EXAMPLES. XIV. b.

[*When required the values of* log 2, log 3, log 7 *given on p.* 145 *may be used.*]

Find the value of

1. $\left(\dfrac{147 \times 375}{126 \times 16}\right)^{\frac{2}{3}}$, given $\log 9{\cdot}076226 = {\cdot}9579053$.

2. $\sqrt[3]{378} \times \sqrt{108} \div (\sqrt[6]{1008} \times \sqrt[3]{486})$,

   given $\log 301824 = 5{\cdot}4797536$.

3. $(1080)^{\frac{1}{2}} \times ({\cdot}24)^{\frac{5}{3}} \times 810$,

   given $\log 2467266 = 6{\cdot}3922160$.

Calculate to two decimal places the values of

4. $\log_{20} 800$. 5. $\log_3 49$. 6. $\log_{125} 4000$.

7. Find how many ciphers there are before the first significant digits in

$$({\cdot}00378)^{\frac{40}{3}} \text{ and } ({\cdot}0\dot{2}5\dot{9})^{50}.$$

8. To what base is 3 the logarithm of 11000?

given $\log 11 = 1{\cdot}0413927$ and $\log 222398 = 5{\cdot}3471309$.

Solve to two decimal places the equations:

9. $2^{x-1} = 5$. 10. $3^{x-4} = 7$. 11. $5^{1-x} = 6^{x-3}$.

12. $5^x = 2^{-y}$ and $5^{2+y} = 2^{2-x}$.

13. $2^x = 3^y$ and $2^{y+1} = 3^{x-1}$.

14. Given $\log 28 = a$, $\log 21 = b$, $\log 25 = c$, find $\log 27$ and $\log 224$ in terms of a, b, c.

15. Given $\log 242 = a$, $\log 80 = b$, $\log 45 = c$, find $\log 36$ and $\log 66$ in terms of a, b, c.

MISCELLANEOUS EXAMPLES. E.

1. Prove that

$$\cos(30°+A)\cos(30°-A)-\cos(60°+A)\cos(60°-A)=\tfrac{1}{2}.$$

2. If $A+B+C=180°$, shew that

$$\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

3. If $a=2$, $c=\sqrt{2}$, $B=15°$, solve the triangle.

4. Shew that $\cos a+\tan\frac{a}{2}\sin a=\cot\frac{a}{2}\sin a-\cos a$.

5. If $b\cos A=a\cos B$, shew that the triangle is isosceles.

6. Prove that

(1) $\sin\theta(\sin 3\theta+\sin 5\theta+\sin 7\theta+\sin 9\theta)=\sin 6\theta\sin 4\theta$;

(2) $\dfrac{\sin a+\sin 3a+\sin 5a+\sin 7a}{\cos a+\cos 3a+\cos 5a+\cos 7a}=\tan 4a$.

7. Shew that $\dfrac{\cos 3a}{\sin a}+\dfrac{\sin 3a}{\cos a}=2\cot 2a$.

8. If $b=a(\sqrt{3}-1)$, $C=30°$, find A and B.

9. Shew that $\tan 4a=\dfrac{4\tan a-4\tan^3 a}{1-6\tan^2 a+\tan^4 a}$.

10. In a triangle, shew that

(1) $a^2\cos 2B+b^2\cos 2A=a^2+b^2-4ab\sin A\sin B$;

(2) $4\left(bc\cos^2\frac{A}{2}+ca\cos^2\frac{B}{2}+ab\cos^2\frac{C}{2}\right)=(a+b+c)^2$.

11. If $a^4+b^4+c^4=2c^2(a^2+b^2)$, prove that $C=45°$ or $135°$.

[*Solve as a quadratic in* c^2.]

12. If in a triangle $\cos 3A+\cos 3B+\cos 3C=1$, shew that one angle must be $120°$.

CHAPTER XV.

THE USE OF LOGARITHMIC TABLES.

171. We shall now explain the use of logarithmic Tables to which reference has been made in the previous chapter.

In a book of Tables there will usually be found the *mantissæ* of the logarithms of all *integers* from 1 to 100000; the *characteristics* can be written down by inspection and are therefore omitted. [Art. 162.]

The logarithm of any number consisting of not more than 5 significant digits can be obtained directly from these Tables. For instance, suppose the logarithm of 336·34 is required. Opposite to 33634 we find the figures 5267785; this, with the decimal point prefixed, is the mantissa for the logarithms of all numbers whose significant digits are the same as 33634. We have therefore only to prefix the characteristic 2, and we obtain

$$\log 336{\cdot}34 = 2{\cdot}5267785.$$

Similarly, $\log 33634 = 4{\cdot}5267785,$

and $\log {\cdot}0033634 = \bar{3}{\cdot}5267785.$

172. Suppose now that we required log 33634·392.

Since this number contains more than 5 significant digits it cannot be obtained directly from the tables; but it lies between the two consecutive numbers 33634 and 33635, and therefore its logarithm lies between the logarithms of these two numbers. If we pass from 33634 to 33635, making an increase of 1 in the number, the corresponding increase in the logarithm as obtained from the tables is ·0000129. If now we pass from 33634 to 33634·392, making an increase of ·392 in the number, the increase in the logarithm will be $\cdot 392 \times \cdot 0000129$, provided that the increase in the logarithm is proportional to the increase in the number.

Now it can be proved that *when the increase made is small in comparison with the number, the increase in the logarithm is very nearly proportional to the increase in the number.*

This principle is known as the **Rule of Proportional Parts**.

The application of this rule will be illustrated in the examples which follow.

173. In order to make the explanations more intelligible we give here an Extract from Chambers' *Mathematical Tables.*

No.	0	1	2	3	4	5	6	7	8	9	Diff.
3361	526 4685	4814	4944	5073	5202	5331	5460	5590	5719	5848	
62	5977	6106	6235	6365	6494	6623	6752	6881	7010	7140	129
63	7269	7398	7527	7656	7785	7914	8043	8173	8302	8431	1 13
64	8560	8689	8818	8947	9076	9205	9334	9463	9593	9722	2 26
65	9851	9980	0109	0238	0367	0496	0625	0754	0883	1012	3 39
											4 52
66	527 1141	1270	1399	1528	1657	1786	1915	2044	2173	2302	5 65
67	2431	2560	2689	2818	2947	3076	3205	3334	3463	3592	6 77
68	3721	3850	3979	4108	4237	4366	4494	4623	4752	4881	7 90
69	5010	5139	5268	5397	5526	5655	5783	5912	6041	6170	8 103
70	6299	6428	6557	6686	6814	6943	7072	7201	7330	7459	9 116

174. Suppose that log 33635 is required.

In the third horizontal line we have the logarithms of numbers beginning with 3363. As the next digit is 5 we choose from this line the mantissa which stands under the column 5. We have now only to prefix the characteristic and we obtain $\log 33635 = 4{\cdot}5267914$.

Similarly, $\log 33651 = 4{\cdot}5269980$,

and $\log 33652 = 4{\cdot}5\overline{270109}$,

the transition in the mantissæ from 526... to 527... being shewn by the bar drawn over 0109. This bar is repeated over each of the subsequent logarithms as far as the end of the line, and in the next line the mantissæ begin with 527.

Example. Find log 33634·392.

$$\begin{array}{rl} \text{From the Tables,} \quad \log 33635 = & 4{\cdot}5267914 \\ \log 33634 = & 4{\cdot}5267785 \\ \hline \text{difference for } 1 = & {\cdot}0000129 \end{array}$$

Now by the Rule of Proportional Parts, log 33634·392 will be greater than log 33634 by ·392 times the difference for 1; hence to 7 places of decimals, we have

$$\begin{array}{r|l} {\cdot}0000129 & \\ {\cdot}392 & \\ \hline & 258 \\ 11 & 61 \\ 38 & 7 \\ \hline {\cdot}0000050 & 568 \end{array}$$

$$\begin{array}{rl} \log 33634 \qquad = & 4{\cdot}5267785 \\ \text{proportional difference for } {\cdot}392 = & {\cdot}0000051 \\ \hline \therefore \log 33634{\cdot}392 = & 4{\cdot}5267836 \end{array}$$

In practice, the difference for 1 is usually quoted without the ciphers; if therefore we *treat the difference* 129 *as a whole number*, on multiplying by ·392 we obtain the product 50·568, and we take the digits given by its integral part (51 approximately) as the proportional increase for ·392.

175. The method of calculating the proportional difference for ·392 which we have explained is that which must be adopted when we have nothing given but the logarithms of two consecutive numbers between which lies the number whose logarithm we are seeking.

But when the Tables are used the calculation is facilitated by means of the proportional differences standing in the column to the right. This gives the differences for *tenths* of unity.

The difference for ·392 is obtained as follows.

$${\cdot}392 \times 129 = \left(\frac{3}{10} + \frac{9}{100} + \frac{2}{1000}\right) \times 129 = 39 + 11{\cdot}6 + {\cdot}26 = 50{\cdot}86.$$

The difference for 9 quoted in the margin (really 9 *tenths*) is 116, and therefore the difference for 9 hundredths is 11·6; and similarly the difference for 2 thousandths is ·26.

In practical work, the following arrangement is adopted.

	log 33634	= 4·5267785	
add for	3	39	
	9	11	6
	2		26
∴	log 33634·392	= 4·5267836	

176. The following example is solved more concisely as a model for the student. In the column on the left we work from the data of the question; in the column on the right we obtain the logarithm by the use of the Tables independently of the two given logarithms.

Example. Find log 33·656208, having given

log 33656 = 4·5270625 and log 33657 = 4·5270754.

log 33·657 =	1·5270754
log 33·656 =	1·5270625
diff. for ·001 =	129
	·208
	1|032
	25|80
diff. for ·000208 =	26|832
log 33·656 =	1·5270625
log 33·656208 =	1·5270652

From the Tables, we have

	log 33·656	= 1·5270625	
add for	2	26	
	0	0	
	8	1	03

log 33·656208 = 1·5270652

177. The Rule of Proportional Parts also enables us to find the number corresponding to a given logarithm.

Example 1. Find the number whose logarithm is $\bar{2}$·5274023, having given log 3·3683 = ·5274108 and log 3·3682 = ·5273979.

Let x be the required number; then

log x =	$\bar{2}$·5274023
log ·033682 =	$\bar{2}$·5273979
diff. =	44

log ·033683 =	$\bar{2}$·5274108
log ·033682 =	$\bar{2}$·5273979
diff. for ·000001 =	129

hence x lies between ·033682 and ·033683, and is greater than ·033682 by $\frac{44}{129} \times$ ·000001, that is by ·00000034.

∴ x = ·03368234.

```
129 ) 440 ( 34
      387
       530
       516
```

In working from the Tables, we proceed as follows.

$$\begin{array}{lcr} \log x & = & \bar{2}\cdot5274023 \\ \log \cdot033682 & = & \bar{2}\cdot5273979 \\ \hline & & 44 \\ & 3 & 39 \\ \hline & & 50 \\ & 4 & 52 \\ \hline \end{array}$$

$\therefore\ x = \cdot03368234.$

We are saved the trouble of the division, as the multiples of 129 which occur during the work are given in the approximate forms 39 and 52 in the difference column opposite to the numbers 3 and 4.

Example 2. Find the fifth root of $\cdot0025612$, having given
$\log 2\cdot5612 = \cdot4084435$, $\log 3\cdot0317 = \cdot4816862$, $\log 3\cdot0318 = \cdot4817005$.

