by the equation
$$a^2 \sin^2 \gamma \cot^2 \theta = b^2 + c^2 + 2bc \cos \gamma.$$

MISCELLANEOUS EXAMPLES. K.

I. (*Including Chapters I—VII.*)

1. Express in degrees and minutes and also in grades the vertical angle of an isosceles triangle in which each of the angles at the base is twelve times the vertical angle.

2. The angles of a triangle are as 4 : 5 : 6. Express them in radians.

3. Prove that $\frac{\cot A - \tan A}{\cot A + \tan A} = 1 - 2\sin^2 A$.

4. If A is an acute angle and $\sin A = \frac{5}{13}$, find the value of $\tan A + \sec A$.

5. The adjacent sides of a parallelogram are 15 ft. and 30 ft., and the included angle is $60°$, find the length of the shorter diagonal to two places of decimals.

6. A tower 50 ft. high stands on the edge of a cliff. From a point in a horizontal plane through the foot of the cliff, the angular elevations of the top and bottom of the tower are observed to be α and β, where $\tan\alpha = 1{\cdot}26$ and $\tan\beta = 1{\cdot}185$. Find the height of the cliff.

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7. Find the length of 10 degrees of a meridian upon a globe 60 ft. in diameter.

8. The sine of an angle is to its cosine as 8 to 15, find their actual values.

9. Find the values of θ from the equation

$$4\sin^2\theta+\sqrt{3}=2(1+\sqrt{3})\sin\theta.$$

10. If $\tan\alpha=\frac{4}{15}$, find the value of $\frac{5\sin\alpha+7\cos\alpha}{6\cos\alpha-3\sin\alpha}$.

11. Prove that

$$(1+\sin A+\cos A)^2=2(1+\sin A)(1+\cos A).$$

12. Simplify the expression

$$2\sec^2 A-\sec^4 A-2\operatorname{cosec}^2 A+\operatorname{cosec}^4 A,$$

giving the result in terms of $\tan A$.

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13. If $\tan\theta=\frac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha}$, prove that

$$\sqrt{2}\sin\theta=\sin\alpha-\cos\alpha.$$

14. Shew that the values of

$$\frac{\sin 45^\circ-\sin 30^\circ}{\cos 45^\circ+\cos 60^\circ} \text{ and } \frac{\sec 45^\circ-\tan 45^\circ}{\operatorname{cosec} 45^\circ+\cot 45^\circ}$$

are the same.

15. Prove that the multiplier which will convert any number of centesimal seconds into English minutes is ·0054.

16. Prove the identities:

(1) $\frac{\tan A-\tan B}{\cot B-\cot A}=\frac{\tan B}{\cot A}$;

(2) $\frac{1+\tan^2\theta}{1+\cot^2\theta}=\left(\frac{1-\tan\theta}{1-\cot\theta}\right)^2$.

17. Solve the equations:

(1) $\sin\theta+\operatorname{cosec}\theta=\frac{3}{\sqrt{2}}$; (2) $\cos\theta+\sec\theta=2\frac{1}{2}$.

18. A man running on a circular track at the rate of 10 miles an hour traverses an arc which subtends 56° at the centre in 36 seconds. Find the diameter of the circle. Take $\pi = \frac{22}{7}$.

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19. If AD is drawn perpendicular to BC, the base of an equilateral triangle, and $BC=2m$, find AD. Thence, from the figure, shew that

$$\cos^2 60° + \cot^2 30° = \frac{13}{4}.$$

20. Prove the identities:

(1) $(\sin^2 A + \cos^2 A)(\tan^2 A - 1) = (\tan^2 A + 1)(\sin^2 A - \cos^2 A)$.

(2) $\sin^2\alpha \cos^2\beta - \cos^2\alpha \sin^2\beta = \sin^2\alpha \sin^2\beta (\operatorname{cosec}^2\beta - \operatorname{cosec}^2\alpha)$.

21. In a triangle, right-angled at C, find c and b, given that

$$a + c = 281, \quad \cos B = \cdot 405.$$

22. On a globe of 6 miles diameter an arc of 2 fur. 55 yds. is measured: find the radian measure of the angle subtended at the centre of the globe.

If this was taken as the unit of measurement, how would a right angle be represented?

23. Shew that

(1) $$\sin\theta\cos\theta\left\{\sin\left(\frac{\pi}{2}-\theta\right)\operatorname{cosec}\theta + \cos\left(\frac{\pi}{2}-\theta\right)\sec\theta\right\} = 1;$$

(2) $$\frac{\tan\left(\frac{\pi}{2}-\theta\right)}{\sec\theta} \cdot \frac{\cot^2\theta}{\sec\left(\frac{\pi}{2}-\theta\right)} \cdot \frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sin^3\theta} = \cot^5\theta.$$

24. From a station two lighthouses A, B, are seen in directions N. and N.E. respectively; but if A were half as far off as it really is, it would appear due W. from B. Compare the distances of A and B from the station.

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25. Find the numerical value of

$$3\tan^2 45° - \sin^2 60° - \frac{1}{2}\cot^2 30° + \frac{1}{8}\sec^2 45°;$$

and find x from the equation

$$\operatorname{cosec}(90° - A) - x\cos A \cot(90° - A) = \sin(90° - A).$$

26. Prove the identities:

(1) $(\sin A - \text{cosec } A)^2 + (\cos A - \sec A)^2 = \cot^2 A + \tan^2 A - 1$;

(2) $(\cot\theta - 3)(3\cot\theta - 1) = (3\,\text{cosec}^2\theta - 10\cot\theta)$.

27. If $\cot A = 4{\cdot}5$, find the value of $\dfrac{2\sin A - \cos A}{2\sin A + 3\cos A}$.

28. Find two values of θ which satisfy

$$2\cos\theta\cot\theta + 1 = \cot\theta + 2\cos\theta.$$

29. If an arc subtends $20^\circ\,17'$ at the centre of a circle whose radius is 6 inches, find in sexagesimal measure the angle it will subtend in a circle whose radius is 8 inches.

30. Looking due South from the top of a cliff the angles of depression of a rock and a life-buoy are found to be 45° and 60°. If these objects are known to be 110 yards apart, find the height of the cliff.

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31. Prove that

$$\frac{1+\cos A}{1-\cos A} - \frac{\sec A - 1}{1+\sec A} - 4\cot^2 A = \frac{4}{1+\sec A}.$$

32. Solve the equations:

(1) $8\sin^2\theta - 2\cos\theta = 5$; (2) $5\tan^2 x - \sec^2 x = 11$.

33. What is the difference in latitude of two places on the same meridian whose distance apart is 11 inches on a globe whose radius is 5 feet? Take $\pi = \dfrac{22}{7}$.

34. Given that $\sec A = \dfrac{25}{7}$, find all the other Trigonometrical ratios of A.

35. Which of the following statements are possible, and which impossible?

(1) $4\sin^2\theta = 5$; (2) $(a^2+b^2)\cos\theta = 2ab$;

(3) $(m^2+n^2)\,\text{cosec}\,\theta = m^2 - n^2$; (4) $\sec\theta = 2{\cdot}375$.

36. Walking down a hill inclined to the horizon at an angle θ a man observes an object in the horizontal plane whose angle of depression is α. Half way down the hill the angle of depression is β. Prove that $\cot\theta = 2\cot\alpha - \cot\beta$.

II. (*After Chapter XII.*)

37. In a triangle $a = 25\sqrt{2}$, $c = 50$, $C = 90°$: find B, b and the perpendicular from C on c.

38. Prove that

$$(2\sec A + 3\sin A)(3\operatorname{cosec} A - 2\cos A)$$
$$= (2\operatorname{cosec} A + 3\cos A)(3\sec A - 2\sin A).$$

39. Find the values of $\sin 960°$, $\operatorname{cosec}(-510°)$, $\tan 570°$.

40. Find all the angles between 0° and 500° which satisfy the equation $\tan^2\theta = 1$.

41. The angle of elevation of the top of a steeple is 58° from a point in the same level as its base, and is 44° from a point 42 feet directly above the former point. Given that $\tan 58° = 1{\cdot}6$ and $\tan 44° = {\cdot}965$, shew that the height of the steeple is 105 ft. approximately.

42. From the formula $\tan A = \dfrac{\sin 2A}{1 + \cos 2A}$ find $\tan 15°$ and $\tan 75°$, and solve the equation $\sec^2\theta = 4\tan\theta$.

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43. Shew that

$$(1 + \sec\theta + \tan\theta)(1 + \operatorname{cosec}\theta + \cot\theta)$$
$$= 2(1 + \tan\theta + \cot\theta + \sec\theta + \operatorname{cosec}\theta).$$

44. In a triangle ABC right-angled at C shew that

$$\frac{\sin^2 A}{\sin^2 B} - \frac{\cos^2 A}{\cos^2 B} = \frac{a^4 - b^4}{a^2 b^2}.$$

45. Find all the angles less than four right angles which satisfy the equation

$$2\cos^2\theta = 1 + \sin\theta.$$

46. Determine the value of $\sin(270^\circ + A)$ when $\sin A = \cdot 6$.

47. Given $\sin \alpha = \frac{5}{13}$, $\sin \beta = \frac{4}{5}$, find the value of $\cos(\alpha + \beta)$, and deduce $\sin(45^\circ + \alpha + \beta) = \frac{79\sqrt{2}}{130}$.

48. Reduce $\frac{\cos A - \cos 3A}{\sin 3A - \sin A}$ to a single term and trace the changes of the expression in sign and magnitude as A increases from 0° to 180°.

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49. If $\cos A = -\frac{\sqrt{3}}{2}$, find $\tan A$, drawing a diagram to explain the two values.

50. From a balloon vertically over a straight road, the angles of depression of two consecutive milestones are observed to be 45° and 60°; find the height of the balloon.

51. Find the value of

(1) $\cot^2 \frac{\pi}{6} - 2\cos^2 \frac{\pi}{3} - \frac{3}{4}\sec^2 \frac{\pi}{4} - 4\sin^2 \frac{\pi}{6}$;

(2) $2\sec^2 180^\circ \sin 0^\circ - \cos 2\pi + \operatorname{cosec} \frac{3\pi}{2}$.

52. Prove the following identities:

(1) $\sin^4 \alpha + 2\sin^2 \alpha \left(1 - \frac{1}{\operatorname{cosec}^2 \alpha}\right) = 1 - \cos^4 \alpha$;

(2) $\dfrac{1 + \tan^2\left(\frac{\pi}{4} - \theta\right)}{1 - \tan^2\left(\frac{\pi}{4} - \theta\right)} = \operatorname{cosec} 2\theta$;

(3) $\cos 10^\circ + \sin 40^\circ = \sqrt{3} \sin 70^\circ$.

53. If $b \tan \theta = a$, find the value of

$$\frac{a \sin \theta - b \cos \theta}{a \sin \theta + b \cos \theta}.$$

54. Prove that

$$4\cos 18^\circ - 3\sec 18^\circ = 2\tan 18^\circ.$$

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55. Find the values of

$$\tan(-240°),\ \cos 3360°,\ \cot(-840°).$$

Prove also that

$$\sin\frac{3\pi}{2} - \cos\frac{\pi}{2} + \cos\pi = \sec\frac{2\pi}{3}.$$

56. A railway train is travelling on a curve of half-a-mile radius at the rate of 20 miles an hour: through what angle has it turned in 10 seconds? Take $\pi = \frac{22}{7}$.

57. If $\sec a = \frac{13}{5}$, find the value of

$$\frac{2 - 3\cot a}{4 - 9\sqrt{\sec^2 a - 1}}.$$

58. Prove

(1) $2 - 2\tan A\cot 2A = \sec^2 A$;

(2) $$\frac{\cos\left(\frac{\pi}{4} - \theta\right) - \cos\left(\frac{\pi}{4} + \theta\right)}{\sin\left(\frac{2\pi}{3} + \theta\right) - \sin\left(\frac{2\pi}{3} - \theta\right)} + \sqrt{2} = 0.$$

59. When $A + B + C = 180°$, simplify

(1) $$\frac{\cos A\cos C + \cos(A+B)\cos(C+B)}{\cos A\sin C - \sin(A+B)\cos(C+B)};$$

(2) $$\frac{\cos A}{\sin B\sin C} + \frac{\cos B}{\sin C\sin A} + \frac{\cos C}{\sin A\sin B}.$$

60. A flagstaff 100 feet high stands vertically at the centre of a horizontal equilateral triangle: if each side of the triangle subtends an angle of 60° at the top of the flagstaff, find the side of the triangle.

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61. Prove that the product of

$$\sin\theta(1 + \sin\theta) + \cos\theta(1 + \cos\theta)$$

and

$$\sin\theta(1 - \sin\theta) + \cos\theta(1 - \cos\theta)$$

is equal to $\sin 2\theta$.

62. Shew that

$$(1-\sin\theta)(1-\sin\phi)=\left\{\sin\frac{\theta+\phi}{2}-\cos\frac{\theta-\phi}{2}\right\}^2.$$

63. Prove that the value of

$$\frac{\sin(a+\theta)-\sin(a-\theta)}{\cos(\beta-\theta)-\cos(\beta+\theta)}$$

is the same for all values of θ.

64. If $A+B+C=180°$, prove that

$$\cos\frac{A}{2}\cos\frac{B-C}{2}+\cos\frac{B}{2}\cos\frac{C-A}{2}+\cos\frac{C}{2}\cos\frac{A-B}{2}$$
$$=\sin A+\sin B+\sin C.$$

65. If $\tan\frac{\theta}{2}=\operatorname{cosec}\theta-\sin\theta$, shew that

$$\cos^2\frac{\theta}{2}=\cos 36° \text{ or } \cos 108°.$$

66. A man stands at a point X on the bank XY of a river with straight and parallel sides, and observes that the line joining X to a point Z on the opposite bank makes with XY an angle of 30°. He then goes 200 yards along the bank to Y and finds that YZ makes with YX an angle of 60°. Find the breadth of the river.

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67. It is found that the driving wheel of a bicycle, 32 inches in diameter, makes very nearly 1000 revolutions in travelling $2792\frac{1}{2}$ yards. Use this observation to calculate (to three places of decimals) the ratio of the circumference of a circle to its diameter.

68. If $a+\beta+\gamma=\frac{\pi}{2}$, prove that

$$\sin^2 a+\sin^2\beta+\sin^2\gamma+2\sin a\sin\beta\sin\gamma=1.$$

69. Prove that

(1) $(\tan A+\tan 2A)(\cos A+\cos 3A)=2\sin 3A$;

(2) $\sin^2 A\cos^4 A=\frac{1}{16}+\frac{1}{32}\cos 2A-\frac{1}{16}\cos 4A-\frac{1}{32}\cos 6A.$

70. If $a=\frac{\pi}{19}$, find the value of $\frac{\sin 23a-\sin 3a}{\sin 16a+\sin 4a}$.

71. If $A+B=225°$, prove that

$$\frac{\cot A}{1+\cot A}\cdot\frac{\cot B}{1+\cot B}=\frac{1}{2}.$$

72. Prove that $\cot\theta-\tan\theta=2\cot 2\theta$: and hence shew that

$$\tan\theta+2\tan 2\theta+4\tan 4\theta=\cot\theta-8\cot 8\theta.$$

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73. Simplify $1-\frac{\sin^2\theta}{1+\cot\theta}-\frac{\cos^2\theta}{1+\tan\theta}$.

74. Eliminate A between the equations

$$x=3\sin A-\sin 3A,\quad y=\cos 3A+3\cos A.$$

75. Two flagstaffs stand on a horizontal plane. A, B are two points on the line joining the bases of the flagstaffs and between them. The angles of elevation of the tops of the flagstaffs as seen from A are 30° and 60°, and as seen from B, 60° and 45°. If the length of AB is 30 ft., find the heights of the flagstaffs and the distance between them.

76. Prove the identities:

(1) $\cos^2 A+\cos^2 B-2\cos A\cos B\cos(A+B)=\sin^2(A+B)$;

(2) $2\sin 5A-\sin 3A-3\sin A=4\sin A\cos^2 A(1-8\sin^2 A)$.

77. A square is inscribed in a circle the circumference of which is 3 feet. Find the number of inches in the length of a side, correct to two places of decimals. Given

$$\frac{1}{\pi}=\cdot 3183,\quad \sqrt{2}=1\cdot 4142.$$

78. Points A, B, C, D are taken on the circumference of a circle so that the arcs AB, BC, and CD subtend respectively at the centre angles of 108°, 60°, and 36°. Shew that

$$AB=BC+CD.$$

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79. Prove that $\cot 15° + \cot 75° + \cot 135° - \operatorname{cosec} 30° = 1$.

