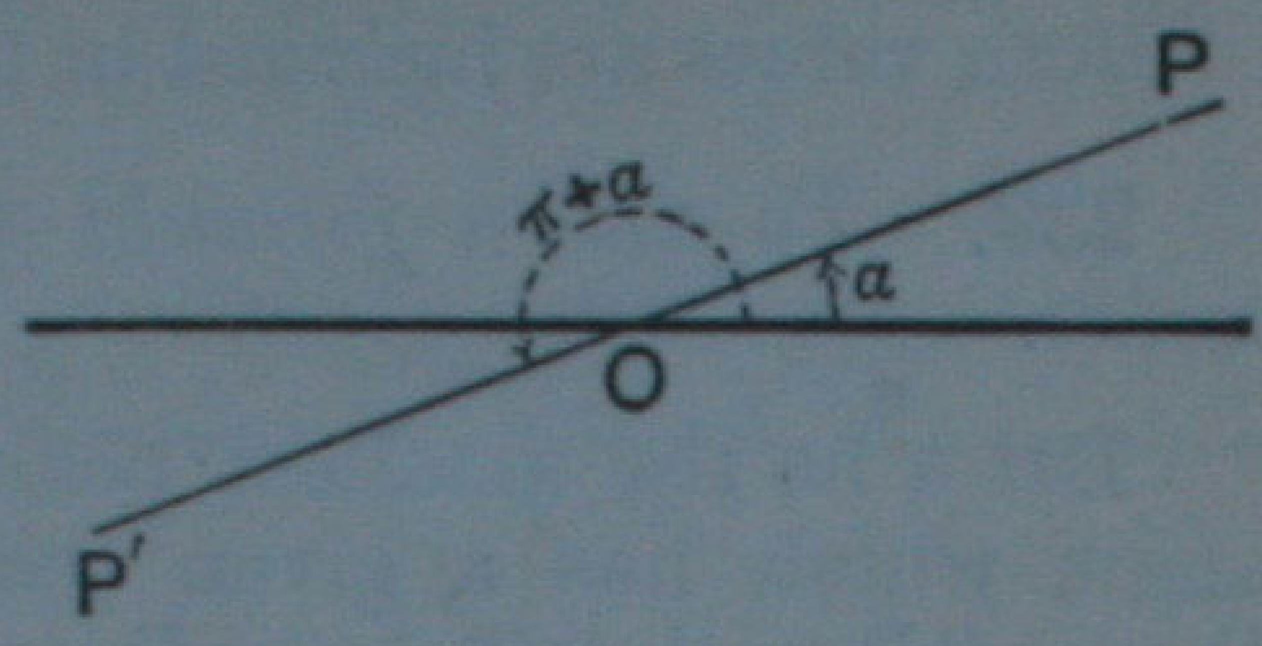
240. *To find a formula for all the angles which have a given tangent.*



Let a be the smallest positive angle which has a given tangent. Draw OP and OP' bounding the angles a and $\pi+a$; then the required angles are those coterminal with OP and OP'.

The positive angles are

$$2p\pi+a \text{ and } 2p\pi+(\pi+a),$$

where p is zero, or any positive integer.

The negative angles are

$$-(\pi-a) \text{ and } -(2\pi-a),$$

and those which may be obtained from them by the addition of any negative multiple of 2π; that is, angles denoted by

$$2q\pi-(\pi-a) \text{ and } 2q\pi-(2\pi-a),$$

where q is zero, or any negative integer.

The angles may be grouped as follows:

$$\left.\begin{array}{r}2p\pi+a,\\(2q-2)\pi+a,\end{array}\right\} \text{ and } \left\{\begin{array}{l}(2p+1)\pi+a,\\(2q-1)\pi+a,\end{array}\right.$$

and it will be noticed that whether the multiple of π is even or odd, it is always followed by $+a$. Thus all angles equi-tangential with a are included in the formula

$$\theta=n\pi+a.$$

This is also the formula for all the angles which have the same cotangent as a.

Example. Solve the equation $\cot 4\theta=\cot\theta$.

The general solution is $4\theta=n\pi+\theta$;

whence $3\theta=n\pi$, or $\theta=\dfrac{n\pi}{3}$.

241. *All angles which are both equi-sinal and equi-cosinal with a are included in the formula $2n\pi+a$.*

All angles equi-cosinal with a are included in the formula $2n\pi\pm a$; so that the multiple of π is even. But in the formula $n\pi+(-1)^n a$, which includes all angles equi-sinal with a, when the multiple of π is even, a must be preceded by the $+$ sign. Thus the formula is $2n\pi+a$.

242. In the solution of equations, the general value of the angle should always be given.

Example. Solve the equation $\cos 9\theta = \cos 5\theta - \cos\theta$.

By transposition, $(\cos 9\theta + \cos\theta) - \cos 5\theta = 0$;

$$\therefore\ 2\cos 5\theta \cos 4\theta - \cos 5\theta = 0;$$

$$\therefore\ \cos 5\theta\,(2\cos 4\theta - 1) = 0;$$

$\therefore$ either $\cos 5\theta = 0$, or $2\cos 4\theta - 1 = 0$.

From the first equation, $5\theta = 2n\pi \pm \dfrac{\pi}{2}$, or $\theta = \dfrac{(4n\pm 1)\pi}{10}$;

and from the second, $4\theta = 2n\pi \pm \dfrac{\pi}{3}$, or $\theta = \dfrac{(6n\pm 1)\pi}{12}$.

EXAMPLES. XIX. a.

Find the general solution of the equations:

1. $\sin\theta = \dfrac{1}{2}$.
2. $\sin\theta = \dfrac{1}{\sqrt{2}}$.
3. $\cos\theta = \dfrac{1}{2}$.
4. $\tan\theta = \sqrt{3}$.
5. $\cot\theta = -\sqrt{3}$.
6. $\sec\theta = -\sqrt{2}$.
7. $\cos^2\theta = \dfrac{1}{2}$.
8. $\tan^2\theta = \dfrac{1}{3}$.
9. $\operatorname{cosec}^2\theta = \dfrac{4}{3}$.
10. $\cos\theta = \cos\alpha$.
11. $\tan^2\theta = \tan^2\alpha$.
12. $\sec^2\theta = \sec^2\alpha$.
13. $\tan 2\theta = \tan\theta$.
14. $\operatorname{cosec} 3\theta = \operatorname{cosec} 3\alpha$.
15. $\cos 3\theta = \cos 2\theta$.
16. $\sin 5\theta + \sin\theta = \sin 3\theta$.
17. $\cos\theta - \cos 7\theta = \sin 4\theta$.
18. $\sin 4\theta - \sin 3\theta + \sin 2\theta - \sin\theta = 0$.
19. $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$.
20. $\sin 5\theta \cos\theta = \sin 6\theta \cos 2\theta$.
21. $\sin 11\theta \sin 4\theta + \sin 5\theta \sin 2\theta = 0$.
22. $\sqrt{2}\cos 3\theta - \cos\theta = \cos 5\theta$.
23. $\sin 7\theta - \sqrt{3}\cos 4\theta = \sin\theta$.
24. $1 + \cos\theta = 2\sin^2\theta$.
25. $\tan^2\theta + \sec\theta = 1$.
26. $\cot^2\theta - 1 = \operatorname{cosec}\theta$.
27. $\cot\theta - \tan\theta = 2$.
28. If $2\cos\theta = -1$ and $2\sin\theta = \sqrt{3}$, find θ.
29. If $\sec\theta = \sqrt{2}$ and $\tan\theta = -1$, find θ.

243. In the following examples, the solution is simplified by the use of some particular artifice.

Example 1. Solve the equation $\cos m\theta = \sin n\theta$.

Here $$\cos m\theta = \cos\left(\frac{\pi}{2} - n\theta\right);$$

$$\therefore m\theta = 2k\pi \pm \left(\frac{\pi}{2} - n\theta\right),$$

where k is zero, or any integer.

By transposition, we obtain

$$(m+n)\theta = \left(2k + \frac{1}{2}\right)\pi, \text{ or } (m-n)\theta = \left(2k - \frac{1}{2}\right)\pi.$$

This equation may also be solved through the medium of the sine. For we have

$$\sin\left(\frac{\pi}{2} - m\theta\right) = \sin n\theta;$$

$$\therefore \frac{\pi}{2} - m\theta = p\pi + (-1)^p n\theta,$$

where p is zero or any integer;

$$\therefore \{m + (-1)^p n\}\theta = \left(\frac{1}{2} - p\right)\pi.$$

Note. The general solution can frequently be obtained in several ways. The various forms which the result takes are merely different modes of expressing the same series of angles.

Example 2. Solve $\sqrt{3}\cos\theta + \sin\theta = 1$.

Multiply every term by $\frac{1}{2}$, then

$$\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta = \frac{1}{2},$$

$$\therefore \cos\frac{\pi}{6}\cos\theta + \sin\frac{\pi}{6}\sin\theta = \frac{1}{2};$$

$$\therefore \cos\left(\theta - \frac{\pi}{6}\right) = \frac{1}{2};$$

$$\therefore \theta - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3};$$

$$\therefore \theta = 2n\pi + \frac{\pi}{2} \text{ or } 2n\pi - \frac{\pi}{6}.$$

NOTE. In examples of this type, it is a common mistake to square the equation; but this process is objectionable, because it introduces solutions which do not belong to the given equation. Thus in the present instance,

$$\sqrt{3}\cos\theta = 1 - \sin\theta;$$

by squaring, $$3\cos^2\theta = (1-\sin\theta)^2.$$

But the solutions of this equation include the solutions of

$$-\sqrt{3}\cos\theta = 1 - \sin\theta,$$

as well as those of the given equation.

Example 3. Solve $\cos 2\theta = \cos\theta + \sin\theta$.

From this equation, $\cos^2\theta - \sin^2\theta = \cos\theta + \sin\theta$;

$$\therefore (\cos\theta + \sin\theta)(\cos\theta - \sin\theta) = \cos\theta + \sin\theta;$$

$\therefore$ either $$\cos\theta + \sin\theta = 0 \quad\ldots\ldots(1),$$

or $$\cos\theta - \sin\theta = 1 \quad\ldots\ldots(2).$$

From (1), $$\tan\theta = -1,$$

$$\therefore \theta = n\pi - \frac{\pi}{4}.$$

From (2), $$\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta = \frac{1}{\sqrt{2}};$$

$$\therefore \cos\theta\cos\frac{\pi}{4} - \sin\theta\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

$$\therefore \cos\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}};$$

$$\therefore \theta + \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4};$$

$$\therefore \theta = 2n\pi \text{ or } 2n\pi - \frac{\pi}{2}.$$

EXAMPLES. XIX. b.

Find the general solution of the equations:

1. $\tan p\theta = \cot q\theta$.
2. $\sin m\theta + \cos n\theta = 0$.
3. $\cos\theta - \sqrt{3}\sin\theta = 1$.
4. $\sin\theta - \sqrt{3}\cos\theta = 1$.
5. $\cos\theta = \sqrt{3}(1 - \sin\theta)$.
6. $\sin\theta + \sqrt{3}\cos\theta = \sqrt{2}$.

Find the general solution of the equations:

7. $\cos\theta - \sin\theta = \frac{1}{\sqrt{2}}$.

8. $\cos\theta + \sin\theta + \sqrt{2} = 0$.

9. $\operatorname{cosec}\theta + \cot\theta = \sqrt{3}$.

10. $\cot\theta - \cot 2\theta = 2$.

11. $2\sin\theta\sin 3\theta = 1$.

12. $\sin 3\theta = 8\sin^3\theta$.

13. $\tan\theta + \tan 3\theta = 2\tan 2\theta$.

14. $\cos\theta - \sin\theta = \cos 2\theta$.

15. $\operatorname{cosec}\theta + \sec\theta = 2\sqrt{2}$.

16. $\sec\theta - \operatorname{cosec}\theta = 2\sqrt{2}$.

17. $\sec 4\theta - \sec 2\theta = 2$.

18. $\cos 3\theta + 8\cos^3\theta = 0$.

19. $1 + \sqrt{3}\tan^2\theta = (1+\sqrt{3})\tan\theta$.

20. $\tan^3\theta + \cot^3\theta = 8\operatorname{cosec}^3 2\theta + 12$.

21. $\sin\theta = \sqrt{2}\sin\phi, \qquad \sqrt{3}\cos\theta = \sqrt{2}\cos\phi$.

22. $\operatorname{cosec}\theta = \sqrt{3}\operatorname{cosec}\phi, \qquad \cot\theta = 3\cot\phi$.

23. $\sec\phi = \sqrt{2}\sec\theta, \qquad \cot\theta = \sqrt{3}\cot\phi$.

24. Explain why the same two series of angles are given by the equations

$$\theta + \frac{\pi}{4} = n\pi + (-1)^n\frac{\pi}{6} \quad \text{and} \quad \theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}.$$

25. Shew that the formulæ

$$\left(2n + \frac{1}{4}\right)\pi \pm a \quad \text{and} \quad \left(n - \frac{1}{4}\right)\pi + (-1)^n\left(\frac{\pi}{2} - a\right)$$

comprise the same angles, and illustrate by a figure.

Inverse Circular Functions.

244. If $\sin\theta = s$, we know that θ may be *any* angle whose sine is s. It is often convenient to express this statement *inversely* by writing $\theta = \sin^{-1}s$.

In this *inverse notation* θ stands alone on one side of the equation, and may be regarded as an angle whose value is only known through the medium of its sine. Similarly, $\tan^{-1}\sqrt{3}$ indicates in a concise form any one of the angles whose tangent is $\sqrt{3}$. But all these angles are given by the formula $n\pi + \frac{\pi}{3}$. Thus

$$\theta = \tan^{-1}\sqrt{3} \quad \text{and} \quad \theta = n\pi + \frac{\pi}{3}$$

are equivalent statements expressed in different forms.

245. Expressions of the form $\cos^{-1}x$, $\sin^{-1}a$, $\tan^{-1}b$ are called **Inverse Circular Functions.**

It must be remembered that these expressions denote angles, and that -1 *is not an algebraical index*; that is,

$$\sin^{-1}x \textit{ is not the same as } (\sin x)^{-1} \textit{ or } \frac{1}{\sin x}.$$

246. From Art. 244, we see that an inverse function has an infinite number of values.

If f denote any one of the circular functions, and $f^{-1}(x)=A$, the **principal value** of $f^{-1}(x)$ is the smallest numerical value of A. Thus the principal values of

$$\cos^{-1}\frac{1}{2},\quad \sin^{-1}\left(-\frac{1}{2}\right),\quad \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right),\quad \tan^{-1}(-1)$$

are $60°$, $-30°$, $135°$, $-45°$.

Hence if x be positive, the principal values of $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ all lie between 0 and 90°.

If x be negative, the principal values of $\sin^{-1}x$ and $\tan^{-1}x$ lie between 0 and $-90°$, and the principal value of $\cos^{-1}x$ lies between 90° and 180°.

In numerical instances we shall usually suppose that the *principal value* is selected.

247. If $\sin\theta=x$, we have $\cos\theta=\sqrt{1-x^2}$.

Expressed in the inverse notation, these equations become

$$\theta=\sin^{-1}x,\quad \theta=\cos^{-1}\sqrt{1-x^2}.$$

In each of these two statements, θ has an infinite number of values; but, as the formulæ for the general values of the sine and cosine are not identical, we cannot assert that the equation

$$\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}$$

is identically true. This will be seen more clearly from a numerical instance. If $x=\frac{1}{2}$, then $\sqrt{1-x^2}=\frac{\sqrt{3}}{2}$.

Here $\sin^{-1}x$ may be any one of the angles

$$30°,\ 150°,\ 390°,\ 510°,\ \ldots;$$

and $\cos^{-1}\sqrt{1-x^2}$ may be any one of the angles

$$30°,\ 330°,\ 390°,\ 690°,\ \ldots.$$

248. From the relations established in the previous chapters, we may deduce corresponding relations connecting the inverse functions. Thus in the identity

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta},$$

let $\tan\theta = a$, so that $\theta = \tan^{-1} a$; then

$$\cos(2\tan^{-1} a) = \frac{1-a^2}{1+a^2};$$

$$\therefore \quad 2\tan^{-1} a = \cos^{-1}\frac{1-a^2}{1+a^2}.$$

Similarly, the formula

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

when expressed in the inverse notation becomes

$$3\cos^{-1} a = \cos^{-1}(4a^3 - 3a).$$

249. *To prove that*

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}.$$

Let $\tan^{-1} x = \alpha$, so that $\tan\alpha = x$;

and $\tan^{-1} y = \beta$, so that $\tan\beta = y$.

We require $\alpha+\beta$ in the form of an inverse tangent.

Now
$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

$$= \frac{x+y}{1-xy};$$

$$\therefore \quad \alpha+\beta = \tan^{-1}\frac{x+y}{1-xy};$$

that is,
$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}.$$

By putting $y = x$, we obtain

$$2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2}.$$

Note. It is useful to remember that

$$\tan(\tan^{-1} x + \tan^{-1} y) = \frac{x+y}{1-xy}.$$

Example 1. Prove that

$$\tan^{-1}5 - \tan^{-1}3 + \tan^{-1}\frac{7}{9} = n\pi + \frac{\pi}{4}.$$

$$\text{The first side} = \tan^{-1}\frac{5-3}{1+15} + \tan^{-1}\frac{7}{9}$$

$$= \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{7}{9}$$

$$= \tan^{-1}\frac{\frac{1}{8}+\frac{7}{9}}{1-\frac{7}{72}} = \tan^{-1}1$$

$$= n\pi + \frac{\pi}{4}.$$

NOTE. The value of n cannot be assigned until we have selected some particular values for the angles $\tan^{-1}5$, $\tan^{-1}3$, $\tan^{-1}\frac{7}{9}$. If we choose the *principal values*, then $n = 0$.

Example 2. Prove that

$$\sin^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}.$$

We may write this identity in the form

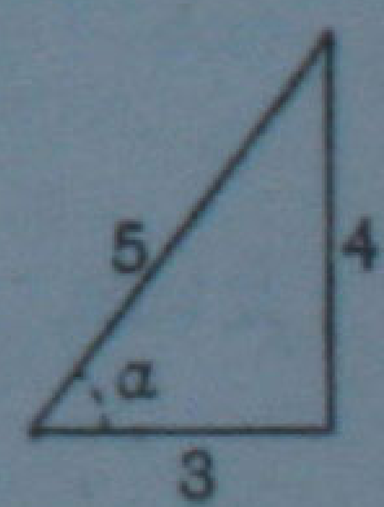
$$\sin^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \frac{\pi}{2} - \sin^{-1}\frac{16}{65} = \cos^{-1}\frac{16}{65}.$$

Let $\alpha = \sin^{-1}\frac{4}{5}$, so that $\sin\alpha = \frac{4}{5}$;

and $\beta = \cos^{-1}\frac{12}{13}$, so that $\cos\beta = \frac{12}{13}$.

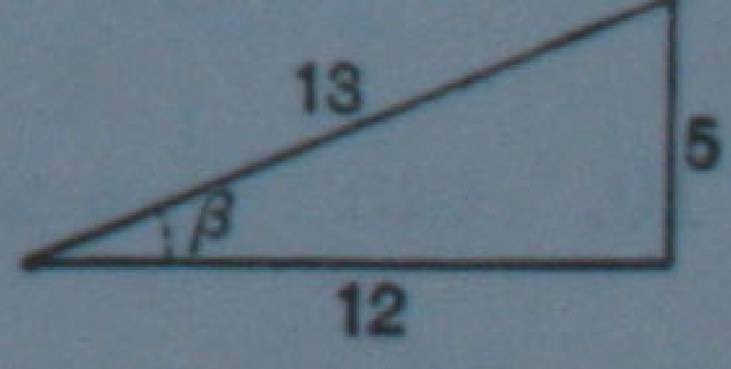
We have to express $\alpha + \beta$ as an inverse cosine.

Now $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$; whence by reading off the values of the functions from the figures in the margin, we have



$$\cos(\alpha+\beta) = \frac{3}{5}\cdot\frac{12}{13} - \frac{4}{5}\cdot\frac{5}{13}$$

$$= \frac{16}{65};$$

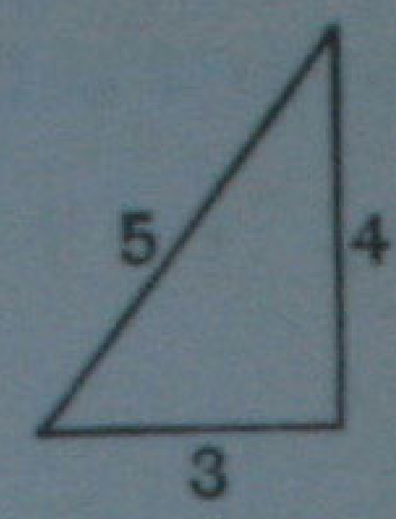


$$\therefore \alpha + \beta = \cos^{-1}\frac{16}{65}.$$

It is sometimes convenient to work entirely in terms of the tangent or cotangent.

Example 3. Prove that

$$2\cot^{-1}7+\cos^{-1}\frac{3}{5}=\operatorname{cosec}^{-1}\frac{125}{117}.$$

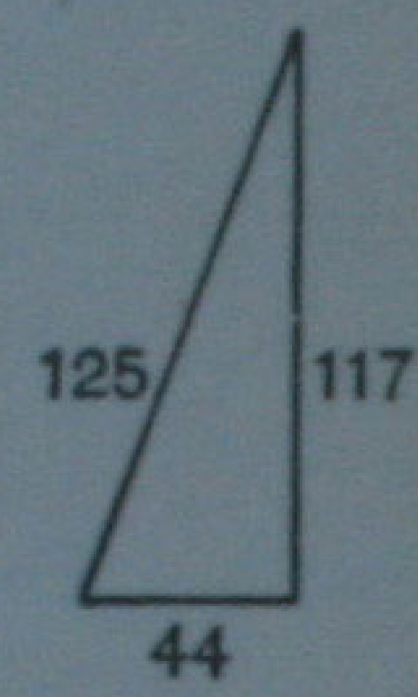


$$\text{The first side}=\cot^{-1}\frac{7^2-1}{2\times 7}+\cot^{-1}\frac{3}{4}$$

$$=\cot^{-1}\frac{24}{7}+\cot^{-1}\frac{3}{4}$$

$$=\cot^{-1}\frac{\frac{24}{7}\times\frac{3}{4}-1}{\frac{24}{7}+\frac{3}{4}}$$

$$=\cot^{-1}\frac{44}{117}=\operatorname{cosec}^{-1}\frac{125}{117}.$$



EXAMPLES. XIX. c.

Prove the following statements :

1. $\sin^{-1}\frac{12}{13}=\cot^{-1}\frac{5}{12}$.

2. $\operatorname{cosec}^{-1}\frac{17}{8}=\tan^{-1}\frac{8}{15}$.

3. $\sec(\tan^{-1}x)=\sqrt{1+x^2}$.

4. $2\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{3}{4}$.

5. $\tan^{-1}\frac{4}{3}-\tan^{-1}1=\tan^{-1}\frac{1}{7}$.

6. $\tan^{-1}\frac{2}{11}+\cot^{-1}\frac{24}{7}=\tan^{-1}\frac{1}{2}$.

7. $\cot^{-1}\frac{4}{3}-\cot^{-1}\frac{15}{8}=\cot^{-1}\frac{84}{13}$.

8. $2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{4}=\tan^{-1}\frac{32}{43}$.

9. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{5}{6}+\tan^{-1}\frac{1}{11}$.

10. $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{18}=\cot^{-1}3$.

11. $\tan^{-1}\frac{3}{5}+\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{27}{11}$.

12. $2\cot^{-1}\frac{5}{4}=\tan^{-1}\frac{40}{9}$. **13.** $2\tan^{-1}\frac{8}{15}=\sin^{-1}\frac{240}{289}$.

14. $\sin(2\sin^{-1}x)=2x\sqrt{1-x^2}$.

15. $\cos^{-1}x=2\sin^{-1}\sqrt{\frac{1-x}{2}}$.

16. $2\tan^{-1}\sqrt{\frac{x}{a}}=\cos^{-1}\frac{a-x}{a+x}$.

17. $2\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{5}=\frac{\pi}{4}$.

18. $\sin^{-1}a-\cos^{-1}b=\cos^{-1}\{b\sqrt{1-a^2}+a\sqrt{1-b^2}\}$.

19. $\sin^{-1}\frac{4}{5}+\cos^{-1}\frac{2}{\sqrt{5}}=\cot^{-1}\frac{2}{11}$.

20. $\cos^{-1}\frac{63}{65}+2\tan^{-1}\frac{1}{5}=\sin^{-1}\frac{3}{5}$.

21. $\tan^{-1}m+\tan^{-1}n=\cos^{-1}\frac{1-mn}{\sqrt{(1+m^2)(1+n^2)}}$.

22. $\cos^{-1}\frac{20}{29}-\tan^{-1}\frac{16}{63}=\cos^{-1}\frac{1596}{1885}$.

23. $\cos^{-1}\sqrt{\frac{2}{3}}-\cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}}=\frac{\pi}{6}$.

24. $\tan(2\tan^{-1}x)=2\tan(\tan^{-1}x+\tan^{-1}x^3)$.

25. $\tan^{-1}a=\tan^{-1}\frac{a-b}{1+ab}+\tan^{-1}\frac{b-c}{1+bc}+\tan^{-1}c$.

26. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, prove that

$$x+y+z=xyz.$$

27. If $u=\cot^{-1}\sqrt{\cos\alpha}-\tan^{-1}\sqrt{\cos\alpha}$, prove that

$$\sin u=\tan^2\frac{\alpha}{2}.$$

250. We shall now shew how to solve equations expressed in the inverse notation.

Example 1. Solve $\tan^{-1}2x+\tan^{-1}3x=n\pi+\frac{3\pi}{4}$.

We have $$\tan^{-1}\frac{2x+3x}{1-6x^2}=n\pi+\frac{3\pi}{4};$$

$$\therefore \frac{2x+3x}{1-6x^2}=\tan\left(n\pi+\frac{3\pi}{4}\right)=-1;$$

$$\therefore 6x^2-5x-1=0, \text{ or } (6x+1)(x-1)=0;$$

$$\therefore x=1, \text{ or } -\frac{1}{6}.$$

Example 2. Solve $\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x$.

By transposition, $\sin^{-1}(1-x)=\cos^{-1}x-\sin^{-1}x$.

Let $\cos^{-1}x=\alpha$, and $\sin^{-1}x=\beta$; then

$$\sin^{-1}(1-x)=\alpha-\beta;$$

$$\therefore 1-x=\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta.$$

But $\cos\alpha=x$, and therefore $\sin\alpha=\sqrt{1-x^2}$;

also $\sin\beta=x$, and therefore $\cos\beta=\sqrt{1-x^2}$;

$$\therefore 1-x=(1-x^2)-x^2=1-2x^2;$$

$$\therefore 2x^2-x=0;$$

whence $$x=0, \text{ or } \frac{1}{2}.$$

EXAMPLES. XIX. d.

Solve the equations:

1. $\sin^{-1}x=\cos^{-1}x$. 2. $\tan^{-1}x=\cot^{-1}x$.

3. $\tan^{-1}(x+1)-\tan^{-1}(x-1)=\cot^{-1}2$.

4. $\cot^{-1}x+\cot^{-1}2x=\frac{3\pi}{4}$.

5. $\sin^{-1}x-\cos^{-1}x=\sin^{-1}(3x-2)$.

6. $\cos^{-1}x-\sin^{-1}x=\cos^{-1}x\sqrt{3}$.

7. $\tan^{-1}\dfrac{x-1}{x-2}+\tan^{-1}\dfrac{x+1}{x+2}=\dfrac{\pi}{4}.$

8. $2\cot^{-1}2+\cos^{-1}\dfrac{3}{5}=\operatorname{cosec}^{-1}x.$

9. $\tan^{-1}x+\tan^{-1}(1-x)=2\tan^{-1}\sqrt{x-x^2}.$

10. $\cos^{-1}\dfrac{1-a^2}{1+a^2}-\cos^{-1}\dfrac{1-b^2}{1+b^2}=2\tan^{-1}x.$

11. $\sin^{-1}\dfrac{2a}{1+a^2}+\tan^{-1}\dfrac{2x}{1-x^2}=\cos^{-1}\dfrac{1-b^2}{1+b^2}.$

12. $\cot^{-1}\dfrac{x^2-1}{2x}+\tan^{-1}\dfrac{2x}{x^2-1}+\dfrac{4\pi}{3}=0.$

13. Shew that we can express

$$\sin^{-1}\frac{2ab}{a^2+b^2}+\sin^{-1}\frac{2cd}{c^2+d^2} \text{ in the form } \sin^{-1}\frac{2xy}{x^2+y^2}$$

where x and y are rational functions of a, b, c, d.

14. If $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}]=0$, find x.

15. If $2\tan^{-1}(\cos\theta)=\tan^{-1}(2\operatorname{cosec}\theta)$, find θ.

16. If $\sin(\pi\cos\theta)=\cos(\pi\sin\theta)$, shew that

$$2\theta=\pm\sin^{-1}\frac{3}{4}.$$

17. If $\sin(\pi\cot\theta)=\cos(\pi\tan\theta)$, and n is any integer, shew that either $\cot 2\theta$ or $\operatorname{cosec} 2\theta$ is of the form $\dfrac{4n+1}{4}$.

18. If $\tan(\pi\cot\theta)=\cot(\pi\tan\theta)$, and n is any integer, shew that

$$\tan\theta=\frac{2n+1}{4}\pm\frac{\sqrt{4n^2+4n-15}}{4}.$$

19. Find all the positive integral solutions of

$$\tan^{-1}x+\cot^{-1}y=\tan^{-1}3.$$

MISCELLANEOUS EXAMPLES. G.

1. If the sines of the angles of a triangle are in the ratio of $4:5:6$, shew that the cosines are in the ratio of $12:9:2$.

2. Solve the equations:

(1) $2\cos^3\theta+\sin^2\theta-1=0$; (2) $\sec^3\theta-2\tan^2\theta=2$.

3. If $\tan\beta=2\sin\alpha\sin\gamma\operatorname{cosec}(\alpha+\gamma)$, prove that $\cot\alpha$, $\cot\beta$, $\cot\gamma$ are in arithmetical progression.

4. In a triangle shew that

$$4r(r_1+r_2+r_3)=2(bc+ca+ab)-(a^2+b^2+c^2).$$

5. Prove that

(1) $\tan^{-1}\frac{1}{3}-\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{3}{11}$;

(2) $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{36}{85}=\frac{\pi}{2}$.

6. Find the greatest angle of the triangle whose sides are 185, 222, 259; given $\log 6=\cdot7781513$,

$$L\cos 39^\circ\,14'=9\cdot8890644,\ \text{diff. for } 1'=1032.$$

7. If $\tan(\alpha+\theta)=n\tan(\alpha-\theta)$, prove that $\dfrac{\sin 2\theta}{\sin 2\alpha}=\dfrac{n-1}{n+1}$.

8. If in a triangle $8R^2=a^2+b^2+c^2$, prove that one of the angles is a right angle.

9. The area of a regular polygon of n sides inscribed in a circle is three-fourths of the area of the circumscribed regular polygon with the same number of sides: find n.

10. $ABCD$ is a straight sea-wall. From B the straight lines drawn to two boats are each inclined at 45° to the direction of the wall, and from C the angles of inclination are 15° and 75°. If $BC=400$ yards, find the distance between the boats, and the distance of each from the sea-wall.

CHAPTER XX.

FUNCTIONS OF SUBMULTIPLE ANGLES.