Let $x = (\cdot0025612)^{\frac{1}{5}}$; then

$$\log x = \frac{1}{5}\log(\cdot0025612) = \frac{1}{5}(\bar{3}\cdot4084435) = \frac{1}{5}(\bar{5} + 2\cdot4084435);$$

$$= \bar{1}\cdot4816887.$$

$$\begin{array}{rr} \log x = & \bar{1}\cdot4816887 \\ \log \cdot30317 = & \bar{1}\cdot4816862 \\ \hline \text{diff.} = & 25 \end{array} \qquad \begin{array}{rr} \log \cdot30318 = & \bar{1}\cdot4817005 \\ \log \cdot30317 = & \bar{1}\cdot4816862 \\ \hline \text{diff. for } \cdot00001 = & 143 \end{array}$$

$\therefore$ proportional increase $= \dfrac{25}{143} \times \cdot00001 = \cdot00000175.$

Thus $x = \cdot30317175.$

$$\begin{array}{r} 143\ )\ 250\ (\ 175 \\ 143 \\ \hline 1070 \\ 1001 \\ \hline 690 \\ 715 \\ \hline \end{array}$$

EXAMPLES. XV. a.

1. Find the value of log 4951634, given that
$\log 49516 = 4\cdot6947456$, $\log 49517 = 4\cdot6947543$.

2. Find $\log 3\cdot4713026$, having given that
$\log 347\cdot13 = 2\cdot5404921$, $\log 34714 = 4\cdot5405047$.

3. Find log 2849614, having given that
$\log 2\cdot8496 = \cdot4547839$, $\log 2\cdot8497 = \cdot4547991$.

4. Find $\log 57{\cdot}63325$, having given that

$$\log 576{\cdot}33 = 2{\cdot}7606712, \quad \log 5763{\cdot}4 = 3{\cdot}7606788.$$

5. Given $\log 60814 = 4{\cdot}7840036$, diff. for $1 = 72$, find

$$\log 6081465.$$

6. Find the number whose logarithm is $4{\cdot}7461735$, given

$$\log 55740 = 4{\cdot}7461670, \quad \log 55741 = 4{\cdot}7461748.$$

7. Find the number whose logarithm is $2{\cdot}8283676$, given

$$\log 6{\cdot}7354 = {\cdot}8283634, \quad \log 67355 = 4{\cdot}8283698.$$

8. Find the number whose logarithm is $\bar{2}{\cdot}0288435$, given

$$\log 1068{\cdot}6 = 3{\cdot}0288152, \quad \log 1{\cdot}0687 = {\cdot}0288558.$$

9. Find the number whose logarithm is $\bar{3}{\cdot}9184377$, given

$$\log 8{\cdot}2877 = {\cdot}9184340, \quad \log 8287{\cdot}8 = 3{\cdot}9184392.$$

10. Given $\log 253{\cdot}19 = 2{\cdot}4034465$, diff. for $1 = 172$, find the number whose logarithm is $\bar{1}{\cdot}4034508$.

11. Given $\log 2{\cdot}0313 = {\cdot}3077741, \quad \log 2{\cdot}0314 = {\cdot}3077954,$
and
$$\log 1{\cdot}4271 = {\cdot}1544544,$$
find the seventh root of $142{\cdot}71$.

12. Find the eighth root of $13{\cdot}89492$, given

$$\log 13894 = 4{\cdot}1428273, \quad \log 138{\cdot}95 = 2{\cdot}1428586.$$

13. Find the value of $\sqrt[14]{242447}$, given

$$\log 2{\cdot}4244 = {\cdot}3846043, \text{ diff. for } 1 = 179.$$

14. Find the twentieth root of 2069138, given

$$\log 20691 = 4{\cdot}3157815, \text{ diff. for } 1 = 210.$$

Tables of Natural and Logarithmic Functions.

178. Tables have been constructed giving the values of the trigonometrical functions of all angles between 0° and 90° at intervals of 10″. These are called the Tables of **natural sines, cosines, tangents,**... In the smaller Tables, such as Chambers', the interval is 1′.

The logarithms of the functions have also been calculated. Since many of the trigonometrical functions are less than unity their logarithms are negative, and as the characteristics are not always evident on inspection they cannot be omitted. To avoid the inconvenience of printing the bars over the characteristics, the logarithms are all increased by 10 and are then registered under the name of **tabular logarithmic sines, cosines,...**

The notation used is $L\cos A$, $L\tan\theta$; thus

$$L\sin A = \log\sin A + 10.$$

For instance,

$$L\sin 45^\circ = 10 + \log\sin 45^\circ = 10 + \log\frac{1}{\sqrt{2}}$$

$$= 10 - \frac{1}{2}\log 2 = 9{\cdot}8494850.$$

179. With certain exceptions that need not be here noticed, the rule of proportional parts holds for the natural sines, cosines,... of all angles, and also for their logarithmic sines, cosines,.... In applying this rule it must be remembered that as the angle increases from 0° to 90° the functions sine, tangent, secant increase, while the co-functions cosine, cotangent, cosecant decrease.

Example 1. Find the value of $\sin 29^\circ\,37'\,42''$.

From the Tables,

$$\sin 29^\circ\,38' = {\cdot}4944476$$
$$\sin 29^\circ\,37' = {\cdot}4941948$$
$$\text{diff. for } 60'' = \quad 2528$$

$$\therefore \text{ prop}^{\text{l}} \text{ increase for } 42'' = \frac{42}{60}\times 2528 = \quad 1770$$

$$\sin 29^\circ\,37' = {\cdot}4941948$$
$$\therefore \sin 29^\circ\,37'\,42'' = {\cdot}4943718.$$

Example 2. Find the angle whose cosine is ·7280843.

Let A be the required angle; then from the Tables,

$$\cos 43^\circ\,16' = {\cdot}7281716$$
$$\cos 43^\circ\,17' = {\cdot}7279722$$
$$\text{diff. for } 60'' \quad 1994$$

$$\cos 43^\circ\,16' = {\cdot}7281716$$
$$\cos A \quad = {\cdot}7280843$$
$$\text{prop}^{\text{l}} \text{ part} \quad 873$$

```
           873
            60
1994 ) 52380 ( 26
        3988
       12500
       11964
```

But $\cos A$ is *less* than $\cos 43^\circ\,16'$; hence A must be *greater* than $43^\circ\,16'$ by $\frac{873}{1994}\times 60''$, that is by $26''$ nearly.

Thus the angle is $43^\circ\,16'\,26''$.

180. In order to illustrate the use of the tabular logarithmic functions we give the following extract from the table of logarithmic sines, cosines,... in Chambers' *Mathematical Tables.*

27 Deg.

′	Sine	Diff.	Cosec.	Secant	D.	Cosine	′
0	9·6570468		10·3429532	10·0501191		9·9498809	60
		2478			644		
1	9·6572946		10·3427054	10·0501835		9·9498165	59
		2477			644		
2	9·6575423		10·3424577	10·0502479		9·9497521	58
		2475			645		
3	9·6577898		10·3422102	10·0503124		9·9496876	57
		2473			646		
4	9·6580371		10·3419629	10·0503770		9·9496230	56
56	9·6706576		10·3293424	10·0537968		9·9462032	4
		2382			670		
57	9·6708958		10·3291042	10·0538638		9·9461362	3
		2380			670		
58	9·6711338		10·3288662	10·0539308		9·9460692	2
		2378			671		
59	9·6713716		10·3286284	10·0539979		9·9460021	1
		2377			672		
60	9·6716093		10·3283907	10·0540651		9·9459349	0
′	Cosine	Diff.	Secant	Cosec.	D.	Sine	′

62 Deg.

181. We have quoted here the logarithmic sines, cosecants, secants, and cosines of the angles differing by $1'$ between $27° \, 0'$ and $27° \, 4'$, and also between $27° \, 56'$ and $27° \, 60'$. The same extract gives the logarithmic functions of the complements of these angles, namely those between $62° \, 0'$ and $62° \, 4'$, and those between $62° \, 56'$ and $62° \, 60'$.

The column of minutes for $27°$ is given on the left and increases downwards, the column for $62°$ is on the right and increases upwards.

The names of the functions printed at the top refer to the angle $27°$, the names printed at the foot refer to the angle $62°$. Thus

$$L\cos 27° \, 3' = 9·9496876, \quad L\operatorname{cosec} 27° \, 58' = 10·3288662,$$
$$L\sin 62° \, 2' = 9·9460692, \quad L\cos 62° \, 59' = 9·6572946.$$

The first *difference column* gives the differences in the logarithms of the sines and cosecants, the second *difference column* gives the differences in the logarithms of the cosines and secants, each difference corresponding to a difference of $1'$ in the angle.

Example 1. Find $L \cos 62° 57' 12''$.

From the Tables,

$$\begin{aligned} L \cos 62° 57' &= 9 \cdot 6577898 \\ L \cos 62° 58' &= 9 \cdot 6575423 \\ \text{diff. for } 60'' & \quad\quad\quad 2475 \end{aligned}$$

$$\therefore \text{ proportional } \textit{decrease} \text{ for } 12'' = \frac{12}{60} \times 2475 = 495.$$

$$\begin{aligned} L \cos 62° 57' &= 9 \cdot 6577898 \\ \textit{Subtract} \text{ for } 12'' & \quad\quad\quad 495 \\ \therefore L \cos 62° 57' 12'' &= 9 \cdot 6577403 \end{aligned}$$

Example 2. Given $L \sec 27° 39' = 10 \cdot 0526648$, diff. for $10'' = 110$, find A when $L \sec A = 10 \cdot 0527253$.

$$\begin{aligned} L \sec A \quad &= 10 \cdot 0527253 \\ L \sec 27° 39' &= 10 \cdot 0526648 \\ \text{diff.} & \quad\quad\quad 605 \end{aligned}$$

$$\therefore \text{ proportional increase} = \frac{605}{110} \times 10'' = 55''.$$

Thus $\quad A = 27° 39' 55''$.

EXAMPLES. XV. b.

1. Find $\sin 38° 3' 35''$, having given that
$\sin 38° 4' = \cdot 6165780, \quad \sin 38° 3' = \cdot 6163489.$

2. Find $\tan 38° 24' 37 \cdot 5''$, having given that
$\tan 38° 25' = \cdot 7930640, \quad \tan 38° 24' = \cdot 7925902.$

3. Find $\operatorname{cosec} 55° 21' 28''$, having given that
$\operatorname{cosec} 55° 22' = 1 \cdot 2153535, \quad \operatorname{cosec} 55° 21' = 1 \cdot 2155978.$

4. Find the angle whose secant is $2 \cdot 1809460$, given
$\sec 62° 43' = 2 \cdot 1815435, \quad \sec 62° 42' = 2 \cdot 1803139.$

5. Find the angle whose cosine is $\cdot 8600931$, given
$\cos 30° 41' = \cdot 8600007, \quad \cos 30° 40' = \cdot 8601491.$

6. Find the angle whose cotangent is $\cdot 8766003$, given
$\cot 48° 46' = \cdot 8764620, \quad \cot 48° 45' = \cdot 8769765.$

7. Find $L \sin 44° 17' 33''$, given
$L \sin 44° 18' = 9 \cdot 8441137, \quad L \sin 44° 17' = 9 \cdot 8439842.$

8. Find $L\cot 36^\circ\ 26'\ 16''$, given

$$L\cot 36^\circ\ 27' = 10{\cdot}1315840, \quad L\cot 36^\circ\ 26' = 10{\cdot}1318483.$$

9. Find $L\cos 55^\circ\ 30'\ 24''$, given

$$L\cos 55^\circ\ 31' = 9{\cdot}7529442, \quad L\cos 55^\circ\ 30' = 9{\cdot}7531280.$$

10. Find the angle whose tabular logarithmic sine is 9·8440018, using the data of example 7.

11. Find the angle whose tabular logarithmic cosine is 9·7530075, using the data of example 9.

12. Given $L\tan 24^\circ\ 50' = 9{\cdot}6653662$, diff. for $1' = 3313$, find

$$L\tan 24^\circ\ 50'\ 52{\cdot}5''.$$

13. Given $L\operatorname{cosec} 40^\circ\ 5' = 10{\cdot}1911808$, diff. for $1' = 1502$, find

$$L\operatorname{cosec} 40^\circ\ 4'\ 17{\cdot}5''.$$

182. Considerable practice in the use of logarithmic Tables will be required before the quickness and accuracy necessary in all practical calculations can be attained. Experience shews that mistakes frequently arise from incorrect quotation from the Tables, and from clumsy arrangement. The student is reminded that care in taking out the logarithms from the Tables is of the first importance, and that in the course of the work he should learn to leave out all needless steps, making his solutions as concise as possible consistent with accuracy.