80. From the equations

$$\cot\theta\,(1+\sin\theta)=4m,$$
$$\cot\theta\,(1-\sin\theta)=4n,$$

derive the relation $(m^2-n^2)^2=mn$.

81. Prove the identities:

(1) $\sin(\alpha+\beta)\cos\beta-\sin(\gamma+\alpha)\cos\gamma=\sin(\beta-\gamma)\cos(\alpha+\beta+\gamma)$;

(2) $(\tan 2A-\tan A)(\sec A+\sec 3A)=2\sin A\sec A\sec 3A$.

82. Prove that $\cos 6°\cos 66°\cos 42°\cos 78°=\dfrac{1}{16}$.

83. From the formula $\cot A=\dfrac{1+\cos 2A}{\sin 2A}$, prove that

$$\cot 22° 30' = \sqrt{2}+1.$$

84. An observer on board a ship sailing due North at the rate of ten miles an hour, sees a lighthouse in the East, and an hour later notices that the same lighthouse bears S.S.E.; find in miles, to two places of decimals, the distance of the ship from the lighthouse at the first observation.

III. (*After Chapter XVI.*)

85. Prove that

(1) $\sin A\sin(B-C)+\sin B\sin(C-A)+\sin C\sin(A-B)=0$;

(2) $\tan\theta=\dfrac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}$.

86. If $\alpha+\beta+\gamma=0$, prove that

$$\cos\alpha+\cos\beta+\cos\gamma=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}-1.$$

87. In any triangle prove that

$$\frac{b^2-c^2}{a}\cos A+\frac{c^2-a^2}{b}\cos B+\frac{a^2-b^2}{c}\cos C=0.$$

88. If
$$\frac{\cos\theta}{a}=\frac{\sin\theta}{b},$$
prove that
$$\frac{a}{\sec 2\theta}+\frac{b}{\operatorname{cosec} 2\theta}=a.$$

89. Prove that $\log_a b \log_b c \log_c a=1$.

Given $\log_{10} 3=\cdot 47712$, $\log_{10} 8=\cdot 90309$, find the values of
$$\log_{10} 2\cdot 4, \quad \log_{10} 5400, \quad L\tan 30^\circ.$$

90. If $A+B+C=90^\circ$, prove that
$$\cot A+\cot B+\cot C=\cot A\cot B\cot C;$$
and if A, B, C are in Arithmetical Progression, shew that this equation gives the value of $\cot 15^\circ$.

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91. Shew that
$$(1+\sin 2A+\cos 2A)^2=4\cos^2 A\,(1+\sin 2A).$$

92. In a triangle where a, b, A are given, shew that c is one of the roots of the equation
$$x^2-2bx\cos A+b^2-a^2=0.$$

93. Prove that $\dfrac{\sin 9^\circ}{\sin 48^\circ}=\dfrac{\sin 12^\circ}{\sin 81^\circ}$.

94. If $A+B+C=180^\circ$, prove that
$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}$$
$$=4\cos\left(45^\circ-\frac{A}{4}\right)\cos\left(45^\circ-\frac{B}{4}\right)\cos\left(45^\circ-\frac{C}{4}\right).$$

95. Given
$$L\sin 27^\circ 45'=9\cdot 6680265,$$
$$L\sin 27^\circ 46'=9\cdot 6682665,$$
$$L\sin\theta \quad =9\cdot 6682007,$$
find θ.

96. Prove that if A, B, C are three angles such that the sum of their cosines is zero, the product of their cosines is one-twelfth of the sum of the cosines of $3A$, $3B$, $3C$.

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97. If A be between 270° and 360°, and $\sin A = -\frac{7}{25}$, find the values of $\sin 2A$ and $\tan\frac{A}{2}$.

98. Solve the equation

$$2\cot\frac{\theta}{2} = (1+\cot\theta)^2.$$

Hence find the value of tan 15°.

99. Given $\log_{10} 2 = \cdot 3010300$, $\log_{10} 360 = 2\cdot 5563025$, find the logarithms of ·04, 24, $\cdot\dot{6}$, and shew that $\log_2 30 = 4\cdot 90689$.

100. Prove that

$$\cos(x-y-z)+\cos(y-z-x)+\cos(z-x-y)-4\cos x\cos y\cos z$$

vanishes when $x+y+z$ is an odd multiple of a right angle.

101. If $\cot\alpha = (x^3+x^2+x)^{\frac{1}{2}}$, $\cot\beta = (x+x^{-1}+1)^{\frac{1}{2}}$,

$$\tan\gamma = (x^{-3}+x^{-2}+x^{-1})^{\frac{1}{2}},$$

shew that $\alpha+\beta=\gamma$.

102. Shew how to solve a right-angled triangle of which one acute angle and the opposite side are given.

Apply this to the triangle in which the side is 28 and the angle 31° 53′ 26·8″, given $\log 2\cdot 8 = \cdot 4471580$, $\log 4\cdot 5 = \cdot 6532127$,

$$L\cot 31°\,53' = 10\cdot 2061805, \quad \text{diff. for } 1' = 2816.$$

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103. If $\tan A = \frac{\sqrt{3}}{4-\sqrt{3}}$ and $\tan B = \frac{\sqrt{3}}{4+\sqrt{3}}$,

prove that $$\tan(A-B) = \cdot 375.$$

104. The sides of a triangle are x, y, and $\sqrt{x^2+xy+y^2}$, find its greatest angle.

105. Prove that $\cos A - \sin A$ is a factor of $\cos 3A + \sin 3A$; and that

$$\cos^2 A + \cos^2\left(A+\frac{2\pi}{3}\right) + \cos^2\left(A-\frac{2\pi}{3}\right) = \frac{3}{2}.$$

106. In any triangle, if $\tan\frac{A}{2}=\frac{5}{6}$, and $\tan\frac{B}{2}=\frac{20}{37}$, find $\tan C$.

Shew also that, in such a triangle, $a+c=2b$.

107. Simplify

$$\left\{\cot\theta+\cot\left(\theta-\frac{\pi}{2}\right)\right\}\left\{\tan\left(\frac{\pi}{4}-\theta\right)+\tan\left(\frac{\pi}{4}+\theta\right)\right\}.$$

108. If $a=40$, $b=51$, $c=43$, find the value of A, given

$$\log 1{\cdot}28=\cdot107210,\quad \log 6{\cdot}03=\cdot780317,$$
$$L\tan 24°\,44'\,16''=9{\cdot}6634465.$$

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109. If
$$\tan B=\frac{n\sin A\cos A}{1-n\sin^2 A},$$
prove that
$$\tan(A-B)=(1-n)\tan A.$$

110. Given $\log 5=\cdot69897$, find $\log 200$, $\log\cdot025$, $\log\sqrt[3]{62{\cdot}5}$, and also $L\sin 30°$ and $L\cos 45°$.

111. Prove the identities:

(1) $(\sec 2A-2)\cot(A-30°)=(\sec 2A+2)\tan(A+30°)$;

(2) $1+\cos 2\alpha\cos 2\beta=2(\cos^2\alpha\cos^2\beta+\sin^2\alpha\sin^2\beta)$.

112. In a triangle, $B=60°$, $C=30°$, $BC=132$ yards. BC is produced to D and the angle $ADB=15°$; find CD and the perpendicular from A on BC, given that $\sqrt{3}=1\frac{8}{11}$ approximately.

113. In any triangle prove that

$$(a+b+c)\tan\frac{C}{2}=a\cot\frac{A}{2}+b\cot\frac{B}{2}-c\cot\frac{C}{2}.$$

114. If the sides of a triangle are 68 ft., 75 ft., 77 ft. respectively, find the least angle of the triangle, given

$$\log 2=\cdot30103,\ L\cos 26°\,34'=9{\cdot}9515389,\ \text{diff. for } 1'=632.$$

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115. If $\sin A = \cdot 6$ and A lies between 90° and 180°, find the values of $\sin(A - 90°)$, $\operatorname{cosec}(270° - A)$.

116. Prove that

$$\log_a d = \log_a b \times \log_b c \times \log_c d.$$

Given $\log_{10} 5 = \cdot 69897$, find $\log_{10} 8$, $\log_8 10$, $\log_{10}(\cdot 032)^5$.

117. Prove that

$$\cos(420° + A) + \cos(60° - A) = \cos A.$$

Deduce the value of $\cos 105° + \cos 15°$.

118. Find the values of $\tan\frac{x}{2}$ from the equation

$$\cos x - \sin\alpha \cot\beta \sin x = \cos\alpha.$$

119. If $\sin A : \sin(2A + B) = n : m$, prove that

$$\cot(A + B) = \frac{m - n}{m + n} \cot A.$$

120. A tower AB stands on a horizontal plane, and AC, AD are the shadows at noon and 6 P.M. If AD is 17 ft. longer than AC, and BC is 53 ft., find the height of the tower and the altitude of the sun at noon, when the altitude at 6 P.M. is 45°; given $\tan 31° 48' = \cdot 6\dot{2}$.

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121. Prove that

(1) $\sin 8\theta + \sin 2\theta = 4 \sin\frac{5\theta}{2} \cos\frac{5\theta}{2} \cos 3\theta$;

(2) $\sin 18° + \cos 18° = \sqrt{2} \cos 27°$.

122. Given $\log 36 = 1\cdot 556302$, $\log 48 = 1\cdot 681241$, find $\log 40$ and $\log\sqrt{\frac{2}{15}}$.

123. Given $b = 9\cdot 5$, $c = \cdot 5$, $A = 144°$, find the remaining angles; given $\log 3 = \cdot 4771213$, $L \cot 72° = 9\cdot 5117760$,

$$L \tan 16° 19' = 9\cdot 4664765, \quad L \tan 16° 18' = 9\cdot 4660078.$$

124. In any triangle prove that

(1) $bc \sin^2 A = a^2 (\cos A + \cos B \cos C)$;

(2) $bc \cos A + ca \cos B + 2ab \cos C = a^2 + b^2$.

125. If $\tan \frac{\beta}{2} = 4 \tan \frac{\alpha}{2}$, prove that

$$\tan \frac{\beta - \alpha}{2} = \frac{3 \sin \alpha}{5 - 3 \cos \alpha}.$$

126. Shew that

$$\sin (36° + A) - \sin (36° - A) + \sin (72° - A) - \sin (72° + A) = \sin A.$$

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127. If $\sin \theta = -\frac{2}{3}$, find $\tan \theta$, and explain by means of a figure why there are two values.

128. Prove that

(1) $\sin 2A + \cos 2B = 2 \sin \left(\frac{\pi}{4} + A - B\right) \cos \left(\frac{\pi}{4} - A - B\right)$;

(2) $(\sin \theta - \sin \phi)(\cos \phi + \cos \theta) = 2 \sin (\theta - \phi) \cos^2 \frac{\theta + \phi}{2}$.

129. In any triangle, if

$(\sin A + \sin B + \sin C)(\sin A + \sin B - \sin C) = 3 \sin A \sin B$,

prove that $C = 60°$.

130. Prove that $\log_a b \times \log_c d = \log_a d \times \log_c b$.

131. If $\log 2001 = 3·3012471$, $\log 2 = ·30103$, find $\log 20·0075$.

132. If $a = 7$, $b = 8$, $c = 9$, shew that the length of line joining B to the middle point of AC is 7.

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133. If $\tan A + \sec A = 2$, prove that $\sin A = \frac{3}{5}$ when A is less than 90°.

134. Prove that

$$\frac{3 - 4 \cos 2A + \cos 4A}{3 + 4 \cos 2A + \cos 4A} = \tan^4 A.$$

135. Shew that

$$\frac{\sin 3A + \cos 3A}{\sin 3A - \cos 3A} = \frac{1 + 2 \sin 2A}{1 - 2 \sin 2A} \tan (A - 45°).$$

136. If $$\frac{x}{y} = \frac{\cos A}{\cos B},$$

prove that $$x \tan A + y \tan B = (x + y) \tan \frac{A + B}{2}.$$

137. Given

$$\log 3{\cdot}5 = {\cdot}544068, \quad \log 3{\cdot}25 = {\cdot}511883, \quad \log 2{\cdot}45 = {\cdot}389166,$$

find $\log 5$, $\log 7$, and $\log 13$.

138. In a triangle, $a = 384$, $b = 330$, $C = 90°$; find the other angles; given

$$\log 11 = 1{\cdot}0413927, \quad L \tan 49° \, 19' = 10{\cdot}0656886;$$
$$\log 20 = 1{\cdot}3010300, \quad L \tan 49° \, 20' = 10{\cdot}0659441.$$

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139. If $\cos \theta = \cos \alpha \cos \beta$, prove that

$$\tan \frac{\theta + \alpha}{2} \tan \frac{\theta - \alpha}{2} = \tan^2 \frac{\beta}{2}.$$

140. Prove that

$$\frac{\sin \theta}{\cos \theta + \sin \phi} + \frac{\sin \phi}{\cos \phi - \sin \theta} = \frac{\sin \theta}{\cos \theta - \sin \phi} + \frac{\sin \phi}{\cos \phi + \sin \theta}.$$

141. If in a triangle $c(a + b) \cos \frac{B}{2} = b(a + c) \cos \frac{C}{2}$, prove that $b = c$.

142. Prove the identities:

(1) $$\frac{\cot A + \operatorname{cosec} A}{\tan A + \sec A} = \cot \left(\frac{\pi}{4} + \frac{A}{2} \right) \cot \frac{A}{2};$$

(2) $$\sin^3 A + \sin^3 (120° + A) + \sin^3 (240° + A) = -\frac{3}{4} \sin 3A.$$

143. Calculate the value of $\sqrt[5]{18 \times {\cdot}0015}$, having given

$$\log 3 = {\cdot}4771213, \quad \log 48559 = 4{\cdot}6862697, \quad \log 48560 = 4{\cdot}6862787.$$

144. Find the other angles of a triangle when one angle is 112° 4′, the side opposite to it 573 yards long, and another side 394 yards long; given

$$\log 573 = 2{\cdot}7581546, \quad \log 394 = 2{\cdot}5954962,$$
$$L\cos 22^\circ 4' = 9{\cdot}9669614, \quad L\sin 39^\circ 35' = 9{\cdot}8042757,$$
$$L\sin 39^\circ 36' = 9{\cdot}8044284.$$

IV. (*After Chapter XVIII.*)

145. In any triangle prove

$$\frac{\cos A}{c\cos B + b\cos C} + \frac{\cos B}{a\cos C + c\cos A} + \frac{\cos C}{b\cos A + a\cos B} = \frac{a^2+b^2+c^2}{2abc}.$$

146. Given $\log 7 = {\cdot}8450980$, and $\log 17 = 1{\cdot}2304489$, find $\log 119$, $\log \frac{17}{7}$, and $\log \frac{289}{343}$.

147. If A, B, C are the angles of a triangle, and if

$$\cos\theta(\sin B + \sin C) = \sin A,$$

prove that $$\tan^2\frac{\theta}{2} = \tan\frac{B}{2}\tan\frac{C}{2}.$$

148. Prove that the diameter of a circle is a mean proportional between the lengths of the sides of the equilateral triangle and the regular hexagon that circumscribe it.

149. Given that the sides a and b of a triangle are respectively $50\sqrt{5}$ feet and 150 ft., and that the angle opposite the side a is 45°, find (*without logarithms*) the two values of c. Also having given

$$\log 3 = {\cdot}4771213, \quad L\sin 71^\circ 33' = 9{\cdot}9770832,$$
$$L\sin 71^\circ 34' = 9{\cdot}9771253,$$

find the two values of the angle B.

150. Prove that

$$2\cos 2x \operatorname{cosec} 3x = \operatorname{cosec} x - \operatorname{cosec} 3x.$$

Thence find the sum to n terms of the series

$$\cos 2x \operatorname{cosec} 3x + \cos(2 \,.\, 3x)\operatorname{cosec} 3^2x + \cos(2 \,.\, 3^2x)\operatorname{cosec} 3^3x + \ldots.$$

151. Prove the identities:

(1) $\cos^2 A + \sin^2 A \cos 2B = \cos^2 B + \sin^2 B \cos 2A$;

(2) $\sin 33^\circ + \cos 63^\circ = \cos 3^\circ$.