251. Trigonometrical ratios of $22\frac{1}{2}°$ or $\frac{\pi}{8}$.

From the identity

$$2\sin^2 22\tfrac{1}{2}° = 1 - \cos 45°,$$

we have $$4\sin^2 22\tfrac{1}{2}° = 2 - 2\cos 45° = 2 - \sqrt{2};$$

$$\therefore\ 2\sin 22\tfrac{1}{2}° = \sqrt{2 - \sqrt{2}} \quad \text{..................(1).}$$

In like manner from

$$2\cos^2 22\tfrac{1}{2}° = 1 + \cos 45°,$$

we obtain $$2\cos 22\tfrac{1}{2}° = \sqrt{2 + \sqrt{2}} \quad \text{..................(2).}$$

In each of these cases the positive sign must be taken before the radical, since $22\frac{1}{2}°$ is an acute angle.

Again, $$\tan 22\tfrac{1}{2}° = \frac{1 - \cos 45°}{\sin 45°} = \operatorname{cosec} 45° - \cot 45°;$$

$$\therefore\ \tan 22\tfrac{1}{2}° = \sqrt{2} - 1.$$

252. We have seen that $2\cos\frac{\pi}{8} = \sqrt{2 + \sqrt{2}}$;

but $$4\cos^2\frac{\pi}{16} = 2 + 2\cos\frac{\pi}{8};$$

$$\therefore\ 4\cos^2\frac{\pi}{16} = 2 + \sqrt{2 + \sqrt{2}};$$

$$\therefore\ 2\cos\frac{\pi}{16} = \sqrt{2 + \sqrt{2 + \sqrt{2}}}.$$

Similarly, $$2\cos\frac{\pi}{32} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}};$$

and so on.

253. Suppose that $\cos A = \frac{1}{2}$ and that it is required to find $\sin \frac{A}{2}$.

$$\sin \frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} = \sqrt{\frac{1}{4}} = \pm\frac{1}{2}.$$

This case differs from those of the two previous articles in that the datum is less precise. All we know of the angle A is contained in the statement that its cosine is equal to $\frac{1}{2}$, and without some further knowledge respecting A we cannot remove the ambiguity of sign in the value found for $\sin \frac{A}{2}$.

We now proceed to a more general discussion.

254. *Given* $\cos A$ *to find* $\sin \frac{A}{2}$ *and* $\cos \frac{A}{2}$ *and to explain the presence of the two values in each case.*

From the identities

$$2\sin^2 \frac{A}{2} = 1 - \cos A, \quad \text{and} \quad 2\cos^2 \frac{A}{2} = 1 + \cos A,$$

we have

$$\sin \frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}, \quad \text{and} \quad \cos \frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}.$$

Thus corresponding to *one* value of $\cos A$, there are *two* values for $\sin \frac{A}{2}$, and *two* values for $\cos \frac{A}{2}$.

The presence of these two values may be explained as follows. If $\cos A$ is given and nothing further is stated about the angle A, all we know is that A belongs to a certain group of *equi-cosinal angles.* Let a be the smallest positive angle belonging to this group, then $A = 2n\pi \pm a$. Thus in finding $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$ we are really finding the values of

$$\sin \frac{1}{2}(2n\pi \pm a) \quad \text{and} \quad \cos \frac{1}{2}(2n\pi \pm a).$$

Now $\sin\frac{1}{2}(2n\pi \pm a) = \sin\left(n\pi \pm \frac{a}{2}\right)$

$$= \sin n\pi \cos\frac{a}{2} \pm \cos n\pi \sin\frac{a}{2}$$

$$= \pm\sin\frac{a}{2},$$

for $\sin n\pi = 0$ and $\cos n\pi = \pm 1$.

Again, $\cos\frac{1}{2}(2n\pi \pm a) = \cos n\pi \cos\frac{a}{2} \mp \sin n\pi \sin\frac{a}{2}$

$$= \pm\cos\frac{a}{2}.$$

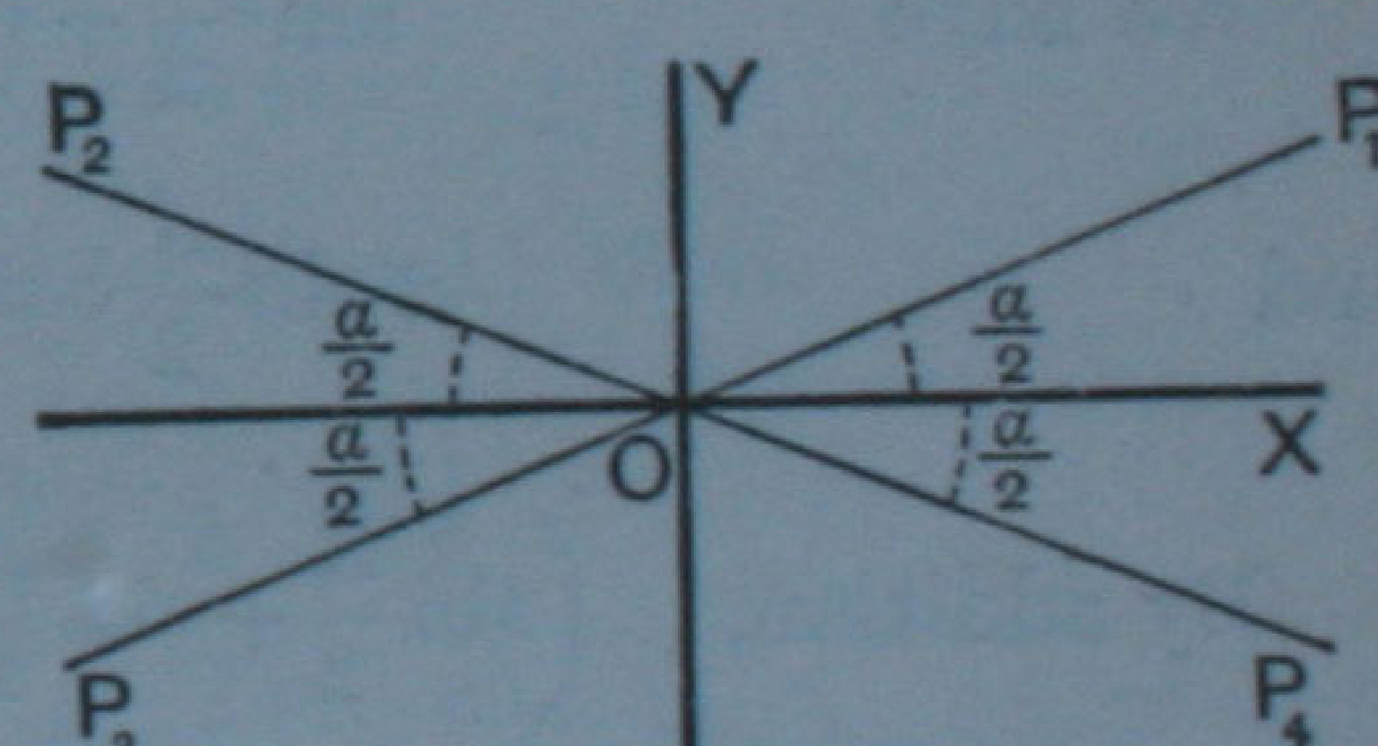
Thus there are two values for $\sin\frac{A}{2}$ and two values for $\cos\frac{A}{2}$ when $\cos A$ is given and nothing further is known respecting A.

255. Geometrical Illustration. Let a be the smallest positive angle which has the same cosine as A; then

$$A = 2n\pi \pm a,$$

and we have to find the sine and cosine of $\frac{A}{2}$, that is of

$$n\pi \pm \frac{a}{2}.$$



Each of the angles denoted by this formula is bounded by one of the lines OP_1, OP_2, OP_3, OP_4. Now

$$\sin XOP_2 = \sin\frac{a}{2}, \quad \sin XOP_3 = -\sin\frac{a}{2}, \quad \sin XOP_4 = -\sin\frac{a}{2},$$

$$\cos XOP_2 = -\cos\frac{a}{2}, \quad \cos XOP_3 = -\cos\frac{a}{2}, \quad \cos XOP_4 = \cos\frac{a}{2}.$$

Thus the values of $\sin\frac{A}{2}$ are $\pm\sin\frac{a}{2}$, and the values of $\cos\frac{A}{2}$ are $\pm\cos\frac{a}{2}$.

256. If $\cos A$ is given, and A lies between certain known limits, the ambiguities of sign in the formulæ of Art. 254 may be removed.

Example. If $\cos A = -\frac{7}{25}$, and A lies between 450° and 540°, find $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$.

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{1}{2}\left(1+\frac{7}{25}\right)} = \sqrt{\frac{16}{25}} = \pm\frac{4}{5};$$

$$\cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}} = \sqrt{\frac{1}{2}\left(1-\frac{7}{25}\right)} = \sqrt{\frac{9}{25}} = \pm\frac{3}{5}.$$

Now $\frac{A}{2}$ lies between 225° and 270°, so that $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ are both negative;

$$\therefore\ \sin\frac{A}{2} = -\frac{4}{5}, \text{ and } \cos\frac{A}{2} = -\frac{3}{5}.$$

257. *To find* $\sin\frac{A}{2}$ *and* $\cos\frac{A}{2}$ *in terms of* $\sin A$ *and to explain the presence of four values in each case.*

We have
$$\sin^2\frac{A}{2} + \cos^2\frac{A}{2} = 1,$$

and
$$2\sin\frac{A}{2}\cos\frac{A}{2} = \sin A.$$

By addition,
$$\left(\sin\frac{A}{2} + \cos\frac{A}{2}\right)^2 = 1 + \sin A;$$

by subtraction,
$$\left(\sin\frac{A}{2} - \cos\frac{A}{2}\right)^2 = 1 - \sin A.$$

$$\therefore\ \sin\frac{A}{2} + \cos\frac{A}{2} = \pm\sqrt{1+\sin A} \quad \text{............(1)},$$

and
$$\sin\frac{A}{2} - \cos\frac{A}{2} = \pm\sqrt{1-\sin A} \quad \text{............(2)}.$$

By addition and subtraction, we obtain $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$; and since there is a double sign before each radical, there are *four* values for $\sin\frac{A}{2}$, and *four* values for $\cos\frac{A}{2}$ corresponding to *one* value of $\sin A$.

The presence of these four values may be explained as follows.

If $\sin A$ is given and nothing else is stated about the angle A all we know is that A belongs to a certain group of *equi-sinal angles*. Let a be the smallest positive angle belonging to this group, then $A = n\pi + (-1)^n a$. Thus in finding $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ we are really finding

$$\sin\frac{1}{2}\{n\pi + (-1)^n a\}, \text{ and } \cos\frac{1}{2}\{n\pi + (-1)^n a\}.$$

First suppose n even and equal to $2m$; then

$$\sin\frac{1}{2}\{n\pi + (-1)^n a\} = \sin\left(m\pi + \frac{a}{2}\right)$$

$$= \sin m\pi \cos\frac{a}{2} + \cos m\pi \sin\frac{a}{2}$$

$$= \pm\sin\frac{a}{2},$$

since $\sin m\pi = 0$, and $\cos m\pi = \pm 1$.

Next suppose n odd and equal to $2m+1$; then

$$\sin\frac{1}{2}\{n\pi + (-1)^n a\} = \sin\left(m\pi + \frac{\pi}{2} - \frac{a}{2}\right)$$

$$= \sin m\pi \cos\left(\frac{\pi}{2} - \frac{a}{2}\right) + \cos m\pi \sin\left(\frac{\pi}{2} - \frac{a}{2}\right)$$

$$= \pm\sin\left(\frac{\pi}{2} - \frac{a}{2}\right).$$

Thus we have *four* values for $\sin\frac{A}{2}$ when $\sin A$ is given and nothing further is known respecting A.

In like manner it may be shewn that

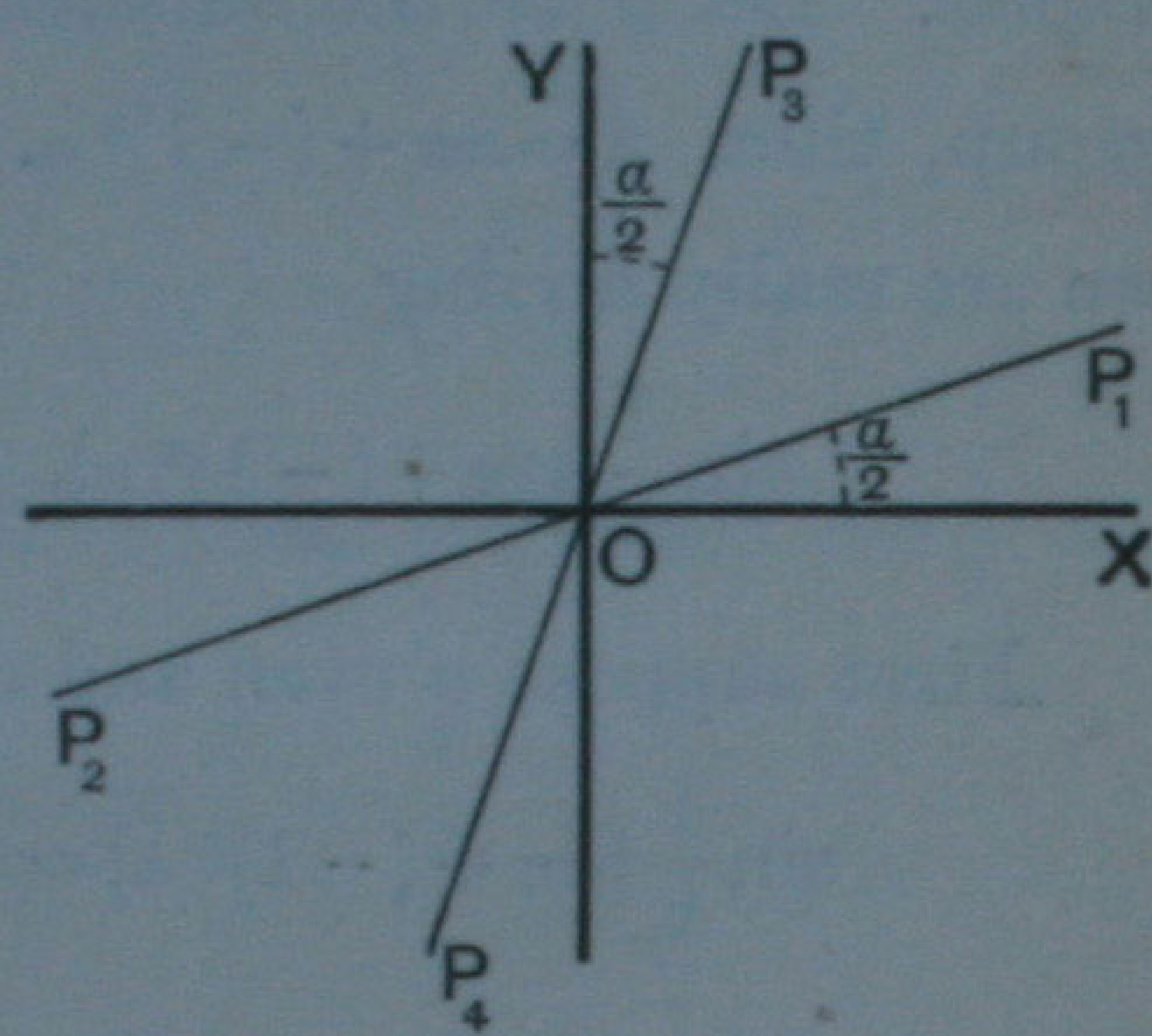
$\cos\frac{A}{2}$ has the *four* values $\pm\cos\frac{a}{2}$, $\pm\cos\left(\frac{\pi}{2} - \frac{a}{2}\right)$.

258. Geometrical Illustration. Let a be the smallest positive angle which has the same sine as A; then

$$A = n\pi + (-1)^n a,$$

and we have to find the sine and cosine of $\frac{A}{2}$, that is of

$$\frac{1}{2}\{n\pi + (-1)^n a\}.$$



If n is even and equal to $2m$, this expression becomes $m\pi + \frac{a}{2}$.

If n is odd and equal to $2m+1$, the expression becomes

$$m\pi + \left(\frac{\pi}{2} - \frac{a}{2}\right).$$

The angles denoted by the formula $m\pi + \frac{a}{2}$ are bounded by one of the lines OP_1 or OP_2; and those denoted by the formula $m\pi + \left(\frac{\pi}{2} - \frac{a}{2}\right)$ are bounded by one of the lines OP_3 or OP_4.

Now $\quad \sin XOP_1 = \sin \frac{a}{2}$;

$$\sin XOP_2 = -\sin XOP_1 = -\sin\frac{a}{2};$$

$$\sin XOP_3 = \sin\left(\frac{\pi}{2} - \frac{a}{2}\right);$$

$$\sin XOP_4 = -\sin XOP_3 = -\sin\left(\frac{\pi}{2} - \frac{a}{2}\right).$$

Thus the values of $\sin\frac{A}{2}$ are

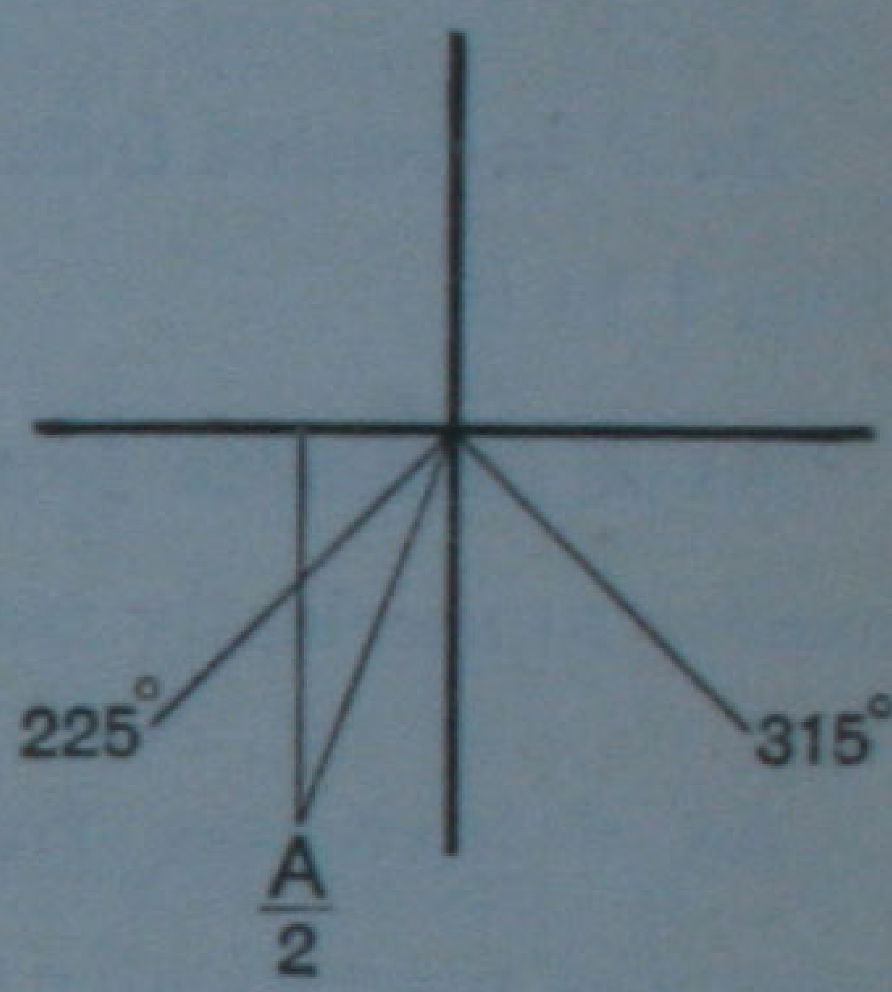
$$\pm\sin\frac{a}{2} \quad\text{and}\quad \pm\sin\left(\frac{\pi}{2} - \frac{a}{2}\right).$$

Similarly the values of $\cos\frac{A}{2}$ are $\pm\cos\frac{a}{2}$ and $\pm\cos\left(\frac{\pi}{2} - \frac{a}{2}\right)$.

259. If in addition to the value of $\sin A$ we know that A lies between certain limits, the ambiguities of sign in equations (1) and (2) of Art. 257 may be removed.

Example 1. Find $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ in terms of $\sin A$ when A lies between 450° and 630°.

In this case $\frac{A}{2}$ lies between 225° and 315°. From the adjoining figure it is evident that between these limits $\sin\frac{A}{2}$ is greater than $\cos\frac{A}{2}$ and is negative.



$$\therefore \sin\frac{A}{2}+\cos\frac{A}{2}=-\sqrt{1+\sin A},$$

and $$\sin\frac{A}{2}-\cos\frac{A}{2}=-\sqrt{1-\sin A}.$$

$$\therefore 2\sin\frac{A}{2}=-\sqrt{1+\sin A}-\sqrt{1-\sin A},$$

and $$2\cos\frac{A}{2}=-\sqrt{1+\sin A}+\sqrt{1-\sin A}.$$

Example 2. Determine the limits between which A must lie in order that

$$2\cos A=-\sqrt{1+\sin 2A}-\sqrt{1-\sin 2A}.$$

The given relation is obtained by combining

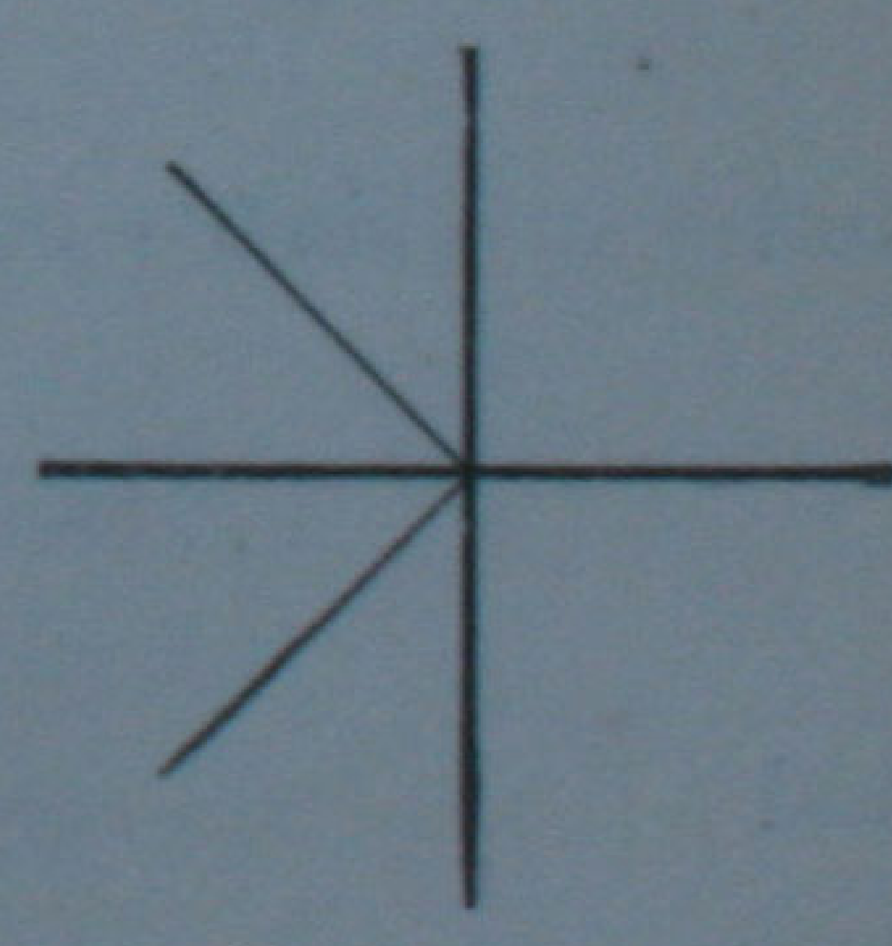
$$\sin A+\cos A=-\sqrt{1+\sin 2A} \quad\text{........ ....(1),}$$

and $$\sin A-\cos A=+\sqrt{1-\sin 2A} \quad\text{..................(2).}$$

From (1), we see that of $\sin A$ and $\cos A$ *the numerically greater is negative.*

From (2), we see that the cosine is the greater.

Hence we have to choose limits between which $\cos A$ is numerically greater than $\sin A$ and is negative. From the figure we see that A lies between $2n\pi+\frac{3\pi}{4}$ and $2n\pi+\frac{5\pi}{4}$.



Example 3. Trace the changes of $\cos\theta - \sin\theta$ in sign and magnitude as θ increases from 0 to 2π.

$$\cos\theta - \sin\theta = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right)$$

$$= \sqrt{2}\left(\cos\theta\cos\frac{\pi}{4} - \sin\theta\sin\frac{\pi}{4}\right)$$

$$= \sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right).$$

As θ increases from 0 to $\frac{\pi}{4}$, the expression is positive and decreases from 1 to 0.

As θ increases from $\frac{\pi}{4}$ to $\frac{3\pi}{4}$, the expression is negative and increases numerically from 0 to $-\sqrt{2}$.

As θ increases from $\frac{3\pi}{4}$ to $\frac{5\pi}{4}$, the expression is negative and decreases numerically from $-\sqrt{2}$ to 0.

As θ increases from $\frac{5\pi}{4}$ to $\frac{7\pi}{4}$, the expression is positive and increases from 0 to $\sqrt{2}$.

As θ increases from $\frac{7\pi}{4}$ to 2π, the expression is positive and decreases from $\sqrt{2}$ to 1.

260. *To find the sine and cosine of* 9°.

Since $\cos 9° > \sin 9°$ and is positive, we have

$$\sin 9° + \cos 9° = +\sqrt{1 + \sin 18°},$$

and

$$\sin 9° - \cos 9° = -\sqrt{1 - \sin 18°}.$$

$$\therefore \sin 9° + \cos 9° = +\sqrt{1 + \frac{\sqrt{5}-1}{4}} = +\frac{1}{2}\sqrt{3+\sqrt{5}},$$

and

$$\sin 9° - \cos 9° = -\sqrt{1 - \frac{\sqrt{5}-1}{4}} = -\frac{1}{2}\sqrt{5-\sqrt{5}}.$$

$$\therefore \sin 9° = \frac{1}{4}\left\{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}\right\},$$

and

$$\cos 9° = \frac{1}{4}\left\{\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}\right\}.$$

EXAMPLES. XX. a.

1. When A lies between $-270°$ and $-360°$, prove that

$$\sin\frac{A}{2} = -\sqrt{\frac{1-\cos A}{2}}.$$

2. If $\cos A = \frac{119}{169}$, find $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ when A lies between $270°$ and $360°$.

3. If $\cos A = -\frac{161}{289}$, find $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ when A lies between $540°$ and $630°$.

4. Find $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ in terms of $\sin A$ when A lies between $270°$ and $450°$.

5. Find $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ in terms of $\sin A$ when $\frac{A}{2}$ lies between $225°$ and $315°$.

6. Find $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ in terms of $\sin A$ when A lies between $-450°$ and $-630°$.

7. If $\sin A = \frac{24}{25}$, find $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ when A lies between $90°$ and $180°$.

8. If $\sin A = -\frac{240}{289}$, find $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ when A lies between $270°$ and $360°$.

9. Determine the limits between which A must lie in order that

(1) $2\sin A = \sqrt{1+\sin 2A} - \sqrt{1-\sin 2A}$;

(2) $2\cos A = -\sqrt{1+\sin 2A} + \sqrt{1-\sin 2A}$;

(3) $2\sin A = -\sqrt{1+\sin 2A} + \sqrt{1-\sin 2A}$.

10. If $A=240°$, is the following statement correct?

$$2\sin\frac{A}{2}=\sqrt{1+\sin A}-\sqrt{1-\sin A}.$$

If not, how must it be modified?

11. Prove that

(1) $\tan 7\frac{1}{2}°=\sqrt{6}-\sqrt{3}+\sqrt{2}-2$;

(2) $\cot 142\frac{1}{2}°=\sqrt{2}+\sqrt{3}-2-\sqrt{6}$.

12. Shew that $\sin 9°$ lies between ·156 and ·157.

13. Prove that

(1) $2\sin 11°\ 15'=\sqrt{2-\sqrt{2+\sqrt{2}}}$;

(2) $\tan 11°\ 15'=\sqrt{4+2\sqrt{2}}-(\sqrt{2}+1)$.

14. When θ varies from 0 to 2π trace the changes in sign and magnitude of

(1) $\cos\theta+\sin\theta$; (2) $\sin\theta-\sqrt{3}\cos\theta$.

15. When θ varies from 0 to π, trace the changes in sign and magnitude of

(1) $\dfrac{\tan\theta+\cot\theta}{\tan\theta-\cot\theta}$; (2) $\dfrac{2\sin\theta-\sin 2\theta}{2\sin\theta+\sin 2\theta}$.

261. *To find* $\tan\frac{A}{2}$ *when* $\tan A$ *is given and to explain the presence of the two values.*

Denote $\tan A$ by t; then

$$t=\tan A=\frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}};$$

$$\therefore\ t\tan^2\frac{A}{2}+2\tan\frac{A}{2}-t=0;$$

$$\therefore\ \tan\frac{A}{2}=\frac{-2\pm\sqrt{4+4t^2}}{2t}=\frac{-1\pm\sqrt{1+t^2}}{t}.$$

The presence of these two values may be explained as follows.

If a be the smallest positive angle which has the given tangent, then $A = n\pi + a$, and we are really finding the value of

$$\tan\frac{1}{2}(n\pi + a).$$

(1) Let n be even and equal to $2m$; then

$$\tan\frac{1}{2}(n\pi + a) = \tan\left(m\pi + \frac{a}{2}\right) = \tan\frac{a}{2}.$$

(2) Let n be odd and equal to $2m+1$; then

$$\tan\frac{1}{2}(n\pi + a) = \tan\left(m\pi + \frac{\pi}{2} + \frac{a}{2}\right) = \tan\left(\frac{\pi}{2} + \frac{a}{2}\right).$$

Thus $\tan\frac{A}{2}$ has the two values $\tan\frac{a}{2}$ and $\tan\left(\frac{\pi}{2} + \frac{a}{2}\right)$.

Example 1. If $A = 170°$, prove that $\tan\frac{A}{2} = \frac{-1 - \sqrt{1 + \tan^2 A}}{\tan A}$.

Here $\frac{A}{2}$ is an acute angle, so that $\tan\frac{A}{2}$ must be positive. Hence in the formula $\frac{-1 \pm \sqrt{1 + \tan^2 A}}{\tan A}$ the numerator must have the same sign as the denominator. But when $A = 170°$, $\tan A$ is negative, and therefore we must choose the sign which will make the numerator negative; thus $\tan\frac{A}{2} = \frac{-1 - \sqrt{1 + \tan^2 A}}{\tan A}$.