Example 1. Divide 6·6425693 by ·3873007.

From the Tables,

log 6·6425	=	·8223316	
6		40	
9		5	9
3			20
log 6·6425693	=	·8223362	
log ·3873007	=	$\bar{1}$·5880483	

log ·38730	=	$\bar{1}$·5880475	
0			0
7		7	8
log ·3873007	=	$\bar{1}$·5880483	

By subtraction, we obtain		1·2342879
From the Tables, log 17·150		=1·2342641
		238
	9	229
		90
	3	76

Thus the quotient is 17·15093.

Example 2. The hypotenuse of a right-angled triangle is 3·141024 and one side is 2·593167; find the other side.

Let c be the hypotenuse, a the given side, and x the side required; then

$$x^2 = c^2 - a^2 = (c+a)(c-a);$$
$$\therefore\ 2\log x = \log(c+a) + \log(c-a).$$

$$\begin{aligned} c &= 3{\cdot}141024 \\ a &= 2{\cdot}593167 \\ c+a &= 5{\cdot}734191 \\ c-a &= {\cdot}547858 \end{aligned}$$

From the Tables,	log 5·7341	= ·7584653	
	9	68	
	1		8
	log ·54785	$= \bar{1}{\cdot}7386617$	
	7	55	
By addition,		·4971394	

Dividing by 2, we have

log x		= ·2485697
log 1·7724		= ·2485617
		80
	3	74
		60
	2	49

Thus the required side is 1·772432.

EXAMPLES. XV. c.

[*In this exercise the logarithms are to be taken from the Tables.*]

1. Multiply 300·2618 by ·0078915194.

2. Find the product of 235·6783 and 357·8438.

3. Find the continued product of

153·2419, 2·8632503, and ·07583646.

4. Divide 1·0304051 by 27·093524.

5. Divide 357·8364 by ·00318973.

6. Find x from the equation

$$\cdot 0178345x = 21{\cdot}85632.$$

7. Find the value of

$$3{\cdot}78956 \times {\cdot}0536872 \div {\cdot}0072916.$$

8. Find the cube of ·83410039.

9. Find the fifth root of 15063·018.

10. Evaluate $\sqrt[5]{384{\cdot}731}$ and $\sqrt[13]{15{\cdot}7324}$.

11. Find the product of the square root of 1034·3963 and the cube root of 353246.

12. Subtract the square of ·7503269 from the square of 1·035627.

13. Find the value of

$$\frac{(34{\cdot}7326)^{\frac{3}{5}} \times \sqrt[6]{2{\cdot}53894}}{\sqrt[5]{4{\cdot}39682}}.$$

Example 3. Find a third proportional to the cube of ·3172564 and the cube root of 23·32873.

Let x be the required third proportional; then

$$({\cdot}3172564)^3 : (23{\cdot}32873)^{\frac{1}{3}} = (23{\cdot}32873)^{\frac{1}{3}} : x;$$

whence
$$x = (23{\cdot}32873)^{\frac{2}{3}} \div ({\cdot}3172564)^3;$$

$$\therefore \log x = \frac{2}{3} \log 23{\cdot}32873 - 3 \log {\cdot}3172564.$$

From the Tables,

log ·31725	=	$\bar{1}{\cdot}5014016$
6		82
4		5 | 5
		$\bar{1}{\cdot}5014103$ | 5
		3
		$\bar{2}{\cdot}5042311$

log 23·328	=	1·3678775
7		130
3		5 | 6
		1·3678910 | 6
		2
	3	2·7357821
		·9119274
		$\bar{2}{\cdot}5042311$

By subtraction,	log x		= 2·4076963
	log 255·67		= 2·4076798
			165
		9	153
			120
		7	119

Thus the third proportional is 255·6797.

14. Find a mean proportional between

$$\cdot 0037258169 \text{ and } \cdot 56301078.$$

15. Find a third proportional to the square of ·43607528 and the square root of ·03751786.

16. Find a fourth proportional to

$$56712\cdot 43, \quad 29\cdot 302564, \quad \cdot 33025107.$$

17. Find the geometric mean between

$$(\cdot 035689)^{\frac{2}{5}} \text{ and } (2\cdot 879432)^{\frac{3}{7}}.$$

18. Find a fourth proportional to

$$\sqrt[3]{32\cdot 7812}, \quad \sqrt[5]{357\cdot 814}, \quad \sqrt[4]{7836\cdot 43}.$$

19. Find the value of

$$\sin 27^\circ\ 13'\ 12'' \times \cos 46^\circ\ 2'\ 15''.$$

20. Find the value of

$$\cot 97^\circ\ 14'\ 16'' \times \sec 112^\circ\ 13'\ 5''.$$

21. Evaluate

$$\sin 20^\circ\ 13'\ 20'' \times \cot 47^\circ\ 53'\ 15'' \times \sec 42^\circ\ 15'\ 30''.$$

22. Find the value of $ab \sin C$, when

$$a = 324\cdot 1368, \quad b = 417\cdot 2431, \quad C = 113^\circ\ 14'\ 16''.$$

23. If $a : b = \sin A : \sin B$, find a, given

$$b = 378\cdot 25, \quad A = 35^\circ\ 15'\ 33'', \quad B = 119^\circ\ 14'\ 18''.$$

24. Find the smallest values of θ which satisfy the equations

(1) $\tan^3 \theta = \frac{5}{12}$; (2) $3 \sin^2 \theta + 2 \sin \theta = 1$.

25. Find x from the equation

$$x \times \sec 28^\circ\ 17'\ 25'' = \sin 23^\circ\ 18'\ 5'' \times \cot 38^\circ\ 15'\ 13''.$$

26. Find θ from the equation

$$\sin^3 \theta = \cos^2 \alpha \cot \beta,$$

where $\alpha = 32^\circ\ 47'$ and $\beta = 41^\circ\ 19'$.

CHAPTER XVI.

SOLUTION OF TRIANGLES WITH LOGARITHMS.

183. The examples on the solution of triangles in Chap. XIII. furnish a useful exercise on the formulæ connecting the sides and angle of a triangle; but in practical work much of the labour of arithmetical calculation is avoided by the use of logarithms.

We shall now shew how the formulæ of Chap. XIII. may be used or adapted for use in connection with logarithmic Tables.

184. *To find the functions of the half-angles in terms of the sides.*

We have $$2\sin^2\frac{A}{2}=1-\cos A$$

$$=1-\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{2bc-b^2-c^2+a^2}{2bc}=\frac{a^2-(b^2-2bc+c^2)}{2bc}$$

$$=\frac{a^2-(b-c)^2}{2bc}=\frac{(a+b-c)(a-b+c)}{2bc}.$$

Let $$a+b+c=2s;$$

then $$a+b-c=2s-2c=2(s-c),$$

and $$a-b+c=2s-2b=2(s-b).$$

$$\therefore\ 2\sin^2\frac{A}{2}=\frac{4(s-c)(s-b)}{2bc}=\frac{2(s-b)(s-c)}{bc};$$

$$\therefore\ \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Again, $2\cos^2\frac{A}{2}=1+\cos A=1+\frac{b^2+c^2-a^2}{2bc}$

$$=\frac{(b+c)^2-a^2}{2bc}=\frac{(b+c+a)(b+c-a)}{2bc};$$

$$\therefore\quad 2\cos^2\frac{A}{2}=\frac{4s(s-a)}{2bc}=\frac{2s(s-a)}{bc};$$

$$\therefore\quad \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}.$$

Also $\tan\frac{A}{2}=\sin\frac{A}{2}\div\cos\frac{A}{2}$

$$=\sqrt{\frac{(s-b)(s-c)}{bc}\times\frac{bc}{s(s-a)}};$$

$$\therefore\quad \tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

185. Similarly it may be proved that

$$\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}},\qquad \sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}};$$

$$\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}},\qquad \cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}};$$

$$\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}},\qquad \tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

In each of these formulæ the positive value of the square root must be taken, for each half angle is less than 90°, so that all its functions are positive.

186. *To find* sin A *in terms of the sides.*

$$\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$=2\sqrt{\frac{(s-b)(s-c)}{bc}\times\frac{s(s-a)}{bc}};$$

$$\therefore\quad \sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

We may also obtain this formula in another way which is instructive.

We have

$$\sin^2 A = 1 - \cos^2 A = (1 + \cos A)(1 - \cos A)$$

$$= \left(1 + \frac{b^2 + c^2 - a^2}{2bc}\right)\left(1 - \frac{b^2 + c^2 - a^2}{2bc}\right)$$

$$= \frac{(b+c)^2 - a^2}{2bc} \times \frac{a^2 - (b-c)^2}{2bc}$$

$$= \frac{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}{4b^2c^2}$$

$$= \frac{16s(s-a)(s-b)(s-c)}{4b^2c^2};$$

$$\therefore \quad \sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

The positive value of the square root must be taken, since the *sine* of an angle of any triangle is always positive.

EXAMPLES. XVI. a.

Prove the following formulæ in any triangle :

1. $b\cos^2\frac{A}{2} + a\cos^2\frac{B}{2} = s.$

2. $s\tan\frac{B}{2}\tan\frac{C}{2} = s - a.$

3. $\frac{\text{vers } A}{\text{vers } B} = \frac{a(a+c-b)}{b(b+c-a)}.$

4. $b\sin^2\frac{A}{2} + a\sin^2\frac{B}{2} = s - c.$

5. $(s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}.$

6. Find the value of $\tan\frac{B}{2}$, when $a = 10$, $b = 17$, $c = 21$.

7. Find $\cot\frac{C}{2}$, when $a = 13$, $b = 14$, $c = 15$.

8. Prove that

$$\frac{1}{a}\cos^2\frac{A}{2} + \frac{1}{b}\cos^2\frac{B}{2} + \frac{1}{c}\cos^2\frac{C}{2} = \frac{s^2}{abc}.$$

9. Prove that

$$\frac{b-c}{a}\cos^2\frac{A}{2} + \frac{c-a}{b}\cos^2\frac{B}{2} + \frac{a-b}{c}\cos^2\frac{C}{2} = 0.$$

187. *To solve a triangle when the three sides are given.*

From the formula

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

$$\log\tan\frac{A}{2}=\frac{1}{2}\{\log(s-b)+\log(s-c)-\log s-\log(s-a)\};$$

whence $\frac{A}{2}$ may be obtained by the help of the Tables.

Similarly B can be found from the formula for $\tan\frac{B}{2}$, and then C from the equation $C=180°-A-B$.

In the above solution, we shall require to look out from the tables *four* logarithms only, namely those of s, $s-a$, $s-b$, $s-c$; whereas if we were to solve from the sine or cosine formulæ we should require *six* logarithms; for

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}} \quad\text{and}\quad \cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}},$$

so that we should have to look out the logarithms of the *six* quantities s, $s-a$, $s-b$, a, b, c.