✓ **152.** Find all the positive angles less than two right angles which satisfy the equation

$$\tan^4 A - 4\tan^2 A + 3 = 0.$$

153. Prove that

$$\cot\frac{\theta}{2} - 3\cot\frac{3\theta}{2} = \frac{4\sin\theta}{1+2\cos\theta}.$$

✓ **154.** The tangents of two angles of a triangle are $\frac{3}{4}$, and $\frac{5}{12}$ respectively. Find the tangent of the third angle, and the cosine of each angle of the triangle. Also find the third angle to the nearest second, having given

$$\log 33 = 1{\cdot}5185139, \quad \log 56 = 1{\cdot}7481880,$$

$$L\tan 59^\circ\, 29' = 10{\cdot}2295627, \quad \text{Diff. for } 1' = 2888.$$

✗ **155.** If in a triangle

$$(a^2+b^2)\sin(A-B) = (a^2-b^2)\sin(A+B),$$

shew that the triangle is either isosceles or right-angled.

156. If r and R are the radii of the in-circle and circum-circle of a triangle, prove that

$$8rR\left\{\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2}\right\} = 2bc + 2ca + 2ab - a^2 - b^2 - c^2.$$

✗ **157.** In any triangle prove that

$$\cot B + \frac{\cos C}{\sin B\cos A} = \cot C + \frac{\cos B}{\sin C\cos A}.$$

158. Given $\log 6 = {\cdot}778151$, $\log 4{\cdot}4 = {\cdot}643453$, $\log 1{\cdot}8 = {\cdot}255273$, find $\log 2$, $\log 3$, $\log 11$.

159. Prove the identities:

✗ (1) $\sin 3A = \sin A\,(2\cos 2A - 1)\tan(60^\circ + A)\tan(60^\circ - A)$;

(2) $(\sin 2A - \sin 2B)\tan(A+B) = 2(\sin^2 A - \sin^2 B)$.

160. Find the greatest angle of a triangle whose sides are 183, 195, and 214 feet respectively; given

$$\log 82 = 1{\cdot}9138139, \qquad \log 296 = 2{\cdot}4712917,$$
$$\log 101 = 2{\cdot}0043214, \qquad L\tan 34^\circ 26' = 9{\cdot}8360513,$$
$$\log 113 = 2{\cdot}0530784, \qquad L\tan 34^\circ 27' = 9{\cdot}8363221.$$

161. A circle and a regular octagon have the same perimeter; compare their areas, given $\sqrt{2} = 1{\cdot}414$, $\pi = 3{\cdot}1416$.

X **162.** If the sides of a triangle be in arithmetical progression, and if a be the least side, then

$$\cos A = \frac{4c - 3b}{2c}.$$

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163. If $a \sin(\theta + \alpha) = b \sin(\theta + \beta)$, prove that

$$\cot\theta = \frac{a\cos\alpha - b\cos\beta}{b\sin\beta - a\sin\alpha}.$$

164. In the ambiguous case shew that the circum-circles of the two triangles are equal.

165. From a point A on a level plain the angle of elevation of a kite is α, and its direction South; and from a place B, which is c yards South of A on the plain, the kite is seen due North at an angle of elevation β. Find the distance of the kite from A and its height above the ground.

166. If $\alpha + \beta + \gamma = 2\pi$, express $\cos\alpha + \cos\beta + \cos\gamma + 1$ in the form of a product.

167. Prove that

$$\cos 10A + \cos 8A + 3\cos 4A + 3\cos 2A = 8\cos A\cos^3 3A.$$

168. In any triangle shew that

$$R = \frac{(r_2 + r_3)(r_3 + r_1)(r_1 + r_2)}{4(r_2 r_3 + r_3 r_1 + r_1 r_2)}.$$

---

169. If $\tan^2\theta = 2\tan^2\phi + 1$, then

$$\cos 2\theta + \sin^2\phi = 0.$$

170. Prove that

$$\tan A \tan(60^\circ + A)\tan(120^\circ + A) = -\tan 3A.$$

171. If in a triangle $A=2B$, then $a^2=b(c+b)$.

172. Shew that the length of a side of an equilateral triangle inscribed in a circle is to that of a square inscribed in the same circle as $\sqrt{3} : \sqrt{2}$.

173. In any triangle prove that $\tan\left(\frac{A}{2}+B\right)=\frac{c+b}{c-b}\tan\frac{A}{2}$.

If $3c=7b$, and $A=6^\circ\,37'\,24''$, find the other angles; given

$L\tan 3^\circ\,18'\,42''=8{\cdot}7624069$, $L\tan 8^\circ\,13'\,50''=9{\cdot}1603083$,

$\log 2 = {\cdot}30103$, diff. for $10''=1486$.

174. If D be the middle point of the side BC of a triangle ABC, and if Δ be the area of the triangle, prove that

$$\cot ADB=\frac{AC^2-AB^2}{4\Delta}.$$

---

175. Prove that $\tan 20^\circ \tan 40^\circ \tan 80^\circ=\tan 60^\circ$.

176. If, in a triangle, $b=\sqrt{3}+1$, $c=2$, and $A=30^\circ$, find B, C, and a.

177. Prove that the rectangle contained by the diameters of the circumscribed and inscribed circles of a triangle is equal to

$$\frac{2abc}{a+b+c}.$$

178. Solve the triangle when $a=7$, $b=8\sqrt{3}$, $A=30^\circ$; given

$\log 2={\cdot}30103$, $L\sin 81^\circ\,47'=9{\cdot}9955188$,

$\log 3={\cdot}4771213$, diff. for $1'=183$.

$\log 7={\cdot}8450980$,

179. If
$$\sin 2\beta=\frac{\sin 2a+\sin 2a'}{1+\sin 2a\sin 2a'},$$

prove that $\tan\left(\frac{\pi}{4}+\beta\right)=\pm\tan\left(\frac{\pi}{4}+a\right)\tan\left(\frac{\pi}{4}+a'\right)$.

180. On a plain at some distance from its base, a mountain is found to have an elevation of 28°. At a station lying 3 miles 77 yards further away from the mountain the angle is reduced to 16°. Find the height of the mountain in feet.

$$\log 1{\cdot}6071 = {\cdot}2060, \quad L\sin 16^\circ = 9{\cdot}4403,$$
$$L\sin 28^\circ = 9{\cdot}6716, \quad L\sin 12^\circ = 9{\cdot}3179.$$

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181. Prove that

(1) $\tan\dfrac{A+B}{2} - \tan\dfrac{A-B}{2} = \dfrac{2\sin B}{\cos A + \cos B}$;

(2) $4\cos^8 A - 4\sin^8 A = 4\cos 2A - \sin 2A \sin 4A$.

182. If $A+B+C=\dfrac{\pi}{2}$, and $\cos A + \cos C = 2\cos B$,

shew that $$1+\tan\frac{A}{2}\tan\frac{C}{2} = 2\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right),$$

or else $A+C$ is an odd multiple of π.

183. Shew that in any triangle

$$\cos A + \cos B - \sin C = 4\sin\frac{C}{2}\sin\left(45^\circ - \frac{A}{2}\right)\sin\left(45^\circ - \frac{B}{2}\right).$$

184. With the usual notation in any triangle, prove that

$$\frac{bc}{r_1}+\frac{ca}{r_2}+\frac{ab}{r_3} = 2R\left\{\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}+\frac{b}{c}-3\right\}.$$

185. The bisector of the angle A meets the side BC in D and the circumscribed circle in E, shew that

$$DE = \frac{a^2\sec\dfrac{A}{2}}{2(b+c)}.$$

186. If $a=4090$, $b=3850$, $c=3811$, find A, given

$$\log 5{\cdot}8755 = {\cdot}7690448, \qquad \log 3{\cdot}85 = {\cdot}5854607,$$
$$\log 1{\cdot}7855 = {\cdot}2517599, \qquad \log 3{\cdot}811 = {\cdot}5810389,$$
$$L\cos 32^\circ 15' = 9{\cdot}9272306, \quad L\cos 32^\circ 16' = 9{\cdot}9271509.$$

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187. Prove that

(1) $\operatorname{cosec}^6\theta - \cot^6\theta = 1 + 3\operatorname{cosec}^2\theta\cot^2\theta$;

(2) $\cos(15^\circ - a)\sec 15^\circ - \sin(15^\circ - a)\operatorname{cosec} 15^\circ = 4\sin a$.

188. Prove that

$$\frac{\sin(A+B+C)}{\cos A \cos B \cos C} = \tan A + \tan B + \tan C - \tan A \tan B \tan C.$$

189. If $\log \frac{1025}{1024} = p$, and $\log 2 = q$,

prove that $$\log 4100 = p + 12q.$$

190. In any triangle prove that

(1) $(a^2 - b^2 - c^2)\tan A + (a^2 - b^2 + c^2)\tan B = 0$;

(2) $\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}$.

191. Find the area of the triangle, whose sides are 68 ft., 75 ft., 77 ft., respectively; and also find the radii of the three escribed circles.

192. If the bisector of the angle A of the triangle ABC meet the opposite side in D, prove that

$$AD = \frac{2bc}{b+c}\cos\frac{A}{2}.$$

V. (*After Chapter XIX.*)

193. Solve the equations:

(1) $\sin 5\theta - \sin 3\theta = \sin\theta \sec 45^\circ$;

(2) $\cot\theta + \cot\left(\frac{\pi}{4} + \theta\right) = 2$.

194. If $2\sec 2\alpha = \tan\beta + \cot\beta$, shew that one value of $\alpha + \beta$ is $\frac{\pi}{4}$.

195. If $$\cos^2\beta \tan(\alpha + \theta) = \sin^2\beta \cot(\alpha - \theta),$$

then $$\tan^2\theta = \tan(\alpha + \beta)\tan(\alpha - \beta).$$

196. If p_1, p_2, p_3 are the perpendiculars from the angular points on the sides of a triangle, prove that

(1) $8R^3 = \dfrac{a^2b^2c^2}{p_1p_2p_3}$;

(2) $\dfrac{1}{p_3^2} = \dfrac{1}{p_1^2} + \dfrac{1}{p_2^2} - \dfrac{2}{p_1p_2} \cos C$.

197. Find the perimeter of a regular quindecagon circumscribed about a circle whose area is 1386 sq. ft.; given

$$\tan 12° = {\cdot}213.$$

198. The top of a pole, placed against a wall at an angle α with the horizon, just touches the coping, and when its foot is moved a feet further from the wall, and its angle of inclination is β, it rests on the sill of a window: prove that the perpendicular distance from the coping to the sill $= a \cot \dfrac{\alpha+\beta}{2}$.

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199. In any triangle prove that

$$\frac{ab - r_1r_2}{r_3} = \frac{bc - r_2r_3}{r_1} = \frac{ca - r_3r_1}{r_2}.$$

200. Prove that

(1) $\cos^{-1}\dfrac{41}{49} = 2\sin^{-1}\dfrac{2}{7}$; (2) $3\tan^{-1}\dfrac{1}{4} = \tan^{-1}\dfrac{47}{52}$.

201. Prove the identities:

(1) $(\tan A + \sec A)\cot\dfrac{A}{2} = (\cot A + \operatorname{cosec} A)\tan\left(45° + \dfrac{A}{2}\right)$;

(2) $\cos 2A + \cos 2B - 4\sin(45° - A)\sin(45° - B)\cos(A+B)$
$= \sin 2(A+B)$.

202. Given $\log 3 = {\cdot}4771213$, $\log 7 = {\cdot}8450980$,

$$L \sin 25\tfrac{5}{7}° = 9{\cdot}6373733;$$

shew that the perimeter of a regular figure of seven sides is greater than 3 times the diameter of the circle circumscribing the figure.

203. If $\tan\phi = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$, in any triangle, prove that

$$c = (a+b)\frac{\sin\frac{C}{2}}{\cos\phi}.$$

204. The sides of a triangle are 237 and 158, and the contained angle is 66° 40′; use the formulæ in the last question to find the base.

$$\log 2 = \cdot 30103, \quad L\cot 33^\circ 20' = 10\cdot 18197,$$
$$\log 79 = 1\cdot 89763, \quad L\sin 33^\circ 20' = 9\cdot 73998,$$
$$\log 22687 = 4\cdot 35578,$$
$$L\tan 16^\circ 54' = 9\cdot 48262, \quad L\sec 16^\circ 54' = 10\cdot 01917,$$
$$L\tan 16^\circ 55' = 9\cdot 48308, \quad L\sec 16^\circ 55' = 10\cdot 01921.$$

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205. Shew that $\sec\theta = \dfrac{2}{\sqrt{2+\sqrt{2+2\cos 4\theta}}}$.

206. Prove that

$$\sin^{-1}\frac{3}{\sqrt{73}} + \cos^{-1}\frac{11}{\sqrt{146}} + \sin^{-1}\frac{1}{2} = \frac{5\pi}{12},$$

and solve the equation

$$\tan^{-1}\frac{x-1}{x+1} + \tan^{-1}\frac{2x-1}{2x+1} = \tan^{-1}\frac{23}{36}.$$

207. If x, y, z are the perpendiculars from the angular points of a triangle upon the opposite sides a, b, c, shew that

$$\frac{bx}{c} + \frac{cy}{a} + \frac{az}{b} = \frac{a^2+b^2+c^2}{2R}.$$

208. If $\sin(a-\theta) = \cos(a+\theta)$, shew that either

$$\theta = m\pi - \frac{\pi}{4} \text{ or } a = m\pi + \frac{\pi}{4},$$

where m is zero or any integer.

209. The vertical angle of an isosceles triangle is 120°; shew that the distance between the centres of the inscribed and circumscribed circles is to the base of the triangle in the ratio

$$\sqrt{3}-1 : \sqrt{3}.$$

210. If in a triangle $3R = 4r$, shew that

$$4(\cos A + \cos B + \cos C) = 7.$$

---

211. If $\dfrac{\sin(\theta+a)}{\cos(\theta-a)}=\dfrac{1-m}{1+m}$, prove that

$$\tan\left(\frac{\pi}{4}-\theta\right)=m\cot\left(\frac{\pi}{4}-a\right).$$

212. Solve the equations:

(1) $\sin 5\theta-\sin 3\theta=\sqrt{2}\cos 4\theta$;

(2) $(1-\tan\theta)(1+\sin 2\theta)=1+\tan\theta$.

213. If $\cos A+\cos B=4\sin^2\dfrac{C}{2}$ in any triangle, prove that $a+b=2c$.

214. A flagstaff standing on the top of a tower 80 feet high subtends an angle $\tan^{-1}\dfrac{1}{9}$ at a point 100 feet from the foot of the tower: find the height of the flagstaff.

215. Prove that

(1) $\cot^{-1}7+\cot^{-1}8+\cot^{-1}18=\cot^{-1}3$;

(2) $4\tan^{-1}\dfrac{1}{5}-\tan^{-1}\dfrac{1}{239}=\dfrac{\pi}{4}$.

216. If $2\sin\dfrac{A}{2}=-\sqrt{1+\sin A}+\sqrt{1-\sin A}$, shew that A lies between $(8n+3)\dfrac{\pi}{2}$ and $(8n+5)\dfrac{\pi}{2}$.

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217. Prove that

$$\sin^2\left(\frac{\pi}{8}+\frac{\theta}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{\theta}{2}\right)=\frac{1}{\sqrt{2}}\sin\theta.$$

218. If $\tan\theta=\dfrac{x\sin\phi}{1-x\cos\phi}$, $\tan\phi=\dfrac{y\sin\theta}{1-y\cos\theta}$,

prove that $$\frac{\sin\theta}{\sin\phi}=\frac{x}{y}.$$

219. Solve the equation

$$\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31};$$

and prove that

$$\sec^2(\tan^{-1}2)+\operatorname{cosec}^2(\cot^{-1}3)=15.$$

220. Prove that in any triangle

$$\sin 10A+\sin 10B+\sin 10C=4\sin 5A\sin 5B\sin 5C;$$

also that the sum of the cotangents of $\frac{5\pi+A}{2^5}$, $\frac{5\pi+B}{2^5}$, $\frac{5\pi+C}{2^5}$ is equal to their product.

221. If d_1, d_2, d_3 are the diameters of the three escribed circles, shew that

$$d_1d_2+d_2d_3+d_3d_1=(a+b+c)^2.$$

222. To determine the breadth AB of a ravine an observer places himself at C in the straight line AB produced through B, and then walks 100 yards at right angles to this line. He then finds that AB and BC subtend angles of 15° and 25° at his eye. Find the breadth of the ravine, given

$$L\cos 25^\circ=9\cdot9572757,\quad L\cos 40^\circ=9\cdot8842540,$$
$$L\cos 75^\circ=9\cdot4129962,$$
$$\log 37279=4\cdot5714643,\quad \log 3728=3\cdot5714759.$$

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223. Prove that

$$(1-\cos\theta)\{\sec\theta+\operatorname{cosec}\theta(1+\sec\theta)\}^2=2\sec^2\theta(1+\sin\theta).$$

224. If in a triangle $C=60^\circ$, prove that

$$\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}.$$

225. Prove that

$$2\cos\frac{A}{2^n}=\sqrt{2+\sqrt{2+\sqrt{\ldots\ldots\sqrt{2+2\cos A}}}};$$

the symbol indicating the extraction of the square root being repeated n times.