Example 2. Given $\cos A = \cdot 6$, find $\tan\frac{A}{2}$, and explain the double answer.

$$\tan^2\frac{A}{2} = \frac{1 - \cos A}{1 + \cos A} = \frac{\cdot 4}{1\cdot 6} = \frac{1}{4};$$

$$\therefore \tan\frac{A}{2} = \pm\frac{1}{2}.$$

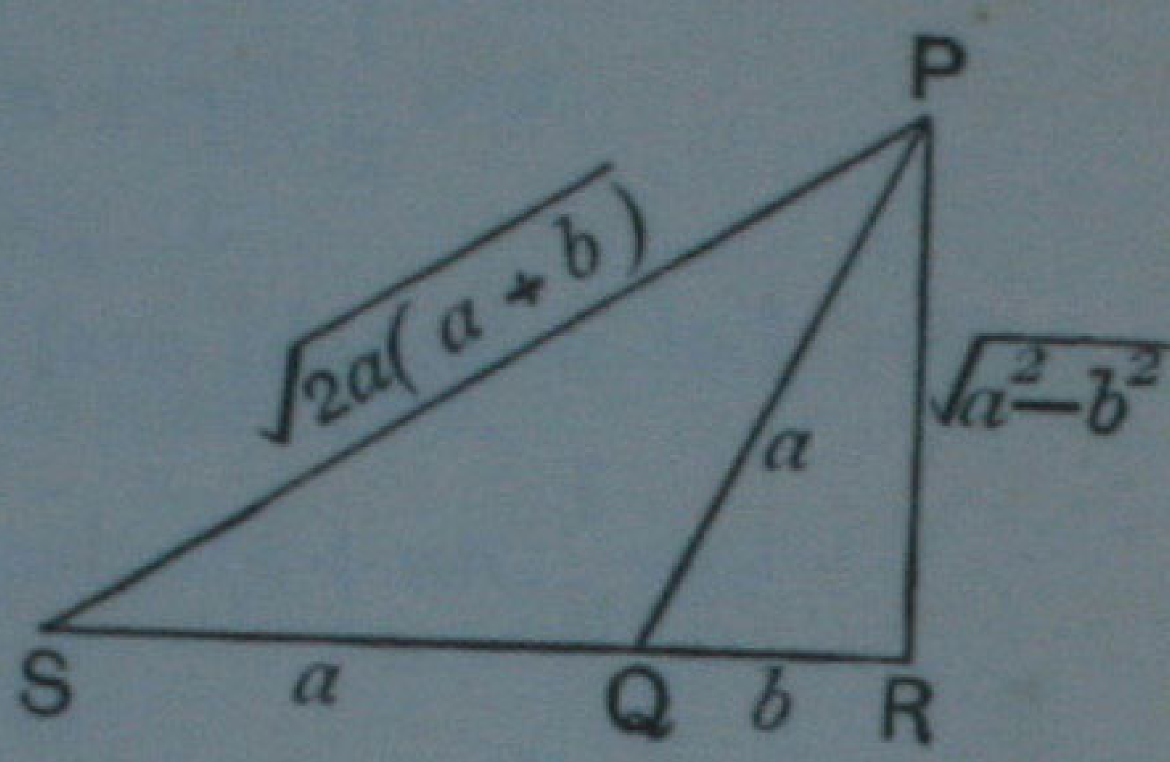
Here all we know of the angle A is that it must be one of a group of equi-cosinal angles. Let a be the smallest positive angle of this group; then $A = 2n\pi \pm a$.

$$\therefore \tan\frac{A}{2} = \tan\left(n\pi \pm \frac{a}{2}\right) = \tan\left(\pm\frac{a}{2}\right) = \pm\tan\frac{a}{2}.$$

Thus we have two values differing only in sign.

262. When any one of the functions of an acute angle A is given, we may in some cases conveniently obtain the functions of $\frac{A}{2}$, as in the following example.

Example. Given $\cos A = \frac{b}{a}$, to find the functions of $\frac{A}{2}$.



Make a right-angled triangle PQR in which the hypotenuse $PQ=a$, and base $QR=b$; then

$$\cos PQR = \frac{QR}{PQ} = \frac{b}{a} = \cos A;$$

$$\therefore \angle PQR = A.$$

Produce RQ to S making $QS=QP$;

$$\therefore \angle PSQ = \angle SPQ = \frac{1}{2}\angle PQR = \frac{A}{2}.$$

Now $\qquad SR=a+b$, and $PR=\sqrt{a^2-b^2}$,

$$\therefore PS^2=(a+b)^2+(a^2-b^2)=2a^2+2ab;$$

$$\therefore PS=\sqrt{2a(a+b)}.$$

The functions of $\frac{A}{2}$ may now be written down in terms of the sides of the triangle PRS.

263. From Art. 125, we have

$$\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3}.$$

Thus it appears that if $\cos A$ be given we have a *cubic* equation to find $\cos\frac{A}{3}$; so that $\cos\frac{A}{3}$ has *three* values.

Similarly, from the equation

$$\sin A = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{3}$$

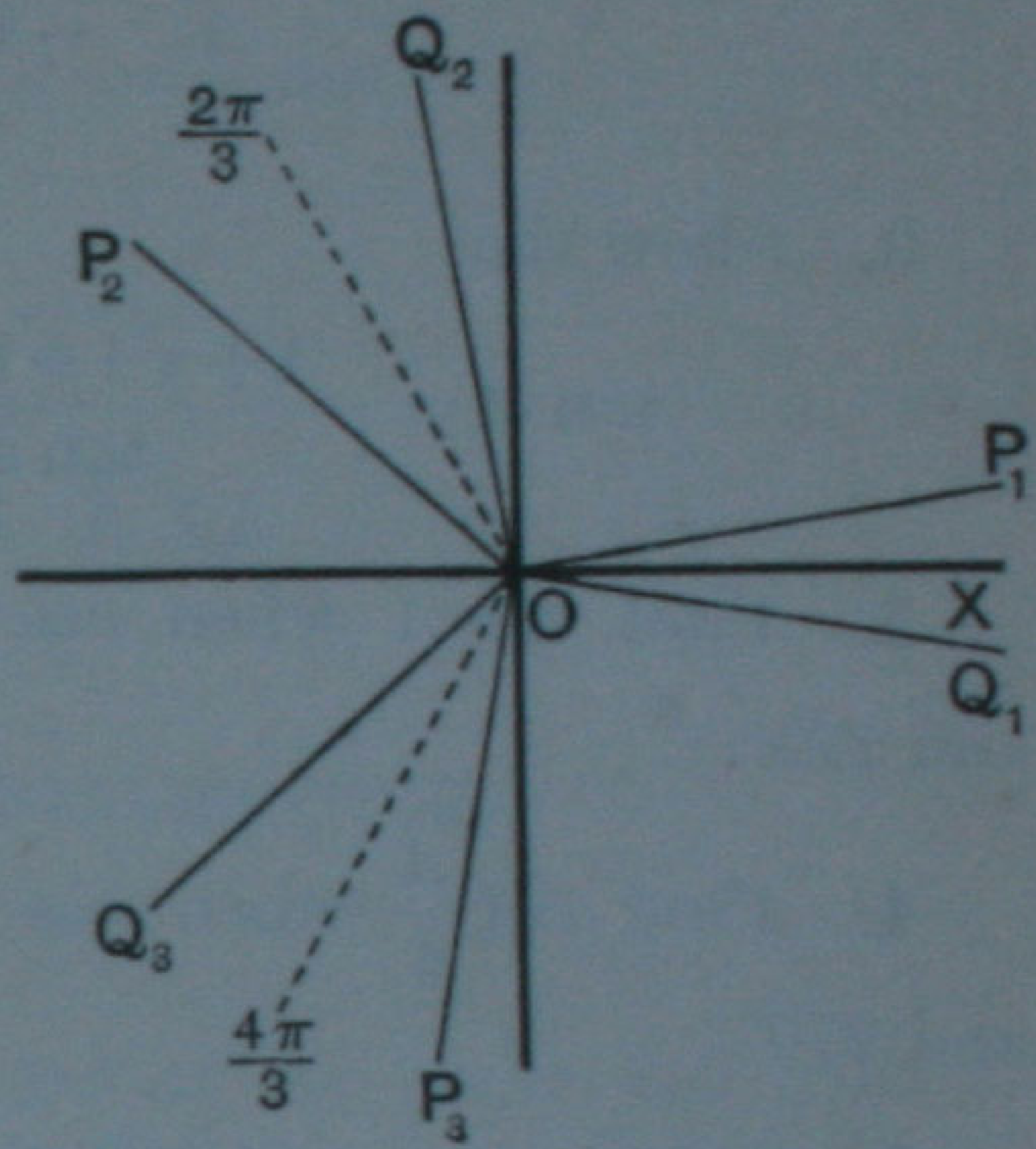
it appears that corresponding to *one* value of $\sin A$ there are *three* values of $\sin\frac{A}{3}$.

It will be a useful exercise to prove these two statements analytically as in Arts. 254 and 257. In the next article we shall give a geometrical explanation for the case of the cosine.

264. *Given* $\cos A$ *to find* $\cos\frac{A}{3}$, *and to explain the presence of the three values.*

Let a be the smallest positive angle with the given cosine; then $A=2n\pi\pm a$, and we have to find all the values of

$$\cos\frac{1}{3}(2n\pi\pm a).$$



Consider the angles denoted by the formula

$$\frac{1}{3}(2n\pi\pm a),$$

and ascribe to n in succession the values 0, 1, 2, 3, ...

When $n=0$, the angles are $\pm\frac{a}{3}$, bounded by OP_1 and OQ_1;

when $n=1$, the angles are $\frac{2\pi}{3}\pm\frac{a}{3}$, bounded by OP_2 and OQ_2

when $n=2$, the angles are $\frac{4\pi}{3}\pm\frac{a}{3}$, bounded by OP_3 and OQ_3.

By giving to n the values 3, 4, 5, ... we obtain a series of angles coterminal with those indicated in the figure.

Thus OP_1, OQ_1, OP_2, OQ_2, OP_3, OQ_3 bound all the angles included in the formula $\frac{1}{3}(2n\pi\pm a)$.

Now $$\cos XOQ_1=\cos XOP_1=\cos\frac{a}{3};$$

$$\cos XOP_3=\cos XOQ_2=\cos\left(\frac{2\pi}{3}-\frac{a}{3}\right);$$

$$\cos XOQ_3=\cos XOP_2=\cos\left(\frac{2\pi}{3}+\frac{a}{3}\right).$$

Thus the values of $\cos\frac{A}{3}$ are $\cos\frac{a}{3}$, $\cos\frac{2\pi+a}{3}$, $\cos\frac{2\pi-a}{3}$.

EXAMPLES. XX. b.

1. If $A=320^\circ$, prove that

$$\tan\frac{A}{2}=\frac{-1+\sqrt{1+\tan^2 A}}{\tan A}.$$

2. Shew that

$$\tan A=-\frac{1+\sqrt{1+\tan^2 2A}}{\tan 2A} \text{ when } A=110^\circ.$$

3. Find $\tan A$ when $\cos 2A=\frac{12}{13}$ and A lies between 180° and 225°.

4. Find $\cot\frac{A}{2}$ when $\cos A=-\frac{4}{5}$ and A lies between 180° and 270°.

5. If $\cot 2\theta=\cot 2\alpha$, shew that $\cot\theta$ has the two values $\cot\alpha$ and $-\tan\alpha$.

6. Given that $\sin\theta=\sin\alpha$, shew that the values of $\sin\frac{\theta}{3}$ are

$$\sin\frac{\alpha}{3},\quad \sin\frac{\pi-\alpha}{3},\quad -\sin\frac{\pi+\alpha}{3}.$$

7. If $\tan\theta=\tan\alpha$, shew that the values of $\tan\frac{\theta}{3}$ are

$$\tan\frac{\alpha}{3},\quad \tan\frac{\pi+\alpha}{3},\quad -\tan\frac{\pi-\alpha}{3}.$$

8. Given that $\cos 3\theta=\cos 3\alpha$, shew that the values of $\sin\theta$ are

$$\pm\sin\alpha,\quad -\sin\left(\frac{\pi}{3}\pm\alpha\right),\quad \sin\left(\frac{2\pi}{3}\pm\alpha\right).$$

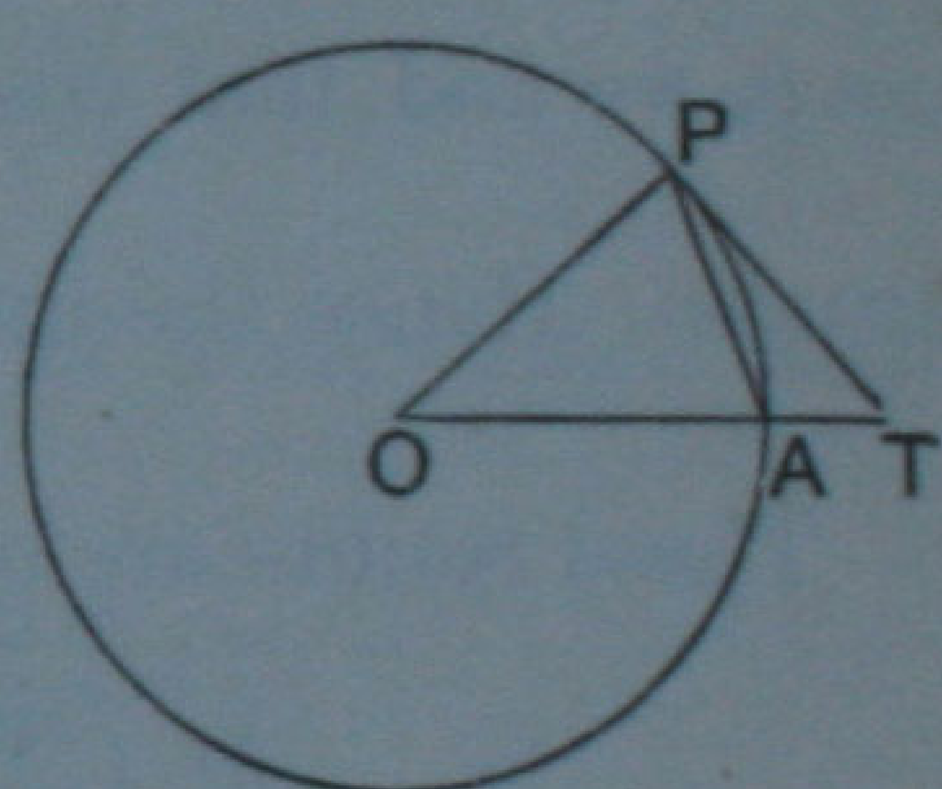
9. Given that $\sin 3\theta=\sin 3\alpha$, shew that the values of $\cos\theta$ are

$$\pm\cos\alpha,\quad \cos\left(\frac{\pi}{3}\pm\alpha\right),\quad \cos\left(\frac{2\pi}{3}\pm\alpha\right).$$

CHAPTER XXI.

LIMITS AND APPROXIMATIONS.

265. *If θ be the radian measure of an angle less than a right angle, to shew that* $\sin\theta$, θ, $\tan\theta$ *are in ascending order of magnitude.*



Let the angle θ be represented by AOP.

With centre O and radius OA describe a circle. Draw PT at right angles to OP to meet OA produced in T, and join PA.

Let r be the radius of the circle.

Area of $\Delta AOP = \frac{1}{2} AO \,.\, OP \sin AOP = \frac{1}{2} r^2 \sin\theta$;

area of sector $AOP = \frac{1}{2} r^2\theta$;

area of $\Delta OPT = \frac{1}{2} OP \,.\, PT = \frac{1}{2} r \,.\, r\tan\theta = \frac{1}{2} r^2 \tan\theta$.

But the areas of the triangle AOP, the sector AOP, and the triangle OTP are in ascending order of magnitude; that is,

$$\frac{1}{2} r^2 \sin\theta, \quad \frac{1}{2} r^2\theta, \quad \frac{1}{2} r^2 \tan\theta$$

are in ascending order of magnitude;

$\therefore$ $\sin\theta$, θ, $\tan\theta$ are in ascending order of magnitude.

266. *When θ is indefinitely diminished, to prove that $\frac{\sin\theta}{\theta}$ and $\frac{\tan\theta}{\theta}$ each have unity for their limit.*

In the last article, we have proved that $\sin\theta$, θ, $\tan\theta$ are in ascending order of magnitude. Divide each of these quantities by $\sin\theta$; then

$$1, \frac{\theta}{\sin\theta}, \frac{1}{\cos\theta} \text{ are in ascending order of magnitude;}$$

that is, $$\frac{\theta}{\sin\theta} \text{ lies between 1 and } \sec\theta.$$

But when θ is indefinitely diminished, the limit of $\sec\theta$ is 1; hence the limit of $\frac{\theta}{\sin\theta}$ is 1; that is, the limit of $\frac{\sin\theta}{\theta}$ is unity.

Again, by dividing each of the quantities $\sin\theta$, θ, $\tan\theta$ by $\tan\theta$, we find that $\cos\theta$, $\frac{\theta}{\tan\theta}$, 1 are in ascending order of magnitude. Hence the limit of $\frac{\tan\theta}{\theta}$ is unity.

These results are often written concisely in the forms

$$\underset{\theta=0}{Lt.}\left(\frac{\sin\theta}{\theta}\right)=1, \qquad \underset{\theta=0}{Lt.}\left(\frac{\tan\theta}{\theta}\right)=1.$$

Example. Find the limit of $n\sin\frac{\theta}{n}$ when $n=\infty$.

$$n\sin\frac{\theta}{n}=\theta\,.\,\frac{n}{\theta}\,.\,\sin\frac{\theta}{n}=\theta\left(\sin\frac{\theta}{n}\div\frac{\theta}{n}\right);$$

but since $\frac{\theta}{n}$ is indefinitely small, the limit of $\sin\frac{\theta}{n}\div\frac{\theta}{n}$ is unity;

$$\therefore \underset{n=\infty}{Lt.}\left(n\sin\frac{\theta}{n}\right)=\theta.$$

Similarly $$\underset{n=\infty}{Lt.}\left(n\tan\frac{\theta}{n}\right)=\theta.$$

267. It is important to remember that the conclusions of the foregoing articles only hold when the angle is expressed in radian measure. If any other system of measurement is used, the results will require modification.

Example. Find the value of $\underset{n=0}{Lt.}\left(\frac{\sin n^\circ}{n}\right)$.

Let θ be the number of radians in n°; then

$$\frac{n}{180}=\frac{\theta}{\pi}, \text{ and } n=\frac{180\theta}{\pi}; \text{ also } \sin n^\circ=\sin\theta;$$

$$\therefore \frac{\sin n^\circ}{n}=\frac{\pi\sin\theta}{180\theta}=\frac{\pi}{180}\cdot\frac{\sin\theta}{\theta}.$$

When n is indefinitely small, θ is indefinitely small;

$$\therefore \underset{n=0}{Lt.}\left(\frac{\sin n^\circ}{n}\right)=\frac{\pi}{180}\cdot\underset{\theta=0}{Lt.}\left(\frac{\sin\theta}{\theta}\right);$$

$$\therefore \underset{n=0}{Lt.}\left(\frac{\sin n^\circ}{n}\right)=\frac{\pi}{180}.$$

268. When θ is the radian measure of a very small angle, we have shewn that

$$\frac{\sin\theta}{\theta}=1, \quad \cos\theta=1, \quad \frac{\tan\theta}{\theta}=1;$$

that is, $\sin\theta=\theta, \quad \cos\theta=1, \quad \tan\theta=\theta.$

Hence $r\tan\theta=r\theta$, and therefore in the figure of Art. 265, the tangent PT is equal to the arc PA, when $\angle AOP$ is very small.

In Art. 270, it will be shewn that these results hold so long as θ is so small that its square may be neglected. When this is the case, we have

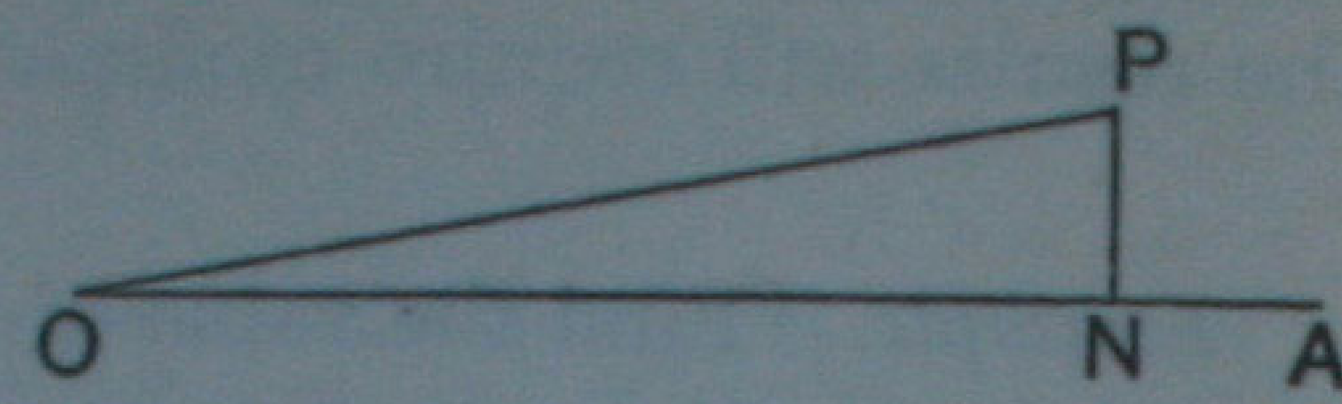
$$\sin(a+\theta)=\sin a\cos\theta+\cos a\sin\theta$$

$$=\sin a+\theta\cos a;$$

$$\cos(a+\theta)=\cos a\cos\theta-\sin a\sin\theta$$

$$=\cos a-\theta\sin a.$$

Example 1. The inclination of a railway to the horizontal plane is 52′ 30″, find how many feet it rises in a mile.



Let OA be the horizontal plane, and OP a mile of the railway. Draw PN perpendicular to OA.

Let $PN = x$ feet, $\angle PON = \theta$;

then $$\frac{PN}{OP} = \sin\theta = \theta \text{ approximately.}$$

But θ = radian measure of $52'\,30'' = \frac{52\frac{1}{2}}{60} \times \frac{\pi}{180} = \frac{7}{8} \times \frac{\pi}{180}$;

$$\therefore \frac{x}{1760 \times 3} = \frac{7}{8} \times \frac{22}{7} \times \frac{1}{180};$$

$$\therefore x = \frac{1760 \times 3 \times 22}{8 \times 180} = \frac{242}{3} = 80\tfrac{2}{3}.$$

Thus the rise is $80\frac{2}{3}$ feet.

Example 2. A pole 6 ft. long stands on the top of a tower 54 ft. high: find the angle subtended by the pole at a point on the ground which is at a distance of 180 yds. from the foot of the tower.

Let A be the point on the ground, BC the tower, CD the pole.

Let $\angle BAC = \alpha$, $\angle CAD = \theta$;

then $\tan\alpha = \frac{BC}{AB} = \frac{54}{540} = \frac{1}{10}$;

$$\tan(\alpha + \theta) = \frac{BD}{AB} = \frac{60}{540} = \frac{1}{9}.$$

But $$\tan(\alpha+\theta) = \frac{\tan\alpha + \tan\theta}{1 - \tan\alpha\tan\theta} = \frac{\tan\alpha + \theta}{1 - \theta\tan\alpha} \text{ approximately;}$$

$$\therefore \frac{1}{9} = \frac{\frac{1}{10} + \theta}{1 - \frac{\theta}{10}} = \frac{1 + 10\theta}{10 - \theta};$$

whence $\theta = \frac{1}{91}$; that is, the angle is $\frac{1}{91}$ of a radian, and therefore contains $\frac{1}{91} \times \frac{180}{\pi}$ degrees.

On reduction, we find that the angle is 37′ 46″ nearly.

269. *If θ be the number of radians in an acute angle, to prove that*

$$\cos\theta > 1 - \frac{\theta^2}{2}, \text{ and } \sin\theta > \theta - \frac{\theta^3}{4}.$$

Since $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$, and $\sin\frac{\theta}{2} < \frac{\theta}{2}$;

$$\therefore \cos\theta > 1 - 2\left(\frac{\theta}{2}\right)^2;$$

that is, $$\cos\theta > 1 - \frac{\theta^2}{2}.$$

Again, $$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2};$$

but $$\tan\frac{\theta}{2} > \frac{\theta}{2};$$

$$\therefore \sin\theta > 2\frac{\theta}{2}\cos^2\frac{\theta}{2};$$

$$\therefore \sin\theta > \theta\left(1 - \sin^2\frac{\theta}{2}\right).$$

But $\sin\frac{\theta}{2} < \frac{\theta}{2}$, and therefore

$$1 - \sin^2\frac{\theta}{2} > 1 - \left(\frac{\theta}{2}\right)^2;$$

$$\therefore \sin\theta > \theta\left\{1 - \left(\frac{\theta}{2}\right)^2\right\};$$

$$\therefore \sin\theta > \theta - \frac{\theta^3}{4}.$$

270. From the propositions established in this chapter, it follows that if θ is an acute angle,

$$\cos\theta \text{ lies between } 1 \text{ and } 1 - \frac{\theta^2}{2},$$

and $$\sin\theta \text{ lies between } \theta \text{ and } \theta - \frac{\theta^3}{4}.$$

Thus $\cos\theta = 1 - k\theta^2$ and $\sin\theta = \theta - k'\theta^3$, where k and k' are proper fractions less than $\frac{1}{2}$ and $\frac{1}{4}$ respectively.

Hence if θ be so small that its square can be neglected,

$$\cos\theta = 1, \quad \sin\theta = \theta.$$

Example. Find the approximate value of sin 10″.

The circular measure of 10″ is $\frac{10\pi}{180 \times 60 \times 60}$ or $\frac{\pi}{64800}$;

$$\therefore \sin 10'' < \frac{\pi}{64800} \text{ and } > \frac{\pi}{64800} - \frac{1}{4}\left(\frac{\pi}{64800}\right)^3.$$

But $$\frac{\pi}{64800} = \frac{3{\cdot}1415926535\ldots}{64800} = {\cdot}000048481368\ldots\ldots;$$

$$\therefore \frac{\pi}{64800} < {\cdot}00005 \text{ and } \left(\frac{\pi}{64800}\right)^3 < {\cdot}000000000000125;$$

$$\therefore \sin 10'' < \frac{\pi}{64800} \text{ and } > \frac{\pi}{64800} - \frac{1}{4}({\cdot}000000000000125).$$

Hence to 12 places of decimals,

$$\sin 10'' = \frac{\pi}{64800} = {\cdot}000048481368\ldots.$$

271. *To shew that when* n *is an indefinitely large integer, the limit of*

$$\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\ldots\cos\frac{\theta}{2^n} = \frac{\sin\theta}{\theta}.$$

We have
$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$
$$= 2^2\sin\frac{\theta}{4}\cos\frac{\theta}{4}\cos\frac{\theta}{2}$$
$$= 2^3\sin\frac{\theta}{8}\cos\frac{\theta}{8}\cos\frac{\theta}{4}\cos\frac{\theta}{2}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$= 2^n\sin\frac{\theta}{2^n}\cos\frac{\theta}{2^n}\ldots\cos\frac{\theta}{8}\cos\frac{\theta}{4}\cos\frac{\theta}{2}.$$

$$\therefore \cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\ldots\cos\frac{\theta}{2^n} = \frac{\sin\theta}{2^n\sin\frac{\theta}{2^n}}.$$

But the limit of $2^n\sin\frac{\theta}{2^n}$ is θ, and thus the proposition is established. [See Art. 266.]

272. *To shew that* $\frac{\sin\theta}{\theta}$ *continually decreases from* 1 *to* $\frac{2}{\pi}$ *as* θ *continually increases from* 0 *to* $\frac{\pi}{2}$.

We shall first shew that the fraction

$$\frac{\sin\theta}{\theta} - \frac{\sin(\theta+h)}{\theta+h} \text{ is positive,}$$

h denoting the radian measure of a small positive angle.

$$\text{This fraction} = \frac{(\theta+h)\sin\theta - \theta(\sin\theta\cos h + \cos\theta\sin h)}{\theta(\theta+h)}$$

$$= \frac{\theta\sin\theta(1-\cos h) + (h\sin\theta - \theta\cos\theta\sin h)}{\theta(\theta+h)}.$$

Now $\tan\theta > \theta$, that is $\sin\theta > \theta\cos\theta$, and $h > \sin h$;

$$\therefore\ h\sin\theta > \theta\cos\theta\sin h.$$

Also $1-\cos h$ is positive; hence the numerator is positive, and therefore the fraction is positive;

$$\therefore\ \frac{\sin(\theta+h)}{\theta+h} < \frac{\sin\theta}{\theta};$$

$\therefore\ \frac{\sin\theta}{\theta}$ continually decreases as θ continually increases.

When $\theta = 0$, $\frac{\sin\theta}{\theta} = 1$; and when $\theta = \frac{\pi}{2}$, $\frac{\sin\theta}{\theta} = \frac{2}{\pi}$.

Thus the proposition is established.

EXAMPLES. XXI. a.

[*In this Exercise take* $\pi = \frac{22}{7}$.]

1. A tower 44 feet high subtends an angle of $35'$ at a point A on the ground: find the distance of A from the tower.

2. From the top of a wall 7 ft. 4 in. high the angle of depression of an object on the ground is $24'\,30''$: find its distance from the wall.

3. Find the height of an object whose angle of elevation at a distance of 840 yards is $1^\circ\ 30'$.

4. Find the angle subtended by a pole 10 ft. 1 in. high at a distance of a mile.

5. Find the angle subtended by a circular target 4 feet in diameter at a distance of 1000 yards.

6. Taking the diameter of a penny as 1·25 inches, find at what distance it must be held from the eye so as just to hide the moon, supposing the diameter of the moon to be half a degree.

7. Find the distance at which a globe 11 inches in diameter subtends an angle of $5'$.

8. Two places on the same meridian are 11 miles apart: find the difference in their latitudes, taking the radius of the earth as 3960 miles.

9. A man 6 ft. high stands on a tower whose height is 120 ft.: shew that at a point 24 ft. from the tower the man subtends an angle of $31{\cdot}5'$ nearly.

10. A flagstaff standing on the top of a cliff 490 feet high subtends an angle of ·04 radians at a point 980 feet from the base of the cliff: find the height of the flagstaff.

11. When $n=0$, find the limit of

$$(1)\quad \frac{\sin n'}{n};\qquad (2)\quad \frac{\sin n''}{n}.$$

12. When $n=\infty$, find the limit of $\frac{1}{2}nr^2\sin\frac{2\pi}{n}$.

When $\theta=0$, find the limit of

13. $\dfrac{1-\cos\theta}{\theta\sin\theta}$.

14. $\dfrac{m\sin m\theta - n\sin n\theta}{\tan m\theta + \tan n\theta}$.

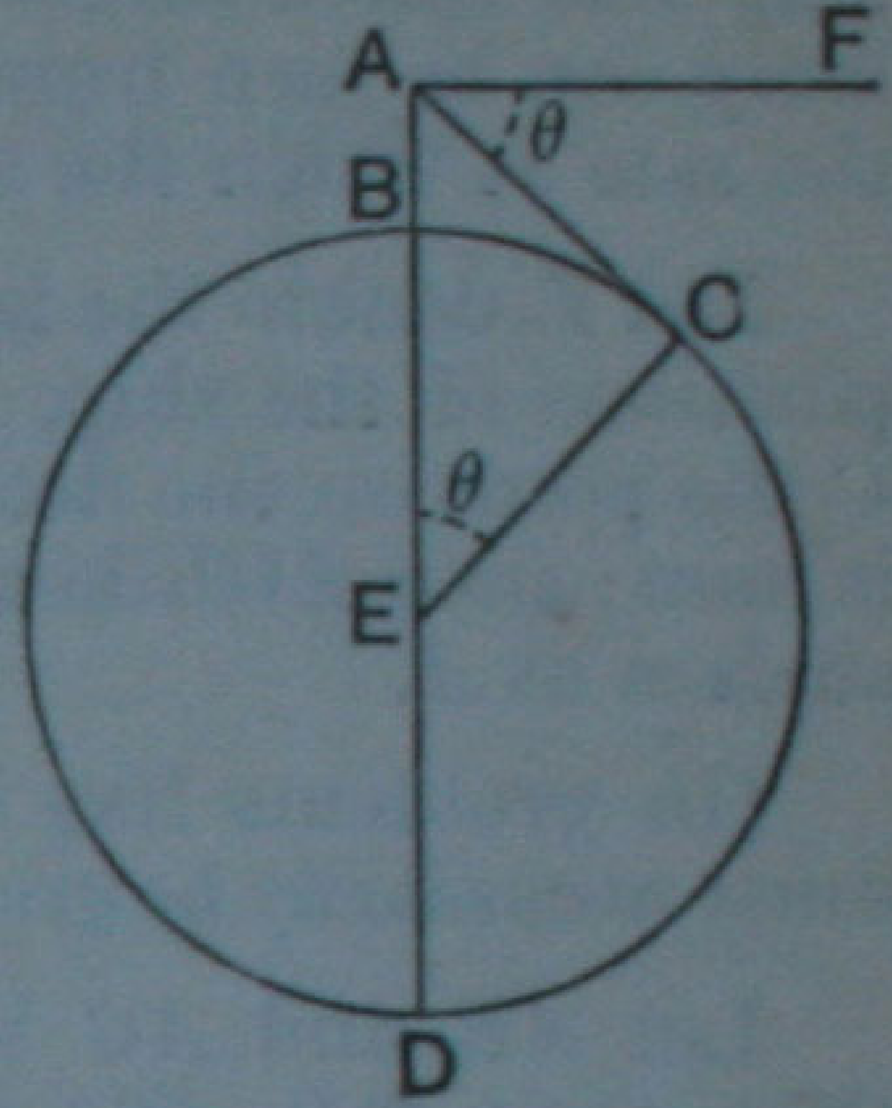
15. If $\theta = {\cdot}01$ of a radian, calculate $\cos\left(\frac{\pi}{3}+\theta\right)$.