If therefore *all* the angles have to be found by the use of the tables it is best to solve from the tangent formulæ; but if *one* angle only is required it is immaterial whether the sine, cosine, or tangent formula is used.

In cases where a solution has to be obtained from certain given logarithms, the choice of formulæ must depend on the data.

NOTE. We shall always find the angles to the nearest second, so that, on account of the multiplication by 2, the half-angles should be found to the nearest tenth of a second.

188. In Art. 178 we have mentioned that 10 is added to each of the logarithmic functions before they are registered as *tabular* logarithms; but this device is introduced only as a convenience for the purposes of tabulation, and in practice it will be found that the work is more expeditious if the tabular logarithms are not used. The 10 should be subtracted mentally in copying down the logarithms. Thus we should write

$$\log\sin 64°\,15'=\bar{1}\cdot9545793, \quad \log\cot 18°\,35'=\cdot4733850,$$

and in the arrangement of the work care must be taken to keep the mantissæ positive.

Example 1. The sides of a triangle are 35, 49, 63; find the greatest angle; given $\log 2 = \cdot 3010300$, $\log 3 = \cdot 4771213$,

$$L\cos 47^\circ 53' = 9\cdot 8264910, \text{ diff. for } 60'' = 1397.$$

Since the *angles* of a triangle depend only on the *ratios* of the sides and not on their actual magnitudes, we may substitute for the sides any lengths proportional to them. Thus in the present case we may take $a=5$, $b=7$, $c=9$; then C is the greatest angle, and

$$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}} = \sqrt{\frac{21}{2}\times\frac{3}{2}\times\frac{1}{5\times 7}} = \sqrt{\frac{9}{20}};$$

$$\therefore \log\cos\frac{C}{2} = \frac{1}{2}(2\log 3 - \log 2 - 1).$$

$$\begin{array}{r} 2\log 3 = \cdot 9542426 \\ 1\cdot 3010300 \\ \hline 2\,)\,\bar{1}\cdot 6532126 \\ \hline \bar{1}\cdot 8266063 \end{array}$$

Thus $\log\cos\dfrac{C}{2} = \bar{1}\cdot 8266063$

$$\log\cos 47^\circ 53' = \bar{1}\cdot 8264910$$

$$\text{diff.} \quad 1153$$

$$\therefore \text{ proportional } \textit{decrease} = \frac{1153}{1397}\times 60'' = 49\cdot 5'';$$

$$\therefore \frac{C}{2} = 47^\circ 52' 10\cdot 5''.$$

$$\begin{array}{r} 1153 \\ 60 \\ \hline 1397\,)\,69180\,(\,49\cdot 5 \\ 5588 \\ \hline 13300 \\ 12573 \\ \hline 7270 \end{array}$$

Thus the greatest angle is $95^\circ 44' 21''$.

Example 2. If $a=283$, $b=317$, $c=428$, find all the angles.

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{197\times 86}{514\times 231}};$$

$$\therefore \log\tan\frac{A}{2} = \frac{1}{2}(\log 197 + \log 86 - \log 514 - \log 231).$$

$$\begin{array}{r} 283 \\ 317 \\ 428 \\ \hline 2\,)\,1028 \\ \hline 514 = s \\ 231 = s-a \\ 197 = s-b \\ 86 = s-c \end{array}$$

From the Tables,

$$\begin{array}{r} \log 197 = 2\cdot 2944662 \\ \log\ \ 86 = 1\cdot 9344985 \\ \hline 4\cdot 2289647 \\ 5\cdot 0745751 \\ \hline 2\,)\,\bar{1}\cdot 1543896 \\ \hline \end{array} \qquad \begin{array}{r} \log 514 = 2\cdot 7109631 \\ \log 231 = 2\cdot 3636120 \\ \hline 5\cdot 0745751 \end{array}$$

$$\log\tan\frac{A}{2} = \bar{1}\cdot 5771948$$

$$\log\tan 20^\circ 41' = \bar{1}\cdot 5769585$$

$$\text{diff.} \quad 2363$$

But diff. for $60''$ is 3822,

$\therefore$ prop^l. increase $=\dfrac{2363}{3822}\times 60''=37\cdot 1''$;

$\therefore\ \dfrac{A}{2}=20^\circ\,41'\,37\cdot 1''$ and $A=41^\circ\,23'\,14''$.

```
        2363
          60
3822 ) 141780 ( 37·1
       11466
        27120
        26754
          3660
```

Again, $\tan\dfrac{B}{2}=\sqrt{\dfrac{(s-c)(s-a)}{s(s-b)}}=\sqrt{\dfrac{86\times 231}{514\times 197}}$;

$\therefore\ \log\tan\dfrac{B}{2}=\dfrac{1}{2}(\log 86+\log 231-\log 514-\log 197)$.

$$\begin{aligned}
\log\ 86 &= 1\cdot 9344985 \\
\log 231 &= 2\cdot 3636120 \\
&\quad 4\cdot 2981105 \\
&\quad 5\cdot 0054293 \\
2\,)\ &\bar{1}\cdot 2926812 \\
\log\tan\frac{B}{2} &= \bar{1}\cdot 6463406 \\
\log\tan 23^\circ\,53' &= \bar{1}\cdot 6461988 \\
\text{diff.} &\qquad\quad 1418
\end{aligned}
\qquad
\begin{aligned}
\log 514 &= 2\cdot 7109631 \\
\log 197 &= 2\cdot 2944662 \\
&\quad 5\cdot 0054293
\end{aligned}$$

But diff. for $60''$ is 3412;

$\therefore$ prop^l. increase $=\dfrac{1418}{3412}\times 60''=24\cdot 9''$.

$\therefore\ \dfrac{B}{2}=23^\circ\,53'\,24\cdot 9''$ and $B=47^\circ\,46'\,50''$.

```
        1418
          60
3412 ) 85080 ( 24·9
       6824
       16840
       13648
        31920
```

Thus $A=41^\circ\,23'\,14''$, $B=47^\circ\,46'\,50''$, $C=90^\circ\,49'\,56''$.

EXAMPLES. XVI. b.

1. The sides of a triangle are 5, 8, 11; find the greatest angle; given $\log 7=\cdot 8450980$,

$L\sin 56^\circ\,47'=9\cdot 9225205$, $L\sin 56^\circ\,48'=9\cdot 9226032$.

2. If $a=40$, $b=51$, $c=43$, find A; given

$L\tan 24^\circ\,44'\,13''=9\cdot 6634464$,

$\log 128=2\cdot 1072100$, $\log 603=2\cdot 7803173$.

3. The sides a, b, c are as $4:5:6$, find B; given $\log 2$,

$L\cos 27^\circ\,53'=9\cdot 9464040$, diff. for $1'=669$.

4. Find the greatest angle of the triangle in which the sides are 5, 6, 7; given $\log 6 = \cdot 7781513$,

$$L\cos 39^\circ\, 14' = 9\cdot 8890644, \quad \text{diff. for } 1' = 1032.$$

5. If $a=3$, $b=1\cdot 75$, $c=2\cdot 75$, find C; given $\log 2$,

$$L\tan 32^\circ\, 18' = 9\cdot 8008365, \quad \text{diff. for } 1' = 2796.$$

× 6. If the sides are 24, 22, 14, find the least angle; given

$$L\tan 17^\circ\, 33' = 9\cdot 500042, \quad \text{diff. for } 1' = 439.$$

7. Find the greatest angle when the sides are 4, 10, 11; given $\log 2$, $\log 3$,

$$L\cos 46^\circ\, 47' = 9\cdot 8355378, \quad \text{diff. for } 1' = 1345.$$

8. If $a : b : c = 15 : 13 : 14$, find the angles; given $\log 2$, $\log 3$, $\log 7$,

$$L\tan 26^\circ\, 33' = 9\cdot 6986847, \quad \text{diff. for } 1' = 3159,$$
$$L\tan 29^\circ\, 44' = 9\cdot 7567587, \quad \text{diff. for } 1' = 2933.$$

9. If $a : b : c = 3 : 4 : 2$, find the angles; given $\log 2$, $\log 3$,

$$L\tan 14^\circ\, 28' = 9\cdot 4116146, \quad \text{diff. for } 10'' = 870,$$
$$L\tan 52^\circ\, 14' = 10\cdot 1108395, \quad \text{diff. for } 10'' = 435.$$

189. *To solve a triangle having given two sides and the included angle.*

Let the given parts be b, c, A, and let

$$k = \frac{\sin B}{b} = \frac{\sin C}{c};$$

then
$$\frac{\sin B - \sin C}{\sin B + \sin C} = \frac{kb - kc}{kb + kc} = \frac{b-c}{b+c};$$

$$\therefore \frac{2\cos\dfrac{B+C}{2}\sin\dfrac{B-C}{2}}{2\sin\dfrac{B+C}{2}\cos\dfrac{B-C}{2}} = \frac{b-c}{b+c};$$

$$\therefore \frac{\tan\dfrac{B-C}{2}}{\tan\dfrac{B+C}{2}} = \frac{b-c}{b+c};$$

$$\therefore \quad \tan\frac{B-C}{2}=\frac{b-c}{b+c}\tan\frac{B+C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2},$$

since $\dfrac{B+C}{2}=90^\circ-\dfrac{A}{2}$.

$$\therefore \quad \log\tan\frac{B-C}{2}=\log(b-c)-\log(b+c)+\log\cot\frac{A}{2},$$

from which equation we can find $\dfrac{B-C}{2}$.

Also $\dfrac{B+C}{2}=90^\circ-\dfrac{A}{2}$, and is therefore known.

By addition and subtraction we obtain B and C.

From the equation $\qquad a=\dfrac{b\sin A}{\sin B}$,

$$\log a=\log b+\log\sin A-\log\sin B;$$

whence a may be found.

Example 1. If the sides a and b are in the ratio of 7 to 3, and the included angle C is 60°, find A and B; given

$$\log 2=\cdot 3010300, \quad \log 3=\cdot 4771213,$$
$$L\tan 34^\circ\,42'=9\cdot 8403776, \text{ diff. for } 1'=2699.$$

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}=\frac{7-3}{7+3}\cot 30^\circ=\frac{4}{10}\sqrt{3};$$

$$\therefore \log\tan\frac{A-B}{2}=2\log 2-1+\frac{1}{2}\log 3;$$

$$2\log 2=\cdot 6020600$$
$$\frac{1}{2}\log 3=\cdot 2385607$$
$$\overline{\cdot 8406207}$$

$$\therefore \log\tan\frac{A-B}{2}=\bar{1}\cdot 8406207$$
$$\log\tan 34^\circ\,42'=\bar{1}\cdot 8403776$$
$$\text{diff.} \qquad 2431$$

$$\therefore \text{prop}^{l}.\text{ increase}=\frac{2431}{2699}\times 60''=54'';$$

$$\begin{array}{r} 2431 \\ 60 \\ \hline 2699\,)\,145860\,(\,54 \\ 13495 \\ \hline 10910 \\ 10796 \\ \hline \end{array}$$

$$\therefore \frac{A-B}{2} \qquad =34^\circ\,42'\,54''.$$

And $\qquad \dfrac{A+B}{2}=90^\circ-\dfrac{C}{2}=60^\circ.$

By addition, $\qquad A=94^\circ\,42'\,54''$,

and by subtraction, $\qquad B=25^\circ\,17'\,6''$.