226. If
$$\frac{m \tan(\alpha-\theta)}{\cos^2\theta} = \frac{n \tan\theta}{\cos^2(\alpha-\theta)},$$
then
$$\theta = \frac{1}{2}\left\{\alpha - \tan^{-1}\left(\frac{n-m}{n+m}\tan\alpha\right)\right\}.$$

227. The sides of a triangle are such that
$$\frac{a}{1+m^2n^2} = \frac{b}{m^2+n^2} = \frac{c}{(1-m^2)(1+n^2)};$$
prove that $A = 2\tan^{-1}\frac{m}{n}$, $B = 2\tan^{-1} mn$, and the area of the triangle $= \frac{mnbc}{m^2+n^2}$.

228. A flagstaff h feet high placed on the top of a tower l feet high subtends the same angle β at two points a feet apart in a horizontal line through the foot of the tower. If θ be the angle subtended by the line a at the top of the flagstaff, shew that
$$h = a \sin\beta \operatorname{cosec}\theta, \text{ and } 2l = a \operatorname{cosec}\theta(\cos\theta - \sin\beta).$$

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229. Prove that
$$\frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{4}\tan\frac{\theta}{4} = \frac{1}{4}\cot\frac{\theta}{4} - \cot\theta.$$

230. A regular polygon is inscribed in a circle such that each side is $\frac{1}{m}$th of the radius; shew that the angle at the centre subtended by each side is equal to $\sec^{-1}\frac{2m^2}{2m^2-1}$.

231. At what distance will an inch subtend an angle of one second?

232. If $\tan^{-1} y = 4\tan^{-1} x$, find y as an algebraical function of x.

Hence prove that $\tan 22° 30'$ is a root of the equation
$$x^4 - 6x^2 + 1 = 0.$$

233. If $\cos 2a = \frac{240}{289}$, find $\tan a$ and explain the double answer.

234. If θ, ϕ be the greatest and least angles of a triangle, the sides of which are in Arithmetical Progression, shew that

$$4(1-\cos\theta)(1-\cos\phi)=\cos\theta+\cos\phi.$$

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235. Solve the equations:

(1) $\sin 7\theta = \sin 4\theta - \sin\theta$;

(2) $\tan x - \sqrt{3}\cot x + 1 = \sqrt{3}$.

236. In any triangle prove that

(1) $\sin 3A \sin(B-C) + \sin 3B \sin(C-A) + \sin 3C \sin(A-B) = 0$;

(2) $a^3 \sin(B-C) + b^3 \sin(C-A) + c^3 \sin(A-B) = 0$.

237. ABC is a triangle and a point P is taken on AB so that $AP : BP = m : n$. If the angle CPB is θ, shew that

$$(m+n)\cot\theta = n\cot A - m\cot B.$$

238. If α, β are unequal values of θ satisfying the equation

$$a\tan\theta + b\sec\theta = 1,$$

find a and b in terms of α and β, and prove that

$$\sin\alpha+\cos\alpha+\sin\beta+\cos\beta=\frac{2b(1-a)}{1+a^2}.$$

239. If $u_n = \sin^n\theta + \cos^n\theta$, prove that

$$\frac{u_3-u_5}{u_1}=\frac{u_5-u_7}{u_3}.$$

240. A building on a square base $ABCD$ has the sides of the base AB and CD, parallel to the banks of a river. An observer standing on the bank of the river furthest from the building in the same straight line as DA finds that the side AB subtends at his eye an angle of 45°, and after walking a yards along the bank he finds that DA subtends the angle whose sine is $\frac{1}{3}$. Prove that the length of each side of the base in yards is $\frac{a\sqrt{2}}{2}$.

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241. Prove the identities:

(1) $(\text{cosec } A - \sin A)(\sec A - \cos A) = (\tan A + \cot A)^{-1}$;

(2) $\dfrac{\tan\theta}{(1+\tan^2\theta)^2} + \dfrac{\cot\theta}{(1+\cot^2\theta)^2} = \dfrac{1}{2}\sin 2\theta.$

242. If $\sin 4\theta \cos\theta = \dfrac{1}{4} + \sin\dfrac{5\theta}{2}\cos\dfrac{5\theta}{2}$, find *one* value of θ.

243. Prove that

$$\tan^{-1}\frac{2mn}{m^2-n^2} + \tan^{-1}\frac{2pq}{p^2-q^2} = \tan^{-1}\frac{2MN}{M^2-N^2},$$

where $$M = mp - nq,\ N = np + mq.$$

244. In any triangle, prove that

$$\frac{\tan\frac{A}{2}}{(a-b)(a-c)} + \frac{\tan\frac{B}{2}}{(b-c)(b-a)} + \frac{\tan\frac{C}{2}}{(c-a)(c-b)} = \frac{1}{\Delta}.$$

245. If r_1, r_2, r_3 be the radii of the three escribed circles, and

$$\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right) = 2,$$

shew that the triangle must be right-angled.

246. The sides of a triangle are 237 and 158, and the contained angle is 58° 40′ 3·9″. Find by the aid of Tables the value of the base, without previously determining the other angles.

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247. If $\tan(A+B) = 3\tan A$, shew that

$$\sin(2A+2B) + \sin 2A = 2\sin 2B.$$

248. Prove that

$$4\sin(\theta-a)\sin(m\theta-a)\cos(\theta-m\theta)$$
$$= 1 + \cos(2\theta-2m\theta) - \cos(2\theta-2a) - \cos(2m\theta-2a).$$

249. Perpendiculars are drawn from the angles A, B, C of an acute-angled triangle on the opposite sides, and produced to meet the circumscribing circle: if those produced parts be α, β, γ respectively, shew that

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=2(\tan A+\tan B+\tan C).$$

250. If A and B are two angles, each positive, and less than 90° and such that

$$3\sin^2 A+2\sin^2 B=1,$$
$$3\sin 2A-2\sin 2B=0,$$

prove that $$A+2B=90^\circ.$$

251. Prove that

(1) $\cot^{-1}(\tan 2x)+\cot^{-1}(-\tan 3x)=x$;

(2) $\tan^{-1}\dfrac{1-x}{1+x}-\tan^{-1}\dfrac{1-y}{1+y}=\sin^{-1}\dfrac{y-x}{\sqrt{1+x^2}\sqrt{1+y^2}}$.

252. In any triangle prove that

$$a\sec\theta=b+c,$$

where $$(b+c)\sin\theta=2\sqrt{bc}\cos\frac{A}{2}.$$

Compass observations are taken from a station to two points distant respectively 1250 yards and 1575 yards. The bearing of one point is 7° 30′ West of North, and that of the other is 42° 15′ East of North. Find the distance between the points by the aid of Tables.

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253. Prove the identities:

(1) $\cos\beta\cos(2\alpha-\beta)=\cos^2\alpha-\sin^2(\alpha-\beta)$;

(2) $(x\tan\alpha+y\cot\alpha)(x\cot\alpha+y\tan\alpha)$
$$=(x+y)^2+4xy\cot^2 2\alpha.$$

254. If $2S=A+B+C$, shew that

$$\cos^2 S+\cos^2(S-A)+\cos^2(S-B)+\cos^2(S-C)$$
$$=2+2\cos A\cos B\cos C.$$

255. If $\sin^{-1}\alpha+\sin^{-1}\beta+\sin^{-1}\gamma=\pi$, prove that

$$\alpha\sqrt{1-\alpha^2}+\beta\sqrt{1-\beta^2}+\gamma\sqrt{1-\gamma^2}=2\alpha\beta\gamma.$$

256. In a triangle $a=36$, $B=73^\circ\ 15'$, $C=45^\circ\ 30'$; find R and r by the aid of Tables.

257. If ρ be the radius of the circle inscribed in the pedal triangle, prove that

$$\rho=R(1-\cos^2 A-\cos^2 B-\cos^2 C).$$

258. A, B, C are the tops of posts at equal intervals by the side of a road; t and t' are the tangents of the angles which AB and BC subtend at any point P; T is the tangent of the angle which the road makes with PB: shew that

$$\frac{2}{T}=\frac{1}{t'}-\frac{1}{t}.$$

---

259. In any triangle prove that

$$\frac{(\cos B+\cos C)(1+2\cos A)}{1+\cos A-2\cos^2 A}=\frac{b+c}{a}.$$

260. With the notation of Art. 219, prove that

$$\frac{AI}{AI_1}+\frac{BI}{BI_2}+\frac{CI}{CI_3}=1.$$

261. Prove that

$$\sin^{-1}\frac{1}{3}+\sin^{-1}\frac{1}{3\sqrt{11}}+\sin^{-1}\frac{3}{\sqrt{11}}=\frac{\pi}{2}.$$

262. Find the relation between α, β, and γ in order that

$$\cot\alpha\cot\beta\cot\gamma-\cot\alpha-\cot\beta-\cot\gamma$$

should vanish.

263. If $A+B+C=\pi$, prove that

$$\frac{\tan A}{\tan B\tan C}+\frac{\tan B}{\tan C\tan A}+\frac{\tan C}{\tan A\tan B}$$
$$=\tan A+\tan B+\tan C-2(\cot A+\cot B+\cot C).$$

264. A man travelling due North along a straight road observes that at a certain milestone two objects lie due N.E. and S.W. respectively, and that when he reaches the next milestone their directions have become S.S.E. and S.S.W. respectively. Find the distance between the two objects, and prove that the sum of their shortest distances from the road is exactly a mile.

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265. Prove that

$$\tan 20^\circ + \tan 40^\circ + \sqrt{3} \tan 20^\circ \tan 40^\circ = \sqrt{3}.$$

266. Solve the equation

$$\cot^3 \theta + 6 \operatorname{cosec} 2\theta - 8 \operatorname{cosec}^3 2\theta = 0.$$

267. If $A+B+C=180^\circ$, prove that

$$1 - 2 \sin B \cos C \cos A + \cos^2 A = \cos^2 B + \cos^2 C,$$

and if $A+B+C=0$, prove that

$$1 + 2 \sin B \sin C \cos A + \cos^2 A = \cos^2 B + \cos^2 C.$$

268. If in a triangle $\cot A$, $\cot B$, $\cot C$ are in A.P., shew that a^2, b^2, c^2 are also in A.P.

269. If α, β, γ are the angles of a triangle, prove that

$$\cos\left(\frac{3\beta}{2} + \gamma - 2\alpha\right) + \cos\left(\frac{3\gamma}{2} + \alpha - 2\beta\right) + \cos\left(\frac{3\alpha}{2} + \beta - 2\gamma\right)$$

$$= 4 \cos \frac{5\alpha - 2\beta - \gamma}{4} \cos \frac{5\beta - 2\gamma - \alpha}{4} \cos \frac{5\gamma - 2\alpha - \beta}{4}.$$

270. If the sum of the pairs of radii of the escribed circles of a triangle taken in order round the triangle be denoted by s_1, s_2, s_3, and the corresponding differences by d_1, d_2, d_3, prove that

$$d_1 d_2 d_3 + d_1 s_2 s_3 + d_2 s_3 s_1 + d_3 s_1 s_2 = 0.$$

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271. If $\cos A = \frac{3}{4}$, shew that

$$32 \sin \frac{A}{2} \sin \frac{5A}{2} = 11.$$

272. Prove that all angles which satisfy the equation

$$\tan^2\theta + 2\tan\theta = 1,$$

are included in the formula $(8n-1)\frac{\pi}{8} \pm \frac{\pi}{4}$ where n is zero or any integer.

273. Prove

(1) $\cos\left(2\tan^{-1}\frac{1}{7}\right) = \sin\left(4\tan^{-1}\frac{1}{3}\right)$;

(2) $\tan^{-1}\frac{3\sin 2a}{5+3\cos 2a} + \tan^{-1}\left(\frac{\tan a}{4}\right) = a.$

274. If in any triangle

$$\cos\frac{A}{2} = \frac{1}{2}\sqrt{\frac{b}{c} + \frac{c}{b}},$$

shew that the square described with one side of the triangle as diagonal is equal to the rectangle contained by the other two sides.

275. Find B and C, having given $A = 50°$, $b = 119$, $a = 97$.

$\log 1{\cdot}19 = {\cdot}0755470$, $L\sin 70° = 9{\cdot}9729858$,

$\log 9{\cdot}7 = {\cdot}9867717$, $L\sin 70°\,1' = 9{\cdot}9730318$,

$L\sin 50° = 9{\cdot}8842540$.

276. Circles are inscribed in the triangles $D_1E_1F_1$, $D_2E_2F_2$, $D_3E_3F_3$, where D_1, E_1, F_1 are the points of contact of the circle escribed to the side BC. Shew that if r_a, r_b, r_c be the radii of these circles

$$\frac{1}{r_a} : \frac{1}{r_b} : \frac{1}{r_c} = 1 - \tan\frac{A}{4} : 1 - \tan\frac{B}{4} : 1 - \tan\frac{C}{4}.$$

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277. Reduce to its simplest form

$$\tan^{-1}\left(\frac{x\cos\theta}{1-x\sin\theta}\right) - \cot^{-1}\left(\frac{\cos\theta}{x-\sin\theta}\right).$$

278. If $\cos A + \cos B = 4\sin^2\frac{C}{2}$ in any triangle, shew that the sides are in A.P.

279. Express $4\cos\alpha\cos\beta\cos\gamma\cos\delta+4\sin\alpha\sin\beta\sin\gamma\sin\delta$ as the sum of four cosines.

280. If I be the in-centre of a triangle and ρ_1, ρ_2, ρ_3 are the circum-radii of the triangles BIC, CIA, AIB, prove that

$$\rho_1\rho_2\rho_3=2rR^2.$$

281. A monument $ABCDE$ stands on level ground. At a point P on the ground the portions AB, AC, AD subtend angles α, β, γ respectively. Supposing that $AB=2$, $AC=16$, $AD=18$, and $\alpha+\beta+\gamma=180^\circ$, find AP.

282. If α and β be two angles both satisfying the equation

$$a\cos 2\theta+b\sin 2\theta=c,$$

prove that

$$\cos^2\alpha+\cos^2\beta=\frac{a^2+ac+b^2}{a^2+b^2}.$$

---

283. If $C=22\tfrac{1}{2}^\circ$, $a=\sqrt{2}$, $b=\sqrt{2+\sqrt{2}}$, solve the triangle.

284. If $A+B+C=180^\circ$, prove that

$$\sin^3 A+\sin^3 B+\sin^3 C$$
$$=3\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+\cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2}.$$

285. In the ambiguous case in which a, b, A are given, if one angle of one triangle be twice the corresponding angle of the other triangle, shew that

$$a\sqrt{3}=2b\sin A,\ \text{or}\ 4b^3\sin^2 A=a^2(a+3b).$$

286. If the roots of $x^3-px^2-r=0$ are $\tan\alpha$, $\tan\beta$, $\tan\gamma$, find the value of $\sec^2\alpha\sec^2\beta\sec^2\gamma$.

287. If $\alpha+\beta+\gamma=\pi$, and

$$\tan\frac{1}{4}(\beta+\gamma-\alpha)\tan\frac{1}{4}(\gamma+\alpha-\beta)\tan\frac{1}{4}(\alpha+\beta-\gamma)=1,$$

prove that $$1+\cos\alpha+\cos\beta+\cos\gamma=0.$$

288. Prove that the side of a regular heptagon inscribed in a circle of radius unity is given by one of the roots of the equation

$$x^6-7x^4+14x^2-7=0,$$

and give the geometrical signification of the other roots.

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289. If in a triangle the angle B is 45°, prove that

$$(1+\cot A)(1+\cot C)=2.$$

290. If twice the square on the diameter of a circle is equal to the sum of the squares on the sides of the inscribed triangle ABC, prove that

$$\sin^2 A+\sin^2 B+\sin^2 C=2,$$

and that the triangle is right-angled.

291. If $\cos A=\tan B$, $\cos B=\tan C$, $\cos C=\tan A$,

prove that $$\sin A=\sin B=\sin C=2\sin 18°.$$

292. In any triangle shew that a, b, c are the roots of the equation

$$x^3-2sx^2+(r^2+s^2+4Rr)\,x-4Rrs=0.$$

293. Shew that $\sin\frac{\pi}{14}$ is a root of the equation

$$8x^3-4x^2-4x+1=0.$$

294. The stones from a circular field (radius r) are collected into n heaps at regular intervals along the hedge. Prove that the distance a labourer will have to travel with a wheelbarrow, which justs holds one heap, in bringing them together to one of the heaps (supposing him to start from this heap) is $4r\cot\frac{\pi}{2n}$.