16. Find the value of $\sin 30^\circ\ 10'\ 30''$.

17. Given $\cos\left(\frac{\pi}{3}+\theta\right) = {\cdot}49$, find the sexagesimal value of θ.

Distance and Dip of the Visible Horizon.

273. Let A be a point above the earth's surface, BCD a section of the earth by a plane passing through its centre E and A.



Let AE cut the circumference in B and D.

From A draw AC to touch the circle BCD in C, and join EC.

Draw AF at right angles to AD; then $\angle FAC$ is called the **dip of the horizon** as seen from A.

Thus the *dip of the horizon* is the angle of depression of any point on the horizon visible from A.

274. *To find the distance of the horizon.*

In the figure of the last article, let

$$AB=h, \quad EB=ED=r, \quad AC=x;$$

then by Euc. III. 36, $\quad AC^2=AB \,.\, AD$;

that is, $\quad x^2=h(2r+h)=2hr+h^2.$

For ordinary altitudes h^2 is very small in comparison with $2hr$; hence approximately

$$x^2=2hr \quad \text{and} \quad x=\sqrt{2hr}.$$

In this formula, suppose the measurements are made in *miles*, and let a be the number of *feet* in AB; then

$$a=1760\times 3\times h.$$

By taking $r=3960$, we have

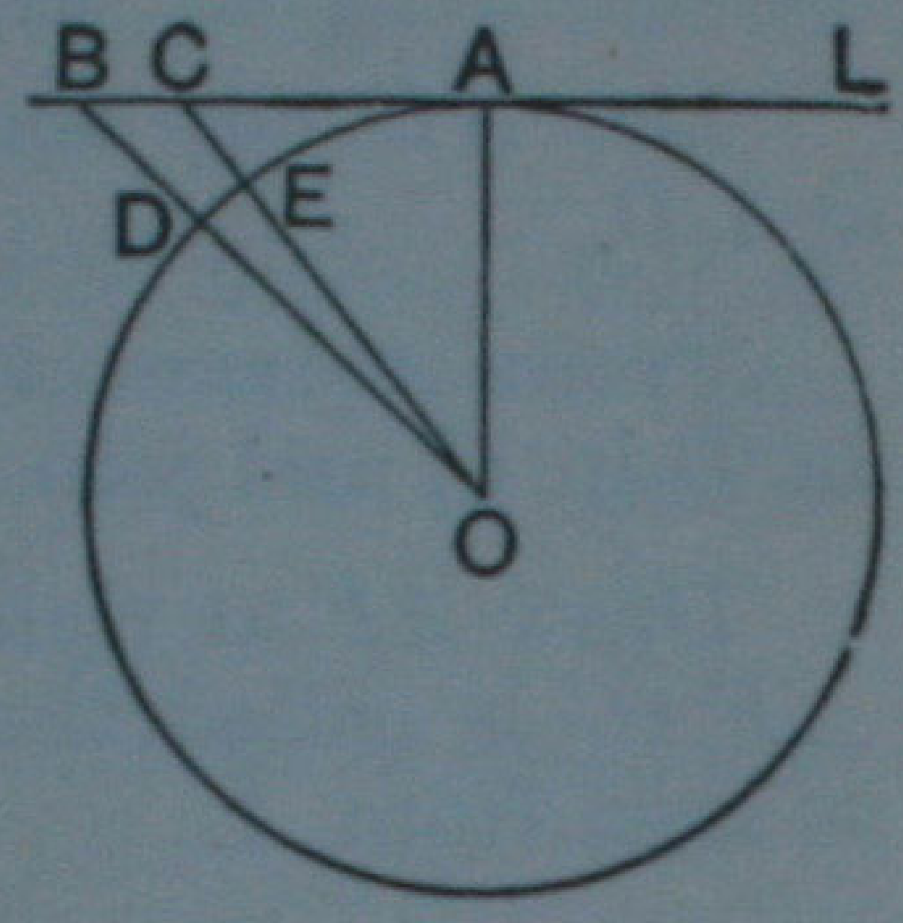
$$x^2=\frac{2\times 3960\times a}{1760\times 3}=\frac{3a}{2}.$$

Thus we have the following rule:

Twice the square of the distance of the horizon measured in miles *is equal to three times the height of the place of observation measured in* feet.

Hence a man whose eye is 6 feet from the ground can see to a distance of 3 miles on a horizontal plane.

Example. The top of a ship's mast is $66\frac{2}{3}$ ft. above the sea-level, and from it the lamp of a lighthouse can just be seen. After the ship has sailed directly towards the lighthouse for half-an-hour the lamp can be seen from the deck, which is 24 ft. above the sea. Find the rate at which the ship is sailing.



Let L denote the lamp, D and E the two positions of the ship, B the top of the mast, C the point on the deck from which the lamp is seen; then LCB is a tangent to the earth's surface at A.

[In problems like this some of the lines must necessarily be greatly out of proportion.]

Let AB and AC be expressed in miles; then since $DB = 66\frac{2}{3}$ feet and $EC = 24$ feet, we have by the rule

$$AB^2 = \frac{3}{2} \times 66\tfrac{2}{3} = 100;$$

$$\therefore\ AB = 10 \text{ miles.}$$

$$AC^2 = \frac{3}{2} \times 24 = 36;$$

$$\therefore\ AC = 6 \text{ miles.}$$

But the angles subtended by AB and AC at O the centre of the earth are very small;

$$\therefore \text{ arc } AD = AB, \text{ and arc } AC = AE. \qquad \text{[Art. 268.]}$$

$$\therefore \text{ arc } DE = AD - AE = AB - AC = 4 \text{ miles.}$$

Thus the ship sails 4 miles in half-an-hour, or 8 miles per hour.

275. Let θ be the number of radians in the dip of the horizon; then with the figure of Art. 273, we have

$$\cos\theta = \frac{EC}{EA} = \frac{r}{h+r} = \left(1 + \frac{h}{r}\right)^{-1};$$

$$\therefore\ 1 - 2\sin^2\frac{\theta}{2} = 1 - \frac{h}{r} + \frac{h^2}{r^2} - \dots;$$

$$\therefore\ 2\sin^2\frac{\theta}{2} = \frac{h}{r} - \frac{h^2}{r^2} + \dots.$$

Since θ and $\frac{h}{r}$ are small, we may replace $\sin\frac{\theta}{2}$ by $\frac{\theta}{2}$ and neglect the terms on the right after the first.

Thus $$\frac{\theta^2}{2}=\frac{h}{r}, \text{ or } \theta=\sqrt{\frac{2h}{r}}.$$

Let N be the number of degrees in θ radians; then

$$N=\frac{180\theta}{\pi}=\frac{180}{\pi}\sqrt{\frac{2h}{r}}.$$

Now $\sqrt{r}=63$ nearly; hence we have approximately

$$N=\frac{180\times 7\times\sqrt{2h}}{22\times 63},$$

or

$$N=\frac{10}{11}\sqrt{2h},$$

a formula connecting the dip of the horizon in degrees and the height of the place of observation in miles.

EXAMPLES. XXI. b.

[Here $\pi=\frac{22}{7}$, *and radius of earth* = 3960 *miles.*]

1. Find the greatest distance at which the lamp of a lighthouse can be seen, the light being 96 feet above the sea-level.

2. If the lamp of a lighthouse begins to be seen at a distance of 15 miles, find its height above the sea-level.

3. The tops of the masts of two ships are 32 ft. 8 in. and 42 ft. 8 in. above the sea-level: find the greatest distance at which one mast can be seen from the other.

4. Find the height of a ship's mast which is just visible at a distance of 20 miles from a point on the mast of another ship which is 54 ft. above the sea-level.

5. From the mast of a ship 73 ft. 6 in. high the lamp of a lighthouse is just visible at a distance of 28 miles: find the height of the lamp.

6. Find the sexagesimal measure of the dip of the horizon from a hill 2640 feet high.

7. Along a straight coast there are lighthouses at intervals of 24 miles: find at what height the lamp must be placed so that the light of one at least may be visible at a distance of $3\frac{1}{2}$ miles from any point of the coast.

8. From the top of a mountain the dip of the horizon is $1{\cdot}\dot{8}\dot{1}^\circ$: find its height in feet.

9. The distance of the horizon as seen from the top of a hill is 30·25 miles: find the height of the hill and the dip of the horizon.

10. If x miles be the distance of the visible horizon and N degrees the dip, shew that

$$N = \frac{x}{66}\sqrt{\frac{10}{11}}.$$

When $\theta = 0$, find the limit of

11. $\dfrac{\sin 4\theta \cot \theta}{\text{vers } 2\theta \cot^2 2\theta}$. 12. $\dfrac{1-\cos\theta+\sin\theta}{1-\cos\theta-\sin\theta}$.

13. When $\theta = a$, find the limit of

(1) $\dfrac{\sin\theta - \sin a}{\theta - a}$; (2) $\dfrac{\cos\theta - \cos a}{\theta - a}$.

14. Two sides of a triangle are 31 and 32, and they include a right angle: find the other angles.

15. A person walks directly towards a distant object P, and observes that at the three points A, B, C, the elevations of P are a, $2a$, $3a$ respectively: shew that $AB = 3BC$ nearly.

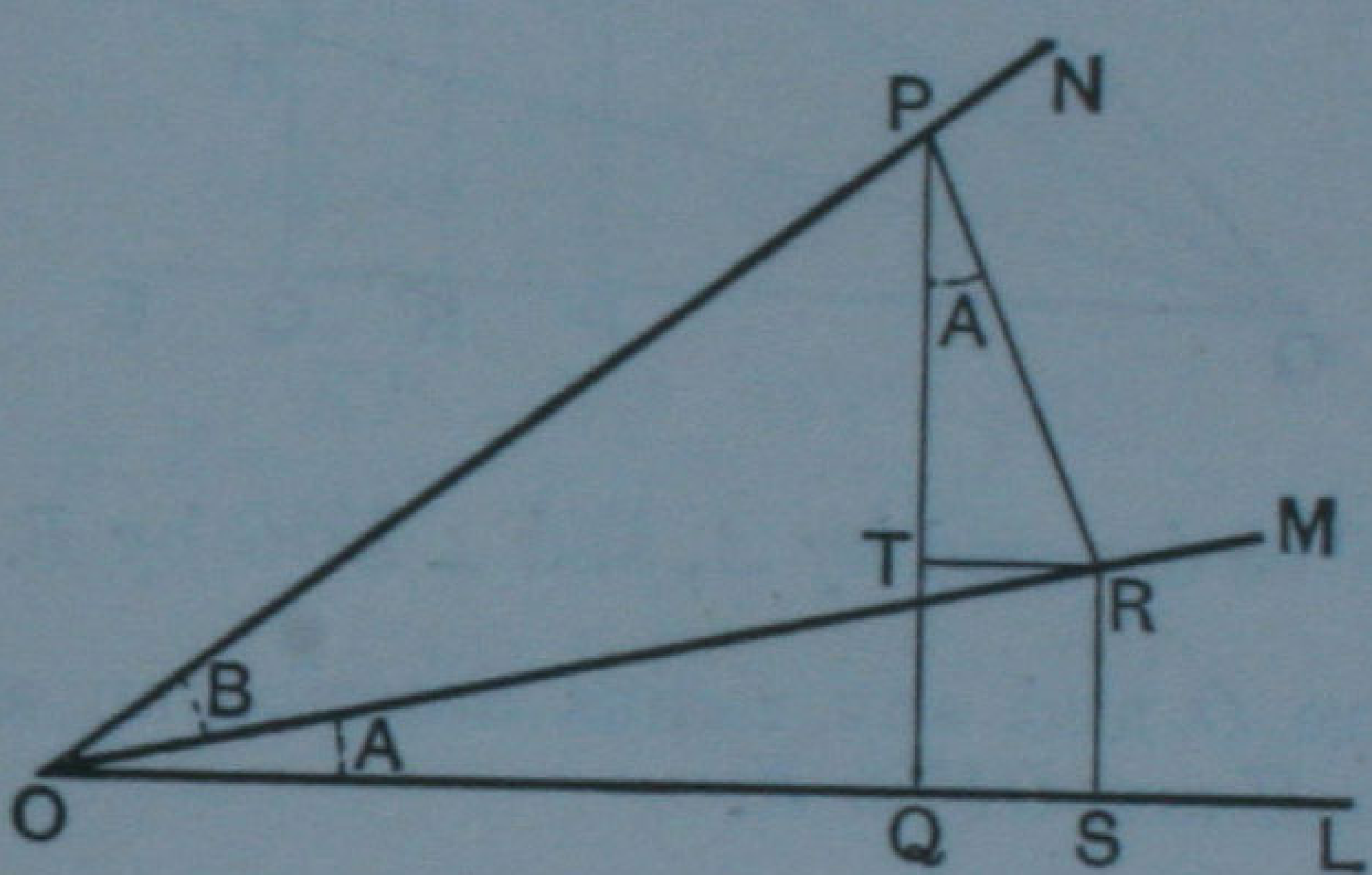
16. Shew that $\dfrac{\tan\theta}{\theta}$ continually increases from 1 to ∞ as θ continually increases from 0 to $\dfrac{\pi}{2}$.

CHAPTER XXII.

GEOMETRICAL PROOFS.

276. *To find the expansion of* $\tan(A+B)$ *geometrically.*

Let $\angle LOM = A$, and $\angle MON = B$; then $\angle LON = A+B$.



In ON take any point P, and draw PQ and PR perpendicular to OL and OM respectively. Also draw RS and RT perpendicular to OL and PQ respectively.

$$\tan(A+B) = \frac{PQ}{OQ} = \frac{RS+PT}{OS-TR}$$

$$= \frac{\frac{RS}{OS}+\frac{PT}{OS}}{1-\frac{TR}{OS}} = \frac{\frac{RS}{OS}+\frac{PT}{OS}}{1-\frac{TR}{TP}\cdot\frac{TP}{OS}}.$$

Now $$\frac{RS}{OS} = \tan A, \text{ and } \frac{TR}{TP} = \tan A;$$

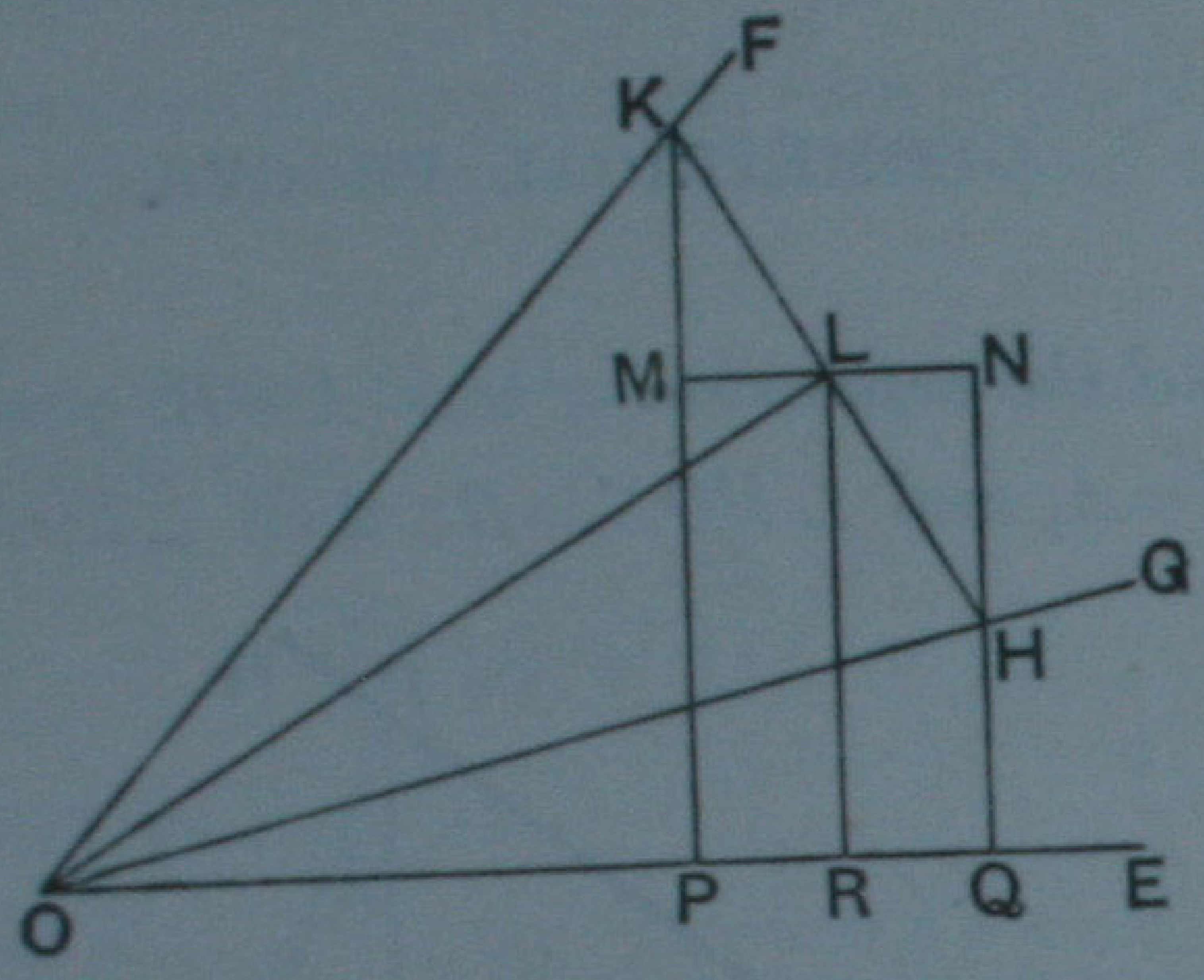
also the triangles ROS and TPR are similar, and therefore

$$\frac{TP}{OS} = \frac{PR}{OR} = \tan B.$$

$$\therefore \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

In like manner, with the help of the figure on page 95, we may obtain the expansion of $\tan(A-B)$ geometrically.

277. *To prove geometrically the formulæ for transformation of sums into products.*



Let $\angle EOF$ be denoted by A, and $\angle EOG$ by B.

With centre O and any radius describe an arc of a circle meeting OG in H and OF in K.

Bisect $\angle KOH$ by OL; then OL bisects HK at right angles.

Draw KP, HQ, LR perpendicular to OE, and through L draw MLN parallel to OE meeting KP in M and QH in N.

It is easy to prove that the triangles MKL and NHL are equal in all respects, so that $KM=NH$, $ML=LN$, $PR=RQ$.

Also $\angle GOF=A-B$, and therefore

$$\angle HOL=\angle KOL=\frac{A-B}{2};$$

$$\therefore \quad \angle EOL=B+\frac{A-B}{2}=\frac{A+B}{2}.$$

$$\sin A+\sin B=\frac{KP}{OK}+\frac{HQ}{OH}=\frac{KP+HQ}{OK}$$

$$=\frac{(KM+LR)+(LR-NH)}{OK}=2\,\frac{LR}{OK};$$

$$\therefore\ \sin A+\sin B=2\frac{LR}{OL}\cdot\frac{OL}{OK}=2\sin ROL\cos KOL$$

$$=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}.$$

$$\cos A+\cos B=\frac{OP}{OK}+\frac{OQ}{OH}=\frac{OP+OQ}{OK}$$

$$=\frac{(OR-PR)+(OR+RQ)}{OK}=2\frac{OR}{OK}$$

$$=2\frac{OR}{OL}\cdot\frac{OL}{OK}=2\cos ROL\cos KOL$$

$$=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}.$$

$$\sin A-\sin B=\frac{KP}{OK}-\frac{HQ}{OH}=\frac{KP-HQ}{OK}$$

$$=\frac{(KM+LR)-(LR-NH)}{OK}=2\frac{KM}{OK}$$

$$=2\frac{KM}{KL}\cdot\frac{KL}{OK}=2\cos LKM\sin KOL$$

$$=2\cos\frac{A+B}{2}\sin\frac{A-B}{2},$$

since $\angle LKM=$ comp^t of $\angle KLM=\angle MLO=\angle LOE=\dfrac{A+B}{2}$.

$$\cos B-\cos A=\frac{OQ}{OH}-\frac{OP}{OK}=\frac{OQ-OP}{OK}$$

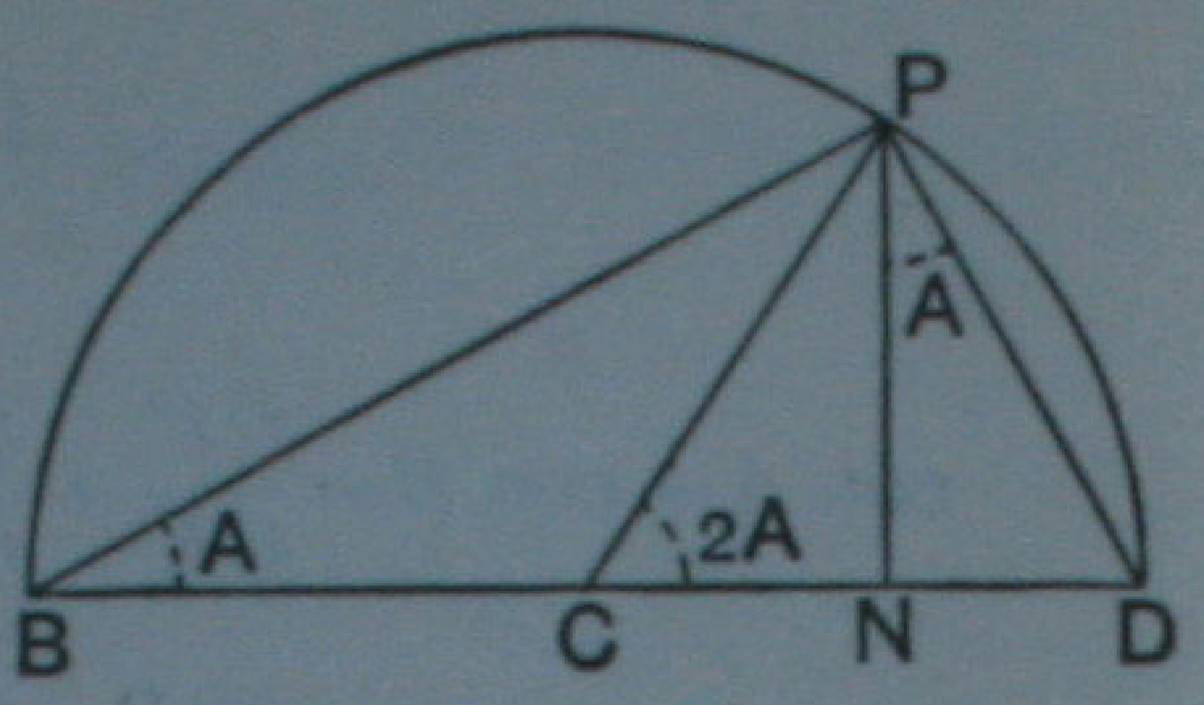
$$=\frac{(OR+RQ)-(OR-PR)}{OK}=2\frac{PR}{OK}=2\frac{ML}{OK}$$

$$=2\frac{ML}{KL}\cdot\frac{KL}{OK}=2\sin LKM\sin KOL$$

$$=2\sin\frac{A+B}{2}\sin\frac{A-B}{2}.$$

278. *Geometrical proof of the 2A formulæ.*

Let BPD be a semicircle, BD the diameter, C the centre.



On the circumference, take any point P, and join PB, PC, PD.

Draw PN perpendicular to BD.

Let $\angle PBD = A$, then

$$\angle PCD = 2A.$$

And $\angle NPD = \text{comp}^{t}$ of $\angle PDN = \angle PBD = A$.

$$\sin 2A = \frac{PN}{CP} = \frac{2PN}{2CP} = \frac{2PN}{BD} = 2\frac{PN}{BP} \cdot \frac{BP}{BD}$$

$$= 2 \sin PBN \cos PBD$$

$$= 2 \sin A \cos A.$$

$$\cos 2A = \frac{CN}{CP} = \frac{2CN}{BD} = \frac{CN + CN}{BD}$$

$$= \frac{(BN - BC) + (CD - ND)}{BD} = \frac{BN - ND}{BD}$$

$$= \frac{BN}{BP} \cdot \frac{BP}{BD} - \frac{ND}{PD} \cdot \frac{PD}{BD}$$

$$= \cos A . \cos A - \sin A . \sin A$$

$$= \cos^2 A - \sin^2 A.$$

$$\cos 2A = \frac{CN}{CP} = \frac{CD - DN}{CP} = 1 - \frac{DN}{CP} = 1 - \frac{2DN}{BD}$$

$$= 1 - 2\frac{DN}{DP} \cdot \frac{DP}{BD} = 1 - 2 \sin A . \sin A$$

$$= 1 - 2 \sin^2 A.$$

$$\cos 2A = \frac{CN}{CP} = \frac{BN - BC}{CP} = \frac{BN}{CP} - 1 = \frac{2BN}{BD} - 1$$

$$= 2\frac{BN}{BP} \cdot \frac{BP}{BD} - 1 = 2 \cos A . \cos A - 1$$

$$= 2 \cos^2 A - 1.$$

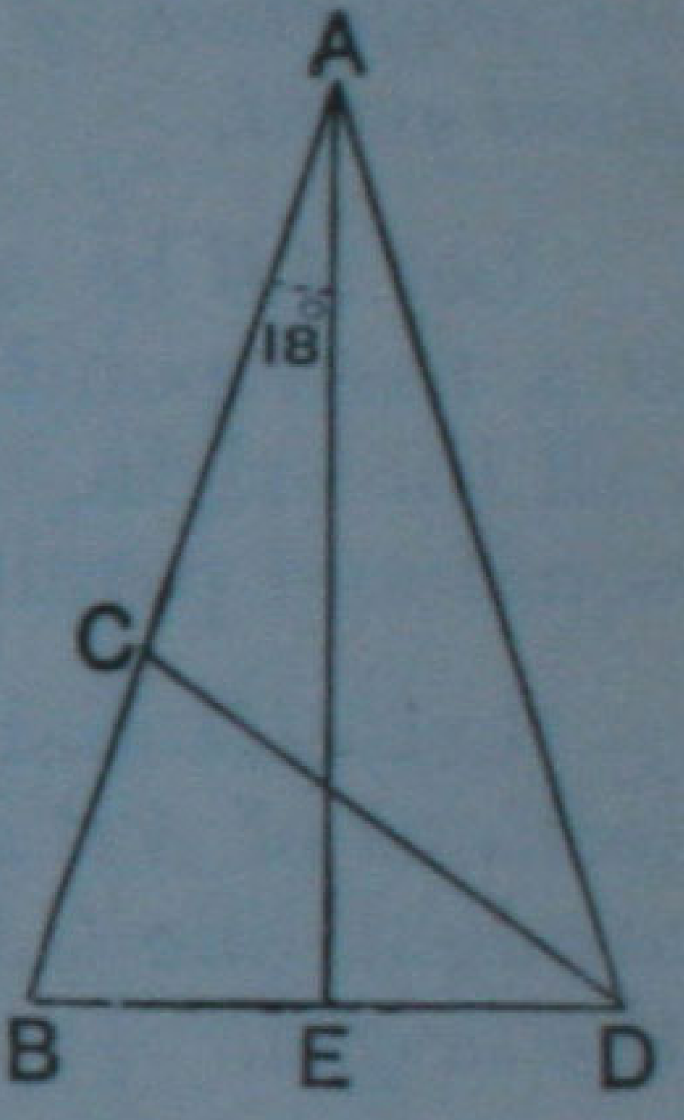
$$\tan 2A = \frac{PN}{CN} = \frac{2PN}{2CN} = \frac{2PN}{BN - ND}$$

$$= \frac{2\dfrac{PN}{BN}}{1 - \dfrac{ND}{BN}} = \frac{2\dfrac{PN}{BN}}{1 - \dfrac{ND}{PN} \cdot \dfrac{PN}{BN}}$$

$$= \frac{2\tan A}{1 - \tan A \,.\, \tan A}$$

$$= \frac{2\tan A}{1 - \tan^2 A}.$$

279. *To find the value of* sin 18° *geometrically.*



Let ABD be an isosceles triangle in which each angle at the base BD is double the vertical angle A; then

$$A + 2A + 2A = 180°,$$

and therefore $A = 36°$.

Bisect $\angle BAD$ by AE; then AE bisects BD at right angles;

$$\therefore \angle BAE = 18°.$$

Thus $$\sin 18° = \frac{BE}{AB} = \frac{x}{a},$$

where $AB = a$, and $BE = x$.

From the construction given in Euc. IV. 10,

$$AC = BD = 2BE = 2x,$$

and $$AB \,.\, BC = AC^2;$$

$$\therefore a(a - 2x) = (2x)^2;$$

$$\therefore 4x^2 + 2ax - a^2 = 0;$$

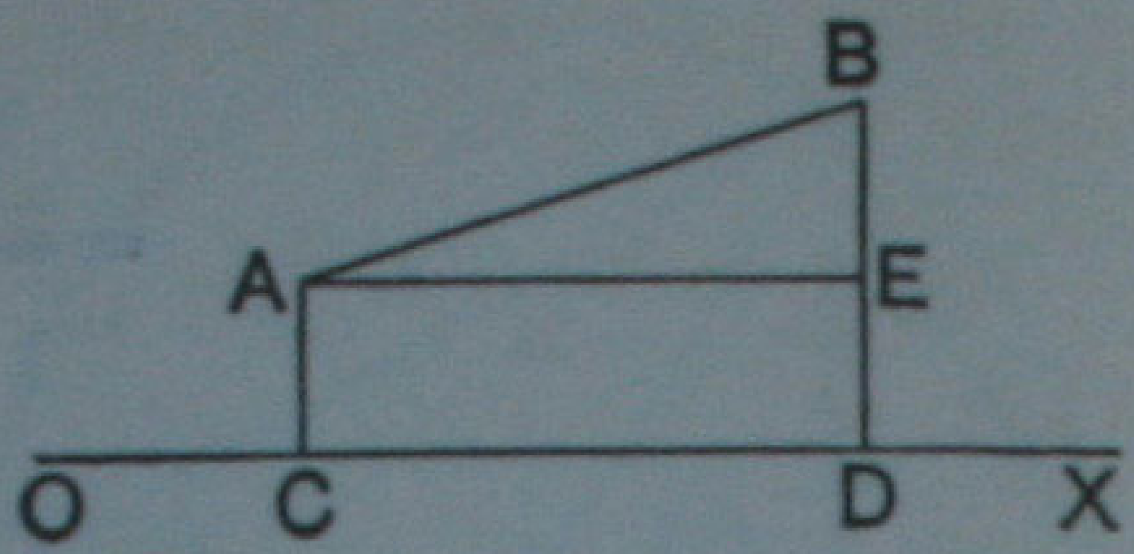
$$\therefore x = \frac{-2a \pm \sqrt{20a^2}}{8} = \frac{-1 \pm \sqrt{5}}{4}a.$$

The upper sign must be taken, since x is positive. Thus

$$\sin 18° = \frac{\sqrt{5} - 1}{4}.$$

Proofs by Projection.