Example 2. If $a=681$, $c=243$, $B=50° 42'$, solve the triangle, by the use of Tables.

$$\tan\frac{A-C}{2}=\frac{a-c}{a+c}\cot\frac{B}{2}=\frac{438}{924}\cot 25° 21';$$

$$\therefore \log\tan\frac{A-C}{2}=\log 438-\log 924 + \log\cot 25° 21'$$

$$\therefore \log\tan\frac{A-C}{2}=\cdot 0002383$$

$$\log\tan 45°=\cdot 0000000$$

$$\text{diff.} \qquad 2383$$

$$\begin{aligned} \log 438 &= 2\cdot 6414741 \\ \log\cot 25° 21' &= \ \ \cdot 3244362 \\ & \overline{2\cdot 9659103} \\ \log 924 &= 2\cdot 9656720 \\ & \overline{\ \ \cdot 0002383} \end{aligned}$$

And diff. for 60″ is 2527;

$$\therefore \text{prop}^{l}.\text{ increase}=\frac{2383}{2527}\times 60''=57'';$$

$$\therefore \frac{A-C}{2} = 45° 0' 57''.$$

$$\begin{array}{r} 2383 \\ 60 \\ 2527\ )\ \overline{142980}\ (\ 56\cdot 6 \\ 12635 \\ \overline{16630} \\ 15162 \\ \overline{14680} \end{array}$$

Also $\dfrac{A+C}{2}=90°-\dfrac{B}{2}=64° 39'$.

By addition, $A=109° 39' 57''$,

and by subtraction, $C= 19° 38' 3''$.

Again, $b=\dfrac{c\sin B}{\sin C}$;

$$\therefore \log b=\log c+\log\sin B-\log\sin C$$

$$=\log 243+\log\sin 50° 42' - \log\sin 19° 38' 3''$$

$$\therefore \log b=2\cdot 7479012$$

$$\log 559\cdot 63 = 2\cdot 7479010$$

$$\therefore b=559\cdot 63.$$

$$\begin{aligned} \log\sin 19° 38' &= \bar{1}\cdot 5263387 \\ \frac{3}{60}\times 3540 &= \qquad\quad 177 \\ \log\sin 19° 38' 3'' &= \bar{1}\cdot 5263564 \\ \log 243 &= 2\cdot 3856063 \\ \log\sin 50° 42' &= \bar{1}\cdot 8886513 \\ & \overline{2\cdot 2742576} \\ \log\sin 19° 38' 3'' &= \bar{1}\cdot 5263564 \\ & \overline{2\cdot 7479012} \end{aligned}$$

Thus $A=109° 39' 57''$, $C=19° 38' 3''$, $b=559\cdot 63$.

190. From the formula

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2},$$

it will be seen that if b, c, and $B-C$ are known A can be found; that is, the triangle can be solved when the given parts are two sides and the difference of the angles opposite to them.

EXAMPLES. XVI. c.

1. If $a=9$, $b=6$, $C=60°$, find A and B; given log 2, log 3,
$L\tan 19°\ 6'=9{\cdot}5394287$, $L\tan 19°\ 7'=9{\cdot}5398371$.

2. If $a=1$, $c=9$, $B=65°$, find A and C; given log 2,
$L\cot 32°\ 30'=10{\cdot}1958127$,
$L\tan 51°\ 28'=10{\cdot}0988763$, diff. for $1'=2592$.

3. If $17a=7b$, $C=60°$, find A and B; given log 2, log 3,
$L\tan 35°\ 49'=9{\cdot}8583357$, diff. for $10''=2662$.

4. If $b=27$, $c=23$, $A=44°\ 30'$, find B and C; given log 2,
$L\cot 22°\ 15'=10{\cdot}3881591$,
$L\tan 11°\ 3'=9{\cdot}2906713$, diff. for $1'=6711$.

5. If $c=210$, $a=110$, $B=34°\ 42'\ 30''$, find C and A; given log 2,
$L\cot 17°\ 21'\ 15''=10{\cdot}5051500$.

6. Two sides of a triangle are as 5 : 3 and include an angle of $60°\ 30'$: find the other angles; given log 2,
$L\cot 30°\ 15'=10{\cdot}23420$,
$L\tan 23°\ 13'=9{\cdot}63240$, diff. for $1'=35$.

7. If $a=327$, $c=256$, $B=56°\ 28'$, find A and C; given
$\log 7{\cdot}1={\cdot}8512583$, $\log 5{\cdot}83={\cdot}7656686$,
$L\tan 61°\ 46'=10{\cdot}2700705$,
$L\tan 12°\ 46'=9{\cdot}3552267$, diff. for $1'=5859$.

8. If $b=4c$, $A=65°$, find B and C; given log 2, log 3,
$L\tan 57°\ 30'=10{\cdot}1958127$,
$L\tan 43°\ 18'=9{\cdot}9742133$, diff. for $1'=2531$.

9. If $a=23031$, $b=7677$, $C=30°\ 10'\ 5''$, find A and B; given log 2,
$L\tan 15°\ 5'=9{\cdot}4305727$, diff. for $10''=838$,
$L\cot 61°\ 41'=9{\cdot}7314436$, diff. for $10''=504$.

191. *To solve a triangle having given two angles and a side.*

Let the given parts be denoted by B, C, a; then the third angle A is found from the equation $A=180°-B-C$,

and
$$b=\frac{a\sin B}{\sin A};$$

$$\therefore\ \log b=\log a+\log\sin B-\log\sin A;$$

whence b may be found.

Similarly, c may be obtained from the equation
$$\log c=\log a+\log\sin C-\log\sin A.$$

Example. If $b=1000$, $A=45°$, $C=68°\,17'\,40''$, find the least side, having given

$$\log 2=\cdot3010300,\ \log 7\cdot6986=\cdot8864118,\ \text{diff. for } 1=57,$$
$$L\sin 66°\,42'=9\cdot9630538,\ \text{diff. for } 1'=544.$$

$B=180°-45°-68°\,17'\,40''=66°\,42'\,20''$.

$$\text{The least side}=a=\frac{b\sin A}{\sin B}=\frac{1000\sin 45°}{\sin 66°\,42'\,20''};$$

$$\therefore\ \log a=3+\log\frac{1}{\sqrt{2}}-\log\sin 66°\,42'\,20''$$
$$=3-\frac{1}{2}\log 2-\log\sin 66°\,42'\,20''$$
$$=3-\cdot1135869$$

$\log\sin 66°\,42'=$	$\bar{1}\cdot9630538$
$\frac{20}{60}\times 544=$	181
$\frac{1}{2}\log 2=$	$\cdot1505150$
	$\cdot1135869$

$$\therefore\ \log a=2\cdot8864131$$
$$\log 769\cdot86=2\cdot8864118$$
$$\text{diff.}\qquad 13$$

$$\therefore\ \text{prop}^{l}.\ \text{increase}=\frac{13}{57}=\cdot22.$$

Thus the least side is $769\cdot8622$.

EXAMPLES. XVI. d.

1. If $B=60°\,15'$, $C=54°\,30'$, $a=100$, find c; given
$L\sin 54°\,30'=9\cdot9106860$, $\log 8\cdot9646162=\cdot9525317$,
$L\sin 65°\,15'=9\cdot9581543$.

2. If $A=55°$, $B=65°$, $c=270$, find a; given log 2, log 3,
$\log 25538=4{\cdot}4071869$, $L\sin 55°=9{\cdot}9133645$,
$\log 25539=4{\cdot}4072039$.

3. If $A=45°\ 41'$, $C=62°\ 5'$, $b=100$, find c; given
$\log 9{\cdot}2788={\cdot}96749$, $L\sin 62°\ 5'=9{\cdot}94627$,
$L\sin 72°\ 14'=9{\cdot}97878$.

4. If $B=70°\ 30'$, $C=78°\ 10'$, $a=102$, find b and c; given
$\log 1{\cdot}02={\cdot}009$, $\log 1{\cdot}85={\cdot}267$, $\log 1{\cdot}92={\cdot}283$,
$L\sin 70°\ 30'=9{\cdot}974$, $L\sin 78°\ 10'=9{\cdot}990$,
$L\sin 31°\ 20'=9{\cdot}716$.

5. If $a=123$, $B=29°\ 17'$, $C=135°$, find c; given log 2,
$\log\ 123=2{\cdot}0899051$, $L\sin 15°\ 43'=9{\cdot}4327777$,
$\log 32110=4{\cdot}5066403$, $D=135$.

6. If $A=44°$, $C=70°$, $b=1006{\cdot}62$, find a and c; given
$L\sin 44°=9{\cdot}8417713$, $\log 100662\ =5{\cdot}0028656$,
$L\sin 66°=9{\cdot}9607302$, $\log 103543\ =5{\cdot}0151212$,
$L\sin 70°=9{\cdot}9729858$, $\log 7654321=6{\cdot}8839067$.

7. If $a=1652$, $B=26°\ 30'$, $C=47°\ 15'$, find b and c;
$L\sin 73°\ 45'=9{\cdot}9822938$, $\log 1{\cdot}652\ ={\cdot}2180100$,
$L\sin 26°\ 30'=9{\cdot}6495274$, $\log 7{\cdot}6780={\cdot}8852481$, $D=57$,
$L\sin 47°\ 15'=9{\cdot}8658868$, $\log 1{\cdot}2636={\cdot}1016096$, $D=344$.

192. *To solve a triangle when two sides and the angle opposite to one of them are given.*

Let a, b, A be given. Then from $\sin B=\frac{b}{a}\sin A$, we have

$$\log\sin B=\log b-\log a+\log\sin A;$$

whence B may be found;

then C is found from the equation $C=180°-A-B$.

Again,
$$c=\frac{a\sin C}{\sin A},$$

$$\therefore\ \log c=\log a+\log\sin C-\log\sin A.$$

If $a<b$, and A is acute the solution is ambiguous and there will be two values of B supplementary to each other, and also two values of C and c. [Art. 147.]

Example. If $b=63$, $c=36$, $C=29°\,23'\,15''$, find B; given

$$\log 2 = \cdot 3010300, \quad \log 7 = \cdot 8450980.$$

$$L \sin 29°\,23' = 9\cdot 6907721, \quad \text{diff. for } 1' = 2243,$$

$$L \sin 59°\,10' = 9\cdot 9338222, \quad \text{diff. for } 1' = 755.$$

$$\sin B = \frac{b}{c}\sin C = \frac{63}{36}\sin C$$

$$= \frac{7}{4}\sin 29°\,23'\,15'';$$

$$\therefore \log\sin B = \log 7 - 2\log 2 + \log\sin 29°\,23'\,15'';$$

$$\therefore \log\sin B = \bar{1}\cdot 9338662$$

$$\log\sin 59°\,10' = \bar{1}\cdot 9338222$$

$$\text{diff.} \quad 440$$

$$\therefore \text{prop}^{l}.\ \text{increase} = \frac{440}{755}\times 60'' = 35'';$$

$$\therefore B = 59°\,10'\,35''.$$

Also since $c < b$ there is another value of B supplementary to the above, namely $B = 120°\,49'\,25''$.

$$\log\sin 29°\,23' = \bar{1}\cdot 6907721$$

$$\frac{15}{60}\times 2243 = \quad 561$$

$$\log 7 = \quad \cdot 8450980$$

$$\cdot 5359262$$

$$2\log 2 = \quad \cdot 6020600$$

$$\bar{1}\cdot 9338662$$

440
60
755) 26400 (35
2265
3750
3775

EXAMPLES. XVI. e.

1. If $a=145$, $b=178$, $B=41°\,10'$, find A; given
$\log 178 = 2\cdot 2504200$, $L\sin 41°\,10' = 9\cdot 8183919$,
$\log 145 = 2\cdot 1613680$, $L\sin 32°\,25'\,35'' = 9\cdot 7293399$.