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295. Shew that

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}=\left(\frac{1}{2}\right)^7.$$

296. If x, y, z are the perpendiculars drawn to the sides from any point within a triangle, shew that $x^2+y^2+z^2$ is a minimum when

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{2\Delta}{a^2+b^2+c^2}.$$

297. If r_a, r_b, r_c, r_d be the radii of the circles which touch each side and the adjacent two sides produced of a quadrilateral, prove that

$$\frac{a}{r_a}+\frac{c}{r_c}=\frac{b}{r_b}+\frac{d}{r_d}.$$

298. If the diameters AA', BB', CC' of the circum-circle cut the sides BC, CA, AB in P, Q, R respectively, prove that

$$\frac{1}{AP}+\frac{1}{BQ}+\frac{1}{CR}=\frac{2}{R},$$

$$\frac{1}{A'P}+\frac{1}{B'Q}+\frac{1}{C'R}=\frac{1}{2R}(4+\sec A\sec B\sec C).$$

299. If α, β, γ are angles, unequal and less than 2π, which satisfy the equation $\frac{a}{\cos\theta}+\frac{b}{\sin\theta}+c=0$, prove that

$$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0.$$

300. Shew that

$$\left(\sec^2\frac{\pi}{7}+\sec^2\frac{2\pi}{7}+\sec^2\frac{3\pi}{7}\right)\left(\operatorname{cosec}^2\frac{\pi}{7}+\operatorname{cosec}^2\frac{2\pi}{7}+\operatorname{cosec}^2\frac{3\pi}{7}\right)=192.$$

ANSWERS.

I. Page 4.

1. $\cdot 75$. **2.** $\cdot 125$. **3.** $\cdot 375$. **4.** $\cdot 024\dot{1}$.

5. $\cdot 089$. **6.** $\cdot 0204045$. **7.** $76^{g}\,91^{`}\,66{\cdot}7^{``}$. **8.** $21^{g}\,12^{`}\,50^{``}$.

9. $56^{g}\,24^{`}\,25^{``}$. **10.** $48^{g}\,75^{`}\,25^{``}$. **11.** $12^{g}\,23^{`}\,40{\cdot}7^{``}$.

12. $158^{g}\,6^{`}\,94{\cdot}4^{``}$. **13.** $22^{`}\,50^{``}$. **14.** $6^{`}\,36{\cdot}7^{``}$.

15. $51^{\circ}\,11'\,15''$. **16.** $35^{\circ}\,9'\,22{\cdot}5''$. **17.** $36^{\circ}\,0'\,40{\cdot}6''$.

18. $55'\,5{\cdot}8''$. **19.** $2^{\circ}\,43'\,6{\cdot}4''$. **20.** $7^{\circ}\,17'\,26{\cdot}1''$.

21. $3'\,22{\cdot}5''$. **22.** $20'\,0{\cdot}4''$. **23.** 45°, 27°. **24.** 72°.

II. Page 11.

1. $\frac{15}{17}, \frac{17}{8}, \frac{8}{15}, \frac{17}{15}$. **2.** $\frac{12}{5}, \frac{13}{12}, \frac{5}{13}, \frac{12}{13}$.

3. $25, \frac{4}{5}, \frac{4}{5}, \frac{3}{4}, \frac{5}{3}$. **4.** $7, \frac{7}{25}, \frac{24}{7}, \frac{25}{24}$.

5. $\frac{37}{35}, \frac{37}{12}, \frac{35}{12}, \frac{35}{37}$. **6.** 12 inches, $\frac{4}{5}, \frac{3}{5}, \frac{4}{3}$.

7. $25, \frac{24}{25}, \frac{7}{25}$. **8.** 40 ft., $\frac{40}{41}, \frac{9}{40}$.

9. 20 ft., sine $=\frac{20}{29}$, cosine $=\frac{21}{29}$, tangent $=\frac{20}{21}$.

10. sine $=\frac{1}{\sqrt{5}}$, cosine $=\frac{2}{\sqrt{5}}$, tangent $=\frac{1}{2}$.

11. $\frac{12}{13}, \frac{13}{12}, \frac{77}{85}, \frac{85}{77}$. **12.** $\frac{3}{5}, \frac{4}{3}, \frac{20}{29}, \frac{29}{20}$.

III. c. Page 23.

1. $\frac{2}{\sqrt{3}}, \sqrt{3}$. **2.** $\frac{4}{5}, \frac{3}{5}$. **3.** $\frac{1}{\sqrt{15}}, \frac{\sqrt{15}}{4}$. **4.** $\frac{\sqrt{5}}{2}, \sqrt{5}$.

5. $\frac{\sqrt{48}}{7}, \frac{1}{\sqrt{48}}$. **6.** $\frac{7}{24}, \frac{25}{24}$. **7.** $\sqrt{1-\cos^2 A}, \frac{\sqrt{1-\cos^2 A}}{\cos A}$.

8. $\sqrt{1+\cot^2 \alpha}, \frac{\cot \alpha}{\sqrt{1+\cot^2 \alpha}}$. **9.** $\frac{\sqrt{\sec^2\theta - 1}}{\sec\theta}, \frac{1}{\sqrt{\sec^2\theta-1}}$.

10. $\operatorname{cosec} A = \frac{1}{\sin A}$, $\cos A = \sqrt{1-\sin^2 A}$, $\sec A = \frac{1}{\sqrt{1-\sin^2 A}}$,

$\tan A = \frac{\sin A}{\sqrt{1-\sin^2 A}}$, $\cot A = \frac{\sqrt{1-\sin^2 A}}{\sin A}$. **11.** $\sqrt{2}$.

13. $\frac{p}{q}$. **14.** $\frac{m^2-1}{2m}, \frac{m^2-1}{m^2+1}$. **15.** $\frac{p^2-q^2}{p^2+q^2}, \frac{p^2+q^2}{2pq}$.

16. 3. **17.** $\frac{p^2-q^2}{p^2+q^2}$.

IV. a. Page 26.

1. 5. **2.** $1\frac{1}{2}$. **3.** 0. **4.** $2\frac{1}{8}$. **5.** $\frac{1}{2}$.

6. $1\frac{1}{2}$. **7.** 9. **8.** 2. **9.** $2\frac{1}{12}$. **10.** $\frac{1}{2}$.

11. $\frac{3}{4}$. **12.** $-\frac{1}{2}$. **13.** $1\frac{11}{12}$. **14.** $\frac{\sqrt{3}}{2}$. **15.** 6.

IV. b. Page 28.

1. $22^\circ 30'$. **2.** $64^\circ 59' 30''$. **3.** $79^\circ 58' 57''$. **4.** $45^\circ + A$.
5. $45^\circ - B$. **6.** $60^\circ + B$. **7.** 50°. **8.** 60°.
9. 18°. **10.** 9°. **11.** $22^\circ 30'$. **12.** 45°.
13. 30°. **14.** 15°. **30.** 1. **31.** $\tan A$.

IV. c. Page 31.

1. 45°. **2.** 60°. **3.** 60°. **4.** 45°. **5.** 60°.
6. 30°. **7.** 45°. **8.** 60°. **9.** 45°. **10.** 60°.
11. 45°. **12.** 60°. **13.** 45°. **14.** 30°. **15.** 45°.
16. 60°. **17.** 30°. **18.** 60°. **19.** 45°. **20.** 60°.
21. 30°. **22.** 30°. **23.** 60°. **24.** 45°.
25. 45° or 30°. [$(2\sin\theta - 1)(\tan\theta - 1) = 0$.] **26.** 60°. **28.** 1 or $\frac{1}{2}$.

MISCELLANEOUS EXAMPLES. A. PAGE 32.

1. (1) ·2537064; (2) ·704. **3.** $\frac{20}{29}, \frac{29}{21}$. **4.** $\frac{15}{8}, \frac{17}{8}$.

6. (1) $15° 28' 7·5''$; (2) $1' 37·2''$. **7.** $41, \frac{9}{40}, \frac{41}{9}, \frac{41}{40}$.

8. (1) possible; (2) impossible; (3) possible.

10. $\frac{\sqrt{1+\cot^2 a}}{\cot a}, \sqrt{1+\cot^2 a}$. **11.** 6.

12. $\frac{m}{\sqrt{m^2+n^2}}, \frac{\sqrt{m^2+n^2}}{n}$. **16.** $\frac{20}{21}, \frac{29}{20}$.

18. 10°. **20.** (1) 30°; (2) 45°. **22.** 30°.

26. $\frac{5}{14}$. **29.** (1) 30°; (2) 30°.

V. a. PAGE 37.

1. $c=2$, $B=60°$, $C=30°$. **2.** $a=6\sqrt{3}$, $A=60°$, $C=30°$.

3. $c=8\sqrt{3}$, $A=30°$, $B=60°$. **4.** $c=30\sqrt{3}$, $B=30°$, $C=60°$.

5. $b=20\sqrt{2}$, $A=C=45°$. **6.** $c=10\sqrt{3}$, $A=30°$, $B=60°$.

7. $a=2\sqrt{2}$, $B=C=45°$. **8.** $a=9$, $A=60°$, $C=30°$.

9. $B=60°$, $b=27$, $c=18\sqrt{3}$. **10.** $C=60°$, $b=2$, $c=2\sqrt{3}$.

11. $B=30°$, $a=4\sqrt{3}$, $b=4$. **12.** $B=90°$, $a=3\sqrt{3}$, $c=3$.

13. $A=30°$, $a=50$, $c=50\sqrt{3}$. **14.** $C=90°$, $a=20$, $c=40$.

15. $A=90°$, $a=4\sqrt{2}$, $b=4$. **16.** $A=90°$, $b=4$, $c=4\sqrt{3}$.

17. 700. **18.** 31. **19.** 86·47. **20.** 97·8.

21. $C=54°$, $a=73$, $b=124$. **22.** $B=68° 17'$, $C=21° 43'$, $b=93$.

23. $C=50° 36'$, $a=39·3875$, $c=30·435$.

24. $c=353$, $A=39° 36'$, $B=50° 24'$.

V. b. PAGE 39.

1. $10\sqrt{3}$. **2.** $a=10\sqrt{2}$, $c=20$.

3. $AB=10\sqrt{3}$ ft., $AC=10$ ft., $AD=5\sqrt{3}$ ft. **4.** 12, 4.

5. $24\sqrt{3}$. **6.** $20(\sqrt{3}-1)$.

7. $20(3+\sqrt{3})$. **8.** $DC=BD=100$.

VI. a. PAGE 42.

1. 173·2 ft. 2. 277·12 ft. 3. 60°. 4. 50 ft.; 100 ft.
5. 22·5 ft.; 38·97 ft. 6. 30 ft. 7. 200 yds.
8. 51 yds., 81 yds. 9. 86·6 yds. 10. 46·19 ft.
11. 273·2 ft. 12. Each = 70·98 ft. 13. 5 miles.
14. 73·2 ft. 15. 64 ft. 16. 300 ft.
17. 1193 yds. 18. 277·12 yds.

VI. b. PAGE 47.

1. 565·6 yds.; 1131·2 yds. 2. 3·464 miles; 6 miles.
3. 29 miles. 4. 10 miles per hour.
5. 10 miles; 24·14 miles. 6. 16 miles; S. 25° W.
7. 9·656 miles. 8. 5·77 miles; 11·54 miles.
9. 295·1 knots. 10. 5·196 miles per hour; 18 miles.
11. 31 minutes past midnight. 12. 38·97 miles per hour.

VII. a. PAGE 54.

1. $\frac{\pi}{4}$. 2. $\frac{\pi}{6}$. 3. $\frac{7\pi}{12}$. 4. $\frac{\pi}{8}$. 5. $\frac{\pi}{10}$.
6. $\frac{23\pi}{72}$. 7. $\frac{2\pi}{25}$. 8. $\frac{7\pi}{16}$. 9. ·4509. 10. ·6545.
11. 1·4399. 12. 1·1999. 13. 2·7489. 14. ·9163.
15. 135°. 16. 28°. 17. 33° 20′. 18. 37° 30′.
19. 22° 30′. 20. 30°. 21. 37° 30′. 22. 165°.
23. ·638. 24. 1·232. 25. 2·0262. 26. 2·9979.

VII. b. PAGE 56.

1. $\frac{3}{4}$. 2. $\frac{1}{3\sqrt{2}}$. 3. $4\frac{1}{4}$. 4. $\frac{3}{\sqrt{2}}$. 5. 9.
6. $\frac{3}{4}$. 7. 1. 13. $\frac{\pi}{4}, \frac{2\pi}{15}$. 14. $\frac{5\pi}{6}, \frac{5\pi}{7}$.

VII. c. PAGE 60.

1. $\frac{1}{5}$. 2. 300 ft. 3. A radian.
4. 5·85 yards. 5. 330. 6. $\frac{1}{44}$ of a second.
7. $58\frac{2}{3}$. 8. 40 yds. 9. 1·15192 miles.
10. 17·904. 11. 2° 6′. 12. 45 feet.

MISCELLANEOUS EXAMPLES. B. PAGE 61.

1. 9°. **2.** 95·26. **3.** 54°. **4.** 3438 inches.
6. 30°. **8.** $22\frac{1}{2}°$, $\frac{\pi}{8}$. **9.** $67\frac{1}{2}°$.
10. $a=6\sqrt{3}$, $c=12$, perp. $=3\sqrt{3}$. **12.** 17·32 ft.
14. 120°, 36°, 24°. **15.** $-\frac{35}{8}$.
17. (1) possible; (2) impossible. **18.** 8·66 miles.
19. $\frac{\pi}{5}$, $\frac{\pi}{3}$, $\frac{7\pi}{15}$. **21.** 90. **24.** 4 miles per hour, 1·732 miles.
25. $\frac{\pi}{8}$. **26.** (1) 30°; (2) 30°. **27.** $\frac{5}{14}$.
29. 200 yards. **30.** 33 feet.

VIII. a. PAGE 69.

1. Second. **2.** Third. **3.** First. **4.** Third.
5. Second. **6.** Second. **7.** Third. **8.** Third.
9. Sine. **10.** Cosine. **11.** Tangent. **12.** Sine.
13. Sine. **14.** Tangent. **15.** Sine. **16.** All.
17. Cosine. **18.** 60°, $\frac{\sqrt{3}}{2}$. **19.** 30°, $\frac{\sqrt{3}}{2}$. **20.** 45°, 1.
21. 45°, $\sqrt{2}$. **22.** 30°, 2. **23.** 60°, $\frac{2}{\sqrt{3}}$. **24.** 45°, 1.
25. 60°, 2. **26.** 60°, $\sqrt{3}$.

VIII. b. PAGE 72.

1. $-\sqrt{3}$. **2.** $\frac{1}{\sqrt{2}}$. **3.** $-\frac{1}{2}$.
4. $-\frac{12}{13}$, $\frac{12}{5}$. **5.** $-\frac{5}{4}$, $-\frac{3}{4}$. **6.** $\frac{4}{3}$, $-\frac{3}{5}$.
7. $\frac{\sqrt{3}}{2}$, $-\frac{1}{\sqrt{3}}$. **8.** 1, $-\sqrt{2}$. **9.** $\pm\frac{5}{13}$, $\pm\frac{5}{12}$.

IX. PAGE 79.

1. $\cot A$ decreases from ∞ to 0, then increases numerically from 0 to $-\infty$, then decreases from ∞ to 0, then increases numerically from 0 to $-\infty$. **2.** $\operatorname{cosec}\theta$ decreases from ∞ to 1, then increases from 1 to ∞. **3.** $\cos\theta$ decreases numerically from -1 to 0, then increases from 0 to 1. **4.** $\tan A$ decreases from ∞ to 0, then increases numerically from 0 to $-\infty$. **5.** $\sec\theta$ decreases numerically from $-\infty$ to -1, then increases numerically from -1 to $-\infty$. **6.** 3. **7.** 1. **8.** -2. **9.** 2.

MISCELLANEOUS EXAMPLES. C. Page 81.

1. $\pm\frac{4}{5}$. 3. $A=60°$, $B=30°$, $a=\frac{21\sqrt{3}}{2}$. 4. $\frac{7}{24}$.

5. 1313 miles, nearly. 6. 301 feet. 7. $3\frac{7}{10}$.

8. 12·003 inches. 10. 200 feet.

X. a. Page 87.

1. $-\frac{1}{\sqrt{2}}$. 2. $\frac{1}{2}$. 3. $\sqrt{3}$. 4. $-\sqrt{2}$.