280. Definition. If from any two points A and B, lines AC and BD are drawn perpendicular to OX, then the intercept CD is called the **projection** of AB upon OX.



Through A draw AE parallel to OX; then

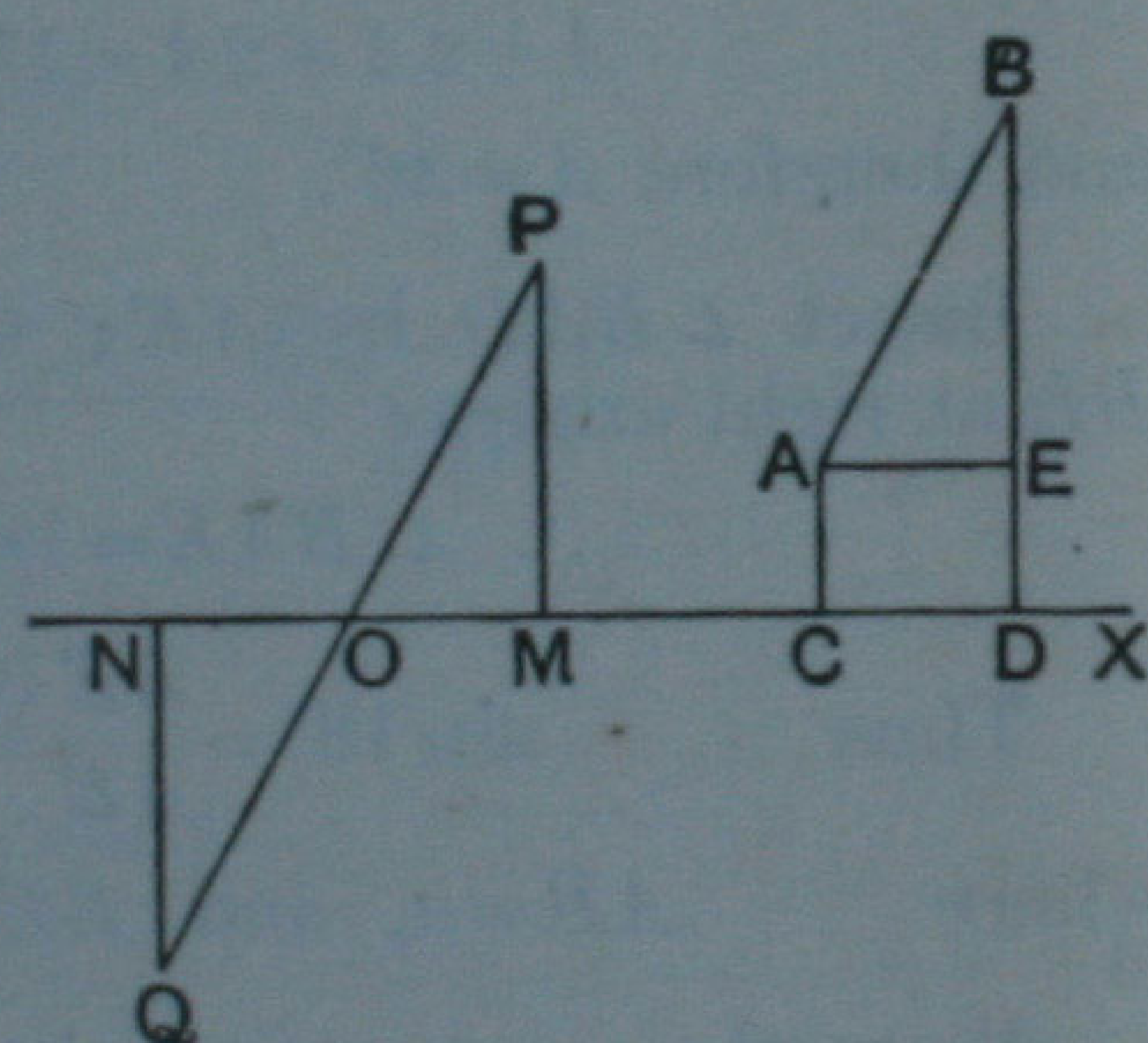
$$CD = AE = AB \cos BAE;$$

that is, $$CD = AB \cos \alpha,$$

where α is the angle of inclination of the lines AB and OX.

281. *To shew that the projection of a straight line is equal to the projection of an equal and parallel straight line drawn from a fixed point.*

Let AB be any straight line, O a fixed point, which we shall call the origin, OP a straight line equal and parallel to AB.



Let CD and OM be the projections of AB and OP upon any straight line OX drawn through the origin.

Draw AE parallel to OX.

The two triangles AEB and OMP are identically equal;

$$\therefore \; OM = AE = CD;$$

that is, projection of OP = projection of AB.

282. In the figure of the last article, *two* straight lines OP and OQ can be drawn from O equal and parallel to AB; it is therefore necessary to have some means of fixing the *direction* in which the line from O is to be drawn. Accordingly it is agreed to consider that

the direction of a line is fixed by the order of the letters.

Thus AB denotes a line drawn from A to B, and BA denotes a line drawn from B to A.

Hence OP denotes a line drawn from the origin parallel to AB, and OQ denotes a line drawn from the origin parallel to BA.

Similarly the direction of a projected line is fixed by the order of the letters.

Thus CD is drawn to the right from C to D and is positive, while DC is drawn to the left from D to C and is negative.

Hence in sign as well as in magnitude

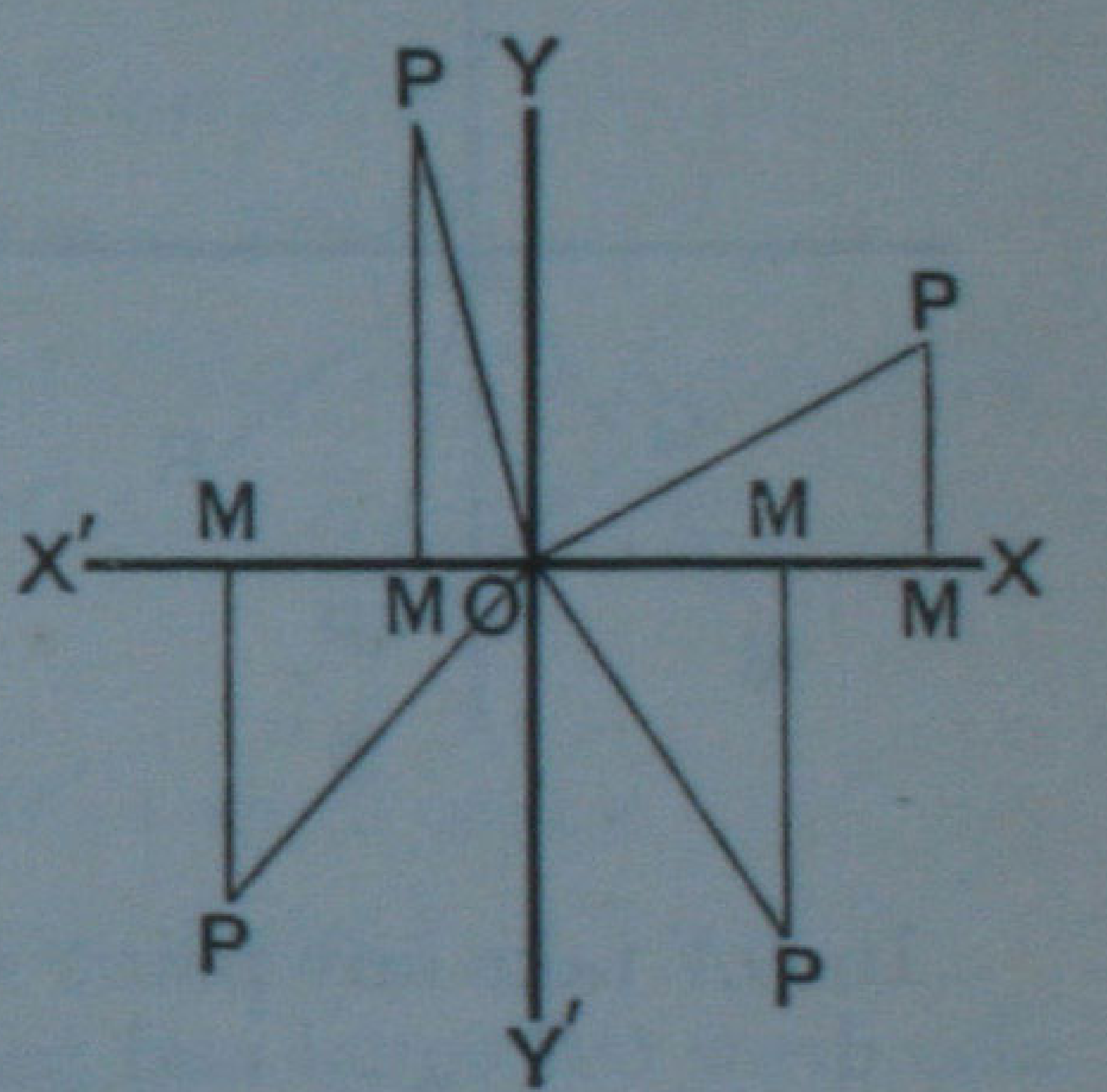
$$OM = CD, \text{ and } ON = DC;$$

that is, projection of OP = projection of AB,

and projection of OQ = projection of BA.

Thus *the projection of a straight line can be represented both in sign and magnitude by the projection of an equal and parallel straight line drawn from the origin.*

283. Whatever be the direction of AB, the line OP will fall within one of the four quadrants.



Also from the definitions given in Art. 75, we have

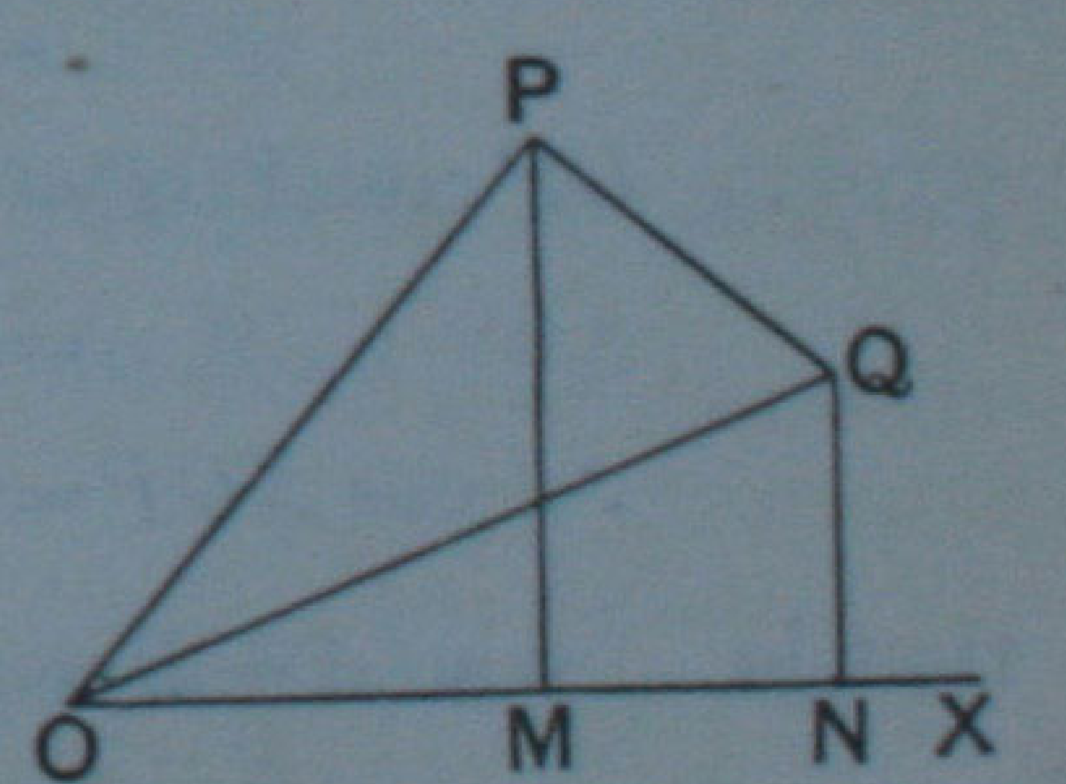
$$\frac{OM}{OP} = \cos XOP,$$

that is,

$$OM = OP \cos XOP,$$

whatever be the magnitude of the angle XOP. We shall always suppose, unless the contrary is stated, that the angles are measured in the positive direction.

284. Let O be the origin, P and Q any two points.



Join OP, OQ, PQ, and draw PM and QN perpendicular to OX.

We have

$$OM = ON + NM,$$

since the line NM is to be regarded as negative; that is,

the projection of OP = projection of OQ + projection of QP.

Hence, *the projection of one side of a triangle is equal to the sum of the projections of the other two sides taken in order.* Thus

projection of OQ = projection of OP + projection of PQ;

projection of QP = projection of QO + projection of OP.

General Proof of the Addition Formulæ.

285. In Fig. 1, let a line starting from OX revolve until it has traced the angle A, taking up the position OM, and then let it further revolve until it has traced the angle B, taking up the final position ON. Thus XON is the angle $A+B$.

Fig. 1.

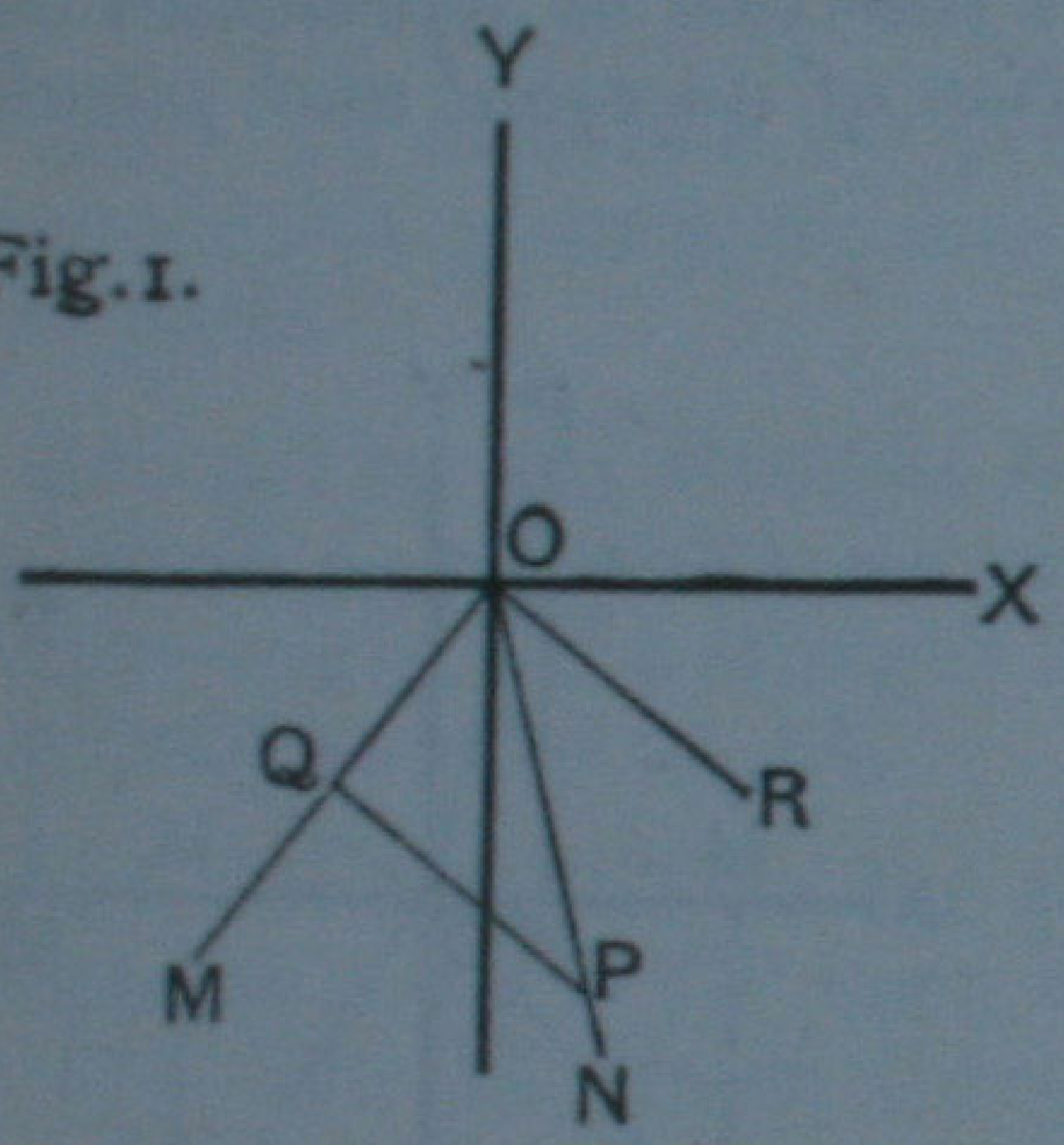
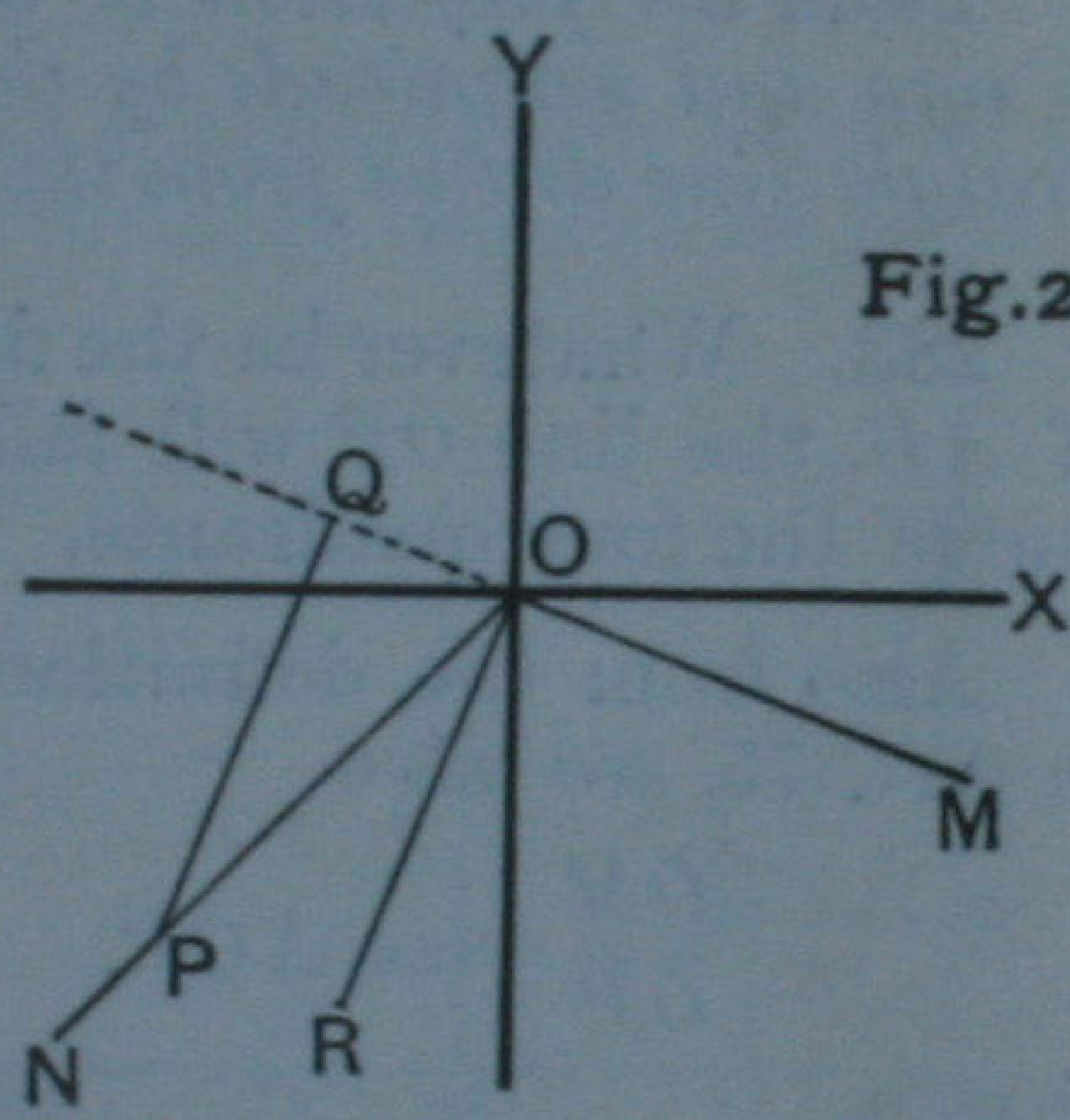


Fig. 2.



In ON take any point P, and draw PQ perpendicular to OM; also draw OR equal and parallel to QP.

Projecting upon OX, we have

projection of OP = projection of OQ + projection of QP

= projection of OQ + projection of OR.

$$\therefore \quad OP\cos XOP = OQ\cos XOQ + OR\cos XOR \text{ ....(1)}$$

$$= OP\cos B\cos XOQ + OP\sin B\cos XOR;$$

$$\therefore \quad \cos XOP = \cos B\cos XOQ + \sin B\cos XOR;$$

that is, $\cos(A+B) = \cos B\cos A + \sin B\cos(90^\circ + A)$

$$= \cos A\cos B - \sin A\sin B.$$

Projecting upon OY, we have only to write Y for X in (1);

thus $OP \cos YOP = OQ \cos YOQ + OR \cos YOR$

$$= OP \cos B \cos YOQ + OP \sin B \cos YOR;$$

$$\therefore \cos YOP = \cos B \cos YOQ + \sin B \cos YOR;$$

that is,

$$\cos(A + B - 90°) = \cos B \cos(A - 90°) + \sin B \cos A;$$

$$\therefore \sin(A + B) = \sin A \cos B + \cos A \sin B.$$

In Fig. 2, let a line starting from OX revolve until it has traced the angle A, taking up the position OM, and then let it revolve *back again* until it has traced the angle B, taking up the final position ON. Thus XON is the angle $A - B$.

In ON take any point P, and draw PQ perpendicular to MO produced; also draw OR equal and parallel to QP.

Projecting upon OX, we have as in the previous case

$$OP \cos XOP = OQ \cos XOQ + OR \cos XOR$$

$$= OP \cos(180° - B) \cos XOQ$$

$$+ OP \sin(180° - B) \cos XOR;$$

$$\therefore \cos XOP = -\cos B \cos XOQ + \sin B \cos XOR;$$

that is,

$$\cos(A - B) = -\cos B \cos(A - 180°) + \sin B \cos(A - 90°)$$

$$= -\cos B(-\cos A) + \sin B \sin A$$

$$= \cos A \cos B + \sin A \sin B.$$

Projecting upon OY, we have

$$OP \cos YOP = OQ \cos YOQ + OR \cos YOR;$$

$$= OP \cos(180° - B) \cos YOQ$$

$$+ OP \sin(180° - B) \cos YOR;$$

$$\therefore \cos YOP = -\cos B \cos YOQ + \sin B \cos YOR;$$

that is,

$$\cos(A - B - 90°) = -\cos B \cos(A - 270°) + \sin B \cos(A - 180°);$$

$$\therefore \sin(A - B) = -\cos B(-\sin A) + \sin B(-\cos A)$$

$$= \sin A \cos B - \cos A \sin B.$$

286. The above method of proof is applicable to every case, and therefore the Addition Formulæ are universally established.

The universal truth of the Addition Formulæ may also be deduced from the special geometrical investigations of Arts. 110 and 111 by analysis, as in the next article.

287. When each of the angles A, B, $A+B$ is less than 90°, we have shewn that

$$\cos(A+B)=\cos A\cos B-\sin A\sin B\text{............(1).}$$

But $\cos(A+B)=\sin(\overline{A+B}+90°)=\sin(\overline{A+90°}+B)$;

also $$\cos A=\sin(A+90°),$$

and $$-\sin A=\cos(A+90°).\quad\text{[Art. 98.]}$$

Hence by substitution in (1), we have

$$\sin(\overline{A+90°}+B)=\sin(A+90°)\cos B+\cos(A+90°)\sin B.$$

In like manner, it may be proved that

$$\cos(\overline{A+90°}+B)=\cos(A+90°)\cos B-\sin(A+90°)\sin B.$$

Thus the formulæ for the sine and cosine of $A+B$ hold when A is increased by 90°. Similarly we may shew that they hold when B is increased by 90°.

By repeated applications of the same process it may be proved that the formulæ are true when either or both of the angles A and B is increased by any multiple of 90°.

Again, $$\cos(A+B)=\cos A\cos B-\sin A\sin B\text{............(1).}$$

But $\cos(A+B)=-\sin(\overline{A+B}-90°)=-\sin(\overline{A-90°}+B)$;

also $$\cos A=-\sin(A-90°),$$

and $$\sin A=\cos(A-90°).\quad\text{[Arts. 99 and 102.]}$$

Hence by substitution in (1), we have

$$\sin(\overline{A-90°}+B)=\sin(A-90°)\cos B+\cos(A-90°)\sin B.$$

Similarly we may shew that

$$\cos(\overline{A-90°}+B)=\cos(A-90°)\cos B-\sin(A-90°)\sin B.$$

Thus the formulæ for the sine and cosine of $A+B$ hold when A is diminished by 90°. In like manner we may prove that they are true when B is diminished by 90°.

By repeated applications of the same process it may be shewn that the formulæ hold when either or both of the angles A and B is diminished by any multiple of 90°. Further, it will be seen that the formulæ are true if either of the angles A or B is increased by a multiple of 90° and the other is diminished by a multiple of 90°.

Thus $\sin(P+Q)=\sin P\cos Q+\cos P\sin Q$,

and $\cos(P+Q)=\cos P\cos Q-\sin P\sin Q$,

where $P=A\pm m\,.\,90°$, and $Q=B\pm n\,.\,90°$,

m and n being any positive integers, and A and B any acute angles.

Thus the Addition Formulæ are true for the algebraical sum of any two angles.

MISCELLANEOUS EXAMPLES. H.

1. If the sides of a right-angled triangle are

$\cos 2\alpha+\cos 2\beta+2\cos(\alpha+\beta)$ and $\sin 2\alpha+\sin 2\beta+2\sin(\alpha+\beta)$,

shew that the hypotenuse is $4\cos^2\dfrac{\alpha-\beta}{2}$.

2. If the in-centre and circum-centre be at equal distances from BC, prove that

$$\cos B+\cos C=1.$$

3. The shadow of a tower is observed to be half the known height of the tower, and some time afterwards to be equal to the height: how much will the sun have gone down in the interval? Given log 2,

$$L\tan 63°\,26'=10{\cdot}3009994,\ \text{diff. for } 1'=3159.$$

4. If $(1+\sin\alpha)(1+\sin\beta)(1+\sin\gamma)$

$$=(1-\sin\alpha)(1-\sin\beta)(1-\sin\gamma),$$

shew that each expression is equal to $\pm\cos\alpha\cos\beta\cos\gamma$.

5. Two parallel chords of a circle lying on the same side of the centre subtend 72° and 144° at the centre: prove that the distance between them is one-half of the radius.

Also shew that the sum of the squares of the chords is equal to five times the square of the radius.

6. Two straight railways are inclined at an angle of 60°. From their point of intersection two trains P and Q start at the same time, one along each line. P travels at the rate of 48 miles per hour, at what rate must Q travel so that after one hour they shall be 43 miles apart?

7. If
$$\alpha=\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b},$$
shew that
$$\sin^2\alpha=\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}.$$

8. If p, q, r denote the sides of the ex-central triangle, prove that
$$\frac{a^2}{p^2}+\frac{b^2}{q^2}+\frac{c^2}{r^2}+\frac{2abc}{pqr}=1.$$

9. A tower is situated within the angle formed by two straight roads OA and OB, and subtends angles α and β at the points A and B where the roads are nearest to it. If $OA=a$, and $OB=b$, shew that the height of the tower is
$$\sqrt{a^2-b^2}\sin\alpha\sin\beta \,/\, \sqrt{\sin(\alpha+\beta)\sin(\alpha-\beta)}.$$

10. In a triangle, shew that
$$r^2+r_1^2+r_2^2+r_3^2=16R^2-a^2-b^2-c^2.$$

11. If AD be a median of the triangle ABC, shew that

(1) $\cot BAD=2\cot A+\cot B$;

(2) $2\cot ADC=\cot B-\cot C$.

12. If p, q, r are the distances of the orthocentre from the sides, prove that
$$4\left(\frac{a}{p}+\frac{b}{q}+\frac{c}{r}\right)=\left(\frac{a}{p}+\frac{b}{q}-\frac{c}{r}\right)\left(\frac{b}{q}+\frac{c}{r}-\frac{a}{p}\right)\left(\frac{c}{r}+\frac{a}{p}-\frac{b}{q}\right).$$

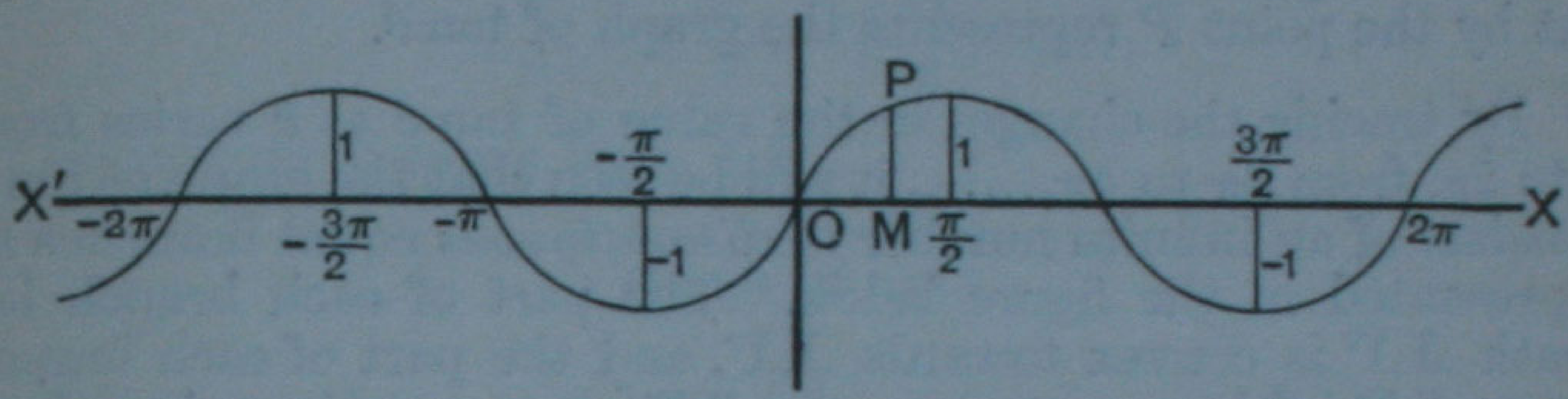
Graphical Representation of the Circular Functions.

288. Definition. Let $f(x)$ be a function of x which has a single value for all values of x, and let the values of x be represented by lines measured from O along OX or OX', and the values of $f(x)$ by lines drawn perpendicular to XX'. Then with the figure of the next article, if OM represent any value of x, and MP the corresponding value of $f(x)$, the curve traced out by the point P is called the **Graph** of $f(x)$.