2. If $A=26°\,26'$, $b=127$, $a=85$, find B; given
$\log 1\cdot 27 = \cdot 1038037$, $L\sin 26°\,26' = 9\cdot 6485124$,
$\log 8\cdot 5 = \cdot 9294189$, $L\sin 41°\,41'\,28'' = 9\cdot 8228972$.

3. If $c=5$, $b=4$, $C=45°$, find A and B; given
$\log 2 = \cdot 30103$, $L\sin 34°\,26' = 9\cdot 7525750$.

4. If $a=1405$, $b=1706$, $A=40°$, find B; given
$\log 1\cdot 405 = \cdot 1476763$, $\log 1\cdot 706 = \cdot 2319790$,
$L\sin 40° = 9\cdot 8080675$, $L\sin 51°\,18' = 9\cdot 8923342$,
diff. for $1' = 1012$.

5. If $B=112^\circ\ 4'$, $b=573$, $c=394$, find A and C; given

$$\log 573=2{\cdot}7581546, \quad \log 394=2{\cdot}5954962,$$
$$L\sin 39^\circ\ 35'=9{\cdot}8042757, \quad \text{diff. for } 60''=1527,$$
$$L\cos 22^\circ\ 4'=9{\cdot}9669614.$$

6. If $b=8{\cdot}4$, $c=12$, $B=37^\circ\ 36'$, find A; given

$$\log 7={\cdot}8450980, \quad L\sin 37^\circ\ 36'=9{\cdot}7854332,$$
$$L\sin 60^\circ\ 39'=9{\cdot}9403381, \quad \text{diff. for } 1'=711.$$

X 7. Supposing the data for the solution of a triangle to be as in the three following cases, point out whether the solution will be ambiguous or not, and find the third side in the obtuse angled triangle in the ambiguous case:

(i) $A=30^\circ$, $a=125$ feet, $c=250$ feet,

(ii) $A=30^\circ$, $a=200$ feet, $c=250$ feet,

(iii) $A=30^\circ$, $a=200$ feet, $c=125$ feet.

Given log 2,

$$\log 6{\cdot}0389={\cdot}7809578, \quad L\sin 38^\circ\ 41'=9{\cdot}7958800,$$
$$\log 6{\cdot}0390={\cdot}7809650, \quad L\sin 8^\circ\ 41'=9{\cdot}1789001.$$

193. Some formulæ which are not primarily suitable for working with logarithms may be adapted to such work by various artifices.

194. *To adapt the formula $c^2=a^2+b^2$ to logarithmic computation.*

We have
$$c^2=a^2\left(1+\frac{b^2}{a^2}\right).$$

Since an angle can always be found whose tangent is equal to a given numerical quantity, we may put $\frac{b}{a}=\tan\theta$, and thus obtain

$$c^2=a^2(1+\tan^2\theta)=a^2\sec^2\theta;$$
$$\therefore\ c=a\sec\theta;$$
$$\therefore\ \log c=\log a+\log\sec\theta.$$

The angle θ is called a **subsidiary angle** and is found from the equation

$$\log\tan\theta=\log b-\log a.$$

Thus *any expression which can be put into the form of the sum of two squares can be readily adapted to logarithmic work.*

195. *To adapt the formula $c^2=a^2+b^2-2ab\cos C$ to logarithmic computation.*

From the identities

$$\cos C=\cos^2\frac{C}{2}-\sin^2\frac{C}{2}, \text{ and } 1=\cos^2\frac{C}{2}+\sin^2\frac{C}{2},$$

we have

$$c^2=(a^2+b^2)\left(\cos^2\frac{C}{2}+\sin^2\frac{C}{2}\right)-2ab\left(\cos^2\frac{C}{2}-\sin^2\frac{C}{2}\right)$$

$$=(a^2+b^2-2ab)\cos^2\frac{C}{2}+(a^2+b^2+2ab)\sin^2\frac{C}{2}$$

$$=(a-b)^2\cos^2\frac{C}{2}+(a+b)^2\sin^2\frac{C}{2}$$

$$=(a-b)^2\cos^2\frac{C}{2}\left\{1+\left(\frac{a+b}{a-b}\right)^2\tan^2\frac{C}{2}\right\}.$$

Take a subsidiary angle θ, such that

$$\tan\theta=\frac{a+b}{a-b}\tan\frac{C}{2},$$

then $c^2=(a-b)^2\cos^2\frac{C}{2}(1+\tan^2\theta)$

$$=(a-b)^2\cos^2\frac{C}{2}\sec^2\theta;$$

$$\therefore\ c=(a-b)\cos\frac{C}{2}\sec\theta;$$

$$\therefore\ \log c=\log(a-b)+\log\cos\frac{C}{2}+\log\sec\theta,$$

where θ is determined from the equation

$$\log\tan\theta=\log(a+b)-\log(a-b)+\log\tan\frac{C}{2}.$$

196. When two sides and the included angle are given, we may solve the triangle by finding the value of the third side first instead of determining the angles first as in Art. 189.

Example. If $a=3$, $c=1$, $B=53°\ 7'\ 48''$ find b; given
$\log 2=\cdot3010300$, $\log 2\cdot5298=\cdot4030862$, diff. for $1=172$,
$L\cos 26°\ 33'\ 54''=9\cdot9515452$, $L\tan 26°\ 33'\ 54''=9\cdot6989700$.

We have $b^2 = c^2 + a^2 - 2ca\cos B$

$$= (a^2+c^2)\left(\cos^2\frac{B}{2} + \sin^2\frac{B}{2}\right) - 2ac\left(\cos^2\frac{B}{2} - \sin^2\frac{B}{2}\right)$$

$$= (a-c)^2\cos^2\frac{B}{2} + (a+c)^2\sin^2\frac{B}{2}$$

$$= (a-c)^2\cos^2\frac{B}{2}\left\{1 + \left(\frac{a+c}{a-c}\right)^2\tan^2\frac{B}{2}\right\}$$

$$= (a-c)^2\cos^2\frac{B}{2}(1+\tan^2\theta)\ldots\ldots\ldots\ldots(1),$$

where $$\tan\theta = \frac{a+c}{a-c}\tan\frac{B}{2} = 2\tan 26^\circ\,33'\,54'';$$

$$\therefore \log\tan\theta = \log 2 + \log\tan 26^\circ\,33'\,54''$$
$$= \cdot3010300 + \bar{1}\cdot6989700$$
$$= 0;$$

whence $\tan\theta = 1$, and $\theta = 45^\circ$.

From (1), $$b = (a-c)\cos\frac{B}{2}\sec\theta$$
$$= 2\sec 45^\circ\cos\frac{B}{2}$$
$$= 2\sqrt{2}\cos 26^\circ\,33'\,54'';$$

$$\therefore \log b = \log 2 + \frac{1}{2}\log 2 + \log\cos 26^\circ\,33'\,54''$$

$\therefore \log b = \cdot4030902$
$\log 2\cdot5298 = \cdot4030862$
diff. $\quad 40$

$\log 2 = \cdot3010300$
$\frac{1}{2}\log 2 = \cdot1505150$
$\log\cos 26^\circ\,33'\,54'' = \bar{1}\cdot9515452$
$\cdot4030902$

But diff. for 1 is 172;

$$\therefore \text{proportional increase} = \frac{40}{172} = \frac{10}{43} = \cdot23.$$

Thus the third side is 2·529823.

197. The formula $c^2 = a^2 + b^2 - 2ab\cos C$ may also be adapted to logarithmic computation in two other ways by making use of the identities $\cos C = 2\cos^2\frac{C}{2} - 1$ and $\cos C = 1 - 2\sin^2\frac{C}{2}$.

We shall take the first of these cases, leaving the other as an exercise.

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$= a^2 + b^2 - 2ab\left(2\cos^2\frac{C}{2} - 1\right)$$
$$= (a+b)^2 - 4ab\cos^2\frac{C}{2}$$
$$= (a+b)^2\left\{1 - \frac{4ab}{(a+b)^2}\cos^2\frac{C}{2}\right\}.$$

Let $$\frac{4ab}{(a+b)^2}\cos^2\frac{C}{2} = \cos^2\theta,$$

then $$c^2 = (a+b)^2(1-\cos^2\theta) = (a+b)^2\sin^2\theta;$$
$$\therefore\ c = (a+b)\sin\theta;$$
$$\therefore\ \log c = \log(a+b) + \log\sin\theta.$$

To determine θ we have the equation

$$\cos\theta = \frac{2\sqrt{ab}}{a+b}\cos\frac{C}{2};$$

$$\therefore\ \log\cos\theta = \log 2 + \frac{1}{2}(\log a + \log b) - \log(a+b) + \log\cos\frac{C}{2}.$$

Since $2\sqrt{ab}$ is never greater than $a+b$ and $\cos\frac{C}{2}$ is positive and less than unity, $\cos\theta$ is positive and less than unity, and thus θ is an acute angle.

EXAMPLES. XVI. f.

1. If $a=8$, $b=7$, $c=9$, find the angles; given $\log 2$, $\log 3$,
$L\tan 24°\ 5' = 9\cdot6502809$, diff. for $60'' = 3390$,
$L\tan 36°\ 41' = 9\cdot8721123$, diff. for $60'' = 2637$.

2. The difference between the angles at the base of a triangle is 24°, and the sides opposite these angles are 175 and 337: find all the angles; given $\log 2$, $\log 3$,
$L\tan 12° = 9\cdot3274745$, $L\cot 56°\ 6'\ 27'' = 9\cdot8272293$.

3. One of the sides of a right-angled triangle is two-sevenths of the hypotenuse: find the greater of the two acute angles; given

$\log 2, \log 7, \quad L\sin 14^\circ\, 11' = 9\cdot455921, \quad L\sin 14^\circ\, 12' = 9\cdot456031.$

4. Find the greatest side when two of the angles are 78° 14′ and 71° 24′ and the side joining them is 2183; given

$\log 2\cdot183 = \cdot3390537, \quad \log 4\cdot2274 = \cdot6260733, \quad D = 103,$
$L\sin 78^\circ\, 14' = 9\cdot9907766, \quad L\sin 30^\circ\, 22' = 9\cdot7037486.$

5. If $b = 2$ ft. 6 in., $c = 2$ ft., $A = 22^\circ\, 20'$, find the other angles; and then shew that the side a is very approximately 1 foot. Given log 2, log 3,

$L\cot 11^\circ\, 10' = 10\cdot70465, \quad L\sin 49^\circ\, 27'\, 34'' = 9\cdot88079,$
$L\sin 22^\circ\, 20' = \ 9\cdot57977, \quad L\tan 29^\circ\, 22'\, 26'' = 9\cdot75041.$

6. If $a = 1\cdot56234$, $b = \cdot43766$, $C = 58^\circ\, 42'\, 6''$, find A and B; given $\log 56234 = 4\cdot75$,

$\log\cot 29^\circ\, 21' = \cdot250015, \quad \log\cot 29^\circ\, 22' = \cdot249715.$

7. If $a = 9$, $b = 12$, $A = 30^\circ$, find the values of c, having given

$\log\ 12 = 1\cdot07918, \quad L\sin\ 30^\circ = 9\cdot69897,$
$\log\ 9 = \cdot95424, \quad L\sin\ 11^\circ\, 48'\, 39'' = 9\cdot31108,$
$\log 171 = 2\cdot23301, \quad L\sin\ 41^\circ\, 48'\, 39'' = 9\cdot82391,$
$\log 368 = 2\cdot56635, \quad L\sin 108^\circ\, 11'\, 21'' = 9\cdot97774.$