5. $-\frac{\sqrt{3}}{2}$. 6. 1. 7. -1. 8. $-\frac{1}{2}$.

9. 2. 10. -1. 11. $-\frac{\sqrt{3}}{2}$. 12. -2.

13. -2. 14. $-\frac{1}{\sqrt{2}}$. 15. $\sqrt{3}$. 16. $\sin A$.

17. $\tan A$. 18. $-\cos A$. 19. $-\sec A$. 20. $-\cos A$.

21. $-\tan A$. 22. $-\cos\theta$. 23. $\tan\theta$. 24. $-\operatorname{cosec}\theta$.

25. 1. 26. $2\sin A$. 27. 1.

X. b. Page 91.

1. $-\frac{1}{2}$. 2. $-\frac{\sqrt{3}}{2}$. 3. $\frac{1}{2}$. 4. $-\frac{1}{2}$. 5. -1.

6. 1. 7. $\frac{2}{\sqrt{3}}$. 8. $-\frac{1}{\sqrt{3}}$. 9. $-\sqrt{2}$. 10. $\frac{1}{\sqrt{2}}$.

11. 0. 12. $\frac{1}{\sqrt{2}}$. 13. $-\sqrt{2}$. 14. $-\frac{1}{2}$. 15. $-\sqrt{2}$.

16. $-\frac{1}{\sqrt{2}}$. 17. -1. 18. 2. 19. $\frac{1}{\sqrt{3}}$. 20. $\frac{2}{\sqrt{3}}$.

21. $\pm 30°$, $\pm 330°$. 22. $210°$, $330°$, $-30°$, $-150°$.

23. $120°$, $300°$, $-60°$, $-240°$. 24. $135°$, $315°$, $-45°$, $-225°$.

30. 3. 31. $\cot^2 A$. 32. -1. 34. -4.

XI. a. Page 97.

4. 1; $\frac{24}{25}$. 5. $\frac{33}{65}$; $\frac{16}{65}$. 6. $-\frac{85}{36}$.

XI. b. Page 100.

1. 1. 2. $\frac{1}{7}$. 3. $0;\ \frac{12}{35}$. 4. $-\frac{278}{29};\ \frac{1}{2}$.

11. $\cos A\cos B\cos C-\cos A\sin B\sin C-\sin A\cos B\sin C-\sin A\sin B\cos C$;
$\sin A\cos B\cos C-\cos A\sin B\cos C+\cos A\cos B\sin C+\sin A\sin B\sin C$.

12. $\dfrac{\tan A-\tan B-\tan C-\tan A\tan B\tan C}{1-\tan B\tan C+\tan C\tan A+\tan A\tan B}$.

13. $\dfrac{\cot A\cot B\cot C-\cot A-\cot B-\cot C}{\cot B\cot C+\cot C\cot A+\cot A\cot B-1}$.

XI. d. Page 104.

1. $-\frac{7}{9}$. 2. $\frac{17}{25}$. 3. $\frac{24}{25}$. 4. $\frac{3}{4}$.

5. $\frac{7}{25};\ \frac{24}{25}$. 6. $\frac{1}{3}$. 7. $\frac{1}{7}$.

XI. e. Page 106.

1. $-\frac{23}{27}$. 2. $\frac{117}{125}$. 3. $\frac{9}{13}$.

XII. a. Page 112.

1. $\sin 4\theta+\sin 2\theta$. 2. $\sin 9\theta-\sin 3\theta$. 3. $\cos 12A+\cos 2A$.

4. $\cos A-\cos 5A$. 5. $\sin 9\theta-\sin\theta$. 6. $\sin 12\theta-\sin 4\theta$.

7. $\cos 6\theta-\cos 12\theta$. 8. $\sin 16\theta-\sin 2\theta$. 9. $\cos 13\alpha+\cos 9\alpha$.

10. $\cos 5\alpha-\cos 15\alpha$. 11. $\frac{1}{2}(\sin 11\alpha-\sin 3\alpha)$.

12. $\frac{1}{2}(\cos 2\alpha-\cos 4\alpha)$. 13. $\frac{1}{2}(\sin 2A+\sin A)$.

14. $\frac{1}{2}(\sin 6A-\sin A)$. 15. $\cos\frac{7\theta}{3}+\cos\theta$.

16. $\frac{1}{2}\left(\cos\frac{\theta}{2}-\cos\theta\right)$. 17. $\cos(\alpha+\beta)+\cos(\alpha-3\beta)$.

18. $\cos(2\alpha-\beta)-\cos(4\alpha+\beta)$. 19. $\sin(3\theta-\phi)+\sin(\theta+3\phi)$.

20. $\sin(4\theta-\phi)-\sin(2\theta+3\phi)$. 21. $\frac{1}{2}\left(\frac{\sqrt{3}}{2}-\sin 2\alpha\right)$.

XII. b. Page 114.

1. $2\sin 6\theta\cos 2\theta$. 2. $2\cos 3\theta\sin 2\theta$. 3. $2\cos 5\theta\cos 2\theta$.

4. $2\sin 10\theta\sin\theta$. 5. $2\cos 6\alpha\sin\alpha$. 6. $2\cos\frac{11\alpha}{2}\cos\frac{5\alpha}{2}$.

7. $2\sin 8\alpha\cos 5\alpha$. 8. $-2\sin 3\alpha\sin 2\alpha$. 9. $2\cos\frac{11A}{2}\cos\frac{7A}{2}$.

10. $-2\cos 7A\sin 4A$. 11. $\sin 20°$. 12. $\sqrt{3}\cos 10°$.

XIII. a. Page 128.

1. 60°. 2. 120°. 3. $A=30°$, $B=120°$, $C=30°$. 4. 45°.

5. 90°. 6. $A=75°$, $B=45°$, $C=60°$.

7. $A=30°$, $B=135°$, $C=15°$. 8. 28° 56′. 9. 101° 33′.

10. $\sqrt{6}$. 11. 7. 12. 8. 13. 14. 14. 9.

15. $b=2\sqrt{6}$, $A=75°$, $C=30°$. 16. $a=\sqrt{5}+1$, $B=36°$, $C=72°$.

17. $C=75°$, $a=c=2\sqrt{3}+2$. 18. $A=105°$, $a=\sqrt{3}+1$, $c=\sqrt{3}-1$.

19. $C=30°$, $a=2$, $b=\sqrt{3}+1$. 21. 2. 22. 6. 23. 60°.

24. 105°, 45°, 30°. 25. 105°, 15°, 60°. 26. $\frac{\sqrt{3}}{2}$, 105°, 15°.

XIII. b. Page 132.

1. $B=60°$, 120°; $C=90°$, 30°; $c=2$, 1.

2. $B=60°$, 120°; $A=75°$, 15°; $a=3+\sqrt{3}$, $3-\sqrt{3}$.

3. $A=45°$, $B=75°$, $b=\sqrt{3}+1$; no ambiguity. 4. Impossible.

5. $C=45°$, 135°; $A=105°$, 15°; $a=3+\sqrt{3}$, $3-\sqrt{3}$.

6. $C=75°$, 105°; $A=45°$, 15°; $a=2\sqrt{3}$, $3-\sqrt{3}$.

7. $A=75°$, 105°; $B=90°$, 60°; $b=2\sqrt{6}$, $3\sqrt{2}$.

8. $B=90°$, $C=72°$, $c=4\sqrt{5+2\sqrt{5}}$; no ambiguity. 9. Impossible.

XIII. d. Page 136.

1. 72°, 72°, 36°; each side $=\sqrt{5}+1$.

2. $A=60°$, $a=9-3\sqrt{3}$, $b=3(\sqrt{6}-\sqrt{2})$, $c=3\sqrt{2}$.

3. $A=105°$, $B=15°$, $C=60°$. 4. $B=54°$, 126°; $C=108°$, 36°.

5. $C=60°$, 120°; $A=90°$, 30°; $a=100\sqrt{3}$. No, for $C=90°$.

6. 18°, 126°. 8. $A=90°$, $B=30°$, $C=60°$; $2c=a\sqrt{3}$.

MISCELLANEOUS EXAMPLES. D. PAGE 138.

2. 43. **3.** ∞, 1. **4.** -1. **6.** $a=2$, $B=30°$, $C=105°$.
9. $A=30°$, $B=75°$, $C=75°$.

XIV. a. PAGE 145.

1. 10, 8, $-\frac{3}{2}, \frac{2}{3}, \frac{1}{2}, -1$ **2.** $\frac{4}{3}, \frac{5}{4}, -\frac{1}{2}, \frac{7}{4}$.

3. 2401, ·5, $\frac{10000}{9}$, 1, $\frac{5}{4}$, 1000, 10000.

4. 5, 3, 3, 4, 0. **5.** 0, 2, $\bar{2}$, 0, $\bar{4}$, 3, $\bar{1}$.

6. $\bar{1}$·8091488, 6·8091488, $\bar{4}$·8091488. **7.** 3·25, 325, ·000325.

8. 2·8853613. **9.** 3·3714373. **10.** 1·5475286.

11. 1·9163822. **12.** $\bar{1}$·4419030. **13.** $\bar{2}$·3380134.

14. $\bar{1}$·6989700. **15.** $\bar{1}$·8125919. **16.** ·0501716.

17. $\log 2 = ·3010300$. **18.** $1-\log 2 = ·6989700$. **19.** 1·320469.

20. ·0260315. **21.** ·2898431. **22.** $\bar{7}$·2621538.

23. 7; 4. **24.** 2058.

XIV. b. PAGE 149.

1. 9·076226. **2.** 3·01824. **3.** 2467·266.

4. 2·23. **5.** 3·54. **6.** 1·72. **7.** 32, 79.

8. 22·2398. **9.** 3·32. **10.** 5·77. **11.** 2·05.

12. $x=2\log 2 = ·60206$, $y=-2\log 5 = -1·39794$.

13. $x=\frac{\log 3}{\log 3-\log 2}=2·71$; $y=\frac{\log 2}{\log 3-\log 2}=x-1=1·71$.

14. $3(b-a-c+2)$, $\frac{1}{2}(2a-3c+6)$.

15. $b+c-2$, $\frac{1}{6}(3a+2b+3c-5)$.

MISCELLANEOUS EXAMPLES. E. PAGE 150.

3. $b=\sqrt{3}-1$, $A=135°$, $C=30°$. **8.** $A=105°$, $B=45°$.

XV. a. PAGE 155.

1. 6·6947486. **2.** ·5404924. **3.** 6·4547860.

4. 1·7606731. **5.** 6·7840083. **6.** 55740·83.

7. 673·5466. **8.** ·0106867. **9.** ·008287771.

10. ·2531925. **11.** 2·031324. **12.** 1·389495.

13. 2·424463. **14.** 2·069138.

XV. b. Page 159.

1. ·6164825. 2. ·7928863. 3. 1·2154838. 4. 62° 42′ 31″.
5. 30° 40′ 23″. 6. 48° 45′ 44″. 7. 9·8440554.
8. 10·1317778. 9. 9·7530545. 10. 44° 17′ 8″.
11. 55° 30′ 39″. 12. 9·6656561. 13. 10·1912872.

XV. c. Page 161.

1. 2·36952. 2. 84336. 3. 33·27475.
4. ·03803142. 5. 112184. 6. 1225·508.
7. 27·90209. 8. ·580303. 9. 6·84829.
10. 3·288754, 1·236122. 11. 2273·54.
12. ·5095328. 13. 7·29889. 14. ·045800373.
15. ·1972945. 16. ·0001706363. 17. ·644065.
18. 9·52912. 19. ·3175271. 20. ·335859.
21. ·4221836. 22. 124272·2. 23. 250·2357.
24. (1) 36° 45′ 22″; (2) 19° 28′ 16″. 25. ·441785. 26. 68° 25′ 6″.

XVI. a. Page 166.

6. $\frac{1}{2}$. 7. $\frac{3}{2}$.

XVI. b. Page 169.

1. 113° 34′ 41″. 2. 49° 28′ 26″. 3. 55° 46′ 16″.
4. 78° 27′ 47″. 5. 64° 37′ 23″. 6. 35° 5′ 49″. 7. 93° 35′.
8. $A=67° 22′ 49″$, $B=53° 7′ 48″$, $C=59° 29′ 23″$.
9. $A=46° 34′ 3″$, $B=104° 28′ 39″$, $C=28° 57′ 18″$.

XVI. c. Page 173.

1. $A=79° 6′ 24″$, $B=40° 53′ 36″$. 2. $A=6° 1′ 54″$, $C=108° 58′ 6″$.
3. $A=24° 10′ 59″$, $B=95° 49′ 1″$. 4. $B=78° 48′ 52″$, $C=56° 41′ 8″$.
5. $A=27° 38′ 45″$, $C=117° 38′ 45″$. 6. 82° 57′ 15″, 36° 32′ 45″.
7. $A=74° 32′ 44″$, $C=48° 59′ 16″$.
8. $B=100° 47′ 1″$, $C=14° 12′ 59″$.
9. $A=136° 35′ 21·8″$, $B=13° 14′ 33·2″$.

XVI. d. Page 174.

1. 89·646162. 2. 255·3864. 3. 92·788. 4. $b=185$, $c=192$.
5. 321·0793. 6. $a=765·4321$, $c=1035·43$.
7. $b=767·792$, $c=1263·58$.

XVI. e. Page 176.

1. $32° 25' 35''$. 2. $41° 41' 28''$ or $138° 18' 32''$.
3. $A=100° 34'$, $B=34° 26'$. 4. $51° 18' 21''$ or $128° 41' 39''$.
5. $A=28° 20' 49''$, $C=39° 35' 11''$. 6. $A=81° 45' 2''$, or $23° 2' 58''$.
7. (1) Not ambiguous, for $C=90°$;
(2) ambiguous, $b=60·3893$ ft.;
(3) not ambiguous.

XVI. f. Page 180.

1. $A=58° 24' 43''$, $B=48° 11' 23''$, $C=73° 23' 54''$.
2. $112° 12' 54''$, $45° 53' 33''$, $21° 53' 33''$. 3. $75° 48' 54''$.
4. 4227·4815. 5. $B=108° 12' 26''$, $C=49° 27' 34''$.
6. $A=105° 38' 57''$, $B=15° 38' 57''$. 7. 17·1 or 3·68.
8. $108° 26' 6''$, $53° 7' 48''$, $18° 26' 6''$. 9. $126° 22'$; $96° 27'$, or $19° 3'$.
10. $B=80° 46' 26·5''$, $C=63° 48' 33·5''$. 11. $70° 0' 56''$, or $109° 59' 4''$.
12. 4·0249. 13. $41° 45' 14''$.
14. $A=42° 0' 14''$, $B=55° 56' 46''$, $C=82° 3'$.
15. $41° 24' 35''$. 16. $A=60° 5' 34''$, $C=29° 54' 26''$.
17. 889·2554 ft. 18. $72° 12' 59''$, $47° 47' 1''$.
19. 44·4878 ft. 20. $A=102° 56' 38''$, $B=42° 3' 22''$.
21. $B=99° 54' 23''$, $C=32° 50' 37''$, $a=18·7254$. 22. $72° 26' 26''$.
23. $A=27° 29' 56''$, $B=98° 55'$, $C=53° 35' 4''$.
24. $B=32° 15' 49''$, $C=44° 31' 17''$, $a=1180·525$.
25. $a=20·9059$, $c=33·5917$. 26. $a=2934·124$, $b=3232·846$.
27. $B=1° 1' 23''$, $C=147° 28' 37''$, $a=4389·8$.
28. $A=26° 24' 23''$, $B=118° 18' 25''$, $b=642·756$.
29. $53° 17' 55''$, or $126° 42' 5''$.
30. $A=31° 39' 33''$, $C=96° 1' 27''$, $a=878·753$.

31. $b=4028{\cdot}38$, $c=2831{\cdot}67$.

32. $B=75^\circ\,53'\,29''$, or $104^\circ\,6'\,31''$; $A=60^\circ\,54'\,19''$, or $32^\circ\,41'\,17''$.

33. Base $=2{\cdot}44845$ ft., altitude $={\cdot}713321$ ft.

34. 90°, nearly. **35.** (1) impossible; (2) ambiguous; (3) 63·996.

36. $\theta=72^\circ\,31'\,53''$, $c=12{\cdot}8255$. **37.** $\theta=60^\circ\,13'\,52''$, $c=19{\cdot}523977$.

XVII. a. Page 185.

1. 146·4 ft. **2.** $880\sqrt{3}=1524$ ft. **5.** $ab/(a-b)$ ft.