Graphs of sin θ and cos θ.

289. Suppose that the unit of length is chosen to represent a radian; then any angle of θ radians will be represented by a line OM which contains θ units of length.

Graph of sin θ.



Let MP, drawn perpendicular to OX, represent the value of $\sin\theta$ corresponding to the value OM of θ; then the curve traced out by the point P represents the graph of $\sin\theta$.

As OM or θ increases from 0 to $\frac{\pi}{2}$, MP or $\sin\theta$ increases from 0 to 1, which is its greatest value.

As OM increases from $\frac{\pi}{2}$ to π, MP decreases from 0 to 1.

As OM increases from π to $\frac{3\pi}{2}$, MP increases numerically from 0 to -1.

As OM increases from $\frac{3\pi}{2}$ to 2π, MP decreases numerically from -1 to 0.

As OM increases from 2π to 4π, from 4π to 6π, from 6π to 8π, ..., MP passes through the same series of values as when OM increases from 0 to 2π.

Since $\sin(-\theta) = -\sin\theta$, the values of MP lying to the left of O are equal in magnitude but are of opposite sign to values of MP lying at an equal distance to the right of O.

Thus the graph of $\sin\theta$ is a *continuous* waving line extending to an infinite distance on each side of O.

The graph of $\cos\theta$ is the same as that of $\sin\theta$, the origin being at the point marked $\frac{\pi}{2}$ in the figure.

Graphs of tan θ and cot θ.

290. As before, suppose that the unit of length is chosen to represent a radian; then any angle of θ radians will be represented by a line OM which contains θ units of length.

Let MP, drawn perpendicular to OX, represent the value of $\tan\theta$ corresponding to the value OM of θ; then the curve traced out by the point P represents the graph of $\tan\theta$.

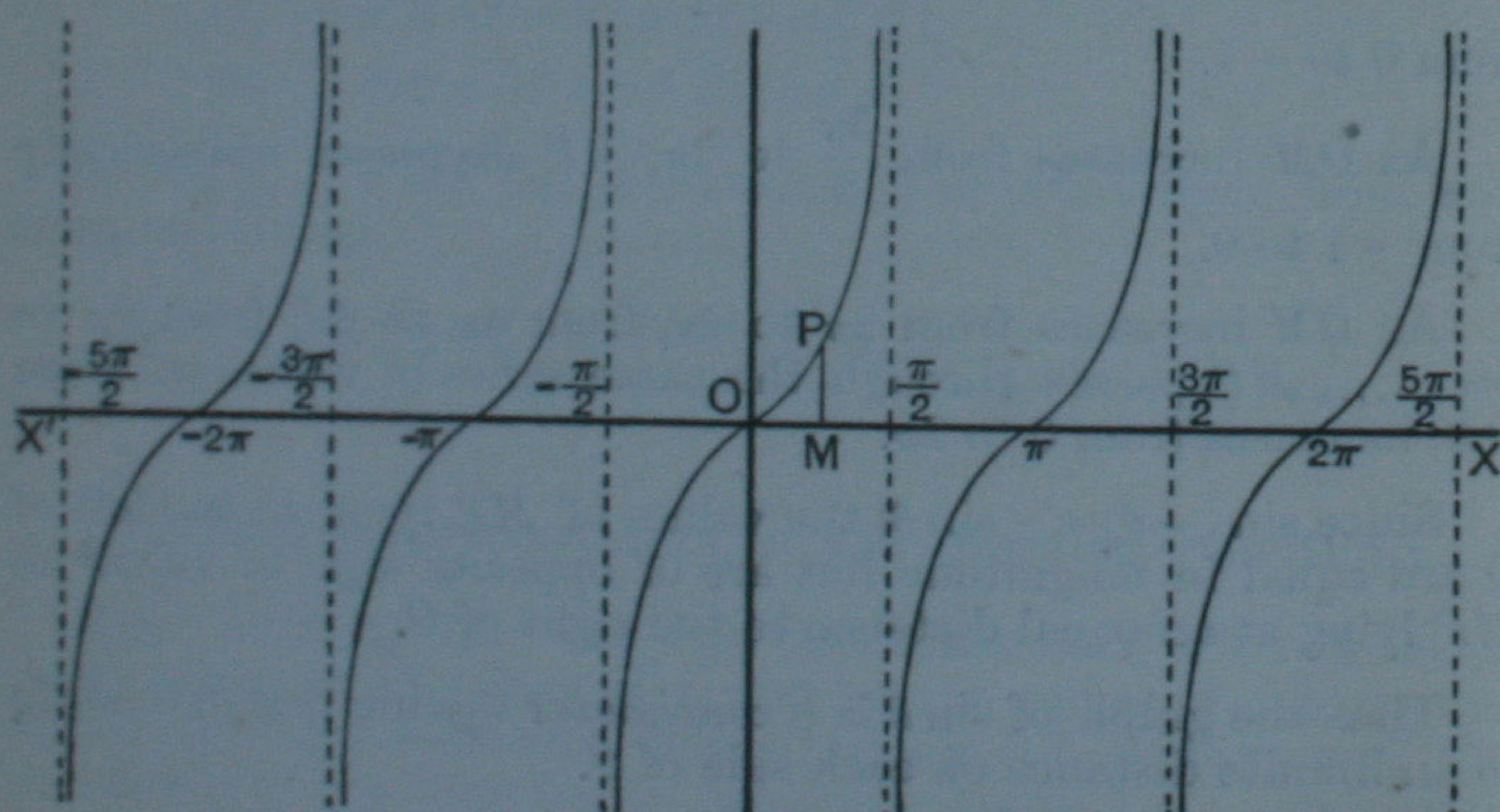
By tracing the changes in the value of $\tan\theta$ as θ varies from 0 to 2π, from 2π to 4π,......, it will be seen that the graph of $\tan\theta$ consists of an infinite number of *discontinuous* equal branches as represented in the figure below. The part of each branch beneath XX' is convex towards XX', and the part of each branch above XX' is also convex towards XX'; hence at the point where any branch cuts XX' there is what is called a *point of inflexion*, where the direction of curvature changes. The proof of these statements is however beyond the range of the present work.

The various branches touch the dotted lines passing through the points marked

$$\pm\frac{\pi}{2}, \quad \pm\frac{3\pi}{2}, \quad \pm\frac{5}{2}, \ldots\ldots,$$

at an infinite distance from XX'.

Graph of tan θ.



The student should draw the graph of $\cot\theta$, which is very similar to that of $\tan\theta$.

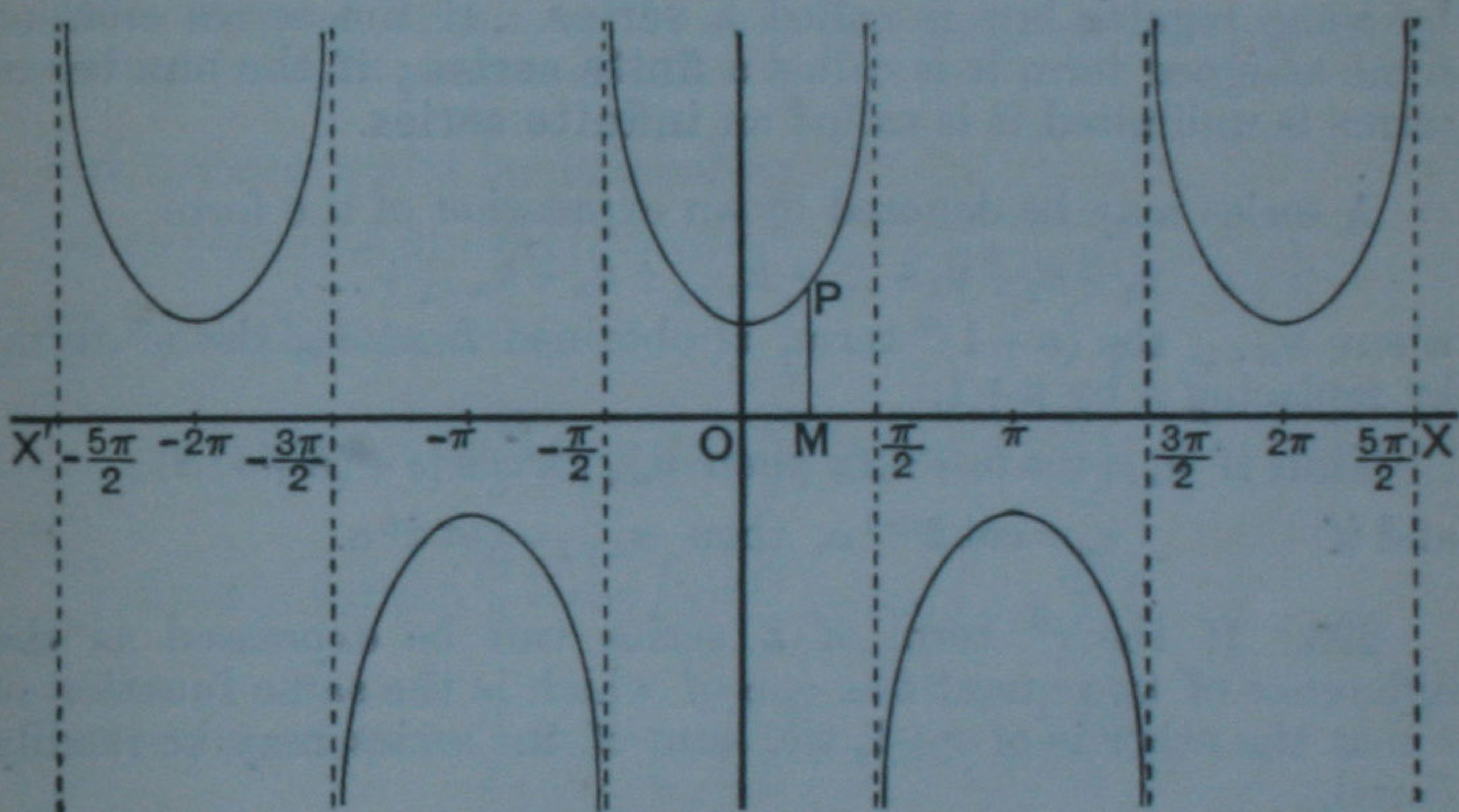
Graphs of sec θ and cosec θ.

291. The graph of $\sec\theta$ is represented in the figure below. It consists of an infinite number of equal festoons lying alternately above and below XX', the vertex of each being at the unit of distance from XX'. The various festoons touch the dotted lines passing through the points marked

$$\pm\frac{\pi}{2},\quad \pm\frac{3\pi}{2},\quad \pm\frac{5\pi}{2},\ldots\ldots,$$

at an infinite distance from XX'.

Graph of sec θ.



The graph of $\operatorname{cosec}\theta$ is the same as that of $\sec\theta$, the origin being at the point marked $-\frac{\pi}{2}$ in the figure.

CHAPTER XXIII.

SUMMATION OF FINITE SERIES.

292. An expression in which the successive terms are formed by some regular law is called a **series**. If the series ends at some assigned term it is called a **finite series**; if the number of terms is unlimited it is called an **infinite series**.

A series may be denoted by an expression of the form

$$u_1+u_2+u_3+\ldots+u_{n-1}+u_n+u_{n+1}+\ldots,$$

where u_{n+1}, the $(n+1)^{\text{th}}$ term, is obtained from u_n, the n^{th} term, by replacing n by $n+1$.

Thus if $u_n=\cos(a+n\beta)$, then $u_{n+1}=\cos\{a+(n+1)\beta\}$;
and if $u_n=\cot 2^{n-1}a$, then $u_{n+1}=\cot 2^n a$.

293. If the r^{th} term of a series can be expressed as the difference of two quantities one of which is the same function of r that the other is of $r+1$, the sum of the series may be readily found.

For let the series be denoted by

$$u_1+u_2+u_3+\ldots\ldots+u_n,$$

and its sum by S, and suppose that any term

$$u_r=v_{r+1}-v_r;$$

then $S=(v_2-v_1)+(v_3-v_2)+(v_4-v_3)+\ldots+(v_n-v_{n-1})+(v_{n+1}-v_n)$
$=v_{n+1}-v_1.$

Example. Find the sum of the series

$$\operatorname{cosec}\alpha+\operatorname{cosec}2\alpha+\operatorname{cosec}4\alpha+\ldots\ldots+\operatorname{cosec}2^{n-1}\alpha.$$

$$\operatorname{cosec}\alpha=\frac{1}{\sin\alpha}=\frac{\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2}\sin\alpha}=\frac{\sin\left(\alpha-\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}\sin\alpha}.$$

Hence $$\operatorname{cosec} \alpha = \cot \frac{\alpha}{2} - \cot \alpha.$$

If we replace α by 2α, we obtain

$$\operatorname{cosec} 2\alpha = \cot \alpha - \cot 2\alpha.$$

Similarly, $$\operatorname{cosec} 4\alpha = \cot 2\alpha - \cot 4\alpha,$$

....

$$\operatorname{cosec} 2^{n-1}\alpha = \cot 2^{n-2}\alpha - \cot 2^{n-1}\alpha.$$

By addition, $$S = \cot \frac{\alpha}{2} - \cot 2^{n-1}\alpha.$$

294. *To find the sum of the sines of a series of* n *angles which are in arithmetical progression.*

Let the sine-series be denoted by

$$\sin \alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots + \sin\{\alpha+(n-1)\beta\}.$$

We have the identities

$$2\sin\alpha\sin\frac{\beta}{2} = \cos\left(\alpha-\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{\beta}{2}\right),$$

$$2\sin(\alpha+\beta)\sin\frac{\beta}{2} = \cos\left(\alpha+\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{3\beta}{2}\right),$$

$$2\sin(\alpha+2\beta)\sin\frac{\beta}{2} = \cos\left(\alpha+\frac{3\beta}{2}\right) - \cos\left(\alpha+\frac{5\beta}{2}\right),$$

....

$$2\sin\{\alpha+(n-1)\beta\}\sin\frac{\beta}{2} = \cos\left(\alpha+\frac{2n-3}{2}\beta\right) - \cos\left(\alpha+\frac{2n-1}{2}\beta\right).$$

By addition,

$$2S\sin\frac{\beta}{2} = \cos\left(\alpha-\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{2n-1}{2}\beta\right)$$

$$= 2\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2};$$

$$\therefore\ S = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha+\frac{n-1}{2}\beta\right).$$

295. In like manner we may shew that the sum of the cosine-series

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos\{\alpha+(n-1)\beta\}$$

$$= \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cos\left(\alpha + \frac{n-1}{2}\beta\right).$$

296. The formulæ of the two last articles may be expressed verbally as follows.

The sum of the sines of a series of n *angles in* A.P.

$$= \frac{\sin\frac{\text{n } \textit{diff.}}{2}}{\sin\frac{\textit{diff.}}{2}} \sin\frac{\textit{first angle} + \textit{last angle}}{2}.$$

The sum of the cosines of a series of n *angles in* A.P.

$$= \frac{\sin\frac{\text{n } \textit{diff.}}{2}}{\sin\frac{\textit{diff.}}{2}} \cos\frac{\textit{first angle} + \textit{last angle}}{2}.$$

Example. Find the sum of the series

$$\cos\alpha + \cos 3\alpha + \cos 5\alpha + \ldots\ldots + \cos(2n-1)\alpha.$$

Here the common difference of the angles is 2α;

$$\therefore S = \frac{\sin n\alpha}{\sin\alpha} \cos\frac{\alpha + (2n-1)\alpha}{2}$$

$$= \frac{\sin n\alpha \cos n\alpha}{\sin\alpha} = \frac{\sin 2n\alpha}{2\sin\alpha}.$$

297. If $\sin\frac{n\beta}{2} = 0$, each of the expressions found in Arts. 294 and 295 for the sum vanishes. In this case

$$\frac{n\beta}{2} = k\pi, \quad \text{or} \quad \beta = \frac{2k\pi}{n}, \text{ where } k \text{ is any integer.}$$

Hence *the sum of the sines and the sum of the cosines of* n *angles in arithmetical progression are each equal to zero, when the common difference of the angles is an even multiple of* $\frac{\pi}{n}$.

298. Some series may be brought under the rule of Art. 296 by a simple transformation.

Example 1. Find the sum of n terms of the series

$$\cos\alpha-\cos(\alpha+\beta)+\cos(\alpha+2\beta)-\cos(\alpha+3\beta)+\ldots\ldots.$$

This series is equal to

$$\cos\alpha+\cos(\alpha+\beta+\pi)+\cos(\alpha+2\beta+2\pi)+\cos(\alpha+3\beta+3\pi)+\ldots\ldots,$$

a series in which the common difference of the angles is $\beta+\pi$, and the last angle is $\alpha+(n-1)(\beta+\pi)$.

$$\therefore\ S=\frac{\sin\dfrac{n(\beta+\pi)}{2}}{\sin\dfrac{\beta+\pi}{2}}\cos\left\{\alpha+\frac{(n-1)(\beta+\pi)}{2}\right\}.$$

Example 2. Find the sum of n terms of the series

$$\sin\alpha+\cos(\alpha+\beta)-\sin(\alpha+2\beta)-\cos(\alpha+3\beta)+\sin(\alpha+4\beta)+\ldots\ldots.$$

This series is equal to

$$\sin\alpha+\sin\left(\alpha+\beta+\frac{\pi}{2}\right)+\sin(\alpha+2\beta+\pi)+\sin\left(\alpha+3\beta+\frac{3\pi}{2}\right)+\ldots\ldots,$$

a series in which the common difference of the angles is $\beta+\dfrac{\pi}{2}$.

$$\therefore\ S=\frac{\sin\dfrac{n(2\beta+\pi)}{4}}{\sin\dfrac{2\beta+\pi}{4}}\sin\left\{\alpha+\frac{(n-1)(2\beta+\pi)}{4}\right\}.$$

EXAMPLES. XXIII. a.

Sum each of the following series to n terms:

1. $\sin\alpha+\sin 3\alpha+\sin 5\alpha+\ldots\ldots.$

2. $\cos\alpha+\cos(\alpha-\beta)+\cos(\alpha-2\beta)+\ldots\ldots.$

3. $\sin\alpha+\sin\left(\alpha-\dfrac{\pi}{n}\right)+\sin\left(\alpha-\dfrac{2\pi}{n}\right)+\ldots\ldots.$

4. $\cos\dfrac{\pi}{k}+\cos\dfrac{2\pi}{k}+\cos\dfrac{3\pi}{k}+\ldots\ldots.$

Find the sum of each of the following series :

5. $\cos\frac{\pi}{19}+\cos\frac{3\pi}{19}+\cos\frac{5\pi}{19}+\ldots\ldots+\cos\frac{17\pi}{19}$.

6. $\cos\frac{2\pi}{21}+\cos\frac{4\pi}{21}+\cos\frac{6\pi}{21}+\ldots\ldots+\cos\frac{20\pi}{21}$.

7. $\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+\ldots\ldots$ to $n-1$ terms.

8. $\cos\frac{\pi}{n}+\cos\frac{3\pi}{n}+\cos\frac{5\pi}{n}+\ldots\ldots$ to $2n-1$ terms.

9. $\sin n\alpha+\sin(n-1)\alpha+\sin(n-2)\alpha+\ldots\ldots$ to $2n$ terms.

Sum each of the following series to n terms :

10. $\sin\theta-\sin 2\theta+\sin 3\theta-\sin 4\theta+\ldots\ldots$.

11. $\cos\alpha-\cos(\alpha-\beta)+\cos(\alpha-2\beta)-\cos(\alpha-3\beta)+\ldots\ldots$.

12. $\cos\alpha-\sin(\alpha-\beta)-\cos(\alpha-2\beta)+\sin(\alpha-3\beta)+\ldots\ldots$.

13. $\sin 2\theta\sin\theta+\sin 3\theta\sin 2\theta+\sin 4\theta\sin 3\theta+\ldots\ldots$.

14. $\sin\alpha\cos 3\alpha+\sin 3\alpha\cos 5\alpha+\sin 5\alpha\cos 7\alpha+\ldots\ldots$.

15. $\sec\alpha\sec 2\alpha+\sec 2\alpha\sec 3\alpha+\sec 3\alpha\sec 4\alpha+\ldots\ldots$.

16. $\operatorname{cosec}\theta\operatorname{cosec}3\theta+\operatorname{cosec}3\theta\operatorname{cosec}5\theta+\operatorname{cosec}5\theta\operatorname{cosec}7\theta+\ldots\ldots$.

17. $\tan\frac{\alpha}{2}\sec\alpha+\tan\frac{\alpha}{2^2}\sec\frac{\alpha}{2}+\tan\frac{\alpha}{2^3}\sec\frac{\alpha}{2^2}+\ldots\ldots$.

18. $\cos 2\alpha\operatorname{cosec}3\alpha+\cos 6\alpha\operatorname{cosec}9\alpha+\cos 18\alpha\operatorname{cosec}27\alpha+\ldots\ldots$.

19. $\sin\alpha\sec 3\alpha+\sin 3\alpha\sec 9\alpha+\sin 9\alpha\sec 27\alpha+\ldots\ldots$.

20. The circumference of a semicircle of radius a is divided into n equal arcs. Shew that the sum of the distances of the several points of section from either extremity of the diameter is

$$a\left(\cot\frac{\pi}{4n}-1\right).$$

21. From the angular points of a regular polygon, perpendiculars are drawn to XX' and YY' the horizontal and vertical diameter of the circumscribing circle : shew that the algebraical sums of each of the two sets of perpendiculars are equal to zero.

299. By means of the identities

$$2\sin^2 a = 1 - \cos 2a, \qquad 2\cos^2 a = 1 + \cos 2a,$$
$$4\sin^3 a = 3\sin a - \sin 3a, \qquad 4\cos^3 a = 3\cos a + \cos 3a,$$

we can find the sum of the squares and cubes of the sines and cosines of a series of angles in arithmetical progression.

Example 1. Find the sum of n terms of the series

$$\sin^2\alpha + \sin^2(\alpha+\beta) + \sin^2(\alpha+2\beta) + \ldots\ldots .$$

$$2S = \{1 - \cos 2\alpha\} + \{1 - \cos(2\alpha + 2\beta)\} + \{1 - \cos(2\alpha + 4\beta)\} + \ldots\ldots$$
$$= n - \{\cos 2\alpha + \cos(2\alpha+2\beta) + \cos(2\alpha+4\beta) + \ldots\ldots\};$$
$$= n - \frac{\sin n\beta}{\sin\beta}\cos\frac{2\alpha + \{2\alpha + (n-1)\,2\beta\}}{2};$$
$$\therefore\ S = \frac{n}{2} - \frac{\sin n\beta}{2\sin\beta}\cos\{2\alpha + (n-1)\,\beta\}.$$

Example 2. Find the sum of the series

$$\cos^3\alpha + \cos^3 3\alpha + \cos^3 5\alpha + \ldots\ldots + \cos^3(2n-1)\,\alpha.$$

$$4S = (3\cos\alpha + \cos 3\alpha) + (3\cos 3\alpha + \cos 9\alpha) + (3\cos 5\alpha + \cos 15\alpha) + \ldots\ldots$$
$$= 3(\cos\alpha + \cos 3\alpha + \cos 5\alpha + \ldots\ldots) + (\cos 3\alpha + \cos 9\alpha + \cos 15\alpha + \ldots\ldots)$$
$$= \frac{3\sin n\alpha}{\sin\alpha}\cos\left\{\frac{\alpha + (2n-1)\,\alpha}{2}\right\} + \frac{\sin 3n\alpha}{\sin 3\alpha}\cos\left\{\frac{3\alpha + (2n-1)\,3\alpha}{2}\right\};$$
$$\therefore\ S = \frac{3\sin n\alpha\cos n\alpha}{4\sin\alpha} + \frac{\sin 3n\alpha\cos 3n\alpha}{4\sin 3\alpha}.$$

300. The following further examples illustrate the principle of Art. 293.

Example 1. Find the sum of the series

$$\tan^{-1}\frac{x}{1+1\,.\,2\,.\,x^2} + \tan^{-1}\frac{x}{1+2\,.\,3\,.\,x^2} + \ldots\ldots + \tan^{-1}\frac{x}{1+n(n+1)\,x^2}.$$

As in Art. 249, we have

$$\tan^{-1}\frac{x}{1+r(r+1)\,x^2} = \tan^{-1}(r+1)\,x - \tan^{-1}rx;$$
$$\therefore\ S = \tan^{-1}(n+1)\,x - \tan^{-1}x.$$

Example 2. Find the sum of n terms of the series

$$\tan\alpha + \frac{1}{2}\tan\frac{\alpha}{2} + \frac{1}{2^2}\tan\frac{\alpha}{2^2} + \frac{1}{2^3}\tan\frac{\alpha}{2^3} + \ldots\ldots.$$

We have $$\tan\alpha = \cot\alpha - 2\cot 2\alpha.$$

Replacing α by $\frac{\alpha}{2}$ and dividing by 2, we obtain

$$\frac{1}{2}\tan\frac{\alpha}{2} = \frac{1}{2}\cot\frac{\alpha}{2} - \cot\alpha.$$

Similarly, $$\frac{1}{2^2}\tan\frac{\alpha}{2^2} = \frac{1}{2^2}\cot\frac{\alpha}{2^2} - \frac{1}{2}\cot\frac{\alpha}{2};$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{1}{2^{n-1}}\tan\frac{\alpha}{2^{n-1}} = \frac{1}{2^{n-1}}\cot\frac{\alpha}{2^{n-1}} - \frac{1}{2^{n-2}}\cot\frac{\alpha}{2^{n-2}}.$$

By addition, $$S = \frac{1}{2^{n-1}}\cot\frac{\alpha}{2^{n-1}} - 2\cot 2\alpha.$$

EXAMPLES. XXIII. b.

Sum each of the following series to n terms:

1. $\cos^2\theta + \cos^2 3\theta + \cos^2 5\theta + \ldots\ldots.$

2. $\sin^2\alpha + \sin^2\left(\alpha + \frac{\pi}{n}\right) + \sin^2\left(\alpha + \frac{2\pi}{n}\right) + \ldots\ldots.$

3. $\cos^2\alpha + \cos^2\left(\alpha - \frac{\pi}{n}\right) + \cos^2\left(\alpha - \frac{2\pi}{n}\right) + \ldots\ldots.$

4. $\sin^3\theta + \sin^3 2\theta + \sin^3 3\theta + \ldots\ldots.$

5. $\sin^3\alpha + \sin^3\left(\alpha + \frac{2\pi}{n}\right) + \sin^3\left(\alpha + \frac{4\pi}{n}\right) + \ldots\ldots.$

6. $\cos^3\alpha + \cos^3\left(\alpha - \frac{2\pi}{n}\right) + \cos^3\left(\alpha - \frac{4\pi}{n}\right) + \ldots\ldots.$

7. $\tan\theta + 2\tan 2\theta + 2^2\tan 2^2\theta + \ldots\ldots$

8. $\dfrac{1}{\cos\alpha + \cos 3\alpha} + \dfrac{1}{\cos\alpha + \cos 5\alpha} + \dfrac{1}{\cos\alpha + \cos 7\alpha} + \ldots\ldots.$

9. $\sin^2\theta\sin 2\theta+\frac{1}{2}\sin^2 2\theta\sin 4\theta+\frac{1}{4}\sin^2 4\theta\sin 8\theta+\ldots\ldots$

10. $2\cos\theta\sin^2\frac{\theta}{2}+2^2\cos\frac{\theta}{2}\sin^2\frac{\theta}{2^2}+2^3\cos\frac{\theta}{2^2}\sin^2\frac{\theta}{2^3}+\ldots\ldots$

11. $\tan^{-1}\frac{x}{1.2+x^2}+\tan^{-1}\frac{x}{2.3+x^2}+\tan^{-1}\frac{x}{3.4+x^2}+\ldots\ldots$

12. $\tan^{-1}\frac{1}{1+1+1^2}+\tan^{-1}\frac{1}{1+2+2^2}+\tan^{-1}\frac{1}{1+3+3^2}+\ldots\ldots$

13. $\tan^{-1}\frac{2}{2+1^2+1^4}+\tan^{-1}\frac{4}{2+2^2+2^4}+\tan^{-1}\frac{6}{2+3^2+3^4}+\ldots$

14. $\tan^{-1}\frac{2}{1-1^2+1^4}+\tan^{-1}\frac{4}{1-2^2+2^4}+\tan^{-1}\frac{6}{1-3^2+3^4}+\ldots$

15. From any point on the circumference of a circle of radius r, chords are drawn to the angular points of the regular inscribed polygon of n sides: shew that the sum of the squares of the chords is $2nr^2$.

16. From a point P within a regular polygon of $2n$ sides, perpendiculars PA_1, PA_2, PA_3, ...PA_{2n} are drawn to the sides: shew that

$$PA_1+PA_3+\ldots+PA_{2n-1}=PA_2+PA_4+\ldots+PA_{2n}=nr,$$

where r is the radius of the inscribed circle.

17. If $A_1A_2A_3\ldots A_{2n+1}$ is a regular polygon and P a point on the circumscribed circle lying on the arc A_1A_{2n+1}, shew that

$$PA_1+PA_3+\ldots+PA_{2n+1}=PA_2+PA_4+\ldots+PA_{2n}.$$

18. From any point on the circumference of a circle, perpendiculars are drawn to the sides of the regular circumscribing polygon of n sides: shew that

(1) the sum of the squares of the perpendiculars is $\frac{3nr^2}{2}$;

(2) the sum of the cubes of the perpendiculars is $\frac{5nr^3}{2}$.

CHAPTER XXIV.

MISCELLANEOUS TRANSFORMATIONS AND IDENTITIES.

Symmetrical Expressions.

301. An expression is said to be *symmetrical* with respect to certain of the letters it contains, if the value of the expression remains unaltered when any pair of these letters are interchanged. Thus

$$\cos\alpha+\cos\beta+\cos\gamma, \qquad \sin\alpha\sin\beta\sin\gamma,$$
$$\tan(\alpha-\theta)+\tan(\beta-\theta)+\tan(\gamma-\theta),$$

are expressions which are symmetrical with respect to the letters α, β, γ.

302. A symmetrical expression involving the *sum* of a number of quantities may be concisely denoted by writing down one of the *terms* and prefixing the symbol Σ. Thus $\Sigma\cos\alpha$ stands for the sum of all the terms of which $\cos\alpha$ is the type, $\Sigma\sin\alpha\sin\beta$ stands for the sum of all the terms of which $\sin\alpha\sin\beta$ is the type; and so on.

For instance, if the expression is symmetrical with respect to the three letters α, β, γ,

$$\Sigma\cos\beta\cos\gamma=\cos\beta\cos\gamma+\cos\gamma\cos\alpha+\cos\alpha\cos\beta;$$
$$\Sigma\sin(\alpha-\theta)=\sin(\alpha-\theta)+\sin(\beta-\theta)+\sin(\gamma-\theta).$$

303. A symmetrical expression involving the *product* of a number of quantities may be denoted by writing down one of the *factors* and prefixing the symbol Π. Thus $\Pi\sin\alpha$ stands for the product of all the factors of which $\sin\alpha$ is the type.