8. The sides of a triangle are 9 and 3, and the difference of the angles opposite to them is 90°: find the angles; having given log 2,

$L\tan 26^\circ\, 33' = 9\cdot6986847, \quad L\tan 26^\circ\, 34' = 9\cdot6990006.$

9. Two sides of a triangle are 1404 and 960 respectively, and an angle opposite to one of them is 32° 15′: find the angle contained by the two sides; having given log 2, log 3,

$\log 13 = 1\cdot1139434, \quad L\operatorname{cosec} 32^\circ\, 15' = 10\cdot2727724,$
$L\sin 21^\circ\, 23' = 9\cdot5621316, \quad L\sin 51^\circ\, 18' = \ 9\cdot8923236.$

10. If $b : c = 11 : 10$ and $A = 35^\circ\, 25'$, use the formula

$$\tan\frac{1}{2}(B - C) = \tan^2\frac{\phi}{2}\cot\frac{A}{2} \text{ to find } B \text{ and } C;$$

given $\log 1\cdot1 = \ \cdot041393, \quad L\tan 12^\circ\, 18'\, 36'' = \ 9\cdot338891,$
$L\cos 24^\circ\, 37'\, 12'' = 9\cdot958607, \quad L\cot 17^\circ\, 42'\, 30'' = 10\cdot495800,$
$L\tan 8^\circ\, 28'\, 56\cdot5'' = 9\cdot173582.$

11. If $A=50°$, $b=1071$, $a=873$, find B; given

$$\log 1{\cdot}071 = {\cdot}029789, \quad \log 8{\cdot}73 = {\cdot}941014,$$
$$L\sin 50° = 9{\cdot}884254, \quad L\sin 70° = 9{\cdot}972986,$$
$$L\sin 70°\ 1' = 9{\cdot}973032.$$

12. If $a=6$, $b=3$, $C=36°\ 52'\ 12''$, find c without determining A and B; given $\log 2 = {\cdot}30103$, $\log 3 = {\cdot}47712$,

$$\log 40249 = 4{\cdot}60476, \quad L\sin 18°\ 26'\ 6'' = 9{\cdot}5,$$
$$L\cot 18°\ 26'\ 6'' = 10{\cdot}47712.$$

(*In the following Examples the necessary Logarithms must be taken from the Tables.*)

13. Given $a=1000$, $b=840$, $c=1258$, find B.

14. Solve the triangle in which $a=525$, $b=650$, $c=777$.

15. Find the least angle when the sides are proportional to 4, 5, and 6.

16. If $B=90°$, $AC=57{\cdot}321$, $AB=28{\cdot}58$, find A and C.

17. Find the hypotenuse of a right-angled triangle in which the smallest angle is 18° 37′ 29″ and the side opposite to it is 284 feet.

18. The sides of a triangle are 9 and 7 and the angle between them is 60°: find the other angles.

19. How long must a ladder be so that when inclined to the ground at an angle of 72° 15′ it may just reach a window 42·37 feet from the ground?

20. If $a=31{\cdot}95$, $b=21{\cdot}96$, $C=35°$, find A and B.

21. Find B, C, a when $b=25{\cdot}12$, $c=13{\cdot}83$, $A=47°\ 15'$.

22. Find the greatest angle of the triangle whose sides are 1837·2, 2385·6, 2173·84.

23. When $a=21{\cdot}352$, $b=45{\cdot}6843$, $c=37{\cdot}2134$, find A, B, and C.

24. If $b=647{\cdot}324$, $c=850{\cdot}273$, $A=103°\ 12'\ 54''$, find the remaining parts.

25. If $b=23{\cdot}2783$, $A=37°\ 57'$, $B=43°\ 13'$, find the remaining sides.

26. Find a and b when $B=72°\ 43'\ 25''$, $C=47°\ 12'\ 17''$, $c=2484{\cdot}3$.

27. If $AB=4517$, $AC=150$, $A=31^\circ\ 30'$, find the remaining parts.

28. Find A, B, and b when

$$a=324{\cdot}68,\quad c=421{\cdot}73,\quad C=35^\circ\ 17'\ 12''.$$

29. Given $a=321{\cdot}7$, $c=435{\cdot}6$, $A=36^\circ\ 18'\ 27''$, find C.

30. If $b=1325$, $c=1665$, $B=52^\circ\ 19'$, solve the obtuse-angled triangle to which the data belong.

31. If $a=3795$, $B=73^\circ\ 15'\ 15''$, $C=42^\circ\ 18'\ 30''$, find the other sides.

32. Find the angles of the two triangles which have $b=17$, $c=12$, and $C=43^\circ\ 12'\ 12''$.

33. Two sides of a triangle are 2·7402 ft. and ·7401 ft. respectively, and contain an angle $59^\circ\ 27'\ 5''$: find the base and altitude of the triangle.

34. The difference between the angles at the base of a triangle is $17^\circ\ 48'$ and the sides subtending these angles are 105·25 ft. and 76·75 ft.: find the angle included by the given sides.

35. From the following data:

(1) $A=43^\circ\ 15'$, $AB=36{\cdot}5$, $BC=20$,

(2) $A=43^\circ\ 15'$, $AB=36{\cdot}5$, $BC=30$,

(3) $A=43^\circ\ 15'$, $AB=36{\cdot}5$, $BC=45$,

point out which solution is impossible and which ambiguous. Find the third side for the triangle the solution of which is neither impossible nor ambiguous.

36. In any triangle prove that $c=(a-b)\sec\theta$, where

$$\tan\theta=\frac{2\sqrt{ab}}{a-b}\sin\frac{C}{2}.$$

If $a=17{\cdot}32$, $b=13{\cdot}47$, $C=47^\circ\ 13'$, find c without finding A and B.

37. If $\tan\phi=\dfrac{a+b}{a-b}\tan\dfrac{C}{2}$, prove that $c=(a-b)\cos\dfrac{C}{2}\sec\phi$.

If $a=27{\cdot}3$, $b=16{\cdot}8$, $C=45^\circ\ 12'$, find ϕ, and thence find c.

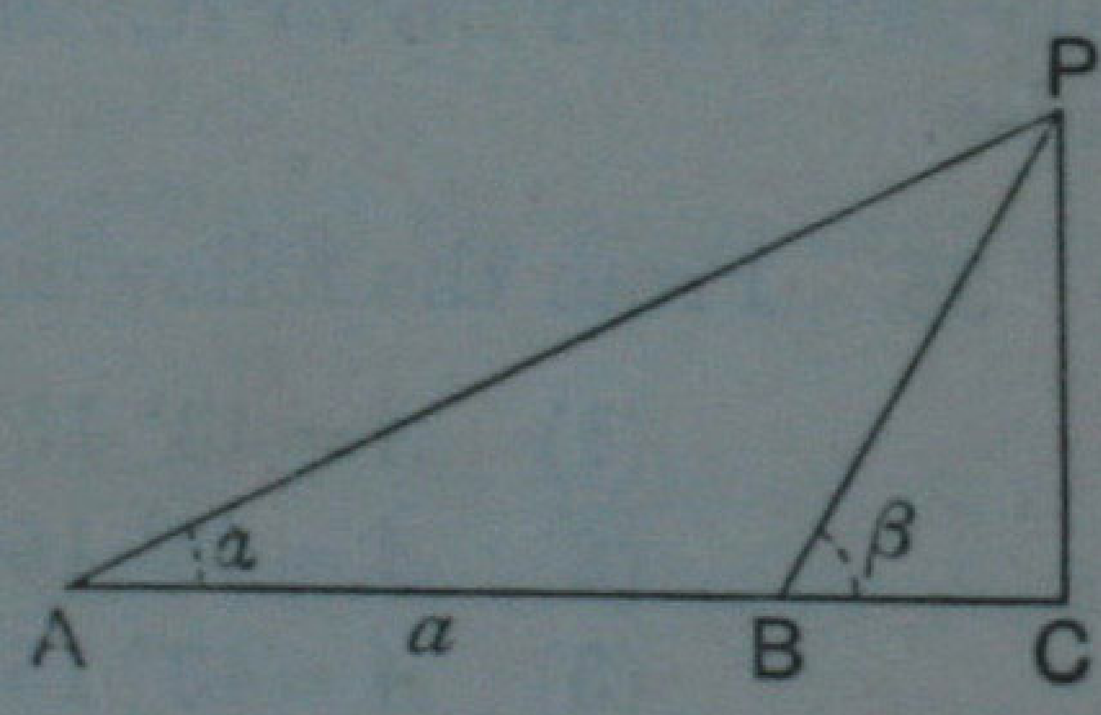
CHAPTER XVII.

HEIGHTS AND DISTANCES.

198. Some easy cases of heights and distances depending only on the solution of right-angled triangles have been already dealt with in Chap. VI. The problems in the present chapter are of a more general character, and require for their solution some geometrical skill as well as a ready use of trigonometrical formulæ.

Measurements in one plane.

199. *To find the height and distance of an inaccessible object on a horizontal plane.*



Let A be the position of the observer, CP the object; from P draw PC perpendicular to the horizontal plane; then it is required to find PC and AC.

At A observe the angle of elevation PAC. Measure a base line AB in a direct line from A towards the object, and at B observe the angle of elevation PBC.

Let $\qquad \angle PAC=\alpha, \quad \angle PBC=\beta, \quad AB=a.$

From $\triangle PBC$, $\qquad PC=PB\sin\beta.$

From $\triangle PAB$,

$$PB=\frac{AB\sin PAB}{\sin APB}=\frac{a\sin\alpha}{\sin(\beta-\alpha)};$$

$$\therefore\ PC=a\sin\alpha\sin\beta\operatorname{cosec}(\beta-\alpha).$$

Also $\qquad AC=PC\cot\alpha=a\cos\alpha\sin\beta\operatorname{cosec}(\beta-\alpha).$

Each of the above expressions is adapted to logarithmic work; thus if $PC=x$, we have

$$\log x=\log a+\log\sin\alpha+\log\sin\beta+\log\operatorname{cosec}(\beta-\alpha).$$

NOTE. Unless the contrary is stated, it will be supposed that the observer's height is disregarded, and that the angles of elevation are measured from the ground.

Example I. A person walking along a straight road observes that at two consecutive milestones the angles of elevation of a hill in front of him are 30° and 75°: find the height of the hill.