6. $\frac{1}{3}\sqrt{6}={\cdot}816$ miles. **7.** $10(\sqrt{10}+\sqrt{2})=45{\cdot}76$ ft.

9. 1 or $\frac{1}{3}$. **10.** $9\frac{2}{7}$ ft. **12.** $48\sqrt{6}=117{\cdot}6$ ft.

14. $750\sqrt{6}=1837$ ft. **15.** $2640(3+\sqrt{3})=12492$ ft.

XVII. b. Page 190.

1. 30 ft. **2.** $a\sqrt{2}$ ft. **5.** 100 ft.

12. $\sqrt{500-200\sqrt{3}}=12{\cdot}4$ ft.

XVII. c. Page 195.

1. 1060·5 ft. **2.** $\frac{500\sqrt{6}}{3}=408$ ft.

3. $120\sqrt{6}=294$ ft. **5.** 106 ft.

10. Height $=40\sqrt{6}=98$ ft.; distance $=40(\sqrt{14}+\sqrt{2})=206$ ft.

11. $50\sqrt{120+30\sqrt{6}}=696$ yds.

XVII. d. Page 197.

1. 5 miles nearly.

2. Height $=19{\cdot}5375$ yds.; distance $=102{\cdot}9093$ yds.

3. 200·017 ft. **4.** Height $=418{\cdot}4045$ ft.; distance $=430$ ft.

5. Height $=916{\cdot}8624$ ft.; distance $={\cdot}984808$ miles.

6. Height $=46{\cdot}14021$ ft.; distance $=99{\cdot}92$ ft.

7. 11·550316 or 25·9733 miles per hour.

8. Height $=159{\cdot}4221$ ft.; distance $=215{\cdot}6762$ ft.

XVIII. a. PAGE 206.

1. 9000 sq. ft.
2. 15390.
3. $\frac{84}{85}$.
4. 24, $\frac{117}{5}$, $\frac{936}{25}$.
5. 225 sq. ft.
6. 672 sq. ft.
7. 36 yds.
8. $r=4$, $R=8\frac{1}{8}$.
9. 12, 6, 28.
10. 12, 16, 20.

XVIII. b. PAGE 210.

1. 26·46 sq. ft.
2. 9·585 yds., 7·18875 sq. yds.
4. 216·23 sq. ft.
5. 128·352 in.
6. 101·78 ft.
7. 57·232 ft.
8. 63·09 sq. ft.

XVIII. c. PAGE 218.

17. $$\frac{\pi}{3}+(-1)^n\frac{1}{2^n}\left(A-\frac{\pi}{3}\right),$$
$$\frac{\pi}{3}+(-1)^n\frac{1}{2^n}\left(B-\frac{\pi}{3}\right),$$
$$\frac{\pi}{3}+(-1)^n\frac{1}{2^n}\left(C-\frac{\pi}{3}\right).$$

XVIII. d. PAGE 223.

1. $1\frac{5}{7}$, $2\frac{1}{2}$.
4. Diagonals 65, 63; area 1764.
5. $2\sqrt{77}+6\sqrt{11}$.

XVIII. e. PAGE 225.

2. 7071 sq. yds.
5. $\sqrt{\frac{x}{y}+\frac{y}{z}+\frac{z}{x}}$.
13. 20, 21, 29.

MISCELLANEOUS EXAMPLES. F. PAGE 228.

3. Expression $=\cot A+\cot B+\cot C$.
4. $B=45°$, $135°$; $C=105°$, $15°$; $c=\sqrt{6}+\sqrt{2}$, $\sqrt{6}-\sqrt{2}$.
6. 126.
7. 68·3 yds., 35·35 yds.
11. $C=45°$, $135°$; $A=105°$, $15°$; $a=2\sqrt{3}$, $4\sqrt{3}-6$.
12. 10 miles; $10\sqrt{2-\sqrt{2}}$ miles.
24. (1) $90°-\frac{A}{2}$, $90°-\frac{B}{2}$, $90°-\frac{C}{2}$;
(2) $180°-2A$, $180°-2B$, $180°-2C$.
25. Expression $=\sin^2(\alpha-\beta)$.
28. 21·3 miles per hour.
29. 1 hr. 30′; 2 hrs. 16′.

XIX. a. PAGE 235.

1. $n\pi+(-1)^n\frac{\pi}{6}$. 2. $n\pi+(-1)^n\frac{\pi}{4}$. 3. $2n\pi\pm\frac{\pi}{3}$.

4. $n\pi+\frac{\pi}{3}$. 5. $n\pi-\frac{\pi}{6}$. 6. $2n\pi\pm\frac{3\pi}{4}$.

7. $n\pi\pm\frac{\pi}{4}$. 8. $n\pi\pm\frac{\pi}{6}$. 9. $n\pi\pm\frac{\pi}{3}$.

10. $2n\pi\pm a$. 11. $n\pi\pm a$. 12. $n\pi\pm a$.

13. $n\pi$. 14. $\frac{n\pi}{3}+(-1)^n a$. 15. $2n\pi$, or $\frac{2n\pi}{5}$.

16. $\frac{n\pi}{3}$, or $n\pi\pm\frac{\pi}{6}$. 17. $\frac{n\pi}{4}$, or $\frac{n\pi}{3}+(-1)^n\frac{\pi}{18}$.

18. $\frac{(2n+1)\pi}{2}$, or $2n\pi$, or $\frac{(2n+1)\pi}{5}$.

19. $\frac{(2n+1)\pi}{2}$, or $\frac{(2n+1)\pi}{4}$, or $\frac{(2n+1)\pi}{8}$.

20. $n\pi$, or $\frac{(2n+1)\pi}{14}$. 21. $\frac{n\pi}{6}$, or $\frac{n\pi}{9}$.

22. $\frac{(2n+1)\pi}{6}$, or $n\pi\pm\frac{\pi}{8}$. 23. $\frac{(2n+1)\pi}{8}$, or $\frac{n\pi}{3}+(-1)^n\frac{\pi}{9}$.

24. $(2n+1)\pi$, or $2n\pi\pm\frac{\pi}{3}$. 25. $2n\pi$, or $2n\pi\pm\frac{2\pi}{3}$.

26. $n\pi+(-1)^n\frac{\pi}{6}$, or $n\pi+(-1)^n\frac{3\pi}{2}$. 27. $\frac{n\pi}{2}+\frac{\pi}{8}$.

28. $2n\pi+\frac{2\pi}{3}$. 29. $2n\pi-\frac{\pi}{4}$.

XIX. b. PAGE 237.

1. $\frac{(2n+1)\pi}{2(p+q)}$. 2. $\frac{(4k+1)\pi}{2(n-m)}$, or $\frac{(4k-1)\pi}{2(n+m)}$. 3. $2n\pi$, or $2n\pi-\frac{2\pi}{3}$.

4. $2n\pi+\frac{\pi}{2}$, or $(2n+1)\pi+\frac{\pi}{6}$. 5. $2n\pi+\frac{\pi}{2}$, or $2n\pi+\frac{\pi}{6}$.

6. $2n\pi+\frac{5\pi}{12}$, or $2n\pi-\frac{\pi}{12}$. 7. $2n\pi+\frac{\pi}{12}$, or $2n\pi-\frac{7\pi}{12}$.

8. $2n\pi+\frac{5\pi}{4}$, or $2n\pi-\frac{3\pi}{4}$. 9. $2n\pi+\frac{\pi}{3}$, or $(2n+1)\pi$.

10. $\frac{n\pi}{2}+(-1)^n\frac{\pi}{12}$. 11. $\frac{(2n+1)\pi}{4}$, or $n\pi\pm\frac{\pi}{6}$.

12. $n\pi$, or $n\pi \pm \frac{\pi}{6}$.

13. $\frac{n\pi}{2}$.

14. $n\pi + \frac{\pi}{4}$, or $2n\pi$, or $2n\pi + \frac{\pi}{2}$.

[*In some of the following examples, the equations have to be squared, so that extraneous solutions are introduced.*]

15. $\frac{2n\pi}{3} + \frac{\pi}{4}$, or $2n\pi + \frac{\pi}{4}$.

16. $n\pi - \frac{\pi}{4}$, or $\frac{n\pi}{2} + (-1)^n \frac{\pi}{12}$.

17. $\frac{(2n+1)\pi}{10}$, or $\frac{(2n+1)\pi}{2}$.

18. $\frac{(2n+1)\pi}{2}$, or $n\pi \pm \frac{\pi}{3}$.

19. $n\pi + \frac{\pi}{4}$, or $n\pi + \frac{\pi}{6}$.

20. $\frac{n\pi}{2} + (-1)^{n+1} \frac{\pi}{12}$, or $\frac{n\pi}{2}$.

21. $\theta = n\pi \pm \frac{\pi}{4}$, $\phi = n\pi \pm \frac{\pi}{6}$.

22. $\theta = n\pi \pm \frac{\pi}{6}$, $\phi = n\pi \pm \frac{\pi}{3}$.

23. $\theta = n\pi \pm \frac{\pi}{4}$, $\phi = n\pi \pm \frac{\pi}{3}$.

XIX. d. Page 244.

1. $\pm \frac{1}{\sqrt{2}}$.

2. ± 1.

3. ± 2.

4. $\frac{-3 \pm \sqrt{17}}{4}$.

5. 1, or $\frac{1}{2}$.

6. 0, or $\pm \frac{1}{2}$.

7. $\pm \frac{1}{\sqrt{2}}$.

8. $\pm \frac{25}{24}$.

9. $\frac{1}{2}$.

10. $\frac{a-b}{1+ab}$.

11. $\frac{b-a}{1+ab}$.

12. $\sqrt{3}$.

13. $x = ac - bd$, $y = bc + ad$.

14. ± 1, or $\pm (1 \pm \sqrt{2})$.

15. $n\pi + \frac{\pi}{4}$.

19. $x = 1$, $y = 2$; $x = 2$, $y = 7$.

MISCELLANEOUS EXAMPLES. G. Page 246.

2. (1) $\frac{(2n+1)\pi}{2}$, $2n\pi \pm \frac{\pi}{3}$; (2) $2n\pi \pm \frac{\pi}{3}$.

6. 78° 27′ 4″.

9. 6.

10. 800 yds, 146·4 yds, 546·4 yds.

XX. a. Page 255.

2. $\sin \frac{A}{2} = \frac{5}{13}$, $\cos \frac{A}{2} = -\frac{12}{13}$.

3. $\sin \frac{A}{2} = -\frac{15}{17}$, $\cos \frac{A}{2} = \frac{8}{17}$.

4. $2\sin\frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A}$;

$2\cos\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}$.

5. $2\sin\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}$;

$2\cos\frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A}$.

6. $2\sin\frac{A}{2} = +\sqrt{1+\sin A} + \sqrt{1-\sin A}$;

$2\cos\frac{A}{2} = +\sqrt{1+\sin A} - \sqrt{1-\sin A}$.

7. $\sin\frac{A}{2} = \frac{4}{5}$, $\cos\frac{A}{2} = \frac{3}{5}$.

8. $\sin\frac{A}{2} = \frac{8}{17}$, $\cos\frac{A}{2} = -\frac{15}{17}$.

9. (1) $2n\pi - \frac{\pi}{4}$ and $2n\pi + \frac{\pi}{4}$; (2) $2n\pi + \frac{5\pi}{4}$ and $2n\pi + \frac{7\pi}{4}$;

(3) $2n\pi + \frac{3\pi}{4}$ and $2n\pi + \frac{5\pi}{4}$.

10. No; $2\sin\frac{A}{2} = \sqrt{1+\sin A} + \sqrt{1-\sin A}$.

14. (1) $= \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$; (2) $= 2\sin\left(\theta - \frac{\pi}{3}\right)$.

15. (1) $= -\sec 2\theta$; (2) $= \tan^2\frac{\theta}{2}$.

XX. b. Page 260.

3. $\frac{1}{5}$.

4. $-\frac{1}{3}$.

XXI. a. Page 267.

1. 1440 yards.
2. $342\frac{6}{7}$ yards.
3. 22 yards.
4. 6′ 34″.
5. 4′35″.
6. 11 ft. 11 in.
7. 210 yards.
8. 9′ 33″.
10. 50 ft.
11. (1) $\frac{\pi}{10800}$; (2) $\frac{\pi}{648000}$.
12. πr^2.
13. $\frac{1}{2}$.
14. $m - n$.
15. $\frac{1}{2} - \frac{\sqrt{3}}{200} = \cdot 491$.
16. $\frac{1}{2} + \frac{11\sqrt{3}}{7200} = \cdot 503$.
17. $\frac{21\sqrt{3}}{55} = 39\cdot 7'$.

XXI. b. PAGE 271.

1. 12 miles. **2.** 150 ft. **3.** 15 miles. **4.** 80 ft. 8 in.

5. 204 ft. 2 in. **6.** $54' 33''$. **7.** 104 ft. 2 in.

8. 10560 ft. **9.** 610 ft., $\frac{5}{2}\sqrt{110}$ minutes $= 26' 13''$.

11. 8. **12.** -1. **13.** (1) $\cos\alpha$; (2) $-\sin\alpha$.

14. $45° 54' 33''$, $44° 5' 27''$.

MISCELLANEOUS EXAMPLES. H. PAGE 283.

3. $18° 26' 6''$. **6.** 35 miles or 13 miles per hour.

XXIII. a. PAGE 291.

1. $\dfrac{\sin^2 n\alpha}{\sin\alpha}$. **2.** $\sin\dfrac{n\beta}{2}\cos\left(\alpha - \dfrac{n-1}{2}\beta\right) \Big/ \sin\dfrac{\beta}{2}$.

3. $-\cos\left(\alpha + \dfrac{\pi}{2n}\right) \Big/ \sin\dfrac{\pi}{2n}$. **4.** $\sin\dfrac{n\pi}{2k}\cos\dfrac{(n+1)\pi}{2k} \Big/ \sin\dfrac{\pi}{2k}$.

5. $\dfrac{1}{2}$. **6.** $-\dfrac{1}{2}$. **7.** $\cot\dfrac{\pi}{2n}$. **8.** $-\cos\dfrac{\pi}{n}$.

9. $\sin n\alpha$. **10.** $\sin\dfrac{n(\theta+\pi)}{2}\sin\left(\dfrac{n+1}{2}\theta + \dfrac{n-1}{2}\pi\right) \Big/ \sin\dfrac{\theta+\pi}{2}$.

11. $\sin\dfrac{n(\pi-\beta)}{2}\cos\left\{\alpha + \dfrac{(n-1)(\pi-\beta)}{2}\right\} \Big/ \sin\dfrac{\pi-\beta}{2}$.

12. $\sin\dfrac{n(\pi-2\beta)}{4}\cos\left\{\alpha + \dfrac{(n-1)(\pi-2\beta)}{4}\right\} \Big/ \sin\dfrac{\pi-2\beta}{4}$.

13. $\dfrac{n\cos\theta}{2} - \dfrac{\sin n\theta\cos(n+2)\theta}{2\sin\theta}$.

14. $\dfrac{\sin 2n\alpha\sin 2(n+1)\alpha}{2\sin 2\alpha} - \dfrac{n\sin 2\alpha}{2}$.

15. $\operatorname{cosec}\alpha\{\tan(n+1)\alpha - \tan\alpha\}$.

16. $\operatorname{cosec} 2\theta\{\cot\theta - \cot(2n+1)\theta\}$. **17.** $\tan\alpha - \tan\dfrac{\alpha}{2^n}$.

18. $\dfrac{1}{2}(\operatorname{cosec}\alpha - \operatorname{cosec} 3^n\alpha)$. **19.** $\dfrac{1}{2}(\tan 3^n\alpha - \tan\alpha)$.

XXIII. b. PAGE 294.

1. $\frac{n}{2}+\frac{\sin 4n\theta}{4\sin 2\theta}$. 2. $\frac{n}{2}$. 3. $\frac{n}{2}$.

4. $\dfrac{3\sin\frac{n\theta}{2}\sin\frac{(n+1)\theta}{2}}{4\sin\frac{\theta}{2}} - \dfrac{\sin\frac{3n\theta}{2}\sin\frac{3(n+1)\theta}{2}}{4\sin\frac{3\theta}{2}}$.

5. 0. 6. 0. 7. $\cot\theta - 2^n\cot 2^n\theta$.

8. $\frac{1}{2}\operatorname{cosec}\alpha\,\{\tan(n+1)\alpha - \tan\alpha\}$. 9. $\frac{\sin 2\theta}{2} - \frac{\sin 2^{n+1}\theta}{2^{n+1}}$.

10. $\sin^2\theta - 2^n\sin^2\frac{\theta}{2^n}$. 11. $\tan^{-1}x - \tan^{-1}\frac{x}{n+1}$.