For instance, if the expression is symmetrical with respect to the three letters α, β, γ,

$$\Pi\tan(\alpha+\theta)=\tan(\alpha+\theta)\tan(\beta+\theta)\tan(\gamma+\theta);$$

$$\Pi(\cos\beta+\cos\gamma)=(\cos\beta+\cos\gamma)(\cos\gamma+\cos\alpha)(\cos\alpha+\cos\beta).$$

304. With the notation just explained, certain theorems in Chap. XII. involving the three angles A, B, C, which are connected by the relation $A+B+C=180°$, may be written more concisely. For instance

$$\Sigma \sin 2A = 4\Pi \sin A;$$

$$\Sigma \sin A = 4\Pi \cos \frac{A}{2};$$

$$\Sigma \tan A = \Pi \tan A;$$

$$\Sigma \tan \frac{B}{2} \tan \frac{C}{2} = 1.$$

Example 1. Find the ratios of $a : b : c$ from the equations

$$a \cos \theta + b \sin \theta = c \quad \text{and} \quad a \cos \phi + b \sin \phi = c.$$

From the given equations, we have

$$a \cos \theta + b \sin \theta - c = 0,$$

and

$$a \cos \phi + b \sin \phi - c = 0;$$

whence by *cross multiplication*

$$\frac{a}{\sin \phi - \sin \theta} = \frac{b}{\cos \theta - \cos \phi} = \frac{c}{\sin \phi \cos \theta - \cos \phi \sin \theta};$$

$$\therefore \frac{a}{2 \cos \dfrac{\phi+\theta}{2} \sin \dfrac{\phi-\theta}{2}} = \frac{b}{2 \sin \dfrac{\phi+\theta}{2} \sin \dfrac{\phi-\theta}{2}} = \frac{c}{\sin (\phi - \theta)}.$$

Dividing each denominator by $2 \sin \dfrac{\phi-\theta}{2}$, we have

$$\frac{a}{\cos \dfrac{\theta+\phi}{2}} = \frac{b}{\sin \dfrac{\theta+\phi}{2}} = \frac{c}{\cos \dfrac{\theta-\phi}{2}}.$$

Note. This result is important in Analytical Geometry.

It should be remarked that $\cos(\theta - \phi)$ is a symmetrical function of θ and ϕ, for $\cos(\theta-\phi) = \cos(\phi-\theta)$; hence the values obtained for $a : b : c$ involve θ and ϕ symmetrically.

Example 2. If α and β are two different values of θ which satisfy the equation $a \cos \theta + b \sin \theta = c$, find the values of

$$4 \cos^2 \frac{\alpha}{2} \cos^2 \frac{\beta}{2}, \quad \sin \alpha + \sin \beta, \quad \sin \alpha \sin \beta.$$

From the given equation, by transposing and squaring,

$$(a\cos\theta - c)^2 = b^2\sin^2\theta = b^2(1-\cos^2\theta);$$

$$\therefore\ (a^2+b^2)\cos^2\theta - 2ac\cos\theta + c^2 - b^2 = 0.$$

The roots of this quadratic in $\cos\theta$ are $\cos\alpha$ and $\cos\beta$;

$$\therefore\ \cos\alpha + \cos\beta = \frac{2ac}{a^2+b^2} \quad\text{.......................(1)},$$

and

$$\cos\alpha\cos\beta = \frac{c^2-b^2}{a^2+b^2} \quad\text{.......................(2)}.$$

And

$$4\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2} = (1+\cos\alpha)(1+\cos\beta)$$

$$= 1 + \frac{2ac}{a^2+b^2} + \frac{c^2-b^2}{a^2+b^2}$$

$$= \frac{(a+c)^2}{a^2+b^2}.$$

From the data, we see that $\frac{\pi}{2}-\alpha$ and $\frac{\pi}{2}-\beta$ are values of θ which satisfy the equation $a\sin\theta + b\cos\theta = c.$

By writing a for b and b for a, equation (1) becomes

$$\cos\left(\frac{\pi}{2}-\alpha\right) + \cos\left(\frac{\pi}{2}-\beta\right) = \frac{2bc}{b^2+a^2},$$

or

$$\sin\alpha + \sin\beta = \frac{2bc}{a^2+b^2}.$$

Similarly, from equation (2) we have

$$\sin\alpha\sin\beta = \frac{c^2-a^2}{a^2+b^2}.$$

These last two results may also be derived from the equation

$$(b\sin\theta - c)^2 = a^2\cos^2\theta = a^2(1-\sin^2\theta).$$

Example 3. If α and β are two different values of θ which satisfy the equation $a\cos\theta + b\sin\theta = c$, prove that $\tan\frac{\alpha+\beta}{2} = \frac{b}{a}$. Also if the values of α and β are equal, shew that $a^2+b^2=c^2$.

By substituting $\cos\theta = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$ and $\sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

in the given equation $a\cos\theta+b\sin\theta=c$, we have

$$a\left(1-\tan^2\frac{\theta}{2}\right)+2b\tan\frac{\theta}{2}=c\left(1+\tan^2\frac{\theta}{2}\right);$$

that is, $$(c+a)\tan^2\frac{\theta}{2}-2b\tan\frac{\theta}{2}+(c-a)=0 \quad \ldots\ldots\ldots\ldots(1).$$

The roots of this equation are $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$;

$$\therefore\ \tan\frac{\alpha}{2}+\tan\frac{\beta}{2}=\frac{2b}{c+a}, \quad \text{and} \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{c-a}{c+a};$$

$$\therefore\ \tan\frac{\alpha+\beta}{2}=\frac{2b}{c+a}\Big/\left(1-\frac{c-a}{c+a}\right)=\frac{b}{a}.$$

If the roots of equation (1) are equal, we have

$$b^2=(c+a)(c-a);$$

whence $$a^2+b^2=c^2.$$

NOTE. The substitution here employed is frequently used in Analytical Geometry.

Example 4. If $\cos\theta+\cos\phi=a$ and $\sin\theta+\sin\phi=b$, find the values of $\cos(\theta+\phi)$ and $\sin 2\theta+\sin 2\phi$.

From the given equations, we have

$$\frac{\sin\theta+\sin\phi}{\cos\theta+\cos\phi}=\frac{b}{a};$$

$$\therefore\ \tan\frac{\theta+\phi}{2}=\frac{b}{a}.$$

For shortness write t instead of $\tan\frac{\theta+\phi}{2}$; then

$$\cos(\theta+\phi)=\frac{1-t^2}{1+t^2}=\frac{a^2-b^2}{a^2+b^2},$$

and $$\sin(\theta+\phi)=\frac{2t}{1+t^2}=\frac{2ab}{a^2+b^2}.$$

Multiplying the two given equations together, we have

$$\sin 2\theta+\sin 2\phi+2\sin(\theta+\phi)=2ab;$$

$$\therefore\ \sin 2\theta+\sin 2\phi=2ab\left(1-\frac{2}{a^2+b^2}\right).$$

Example 5. Resolve into factors the expression

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma-1,$$

and shew that it vanishes if any one of the four angles $\alpha\pm\beta\pm\gamma$ is an odd multiple of two right angles.

The expression $=\cos^2\alpha+(\cos^2\beta+\cos^2\gamma-1)+2\cos\alpha\cos\beta\cos\gamma$

$$=\cos^2\alpha+(\cos^2\beta-\sin^2\gamma)+2\cos\alpha\cos\beta\cos\gamma$$

$$=\cos^2\alpha+\cos(\beta+\gamma)\cos(\beta-\gamma)+\cos\alpha\{\cos(\beta+\gamma)+\cos(\beta-\gamma)\}$$

$$=\{\cos\alpha+\cos(\beta+\gamma)\}\{\cos\alpha+\cos(\beta-\gamma)\}$$

$$=4\cos\frac{\alpha+\beta+\gamma}{2}\cos\frac{\alpha-\beta-\gamma}{2}\cos\frac{\alpha+\beta-\gamma}{2}\cos\frac{\alpha-\beta+\gamma}{2}.$$

The expression vanishes if one of the quantities $\cos\dfrac{\alpha\pm\beta\pm\gamma}{2}=0$;

that is, if one of the four angles $\dfrac{\alpha\pm\beta\pm\gamma}{2}=(2n+1)\dfrac{\pi}{2}$;

that is, if $\alpha\pm\beta\pm\gamma=(2n+1)\pi$, where n is any integer.

Example 6. If $\qquad \tan\theta=\dfrac{\sin\alpha\sin\beta}{\cos\alpha+\cos\beta}$,

prove that one value of $\tan\dfrac{\theta}{2}$ is $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}$.

From the given equation, we have

$$\sec^2\theta=1+\frac{\sin^2\alpha\sin^2\beta}{(\cos\alpha+\cos\beta)^2}=\frac{(\cos\alpha+\cos\beta)^2+(1-\cos^2\alpha)(1-\cos^2\beta)}{(\cos\alpha+\cos\beta)^2}$$

$$=\frac{1+2\cos\alpha\cos\beta+\cos^2\alpha\cos^2\beta}{(\cos\alpha+\cos\beta)^2}.$$

Taking the positive root, $\sec\theta=\dfrac{1+\cos\alpha\cos\beta}{\cos\alpha+\cos\beta}$;

$$\therefore\ \cos\theta=\frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}.$$

$$\therefore\ \frac{1-\cos\theta}{1+\cos\theta}=\frac{1-\cos\alpha-\cos\beta+\cos\alpha\cos\beta}{1+\cos\alpha+\cos\beta+\cos\alpha\cos\beta}=\frac{(1-\cos\alpha)(1-\cos\beta)}{(1+\cos\alpha)(1+\cos\beta)};$$

$$\therefore\ \tan^2\frac{\theta}{2}=\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2};$$

and therefore one value of $\tan\dfrac{\theta}{2}$ is $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}$.

Example 7. In any triangle, shew that

$$\Sigma a^3 \cos A = abc\,(1 + 4\Pi \cos A).$$

Let $$k = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C};$$

so that $$a = k \sin A, \quad b = k \sin B, \quad c = k \sin C.$$

By substituting these values in the given identity, and dividing by k^3, we have to prove that

$$\Sigma \sin^3 A \cos A = \sin A \sin B \sin C\,(1 + 4\Pi \cos A).$$

Now $$8\Sigma \sin^3 A \cos A = 4\Sigma \sin^2 A \sin 2A$$
$$= 2\Sigma\,(1 - \cos 2A) \sin 2A$$
$$= 2\Sigma \sin 2A - \Sigma \sin 4A;$$

and it has been shewn in Example 1, Art. 135, that

$$\Sigma \sin 2A = 4\Pi \sin A;$$

and it is easy to prove that

$$\Sigma \sin 4A = -4\Pi \sin 2A = -32\Pi \sin A \,.\, \Pi \cos A;$$

$$\therefore\ 8\,\Sigma \sin^3 A \cos A = 8\Pi \sin A + 32\Pi \sin A \,.\, \Pi \cos A;$$

$$\therefore\ \Sigma \sin^3 A \cos A = \Pi \sin A\,(1 + 4\Pi \cos A).$$

EXAMPLES. XXIV. a.

1. If $\theta = \alpha$, and $\theta = \beta$ satisfy the equation

$$\frac{1}{a}\cos\theta + \frac{1}{b}\sin\theta = \frac{1}{c},$$

prove that $$a \cos\frac{\alpha+\beta}{2} = b \sin\frac{\alpha+\beta}{2} = c \cos\frac{\alpha-\beta}{2}.$$

Solve the simultaneous equations:

2. $\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1, \qquad \frac{x}{a}\cos\beta + \frac{y}{b}\sin\beta = 1.$

3. $\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1, \qquad \frac{x}{a}\sin\alpha - \frac{y}{b}\cos\alpha = 1.$

If α and β are two different solutions of $a\cos\theta+b\sin\theta=c$, prove that

4. $\cos(\alpha+\beta)=\dfrac{a^2-b^2}{a^2+b^2}$. 5. $\cos^2\dfrac{\alpha-\beta}{2}=\dfrac{c^2}{a^2+b^2}$.

6. $\sin 2\alpha+\sin 2\beta=\dfrac{4ab(2c^2-a^2-b^2)}{(a^2+b^2)^2}$.

7. $\sin^2\alpha+\sin^2\beta=\dfrac{2a^2(a^2+b^2)-2c^2(a^2-b^2)}{(a^2+b^2)^2}$.

8. If $a\cos\alpha+b\sin\alpha=a\cos\beta+b\sin\beta=c$, prove that

$$\sin(\alpha+\beta)=\frac{2ab}{a^2+b^2}, \text{ and } \cot\alpha+\cot\beta=\frac{2ab}{c^2-a^2}.$$

If $\cos\theta+\cos\phi=a$ and $\sin\theta+\sin\phi=b$, prove that

9. $\cos\theta\cos\phi=\dfrac{(a^2+b^2)^2-4b^2}{4(a^2+b^2)}$.

10. $\cos 2\theta+\cos 2\phi=\dfrac{(a^2-b^2)(a^2+b^2-2)}{a^2+b^2}$.

11. $\tan\theta+\tan\phi=\dfrac{8ab}{(a^2+b^2)^2-4b^2}$.

12. $\tan\dfrac{\theta}{2}+\tan\dfrac{\phi}{2}=\dfrac{4b}{a^2+b^2+2a}$.

13. Express

$$1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma$$

as the product of four sines, and shew that it vanishes if any one of the four angles $\alpha\pm\beta\pm\gamma$ is zero or an even multiple of π.

14. Express

$$\sin^2\alpha+\sin^2\beta-\sin^2\gamma+2\sin\alpha\sin\beta\cos\gamma$$

as the product of two sines and two cosines.

15. Express

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma-2\sin\alpha\sin\beta\sin\gamma-1$$

as the product of four cosines.

16. If
$$\cos\theta=\frac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta},$$
prove that one value of $\tan\frac{\theta}{2}$ is $\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$.

17. If
$$\tan^2\theta\cos^2\frac{\alpha+\beta}{2}=\sin\alpha\sin\beta,$$
prove that one value of $\tan^2\frac{\theta}{2}$ is $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$.

18. If
$$\tan\theta(\cos\alpha+\sin\beta)=\sin\alpha\cos\beta,$$
prove that one value of $\tan\frac{\theta}{2}$ is $\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)$.

In any triangle, shew that

19. $\Sigma a^3\sin B\sin C=2abc(1+\cos A\cos B\cos C)$.

20. $\Sigma a\cos^3 A=\frac{abc}{4R^2}(1-4\cos A\cos B\cos C)$.

21. $\Sigma a^3\cos(B-C)=3abc$.

22. If α and β are roots of the equation $a\cos\theta+b\sin\theta=c$, form the equations whose roots are

(1) $\sin\alpha$ and $\sin\beta$; (2) $\cos 2\alpha$ and $\cos 2\beta$.

Alternating Expressions.

305. An expression is said to be *alternating* with respect to certain of the letters it contains, if the sign of the expression but not its numerical value is altered when any pair of these letters are interchanged.

Thus $\cos\alpha-\cos\beta$, $\sin(\alpha-\beta)$, $\tan(\alpha-\beta)$,
$$\cos^2\alpha\sin(\beta-\gamma)+\cos^2\beta\sin(\gamma-\alpha)+\cos^2\gamma\sin(\alpha-\beta)$$
are alternating expressions.

306. Alternating expressions may be abridged by means of the symbols Σ and Π. Thus
$$\Sigma\sin^2\alpha\sin(\beta-\gamma)=\sin^2\alpha\sin(\beta-\gamma)+\sin^2\beta\sin(\gamma-\alpha)+\sin^2\gamma\sin(\alpha-\beta);$$
$$\Pi\tan(\beta-\gamma)=\tan(\beta-\gamma)\tan(\gamma-\alpha)\tan(\alpha-\beta).$$

We shall confine our attention chiefly to alternating expressions involving the three letters α, β, γ, and we shall adopt the *cyclical arrangement* $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$ in which β follows α, γ follows β, and α follows γ.

Example 1. Prove that $\Sigma\cos(\alpha+\theta)\sin(\beta-\gamma)=0$.

$$\Sigma\cos(\alpha+\theta)\sin(\beta-\gamma)=\Sigma(\cos\alpha\cos\theta-\sin\alpha\sin\theta)\sin(\beta-\gamma)$$
$$=\cos\theta\,\Sigma\cos\alpha\sin(\beta-\gamma)-\sin\theta\,\Sigma\sin\alpha\sin(\beta-\gamma)$$
$$=0,$$

since $\Sigma\cos\alpha\sin(\beta-\gamma)=0$ and $\Sigma\sin\alpha\sin(\beta-\gamma)=0$.

Example 2. Shew that $\Sigma\sin 2(\beta-\gamma)=-4\Pi\sin(\beta-\gamma)$.

$$\sin 2(\beta-\gamma)+\sin 2(\gamma-\alpha)+\sin 2(\alpha-\beta)$$
$$=2\sin(\beta-\alpha)\cos(\alpha+\beta-2\gamma)+2\sin(\alpha-\beta)\cos(\alpha-\beta)$$
$$=2\sin(\alpha-\beta)\{\cos(\alpha-\beta)-\cos(\alpha+\beta-2\gamma)\}$$
$$=4\sin(\alpha-\beta)\sin(\alpha-\gamma)\sin(\beta-\gamma)$$
$$=-4\Pi\sin(\beta-\gamma).$$

Example 3. Prove that

(1) $\Sigma\tan(\beta-\gamma)=\Pi\tan(\beta-\gamma)$;

(2) $\Sigma\tan\beta\tan\gamma\tan(\beta-\gamma)=-\Pi\tan(\beta-\gamma)$.

(1) From Art. 118, if $A+B+C=0$, we see that

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

Hence by writing $A=\beta-\gamma$, $B=\gamma-\alpha$, $C=\alpha-\beta$, we have

$$\Sigma\tan(\beta-\gamma)=\Pi\tan(\beta-\gamma).$$

(2) From the formulæ for $\tan(\beta-\gamma)$, $\tan(\gamma-\alpha)$, $\tan(\alpha-\beta)$, we have

$$\Sigma(1+\tan\beta\tan\gamma)\tan(\beta-\gamma)=\Sigma(\tan\beta-\tan\gamma)=0;$$

whence by transposition

$$\Sigma\tan\beta\tan\gamma\tan(\beta-\gamma)=-\Sigma\tan(\beta-\gamma)$$
$$=-\Pi\tan(\beta-\gamma).$$

Example 4. Shew that

$$\Sigma \cos 3\alpha \sin(\beta-\gamma)=4\cos(\alpha+\beta+\gamma)\,\Pi \sin(\beta-\gamma).$$

Since $\quad 2\cos 3\alpha \sin(\beta-\gamma)=\sin(3\alpha+\beta-\gamma)-\sin(3\alpha-\beta+\gamma),$

we have

$$2\Sigma\cos 3\alpha\sin(\beta-\gamma)=\sin(3\alpha+\beta-\gamma)-\sin(3\alpha-\beta+\gamma)+\sin(3\beta+\gamma-\alpha)$$
$$-\sin(3\beta-\gamma+\alpha)+\sin(3\gamma+\alpha-\beta)-\sin(3\gamma-\alpha+\beta).$$

Combining the second and third terms, the fourth and fifth terms, the sixth and first terms, and dividing by 2, we have

$$\Sigma\cos 3\alpha\sin(\beta-\gamma)$$
$$=\cos(\alpha+\beta+\gamma)\{\sin 2(\beta-\alpha)+\sin 2(\gamma-\beta)+\sin 2(\alpha-\gamma)\}$$
$$=4\cos(\alpha+\beta+\gamma)\,\Pi\sin(\beta-\gamma).$$ [See Example 2.]

307. The following example is given as a specimen of a concise solution.

Example. If $(y+z)\tan\alpha+(z+x)\tan\beta+(x+y)\tan\gamma=0,$

and $\quad x\tan\beta\tan\gamma+y\tan\gamma\tan\alpha+z\tan\alpha\tan\beta=x+y+z,$

prove that $\quad x\sin 2\alpha+y\sin 2\beta+z\sin 2\gamma=0.$

From the given equations, we have

$$x(1-\tan\beta\tan\gamma)+y(1-\tan\gamma\tan\alpha)+z(1-\tan\alpha\tan\beta)=0,$$

and $$x(\tan\beta+\tan\gamma)+y(\tan\gamma+\tan\alpha)+z(\tan\alpha+\tan\beta)=0.$$

If we find the values of $x:y:z$ by cross multiplication, the denominator of x

$$=(1-\tan\gamma\tan\alpha)(\tan\alpha+\tan\beta)-(1-\tan\alpha\tan\beta)(\tan\gamma+\tan\alpha)$$
$$=(\tan\beta-\tan\gamma)+\tan^2\alpha(\tan\beta-\tan\gamma)$$
$$=(1+\tan^2\alpha)(\tan\beta-\tan\gamma)$$
$$=\sec^2\alpha(\tan\beta-\tan\gamma)$$
$$=\frac{\sec\alpha\sin(\beta-\gamma)}{\cos\alpha\cos\beta\cos\gamma}.$$

Hence $$\frac{x}{\sec\alpha\sin(\beta-\gamma)}=\frac{y}{\sec\beta\sin(\gamma-\alpha)}=\frac{z}{\sec\gamma\sin(\alpha-\beta)}=k \text{ say.}$$

$$\therefore\ x\sin 2\alpha+y\sin 2\beta+z\sin 2\gamma=k\Sigma\sin 2\alpha\sec\alpha\sin(\beta-\gamma)$$
$$=2k\Sigma\sin\alpha\sin(\beta-\gamma)$$
$$=0.$$

Allied formulæ in Algebra and Trigonometry.

308. From well-known algebraical identities we can deduce some interesting trigonometrical identities.

Example 1. In the identity

$$(x-a)(b-c)+(x-b)(c-a)+(x-c)(a-b)=0,$$

put $\qquad x=\cos 2\theta, \quad a=\cos 2\alpha, \quad b=\cos 2\beta, \quad c=\cos 2\gamma;$

then $\qquad x-a=\cos 2\theta-\cos 2\alpha=2\sin(\alpha+\theta)\sin(\alpha-\theta),$

and $\qquad b-c=\cos 2\beta-\cos 2\gamma=-2\sin(\beta+\gamma)\sin(\beta-\gamma);$

$$\therefore\ \Sigma\sin(\alpha+\theta)\sin(\alpha-\theta)\sin(\beta+\gamma)\sin(\beta-\gamma)=0.$$

Example 2. In the identity

$$\Sigma a^2(b-c)=-\Pi(b-c),$$

put $\qquad a=\sin^2\alpha, \quad b=\sin^2\beta, \quad c=\sin^2\gamma;$

then $\qquad b-c=\sin^2\beta-\sin^2\gamma=\sin(\beta+\gamma)\sin(\beta-\gamma);$

$$\therefore\ \Sigma\sin^4\alpha\sin(\beta+\gamma)\sin(\beta-\gamma)=-\Pi\sin(\beta+\gamma)\,.\,\Pi\sin(\beta-\gamma).$$

Example 3. In the identity

$$\Sigma a^3(b-c)=-(a+b+c)\,\Pi(b-c),$$

put $\qquad a=\cos\alpha, \quad b=\cos\beta, \quad c=\cos\gamma;$

$$\therefore\ \Sigma\cos^3\alpha(\cos\beta-\cos\gamma)=-(\cos\alpha+\cos\beta+\cos\gamma)\,\Pi(\cos\beta-\cos\gamma).$$

But $\qquad \Sigma\cos\alpha(\cos\beta-\cos\gamma)=0;$

$$\therefore\ \Sigma(4\cos^3\alpha-3\cos\alpha)(\cos\beta-\cos\gamma)$$
$$=-4(\cos\alpha+\cos\beta+\cos\gamma)\,\Pi(\cos\beta-\cos\gamma);$$

that is,

$$\Sigma\cos 3\alpha(\cos\beta-\cos\gamma)=-4(\cos\alpha+\cos\beta+\cos\gamma)\,\Pi(\cos\beta-\cos\gamma).$$

Example 4. If $a+b+c=0$, then $a^3+b^3+c^3=3abc$.

Here a, b, c may be any three quantities whose sum is zero; this condition is satisfied if we put $a=\cos(\alpha+\theta)\sin(\beta-\gamma)$, and b and c equal to corresponding quantities.

Thus $\quad \Sigma\cos^3(\alpha+\theta)\sin^3(\beta-\gamma)=3\Pi\cos(\alpha+\theta)\sin(\beta-\gamma).$

309. An algebraical identity may sometimes be established by the aid of Trigonometry.

Example. If $x+y+z=xyz$, prove that

$$x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)=4xyz.$$

By putting $x=\tan\alpha$, $y=\tan\beta$, $z=\tan\gamma$, we have

$$\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma;$$

whence
$$\tan\alpha=-\frac{\tan\beta+\tan\gamma}{1-\tan\beta\tan\gamma}=-\tan(\beta+\gamma);$$

$$\therefore\ \alpha=n\pi-(\beta+\gamma), \text{ where } n \text{ is an integer};$$

$$\therefore\ \alpha+\beta+\gamma=n\pi;$$

$$\therefore\ 2\alpha+2\beta+2\gamma=2n\pi.$$

From this relation it is easy to shew that

$$\tan 2\alpha+\tan 2\beta+\tan 2\gamma=\tan 2\alpha\tan 2\beta\tan 2\gamma;$$

$$\therefore\ \frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)};$$

$$\therefore\ x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)=4xyz.$$

EXAMPLES. XXIV. b.

Prove the following identities:

1. $\Sigma\sin(\alpha-\theta)\sin(\beta-\gamma)=0.$
2. $\Sigma\cos\beta\cos\gamma\sin(\beta-\gamma)=\Sigma\sin\beta\sin\gamma\sin(\beta-\gamma).$
3. $\Sigma\sin(\beta-\gamma)\cos(\beta+\gamma+\theta)=0.$
4. $\Sigma\cos 2(\beta-\gamma)=4\Pi\cos(\beta-\gamma)-1.$
5. $\Sigma\sin\beta\sin\gamma\sin(\beta-\gamma)=-\Pi\sin(\beta-\gamma).$
6. $\Sigma\cot(\alpha-\beta)\cot(\alpha-\gamma)+1=0.$
7. $\Sigma\sin 3\alpha\sin(\beta-\gamma)=4\sin(\alpha+\beta+\gamma)\,\Pi\sin(\beta-\gamma).$
8. $\Sigma\cos^3\alpha\sin(\beta-\gamma)=\cos(\alpha+\beta+\gamma)\,\Pi\sin(\beta-\gamma).$
9. $\Sigma\cos(\theta+\alpha)\cos(\beta+\gamma)\sin(\theta-\alpha)\sin(\beta-\gamma)=0.$
10. $\Sigma\sin^2\beta\sin^2\gamma\sin(\beta+\gamma)\sin(\beta-\gamma)$
$$=-\Pi\sin(\beta+\gamma)\,.\,\Pi\sin(\beta-\gamma).$$

Prove the following identities :

11. $\Sigma \cos 2\beta \cos 2\gamma \sin(\beta+\gamma)\sin(\beta-\gamma)$
$$= -4\Pi \sin(\beta+\gamma) \,.\, \Pi \sin(\beta-\gamma).$$

12. $\Sigma \cos 4\alpha \sin(\beta+\gamma)\sin(\beta-\gamma)$
$$= -8\Pi \sin(\beta+\gamma) \,.\, \Pi \sin(\beta-\gamma).$$

13. $\Sigma \sin 3\alpha (\sin\beta - \sin\gamma)$
$$= 4(\sin\alpha + \sin\beta + \sin\gamma)\,\Pi(\sin\beta - \sin\gamma).$$

14. $\Sigma \sin^3(\beta+\gamma)\sin^3(\beta-\gamma) = 3\Pi \sin(\beta+\gamma) \,.\, \Pi \sin(\beta-\gamma).$

15. $\Sigma \cos^3(\beta+\gamma+\theta)\sin^3(\beta-\gamma)$
$$= 3\Pi \cos(\beta+\gamma+\theta) \,.\, \Pi \sin(\beta-\gamma).$$

16. If $x+y+z=xyz$, prove that
$$\Sigma \frac{3x-x^3}{1-3x^2} = \Pi \frac{3x-x^3}{1-3x^2}.$$

17. If $yz+zx+xy=1$, prove that
$$\Sigma x(1-y^2)(1-z^2) = 4xyz.$$

310. From a trigonometrical identity many others may be derived by various substitutions.

For instance, if A, B, C are *any* angles, positive or negative, connected by the relation $A+B+C=\pi$, we know that
$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Let $\qquad A=\pi-2\alpha, \quad B=\pi-2\beta, \quad C=\pi-2\gamma;$

then $\qquad \sin A = \sin 2\alpha, \text{ and } \cos\frac{A}{2} = \sin\alpha.$

Also $\qquad 2(\alpha+\beta+\gamma) = 3\pi - (A+B+C) = 2\pi;$
$$\therefore\ \alpha+\beta+\gamma=\pi,$$
and $\qquad \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma.$

Again, let $\qquad A=\frac{\pi}{2}-\frac{\alpha}{2}, \quad B=\frac{\pi}{2}-\frac{\beta}{2}, \quad C=\frac{\pi}{2}-\frac{\gamma}{2};$

then $\qquad \sin A = \cos\frac{\alpha}{2}, \text{ and } \cos\frac{A}{2} = \cos\frac{\pi-\alpha}{4}.$

Also $\alpha+\beta+\gamma=3\pi-2(A+B+C)=3\pi-2\pi$;

$$\therefore\ \alpha+\beta+\gamma=\pi,$$

and $\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}=4\cos\frac{\pi-\alpha}{4}\cos\frac{\pi-\beta}{4}\cos\frac{\pi-\gamma}{4}$

Example. If $A+B+C=\pi$, shew that

$$\cos\frac{A}{2}+\cos\frac{B}{2}-\cos\frac{C}{2}=4\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi-C}{4}.$$

Put $\frac{\pi+A}{4}=\frac{\alpha}{2},\quad \frac{\pi+B}{4}=\frac{\beta}{2},\quad \frac{C-\pi}{4}=\frac{\gamma}{2};$

then $\cos\frac{A}{2}=\cos\left(\alpha-\frac{\pi}{2}\right)=\sin\alpha$, and $\cos\frac{C}{2}=\cos\left(\gamma+\frac{\pi}{2}\right)=-\sin\gamma$, so that the above identity becomes

$$\sin\alpha+\sin\beta+\sin\gamma=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$$

which is clearly true since

$$\alpha+\beta+\gamma=\frac{\pi}{2}+\frac{A+B+C}{2}=\frac{\pi}{2}+\frac{\pi}{2}=\pi.$$

311. When $A+B+C=n\pi$,

$$\tan(A+B)=\tan(n\pi-C)=-\tan C;$$

whence we obtain $\Sigma\tan A=\Pi\tan A.$

When $n=0$, the given condition is satisfied in the case of any three angles whose sum is 0; as for instance if

$$A=\beta+\gamma-2\alpha,\quad B=\gamma+\alpha-2\beta,\quad C=\alpha+\beta-2\gamma.$$

Hence $\Sigma\tan(\beta+\gamma-2\alpha)=\Pi\tan(\beta+\gamma-2\alpha).$

Example. If $\alpha+\beta+\gamma=0$, shew that

$$\Sigma\cot(\gamma+\alpha-\beta)\cot(\alpha+\beta-\gamma)=1.$$

Put $\beta+\gamma-\alpha=A,\quad \gamma+\alpha-\beta=B,\quad \alpha+\beta-\gamma=C;$

then, by addition,

$$A+B+C=\alpha+\beta+\gamma=0;$$

$$\therefore\ \cot(A+B)=-\cot C;$$

whence $\Sigma\cot A\cot B=1,$

that is, $\Sigma\cot(\gamma+\alpha-\beta)\cot(\alpha+\beta-\gamma)=1.$

312. The following example is a further illustration of the manner in which an identity may be established by appropriate substitutions in some simpler identity.

Example. Prove that

$$2\Pi \cos(\beta+\gamma) + \Pi \cos 2\alpha = \Sigma \cos 2\alpha \cos^2(\beta+\gamma).$$

In Example 5, Art. 133, we have proved that

$$4 \cos\alpha \cos\beta \cos\gamma = \Sigma \cos(\beta+\gamma-\alpha) + \cos(\alpha+\beta+\gamma).$$

In this identity first replace α, β, γ by $\beta+\gamma$, $\gamma+\alpha$, $\alpha+\beta$ respectively, and secondly replace α, β, γ by 2α, 2β, 2γ respectively.