12. $\tan^{-1}(n+1) - \frac{\pi}{4}$. 13. $\tan^{-1}\{1+n(n+1)\} - \frac{\pi}{4}$.

14. $\tan^{-1}n(n+1)$.

XXIV. a. PAGE 301.

2. $x = a\cos\frac{\alpha+\beta}{2}\Big/\cos\frac{\alpha-\beta}{2}$, $y = b\sin\frac{\alpha+\beta}{2}\Big/\cos\frac{\alpha-\beta}{2}$.

3. $x = a(\cos\alpha+\sin\alpha)$, $y = b(\sin\alpha-\cos\alpha)$.

13. $4\sin\frac{\alpha+\beta+\gamma}{2}\sin\frac{\beta+\gamma-\alpha}{2}\sin\frac{\gamma+\alpha-\beta}{2}\sin\frac{\alpha+\beta-\gamma}{2}$.

14. $4\sin\frac{\alpha+\beta+\gamma}{2}\sin\frac{\alpha+\beta-\gamma}{2}\cos\frac{\beta+\gamma-\alpha}{2}\cos\frac{\gamma+\alpha-\beta}{2}$.

15. $-4\cos\left(\frac{\alpha+\beta+\gamma}{2}-\frac{\pi}{4}\right)\Pi\cos\left(\frac{\beta+\gamma-\alpha}{2}+\frac{\pi}{4}\right)$.

22. (1) $(a^2+b^2)x^2 - 2bcx + c^2 - a^2 = 0$;

(2) $(a^2+b^2)^2x^2 - 2(a^2-b^2)(2c^2-a^2-b^2)x + a^4 + b^4 + 4c^4 - 2a^2b^2$
$- 4a^2c^2 - 4b^2c^2 = 0$.

[*Use* $\cos 2\alpha\cos 2\beta = \cos^2(\alpha-\beta) - \sin^2(\alpha+\beta)$.]

XXV. a. Page 318.

1. $2\sqrt{pq}$. 2. 4. 3. 24. 4. 2. 7. $\sqrt{2}$. 8. 2.

9. $\sqrt{a^2-2ab\sin\alpha+b^2}$. 10. $\sqrt{p^2+2pq\sin\alpha+q^2}$.

11. Maximum $=2\sin\frac{\sigma}{2}$. 12. Maximum $=\sin^2\frac{\sigma}{2}$.

13. Minimum $=2\tan\frac{\sigma}{2}$. 14. Minimum $=2\operatorname{cosec}\frac{\sigma}{2}$.

15. Maximum $=\frac{1}{8}$. 16. Minimum $=\sqrt{3}$.

17. Minimum $=\frac{3}{4}$. 18. Minimum $=6$.

19. Minimum $=1$. 20. Minimum $=1$.

21. $\frac{1}{2}(a+c)\pm\frac{1}{2}\sqrt{b^2+(a-c)^2}$. 25. $\frac{5}{3}$.

26. $k^2/(a^2+b^2+c^2)$; $k^2/(a+b+c)$.

XXV. b. Page 324.

1. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$. 2. $x^2+y^2=a^2+b^2$. 3. $b^2=a^2(2-a^2)$.

4. $y(x^2-1)=2$. 5. $(a^2-b^2)^2=16ab$. 6. $x^{\frac{4}{3}}y^{\frac{2}{3}}-x^{\frac{2}{3}}y^{\frac{4}{3}}=1$.

7. $a^2b^2(a^2+b^2)=1$. 8. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. 9. $x^{\frac{4}{5}}y^{\frac{6}{5}}-x^{\frac{6}{5}}y^{\frac{4}{5}}=a^2$.

10. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. 12. $x^{\frac{1}{2}}+y^{\frac{1}{2}}=2$. 13. $(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2$.

16. $a^2+b^2=2c^2$. 20. $(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}}$. 21. $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$.

22. $\frac{x^2}{a}+\frac{y^2}{b}=a+b$, or $\{a(y^2-b^2)-b(x^2-a^2)\}^2=-4abx^2y^2$.

24. $xy=(y-x)\tan\alpha$. 25. $a^2+b^2-2\cos\alpha=2$. 26. $a+b=2ab$.

29. $(a+b)(m+n)=2mn$. 30. $x^2+y^2=16a^2$.

31. $(a-b)\{c^2-(a+b)^2\}=4abcm$.

XXV. c. Page 334.

1. $2\cos 20°$, $-2\cos 40°$, $-2\cos 80°$.

2. $2\sin 10°$, $2\sin 50°$, $-2\sin 70°$.

3. $2\cos 10°$, $-2\cos 50°$, $-2\cos 70°$.

4. $\sin 15°$, $\sin 45°$, $-\sin 75°$.

5. $\frac{1}{a}\sin A$, $\frac{1}{a}\sin(60°-A)$, $-\frac{1}{a}\sin(60°+A)$.

6. $2a\cos A$, $2a\cos(120° \pm A)$.

16. (1) $8x^3-4x^2-4x+1=0$; (2) $64y^3-80y^2+24y-1=0$.

17. $64y^3-112y^2+56y-7=0$.

18. (1) $16x^4+8x^3-12x^2-4x+1=0$; (2) $16x^4-8x^3-12x^2+4x+1=0$.

19. $256y^4-448y^3+240y^2-40y+1=0$.

20. $t^8-36t^6+126t^4-84t^2+9=0$.

MISCELLANEOUS EXAMPLES. K. Page 337.

1. $7° 12'$: 8 grades.
2. $\frac{4\pi}{15}$, $\frac{\pi}{3}$, $\frac{2\pi}{5}$.
4. $\frac{3}{2}$.
5. $15\sqrt{3}=25{\cdot}98$ ft.
6. 790 ft.
7. $5{\cdot}236$ ft.
8. $\frac{8}{17}$, $\frac{15}{17}$.
9. 30°, 60°.
10. $\frac{125}{78}$.
12. $\frac{1-\tan^8 A}{\tan^4 A}$.
17. (1) 45°; (2) 60°.
18. 360 yards.
21. 200, 183 nearly.
22. $\cdot 09375$; $16{\cdot}7552$.
24. $\sqrt{2}:1$.
25. 1; $\tan A$.
27. $-\frac{5}{31}$.
28. 45° or 60°.
29. $15° 12' 45''$.
30. $260{\cdot}26$ yards.
32. (1) 60°; (2) 60°.
33. $10\frac{1}{2}°$.
34. sine $=\frac{24}{25}$, cosine $=\frac{7}{25}$, tangent $=\frac{24}{7}$.
35. (1) and (3) are impossible, (2) and (4) possible.
37. $B=45°$, $b=25\sqrt{2}$, $p=25$.
39. $-\frac{\sqrt{3}}{2}$, -2, $\frac{1}{\sqrt{3}}$.
40. 45°, 135°, 225°, 315°, 405°, 495°.
42. 15°, 75°.
45. 30°, 150°, 270°.
46. $\pm\cdot 8$.
47. $\frac{16}{65}$.

48. $\tan 2A$. **49.** $\pm\dfrac{1}{\sqrt{3}}$. **50.** $880(3+\sqrt{3})=4164{\cdot}16$ yds.

51. (1) 0; (2) -2. **53.** $\dfrac{a^2-b^2}{a^2+b^2}$. **55.** $-\sqrt{3},\ -\dfrac{1}{2},\ \dfrac{1}{\sqrt{3}}$.

56. $6\tfrac{4}{11}°$. **57.** $-\dfrac{15}{352}$. **59.** (1) $\cot C$; (2) 2.

60. $50\sqrt{6}$ ft. **66.** $50\sqrt{3}=86{\cdot}6$ yds. **67.** $3{\cdot}141$.

70. -1. **73.** $\dfrac{1}{2}\sin 2\theta$. **74.** $x^{\frac{2}{3}}+y^{\frac{2}{3}}=4^{\frac{2}{3}}$.

75. $15\sqrt{3}$ ft., $15(3+\sqrt{3})$ ft., $60+15\sqrt{3}$ ft. **77.** $8{\cdot}10$.

84. $4{\cdot}14$. **89.** $\cdot38021$, $3{\cdot}73239$, $9{\cdot}76143$

95. $27°\,45'\,44''$. **97.** $\sin 2A=-\dfrac{336}{625}$; $\tan\dfrac{A}{2}=-\dfrac{1}{7}$.

98. $2-\sqrt{3}$. **99.** $\bar{2}{\cdot}60206$, $1{\cdot}3802113$, $\bar{1}{\cdot}8239087$.

102. 45, 53; $58°\,6'\,33{\cdot}2''$. **104.** $120°$. **106.** $\dfrac{20}{21}$.

107. $4\operatorname{cosec} 2\theta$. **108.** $49°\,28'\,32''$.

110. $2{\cdot}30103$, $\bar{2}{\cdot}39794$, $\cdot598626$, $9{\cdot}69897$, $9{\cdot}849485$.

112. 114 yds., 57 yds. **114.** $53°\,7'\,48''$. **115.** $\cdot8$, $1{\cdot}25$.

116. $\cdot90309$, $1{\cdot}10739$, $\bar{8}{\cdot}52575$. **117.** $\dfrac{1}{\sqrt{2}}$.

118. $-\tan\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}$, $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}$. **120.** 45 ft.; $58°\,12'$.

122. $1{\cdot}60206$, $\bar{1}{\cdot}562469$. **123.** $34°\,18'\,1''$, $1°\,41'\,59''$.

127. $\pm\dfrac{2\sqrt{5}}{5}$. **131.** $1{\cdot}3011928$.

137. $\cdot69897$, $\cdot845098$, $1{\cdot}113943$.

138. $49°\,19'\,30''$, $40°\,40'\,30''$. **143.** $\cdot4855934$.

144. $39°\,35'\,11''$, $28°\,20'\,49''$.

146. $2{\cdot}0755469$, $\cdot3853509$, $\bar{1}{\cdot}9256038$.

149. $100\sqrt{2}$, $50\sqrt{2}$; $71°\,33'\,54''$, $108°\,26'\,6''$.

150. $\operatorname{cosec} x-\operatorname{cosec} 3^n x$. **152.** $45°$, $60°$, $120°$, $135°$.

154. $-\dfrac{56}{33}$; $\dfrac{4}{5}$, $\dfrac{12}{13}$, $-\dfrac{33}{65}$; $120°\,30'\,37''$.

158. $\cdot30103$, $\cdot477121$, $1{\cdot}041393$. **160.** $68°\,52'\,42''$.

161. $\dfrac{\text{area of circle}}{\text{area of octagon}}=\dfrac{1380}{1309}$. **165.** $\dfrac{c\sin\beta}{\sin(\alpha+\beta)}$, $\dfrac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$.

166. $-4\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}$. **173.** $B=4°\,55'\,11''$, $C=168°\,27'\,25''$.

176. $B=105^\circ$, $C=45^\circ$, $a=\sqrt{2}$.

178. $B=81^\circ\,47'\,12''$ or $98^\circ\,12'\,48''$; $c=13$ or 11;
$C=68^\circ\,12'\,48''$ or $51^\circ\,47'\,12''$.

180. 10000 ft. **186.** $64^\circ\,31'\,58''$.

191. 2310 sq. ft.; 55 ft., 66 ft., 70 ft.

193. (1) $n\pi$, $\frac{n\pi}{2} \pm \frac{\pi}{16}$; (2) $n\pi \pm \frac{\pi}{6}$.

197. 134·19 ft. **204.** 226·87. **206.** $\frac{4}{3}$, $-\frac{3}{8}$.

212. (1) $(2n+1)\frac{\pi}{8}$, $n\pi+(-1)^n\frac{\pi}{4}$; (2) $n\pi$, $n\pi+\frac{3\pi}{4}$.

214. 20 ft. **219.** $\frac{1}{4}$ or -8. **222.** 37·27919.

231. $\frac{1800}{176\pi}=3\frac{1}{4}$ miles nearly. **233.** $\pm\frac{7}{23}$.

235. (1) $\frac{n\pi}{4}$, $\frac{2n\pi}{3} \pm \frac{\pi}{9}$; (2) $n\pi+\frac{\pi}{3}$, $n\pi+\frac{3\pi}{4}$.

242. 10°. **246.** 205·4. **252.** 1224·35 yards.

256. 9·65146, 20·5309. **262.** $\alpha+\beta+\gamma=(2n+1)\frac{\pi}{2}$.

264. $\sqrt{2}$ miles. **266.** $\theta=n\pi$.

275. $B=70^\circ\,0'\,57''$ or $109^\circ\,59'\,3''$;
$C=59^\circ\,59'\,3''$ or $20^\circ\,0'\,57''$. **277.** θ.

279. $\cos(\alpha+\beta+\gamma+\delta)+\cos(\alpha+\beta-\gamma-\delta)+\cos(\alpha+\gamma-\beta-\delta)$
$+\cos(\alpha+\delta-\beta-\gamma)$.

281. 4. **283.** $A=45^\circ$, $B=112\frac{1}{2}^\circ$, $c=\sqrt{2-\sqrt{2}}$.

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D/4

$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$

$\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$

$\sin 18^\circ = \frac{\sqrt{5}-1}{4}$

Trigonometry.

$\cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}$

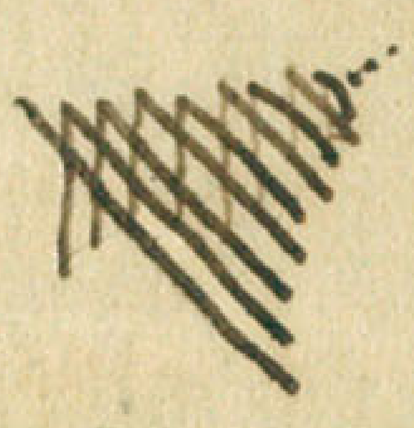
FERNANDO

FERNANDO.

$\sin 36^\circ =$

FERNANDO.

$\cos 36^\circ = \frac{\sqrt{5}+1}{4}$



$\sin 22\frac{1}{2}^\circ = \frac{\sqrt{2-\sqrt{2}}}{2}$

$\cos 22\frac{1}{2}^\circ = \frac{\sqrt{2+\sqrt{2}}}{2}$

$\tan 22\frac{1}{2}^\circ = \sqrt{2}-1$

$\tan 67\frac{1}{2} = \sqrt{2}+1$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

$= \cos^2 A - \sin^2 A$

FERNANDO

$\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

FERNANDO

$\sin 3A = 3\sin A - 4\sin^3 A$

$\cos 3A = 4\cos^3 A - 3\cos A$

$\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

HENRY!

Angles

HENRY

HENRY

HENRY

HENRY

FERNANDO

$\sin(90° - A) = \cos A$

$\cos(90° - A) = \sin A$

$\tan(90° - A) = \cot A$ . . . . and so on.

$\sin 45° = \frac{1}{\sqrt{2}}$

$\cos 45° = \frac{1}{\sqrt{2}}$

$\tan 45° = 1$ . . . + so on

$\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 60° = \frac{1}{2}$

$\tan 60° = \sqrt{3}$

+ so on

$\sin 30° = \frac{1}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$

$\tan 30° = \frac{1}{\sqrt{3}}$

$\sin A = \sin(360° + A)$

$\cos A = \cos(360° + A)$

$\tan A = \tan(360° + A)$

. . . . + so on

$\sin 0° = 0$

$\cos 0° = 1$

$\tan 0° = 0$

+ so on (∞ being reciprocal of 0)

$\sin 90° = 1$

$\cos 90° = 0$

$\tan 90° = \infty$

+ so on

$\sin(180° - A) = \sin A$

$\cos(180° - A) = -\cos A$

$\tan(180° - A) = -\tan A$

+ so on

$\sin(180° + A) = -\sin A$

$\cos(180° + A) = -\cos A$

$\tan(180° + A) = \tan A$

+ so on

$\sin(90° + A) = \cos A$

$\cos(90° + A) = -\sin A$

$\tan(90° + A) = -\cot A$

. . . . + so on.

$\sin(-A) = -\sin A$

$\cos(-A) = \cos A$

$\tan(-A) = -\tan A$

. . . . + so on

$\sin(360° - A) = -\sin A$

$\cos(360° - A) = \cos A$

$\tan(360° - A) = -\tan A$

. . . . + so on

Compound Angles

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B.$

$\tan(A+B) = \frac{\tan + \tan B}{1 - \tan A \tan B}$

$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$

$\sin(A-B)$
$= \sin A \cos B - \cos A \sin B$

$\cos(A-B)$
$= \cos A \cos B + \sin A \sin B$

$\tan(A-B)$
$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$