Thus $\quad 8\Pi \cos(\beta+\gamma) = 2\Sigma \cos 2\alpha + 2 \cos 2(\alpha+\beta+\gamma),$

and $\quad 4\Pi \cos 2\alpha = \Sigma \cos 2(\beta+\gamma-\alpha) + \cos 2(\alpha+\beta+\gamma);$

whence by addition

$$\begin{aligned}
8\Pi \cos(\beta+\gamma) + 4\Pi \cos 2\alpha &= 2\Sigma \cos 2\alpha + \Sigma \cos 2(\beta+\gamma-\alpha) + 3 \cos 2(\alpha+\beta+\gamma) \\
&= 2\Sigma \cos 2\alpha + \Sigma\{\cos 2(\beta+\gamma-\alpha) + \cos 2(\alpha+\beta+\gamma)\} \\
&= 2\Sigma \cos 2\alpha + 2\Sigma \cos 2(\beta+\gamma) \cos 2\alpha \\
&= 2\Sigma \cos 2\alpha \{1 + \cos 2(\beta+\gamma)\} \\
&= 4\Sigma \cos 2\alpha \cos^2(\beta+\gamma);
\end{aligned}$$

$$\therefore \; 2\Pi \cos(\beta+\gamma) + \Pi \cos 2\alpha = \Sigma \cos 2\alpha \cos^2(\beta+\gamma).$$

313. Suppose that $A'B'C'$ is the pedal triangle of ABC, and let the sides and angles of the pedal triangle be denoted by a', b', c', and A', B', C', and its circum-radius by R'. Then from Arts. 224 and 225, we have

$$a' = a \cos A, \qquad b' = b \cos B, \qquad c' = c \cos C, \qquad R' = \frac{R}{2},$$

$$A' = 180° - 2A, \quad B' = 180° - 2B, \quad C' = 180° - 2C.$$

By means of these relations, we may from any identity proved for the triangle ABC derive another, as in the following case.

In the triangle ABC, we know that

$$\Sigma a \cos A = 4R \sin A \sin B \sin C;$$

hence in the pedal triangle $A'B'C'$,

$$\Sigma a' \cos A' = 4R' \sin A' \sin B' \sin C';$$

$$\therefore \; \Sigma a \cos A \cos(180° - 2A) = 2R\Pi \sin(180° - 2A);$$

that is, $\quad -\Sigma a \cos A \cos 2A = 2R \sin 2A \sin 2B \sin 2C.$

Example. In any triangle ABC, shew that

$$\frac{a^2\cos^2 A - b^2\cos^2 B - c^2\cos^2 C}{2bc\cos B\cos C} = \cos 2A.$$

In the pedal triangle $A'B'C'$, we have

$$\frac{b'^2+c'^2-a'^2}{2b'c'} = \cos A';$$

hence, by substituting the equivalents of a', b', c', A', we have

$$\frac{b^2\cos^2 B + c^2\cos^2 C - a^2\cos^2 A}{2bc\cos B\cos C} = \cos(180° - 2A) = -\cos 2A;$$

whence the required identity follows at once.

314. If $A_1B_1C_1$ be the ex-central triangle of ABC, we may, as in the preceding article, from any identity proved for the triangle ABC derive another by means of the relations

$$a_1 = a\,\text{cosec}\,\frac{A}{2},\quad b_1 = b\,\text{cosec}\,\frac{B}{2},\quad c_1 = c\,\text{cosec}\,\frac{C}{2},\quad R_1 = 2R,$$

$$A_1 = 90° - \frac{A}{2},\quad B_1 = 90° - \frac{B}{2},\quad C_1 = 90° - \frac{C}{2}.$$

315. The following Exercise consists of miscellaneous questions on the subject of this Chapter.

EXAMPLES. XXIV. c.

1. Shew that

$$\Sigma\cot(2\alpha+\beta-3\gamma)\cot(2\beta+\gamma-3\alpha) = 1.$$

2. Shew that

(1) $2\Pi\sin(\beta+\gamma) + \Pi\sin 2\alpha = \Sigma\sin 2\alpha\sin^2(\beta+\gamma);$

(2) $\Pi\sin(\beta+\gamma-\alpha) + 2\Pi\sin\alpha = \Sigma\sin^2\alpha\sin(\beta+\gamma-\alpha).$

3. In any triangle, prove that

(1) $a^2\cos^2 A - b^2\cos^2 B = Rc\cos C\sin 2(B-A);$

(2) $a^2\,\text{cosec}^2\frac{A}{2} - b^2\,\text{cosec}^2\frac{B}{2} = 4Rc\,\text{cosec}\,\frac{C}{2}\sin\frac{B-A}{2};$

(3) $\Sigma(b\cos B + c\cos C)\cot A = -2R\Sigma\cos 2A.$

4. If $$\sin 2\theta = 2 \sin \alpha \sin \gamma,$$
and $$\cos 2\theta = \cos 2\alpha \cos 2\beta = \cos 2\gamma \cos 2\delta,$$
prove that one value of $\tan \theta$ is $\tan \beta \tan \delta$.

5. If $$\tan \frac{\theta}{2} \tan \frac{\phi}{2} = \tan \frac{\gamma}{2},$$
and $$\sec \alpha \cos \theta = \sec \beta \cos \phi = \cos \gamma,$$
prove that $$\sin^2 \gamma = (\sec \alpha - 1)(\sec \beta - 1).$$

6. If $$\frac{\cos \theta - \cos \alpha}{\cos \theta - \cos \beta} = \frac{\sin^2 \alpha \cos \beta}{\sin^2 \beta \cos \alpha},$$
prove that one value of $\tan \frac{\theta}{2}$ is $\tan \frac{\alpha}{2} \tan \frac{\beta}{2}$.

7. If $\sin \theta = \cot \alpha \tan \gamma$ and $\tan \theta = \cos \alpha \tan \beta$, prove that one value of $\cos \theta$ is $\cos \beta \sec \gamma$.

8. If α and β are two different values of θ which satisfy
$$bc \cos \theta \cos \phi + ac \sin \theta \sin \phi = ab,$$
prove that
$$(b^2 + c^2 - a^2) \cos \alpha \cos \beta + (c^2 + a^2 - b^2) \sin \alpha \sin \beta = a^2 + b^2 - c^2.$$

9. If β and γ are two different values of θ which satisfy
$$\sin \alpha \cos \theta + \cos \alpha \sin \theta = \cos \alpha \sin \alpha,$$
prove that $$\frac{\cos \beta \cos \gamma}{\cos^2 \alpha} + \frac{\sin \beta \sin \gamma}{\sin^2 \alpha} = 1.$$

10. If β and γ are two different values of θ which satisfy
$$k^2 \cos \alpha \cos \theta + k (\sin \alpha + \sin \theta) + 1 = 0,$$
prove that $$k^2 \cos \beta \cos \gamma + k (\sin \beta + \sin \gamma) + 1 = 0.$$

11. If β and γ are two different values of θ which satisfy
$$\frac{\cos \theta \cos \phi}{\cos^2 \alpha} + \frac{\sin \theta \sin \phi}{\sin^2 \alpha} + 1 = 0,$$
prove that $$\frac{\cos \beta \cos \gamma}{\cos^2 \alpha} + \frac{\sin \beta \sin \gamma}{\sin^2 \alpha} + 1 = 0.$$

CHAPTER XXV.

MISCELLANEOUS THEOREMS AND EXAMPLES.

Inequalities. Maxima and Minima.

316. THE methods of proving trigonometrical inequalities are in many cases identical with those by which algebraical inequalities are established.

Example 1. Shew that $a^2 \tan^2\theta + b^2 \cot^2\theta > 2ab$.

We have $a^2 \tan^2\theta + b^2 \cot^2\theta = (a \tan\theta - b \cot\theta)^2 + 2ab$;

$$\therefore\ a^2 \tan^2\theta + b^2 \cot^2\theta > 2ab,$$

unless $\qquad a \tan\theta - b \cot\theta = 0, \text{ or } a \tan^2\theta = b.$

In this case the inequality becomes an equality.

This proposition may be otherwise expressed by saying that the *minimum value* of $a^2 \tan^2\theta + b^2 \cot^2\theta$ is $2ab$.

Example 2. Shew that

$$1 + \sin^2\alpha + \sin^2\beta > \sin\alpha + \sin\beta + \sin\alpha \sin\beta.$$

Since $(1 - \sin\alpha)^2$ is positive,

$$1 + \sin^2\alpha > 2 \sin\alpha;$$

similarly $\qquad 1 + \sin^2\beta > 2 \sin\beta,$

and $\qquad \sin^2\alpha + \sin^2\beta > 2 \sin\alpha \sin\beta.$

Adding and dividing by 2, we have

$$1 + \sin^2\alpha + \sin^2\beta > \sin\alpha + \sin\beta + \sin\alpha \sin\beta.$$

Example 3. When is $12 \sin\theta - 9 \sin^2\theta$ a maximum?

The expression $= 4 - (2 - 3 \sin\theta)^2$, and is therefore a maximum when $2 - 3 \sin\theta = 0$, so that its maximum value is 4.

317. *To find the numerically greatest values of*

$$a\cos\theta + b\sin\theta.$$

Let $\quad a = r\cos\alpha$ and $b = r\sin\alpha$,

so that $\quad r^2 = a^2 + b^2$ and $\tan\alpha = \dfrac{b}{a}$;

then

$$a\cos\theta + b\sin\theta = r(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$
$$= r\cos(\theta - \alpha).$$

Thus the expression is numerically greatest when

$$\cos(\theta - \alpha) = \pm 1;$$

that is, the greatest positive value $= r = \sqrt{a^2+b^2}$,
and the numerically greatest negative value $= -r = -\sqrt{a^2+b^2}$.

Hence, if $\quad c^2 > a^2 + b^2$,

the maximum value of $a\cos\theta + b\sin\theta + c$ is $c + \sqrt{a^2+b^2}$, and the minimum value is $c - \sqrt{a^2+b^2}$.

318. The expression $a\cos(\alpha+\theta) + b\cos(\beta+\theta)$

$$= (a\cos\alpha + b\cos\beta)\cos\theta - (a\sin\alpha + b\sin\beta)\sin\theta;$$

and therefore its numerically greatest values are equal to the positive and negative square roots of

$$(a\cos\alpha + b\cos\beta)^2 + (a\sin\alpha + b\sin\beta)^2;$$

that is, are equal to

$$\pm\sqrt{a^2 + b^2 + 2ab\cos(\alpha - \beta)}.$$

In like manner, we may find the maximum and minimum values of the sum of any number of expressions of the form $a\cos(\alpha+\theta)$ or $a\sin(\alpha+\theta)$.

319. *If* α *and* β *are two angles, each lying between* 0 *and* $\dfrac{\pi}{2}$, *whose sum is given, to find the maximum value of* $\cos\alpha\cos\beta$ *and of* $\cos\alpha + \cos\beta$.

Suppose that $\quad \alpha + \beta = \sigma$;

then

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$
$$= \cos\sigma + \cos(\alpha - \beta),$$

and is therefore a maximum when $\alpha-\beta=0$, or $\alpha=\beta=\frac{\sigma}{2}$.

Thus the maximum value of $\cos\alpha\cos\beta$ is $\cos^2\frac{\sigma}{2}$.

Again,
$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$=2\cos\frac{\sigma}{2}\cos\frac{\alpha-\beta}{2},$$

and is therefore a maximum when $\alpha=\beta=\frac{\sigma}{2}$.

Thus the maximum value of $\cos\alpha+\cos\beta$ is $2\cos\frac{\sigma}{2}$.

Similar theorems hold in case of the sine.

Example 1. If A, B, C are the angles of a triangle, find the maximum value of

$$\sin A+\sin B+\sin C \text{ and of } \sin A\sin B\sin C.$$

Let us suppose that C remains constant, while A and B vary.

$$\sin A+\sin B+\sin C=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+\sin C$$
$$=2\cos\frac{C}{2}\cos\frac{A-B}{2}+\sin C.$$

This expression is a maximum when $A=B$.

Hence, so long as any two of the angles A, B, C are unequal, the expression $\sin A+\sin B+\sin C$ is not a maximum; that is, the expression is a maximum when $A=B=C=60°$.

Thus the maximum value $=3\sin 60°=\frac{3\sqrt{3}}{2}$.

Again,
$$2\sin A\sin B\sin C=\{\cos(A-B)-\cos(A+B)\}\sin C$$
$$=\{\cos(A-B)+\cos C\}\sin C.$$

This expression is a maximum when $A=B$.

Hence, by reasoning as before, $\sin A\sin B\sin C$ has its maximum value when $A=B=C=60°$.

Thus the maximum value $=\sin^3 60°=\frac{3\sqrt{3}}{8}$.

Example 2. If α and β are two angles, each lying between 0 and $\frac{\pi}{2}$, whose sum is constant, find the minimum value of $\sec\alpha+\sec\beta$.

We have $$\sec\alpha+\sec\beta=\frac{1}{\cos\alpha}+\frac{1}{\cos\beta}=\frac{\cos\alpha+\cos\beta}{\cos\alpha\cos\beta}$$

$$=\frac{4\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{\cos(\alpha+\beta)+\cos(\alpha-\beta)}=\frac{2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{\cos^2\frac{\alpha-\beta}{2}-\sin^2\frac{\alpha+\beta}{2}}$$

$$=\cos\frac{\alpha+\beta}{2}\left(\frac{1}{\cos\frac{\alpha-\beta}{2}+\sin\frac{\alpha+\beta}{2}}+\frac{1}{\cos\frac{\alpha-\beta}{2}-\sin\frac{\alpha+\beta}{2}}\right).$$

Since $\alpha+\beta$ is constant, this expression is least when the denominators are greatest; that is, when $\alpha=\beta=\frac{\alpha+\beta}{2}$.

Thus the minimum value is $2\sec\frac{\alpha+\beta}{2}$.

320. *If $\alpha, \beta, \gamma, \delta, \ldots\ldots$ are n angles, each lying between* 0 *and $\frac{\pi}{2}$, whose sum is constant, to find the maximum value of*

$$\cos\alpha\cos\beta\cos\gamma\cos\delta\ldots\ldots$$

Let $$\alpha+\beta+\gamma+\delta+\ldots\ldots=\sigma.$$

Suppose that any two of the angles, say α and β, are unequal; then if in the given product we replace the two unequal factors $\cos\alpha$ and $\cos\beta$ by the two equal factors $\cos\frac{\alpha+\beta}{2}$ and $\cos\frac{\alpha+\beta}{2}$, the value of the product is increased while the sum of the angles remains unaltered. Hence so long as any two of the angles $\alpha, \beta, \gamma, \delta, \ldots$ are unequal the product is not a maximum; that is, the product is a maximum when all the angles are equal. In this case each angle $=\frac{\sigma}{n}$.

Thus the maximum value is $\cos^n\frac{\sigma}{n}$.

In like manner we may shew that

the maximum value of $\cos\alpha+\cos\beta+\cos\gamma+\ldots\ldots=n\cos\frac{\sigma}{n}$.

321. The methods of solution used in the following examples are worthy of notice.

Example 1. Shew that $\tan 3\alpha \cot \alpha$ cannot lie between 3 and $\frac{1}{3}$.

We have $\tan 3\alpha \cot \alpha = \dfrac{\tan 3\alpha}{\tan \alpha} = \dfrac{3 - \tan^2 \alpha}{1 - 3\tan^2 \alpha} = n$ say;

$$\therefore \tan^2 \alpha = \frac{n-3}{3n-1} = \frac{3-n}{1-3n}.$$

These two fractional values of $\tan^2 \alpha$ must be positive, and therefore n must be greater than 3 or less than $\frac{1}{3}$.

Example 2. If a and b are positive quantities, of which a is the greater, find the minimum value of $a \sec \theta - b \tan \theta$.

Denote the expression by x, and put $\tan \theta = t$;

then
$$x = a\sqrt{1+t^2} - bt;$$
$$\therefore b^2t^2 + 2bxt + x^2 = a^2(1+t^2);$$
$$\therefore t^2(b^2 - a^2) + 2bxt + x^2 - a^2 = 0.$$

In order that the values of t found from this equation may be real,
$$b^2x^2 > (b^2 - a^2)(x^2 - a^2);$$
$$\therefore 0 > a^2(a^2 - b^2 - x^2);$$
$$\therefore x^2 > a^2 - b^2.$$

Thus the minimum value is $\sqrt{a^2 - b^2}$.

Example 3. If a, b, c, k are constant quantities and α, β, γ variable quantities subject to the relation $a \tan \alpha + b \tan \beta + c \tan \gamma = k$, find the minimum value of $\tan^2 \alpha + \tan^2 \beta + \tan^2 \gamma$.

By multiplying out and re-arranging the terms, we have
$$(a^2+b^2+c^2)(\tan^2\alpha + \tan^2\beta + \tan^2\gamma) - (a\tan\alpha + b\tan\beta + c\tan\gamma)^2$$
$$= (b\tan\gamma - c\tan\beta)^2 + (c\tan\alpha - a\tan\gamma)^2 + (a\tan\beta - b\tan\alpha)^2.$$

But the minimum value of the right side of this equation is zero; hence the minimum value of
$$(a^2+b^2+c^2)(\tan^2\alpha + \tan^2\beta + \tan^2\gamma) - k^2 = 0;$$
that is, the minimum value of
$$\tan^2\alpha + \tan^2\beta + \tan^2\gamma = \frac{k^2}{a^2+b^2+c^2}.$$

EXAMPLES. XXV. a.

When θ is variable find the minimum value of the following expressions:

1. $p \cot\theta + q \tan\theta$.
2. $4\sin^2\theta + \operatorname{cosec}^2\theta$.
3. $8\sec^2\theta + 18\cos^2\theta$.
4. $3 - 2\cos\theta + \cos^2\theta$.

Prove the following inequalities:

5. $\tan^2\alpha + \tan^2\beta + \tan^2\gamma > \tan\beta\tan\gamma + \tan\gamma\tan\alpha + \tan\alpha\tan\beta$.
6. $\sin^2\alpha + \sin^2\beta > 2(\sin\alpha + \sin\beta - 1)$.

When θ is variable, find the maximum value of

7. $\sin\theta + \cos\theta$.
8. $\cos\theta + \sqrt{3}\sin\theta$.
9. $a\cos(\alpha + \theta) + b\sin\theta$.
10. $p\cos\theta + q\sin(\alpha + \theta)$.

If $\sigma = \alpha + \beta$, where α and β are two angles each lying between 0 and $\frac{\pi}{2}$, and σ is constant, find the maximum or minimum value of

11. $\sin\alpha + \sin\beta$.
12. $\sin\alpha\sin\beta$.
13. $\tan\alpha + \tan\beta$.
14. $\operatorname{cosec}\alpha + \operatorname{cosec}\beta$.

If A, B, C are the angles of a triangle, find the maximum or minimum value of

15. $\cos A\cos B\cos C$.
16. $\cot A + \cot B + \cot C$.
17. $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2}$.
18. $\sec A + \sec B + \sec C$.
19. $\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2}$. [*Use* $\Sigma \tan\frac{B}{2}\tan\frac{C}{2} = 1$.]
20. $\cot^2 A + \cot^2 B + \cot^2 C$. [*Use* $\Sigma \cot B\cot C = 1$.]
21. If $b^2 < 4ac$, find the maximum and minimum values of
$$a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta.$$

22. If α, β, γ lie between 0 and $\frac{\pi}{2}$, shew that

$$\sin\alpha+\sin\beta+\sin\gamma>\sin(\alpha+\beta+\gamma).$$

23. If a and b are two positive quantities of which a is the greater, shew that $a\operatorname{cosec}\theta>b\cot\theta+\sqrt{a^2-b^2}$.

24. Shew that $\dfrac{\sec^2\theta-\tan\theta}{\sec^2\theta+\tan\theta}$ lies between 3 and $\dfrac{1}{3}$.

25. Find the maximum value of $\dfrac{\tan^2\theta-\cot^2\theta+1}{\tan^2\theta+\cot^2\theta-1}$.

26. If a, b, c, k are constant positive quantities, and α, β, γ variable quantities subject to the relation

$$a\cos\alpha+b\cos\beta+c\cos\gamma=k,$$

find the minimum value of

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma \text{ and of } a\cos^2\alpha+b\cos^2\beta+c\cos^2\gamma.$$

Elimination.

322. No general rules can be given for the elimination of some assigned quantity or quantities from two or more trigonometrical equations. The form of the equations will often suggest special methods, and in addition to the usual algebraical artifices we shall always have at our disposal the identical relations subsisting between the trigonometrical functions. Thus suppose it is required to eliminate θ from the equations

$$x\cos\theta=a, \quad y\cot\theta=b.$$

Here $$\sec\theta=\frac{x}{a}, \text{ and } \tan\theta=\frac{y}{b};$$

but for all values of θ, we have

$$\sec^2\theta-\tan^2\theta=1.$$

$\therefore$ by substitution,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

From this example we see that since θ satisfies *two* equations (either of which is sufficient to determine θ) there is a relation, independent of θ, which subsists between the coefficients and constants of the equations. To determine this relation we *eliminate* θ, and the result is called the *eliminant* of the given equations.

323. The following examples will illustrate some useful methods of elimination.

Example 1. Eliminate θ between the equations

$$l\cos\theta + m\sin\theta + n = 0 \text{ and } p\cos\theta + q\sin\theta + r = 0.$$

From the given equations, we have by cross multiplication

$$\frac{\cos\theta}{mr - nq} = \frac{\sin\theta}{np - lr} = \frac{1}{lq - mp};$$

$$\therefore \cos\theta = \frac{mr - nq}{lq - mp}, \text{ and } \sin\theta = \frac{np - lr}{lq - mp};$$

whence by squaring, adding, and clearing of fractions, we obtain

$$(mr - nq)^2 + (np - lr)^2 = (lq - mp)^2.$$

The particular instance in which $q = l$ and $p = -m$ is of frequent occurrence in Analytical Geometry. In this case the eliminant may be written down at once; for we have

$$l\cos\theta + m\sin\theta = -n,$$

and

$$l\sin\theta - m\cos\theta = -r;$$

whence by squaring and adding, we obtain

$$l^2 + m^2 = n^2 + r^2.$$

Example 2. Eliminate θ between the equations

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = c^2 \text{ and } l\tan\theta = m.$$

From the second equation, we have

$$\frac{\sin\theta}{m} = \frac{\cos\theta}{l} = \frac{\sqrt{\sin^2\theta + \cos^2\theta}}{\sqrt{m^2 + l^2}} = \frac{1}{\sqrt{m^2 + l^2}};$$

$$\therefore \sin\theta = \frac{m}{\sqrt{m^2 + l^2}}, \text{ and } \cos\theta = \frac{l}{\sqrt{m^2 + l^2}}.$$

By substituting in the first equation, we obtain

$$\frac{ax}{l} - \frac{by}{m} = \frac{c^2}{\sqrt{m^2 + l^2}}.$$

Example 3. Eliminate θ between the equations

$$x=\cot\theta+\tan\theta \text{ and } y=\sec\theta-\cos\theta.$$

From the given equations, we have

$$x=\frac{1}{\tan\theta}+\tan\theta=\frac{1+\tan^2\theta}{\tan\theta}$$

$$=\frac{\sec^2\theta}{\tan\theta},$$

and

$$y=\sec\theta-\frac{1}{\sec\theta}=\frac{\sec^2\theta-1}{\sec\theta}$$

$$=\frac{\tan^2\theta}{\sec\theta}.$$

From these values of x and y we obtain

$$x^2y=\sec^3\theta \text{ and } xy^2=\tan^3\theta.$$

But

$$\sec^2\theta-\tan^2\theta=1;$$

$$\therefore (x^2y)^{\frac{2}{3}}-(xy^2)^{\frac{2}{3}}=1;$$

that is,

$$x^{\frac{4}{3}}y^{\frac{2}{3}}-x^{\frac{2}{3}}y^{\frac{4}{3}}=1.$$

Example 4. Eliminate θ from the equations

$$\frac{x}{a}=\cos\theta+\cos 2\theta \text{ and } \frac{y}{b}=\sin\theta+\sin 2\theta.$$

From the given equations, we have

$$\frac{x}{a}=2\cos\frac{3\theta}{2}\cos\frac{\theta}{2},$$

and

$$\frac{y}{b}=2\sin\frac{3\theta}{2}\cos\frac{\theta}{2};$$

whence by squaring and adding, we obtain

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=4\cos^2\frac{\theta}{2}.$$

But

$$\frac{x}{a}=2\cos\frac{\theta}{2}\left(4\cos^3\frac{\theta}{2}-3\cos\frac{\theta}{2}\right)$$

$$=2\cos^2\frac{\theta}{2}\left(4\cos^2\frac{\theta}{2}-3\right);$$

$$\therefore \frac{2x}{a}=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-3\right).$$

324. The following examples are instances of the elimination of two quantities.

Example 1. Eliminate θ and ϕ from the equations

$$a\sin^2\theta+b\cos^2\theta=m,\quad b\sin^2\phi+a\cos^2\phi=n,\quad a\tan\theta=b\tan\phi.$$

From the first equation, we have

$$a\sin^2\theta+b\cos^2\theta=m(\sin^2\theta+\cos^2\theta);$$

$$\therefore\ (a-m)\sin^2\theta=(m-b)\cos^2\theta;$$

$$\therefore\ \tan^2\theta=\frac{m-b}{a-m}.$$

From the second equation, we have

$$b\sin^2\phi+a\cos^2\phi=n(\sin^2\phi+\cos^2\phi);$$

$$\therefore\ \tan^2\phi=\frac{n-a}{b-n}.$$

From the third equation,

$$a^2\tan^2\theta=b^2\tan^2\phi;$$

$$\therefore\ \frac{a^2(m-b)}{a-m}=\frac{b^2(n-a)}{b-n};$$

$$\therefore\ a^2(bm-b^2-mn+bn)=b^2(an-a^2-mn+am);$$

$$\therefore\ mab(a-b)+nab(a-b)=mn(a^2-b^2);$$

$$\therefore\ mab+nab=mn(a+b);$$

$$\therefore\ \frac{1}{n}+\frac{1}{m}=\frac{1}{a}+\frac{1}{b}.$$

Example 2. Eliminate θ and ϕ from the equations

$$x\cos\theta+y\sin\theta=x\cos\phi+y\sin\phi=2a,\quad 2\sin\frac{\theta}{2}\sin\frac{\phi}{2}=1.$$

From the data, we see that θ and ϕ are the roots of the equation

$$x\cos\alpha+y\sin\alpha=2a;$$

$$\therefore\ (x\cos\alpha-2a)^2=y^2\sin^2\alpha=y^2(1-\cos^2\alpha);$$

$$\therefore\ (x^2+y^2)\cos^2\alpha-4ax\cos\alpha+4a^2-y^2=0,$$

which is a quadratic in $\cos\alpha$ with roots $\cos\theta$ and $\cos\phi$.

But $$1=4\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2}=(1-\cos\theta)(1-\cos\phi);$$

whence $$\cos\theta+\cos\phi=\cos\theta\cos\phi;$$

$$\therefore\ \frac{4ax}{x^2+y^2}=\frac{4a^2-y^2}{x^2+y^2};$$

$$\therefore\ y^2=4a(a-x).$$

325. The method exhibited in the following example is one frequently used in Analytical Geometry.

Example. If a, b, c are unequal, find the relations that hold between the coefficients, when

$$a\cos\theta+b\sin\theta=c,$$

and $$a\cos^2\theta+2a\cos\theta\sin\theta+b\sin^2\theta=c.$$

The required relation will be obtained by eliminating θ from the given equations. This is most conveniently done by making each equation homogeneous in $\sin\theta$ and $\cos\theta$.

From the first equation, we have

$$a\cos\theta+b\sin\theta=c\sqrt{\cos^2\theta+\sin^2\theta};$$

whence, by squaring and transposing,

$$(a^2-c^2)\cos^2\theta+2ab\cos\theta\sin\theta+(b^2-c^2)\sin^2\theta=0\ \ldots\ldots(1).$$

From the second equation, we have

$$a\cos^2\theta+2a\cos\theta\sin\theta+b\sin^2\theta=c(\cos^2\theta+\sin^2\theta);$$

$$\therefore\ (a-c)\cos^2\theta+2a\cos\theta\sin\theta+(b-c)\sin^2\theta=0\ \ldots\ldots(2).$$

From (1) and (2) we have by cross-multiplication,

$$\frac{\cos^2\theta}{2ab(b-c)-2a(b^2-c^2)}=\frac{\cos\theta\sin\theta}{(b^2-c^2)(a-c)-(a^2-c^2)(b-c)}$$

$$=\frac{\sin^2\theta}{2a(a^2-c^2)-2ab(a-c)};$$

or $$\frac{\cos^2\theta}{-2ac(b-c)}=\frac{\cos\theta\sin\theta}{(b-c)(a-c)(b-a)}=\frac{\sin^2\theta}{2a(a-c)(a+c-b)};$$

$$\therefore\ -4a^2c(b-c)(a-c)(a+c-b)=(b-c)^2(a-c)^2(b-a)^2.$$

By supposition, the quantities a, b, c are unequal; hence dividing by $(b-c)(a-c)$, we obtain

$$4a^2c(a+c-b)+(b-c)(a-c)(a-b)^2=0.$$

EXAMPLES. XXV. b.

Eliminate θ between the equations:

1. $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1, \quad \frac{x}{a}\sin\theta-\frac{y}{b}\cos\theta=1.$

2. $a\sec\theta-x\tan\theta=y, \quad b\sec\theta+y\tan\theta=x.$

3. $\cos\theta+\sin\theta=a, \quad \cos 2\theta=b,$

4. $x=\sin\theta+\cos\theta, \quad y=\tan\theta+\cot\theta.$

5. $a=\cot\theta+\cos\theta, \quad b=\cot\theta-\cos\theta.$

Find the eliminant in each of the following cases:

6. $x=\cot\theta+\tan\theta, \quad y=\operatorname{cosec}\theta-\sin\theta.$

7. $\operatorname{cosec}\theta-\sin\theta=a^3, \quad \sec\theta-\cos\theta=b^3.$

8. $4x=3a\cos\theta+a\cos 3\theta, \quad 4y=3a\sin\theta-a\sin 3\theta.$

9. $x=\tan^2\theta\,(a\tan\theta-x), \quad y=\sec^2\theta\,(y-a\sec\theta).$

10. $x=a\cos\theta\,(2\cos 2\theta-1), \quad y=b\sin\theta\,(4\cos^2\theta-1).$

11. If $\cos(\theta-\alpha)=a$, and $\sin(\theta-\beta)=b$,
shew that $a^2-2ab\sin(\alpha-\beta)+b^2=\cos^2(\alpha-\beta).$

Find the relation that must hold between x and y if

12. $x+y=3-\cos 4\theta, \quad x-y=4\sin 2\theta.$

13. $x=\sin\theta+\cos\theta\sin 2\theta, \quad y=\cos\theta+\sin\theta\sin 2\theta.$

14. If $\sin\theta+\cos\theta=a$, and $\sin 2\theta+\cos 2\theta=b$,
shew that $(a^2-b-1)^2=a^2(2-a^2).$

15. If $\cos\theta-\sin\theta=b$, and $\cos 3\theta+\sin 3\theta=a$,
shew that $a=3b-2b^3.$

16. Eliminate θ from the equations:
$$a\cos\theta-b\sin\theta=c, \quad 2ab\cos 2\theta+(a^2-b^2)\sin 2\theta=2c^2.$$

17. If $x=a\cos\theta+b\cos 2\theta$, and $y=a\sin\theta+b\sin 2\theta$,
shew that $a^2\{(x+b)^2+y^2\}=(x^2+y^2-b^2)^2.$