In the adjoining figure,

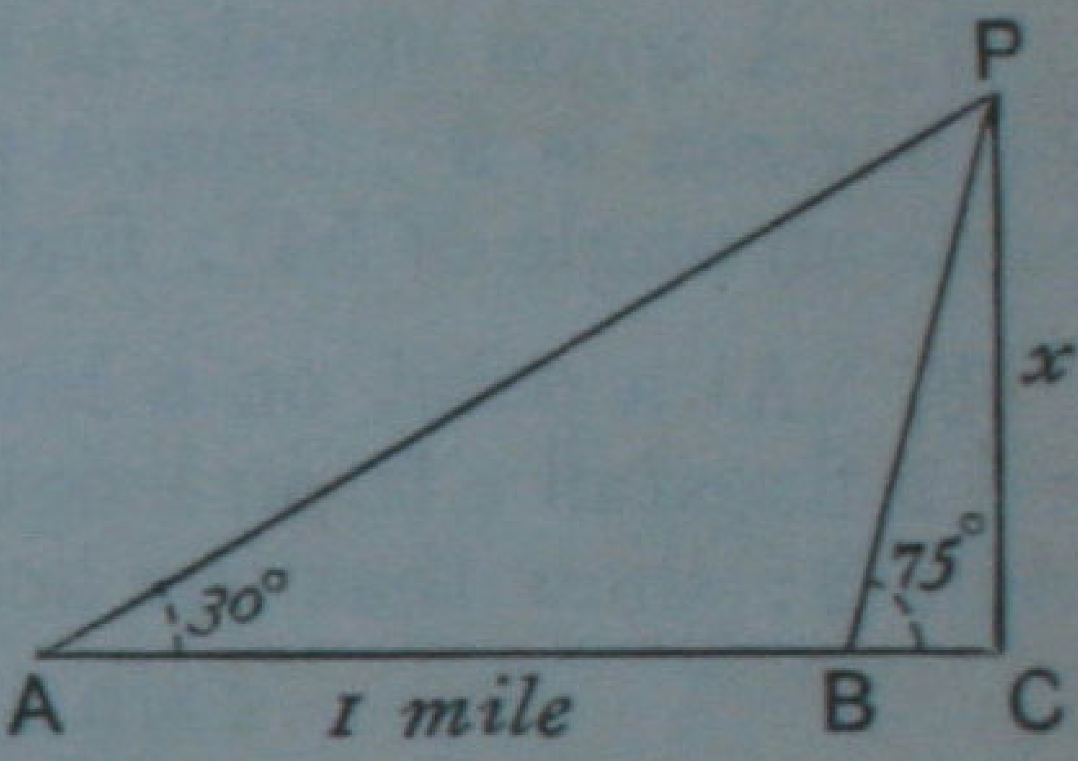
$\angle PAC = 30°$, $\angle PBC = 75°$, $AB = 1$ mile;

$$\angle APB = 75° - 30° = 45°.$$

Let x be the height in yards; then

$$x = PB \sin 75°;$$

but $PB = \dfrac{AB \sin PAB}{\sin APB} = \dfrac{1760 \sin 30°}{\sin 45°}$;



$$\therefore\ x = \frac{1760 \sin 30° \sin 75°}{\sin 45°}$$

$$= 1760 \times \frac{1}{2} \times \sqrt{2} \times \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$= 440(\sqrt{3}+1).$$

If we take $\sqrt{3} = 1{\cdot}732$ and reduce to feet, we find that the height is 3606·24 ft.

EXAMPLES. XVII. a.

1. From the top of a cliff 200 ft. above the sea-level the angles of depression of two boats in the same vertical plane as the observer are 45° and 30°: find their distance apart.

2. A person observes the elevation of a mountain top to be 15°, and after walking a mile directly towards it on level ground the elevation is 75°: find the height of the mountain in feet.

3. From a ship at sea the angle subtended by two forts A and B is 30°. The ship sails 4 miles towards A and the angle is then 48°: prove that the distance of B at the second observation is 6·472 miles.

4. From the top of a tower h ft. high the angles of depression of two objects on the horizontal plane and in a line passing through the foot of the tower are $45° - A$ and $45° + A$. Shew that the distance between them is $2h \tan 2A$.

5. An observer finds that the angular elevation of a tower is A. On advancing a feet towards the tower the elevation is $45°$ and on advancing b feet nearer the elevation is $90° - A$: find the height of the tower.

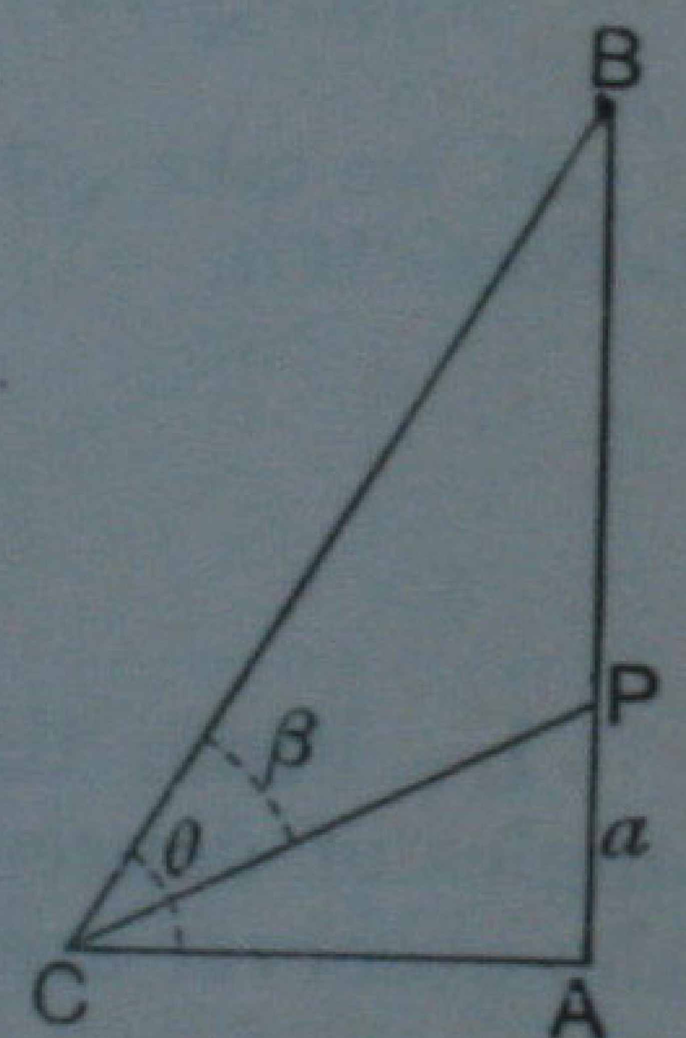
6. A person observes that two objects A and B bear due N. and N. $30°$ W. respectively. On walking a mile in the direction N.W. he finds that the bearings of A and B are N.E. and due E. respectively: find the distance between A and B.

7. A tower stands at the foot of a hill whose inclination to the horizon is $9°$; from a point 40 ft. up the hill the tower subtends an angle of $54°$: find its height.

8. At a point on a level plane a tower subtends an angle α and a flagstaff c ft. in length at the top of the tower subtends an angle β: shew that the height of the tower is

$$c \sin \alpha \operatorname{cosec} \beta \cos (\alpha+\beta).$$

Example II. The upper three-fourths of a ship's mast subtends at a point on the deck an angle whose tangent is ·6; find the tangent of the angle subtended by the whole mast at the same point.



Let C be the point of observation, and let APB be the mast, AP being the lower fourth of it.

Let $AB=4a$, so that $AP=a$;
also let $AC=b$, $\angle ACB=\theta$, $\angle BCP=\beta$,
so that $\tan \beta = \cdot 6$.

From $\triangle PCA$, $\tan(\theta-\beta)=\frac{a}{b}$;

from $\triangle BCA$, $\tan\theta = \frac{4a}{b}$;

$$\therefore \tan\theta = 4\tan(\theta-\beta) = \frac{4(\tan\theta - \tan\beta)}{1+\tan\theta\tan\beta};$$

$$\therefore \tan\theta = \frac{4\left(\tan\theta - \frac{3}{5}\right)}{1+\frac{3}{5}\tan\theta} = \frac{4(5\tan\theta-3)}{5+3\tan\theta}.$$

On reduction, $\tan^2\theta - 5\tan\theta + 4 = 0$;

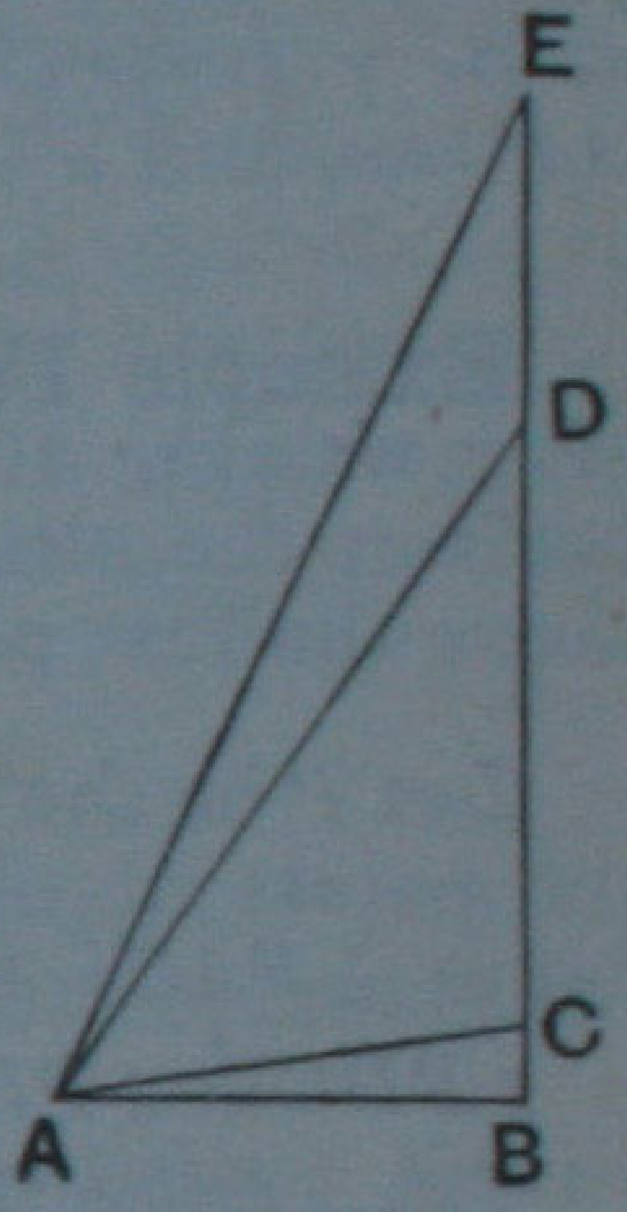
whence $\tan\theta = 1$ or 4.

Note. The student should observe that in examples of this class we make use of right-angled triangles in which the horizontal base line forms one side.

Example III. A tower BCD surmounted by a spire DE stands on a horizontal plane. From the extremity A of a horizontal line BA, it is found that BC and DE subtend equal angles. If $BC=9$ ft., $CD=72$ ft., and $DE=36$ ft., find BA.

Let $$\angle BAC = \angle DAE = \theta,$$
$$\angle DAB = \alpha, \quad AB = x \text{ ft.}$$

Now $BC=9$ ft., $BD=81$ ft., $BE=117$ ft.



$$\therefore \tan(\alpha+\theta) = \frac{BE}{AB} = \frac{117}{x};$$
$$\tan\alpha = \frac{BD}{AB} = \frac{81}{x};$$
$$\tan\theta = \frac{BC}{AB} = \frac{9}{x}.$$

But $$\tan(\alpha+\theta) = \frac{\tan\alpha + \tan\theta}{1-\tan\alpha\tan\theta};$$

$$\therefore \frac{117}{x} = \frac{\frac{81}{x}+\frac{9}{x}}{1-\frac{81}{x}\cdot\frac{9}{x}} = \frac{90}{x}\cdot\frac{x^2}{x^2-81\times 9}.$$

$$117x^2 - 81\times 9\times 117 = 90x^2;$$
$$\therefore 27x^2 = 81\times 9\times 117;$$
$$\therefore x^2 = 81\times 39;$$
$$\therefore x = 9\sqrt{39}.$$

But $\sqrt{39} = 6{\cdot}245$ nearly; $\therefore x = 56{\cdot}205$ nearly.

Thus $AB = 56{\cdot}2$ ft. nearly.

√ **9.** A flagstaff 20 ft. long standing on a wall 10 ft. high subtends an angle whose tangent is ·5 at a point on the ground: find the tangent of the angle subtended by the wall at this point.

10. A statue standing on the top of a pillar 25 feet high subtends an angle whose tangent is ·125 at a point 60 feet from the foot of the pillar: find the height of the statue.

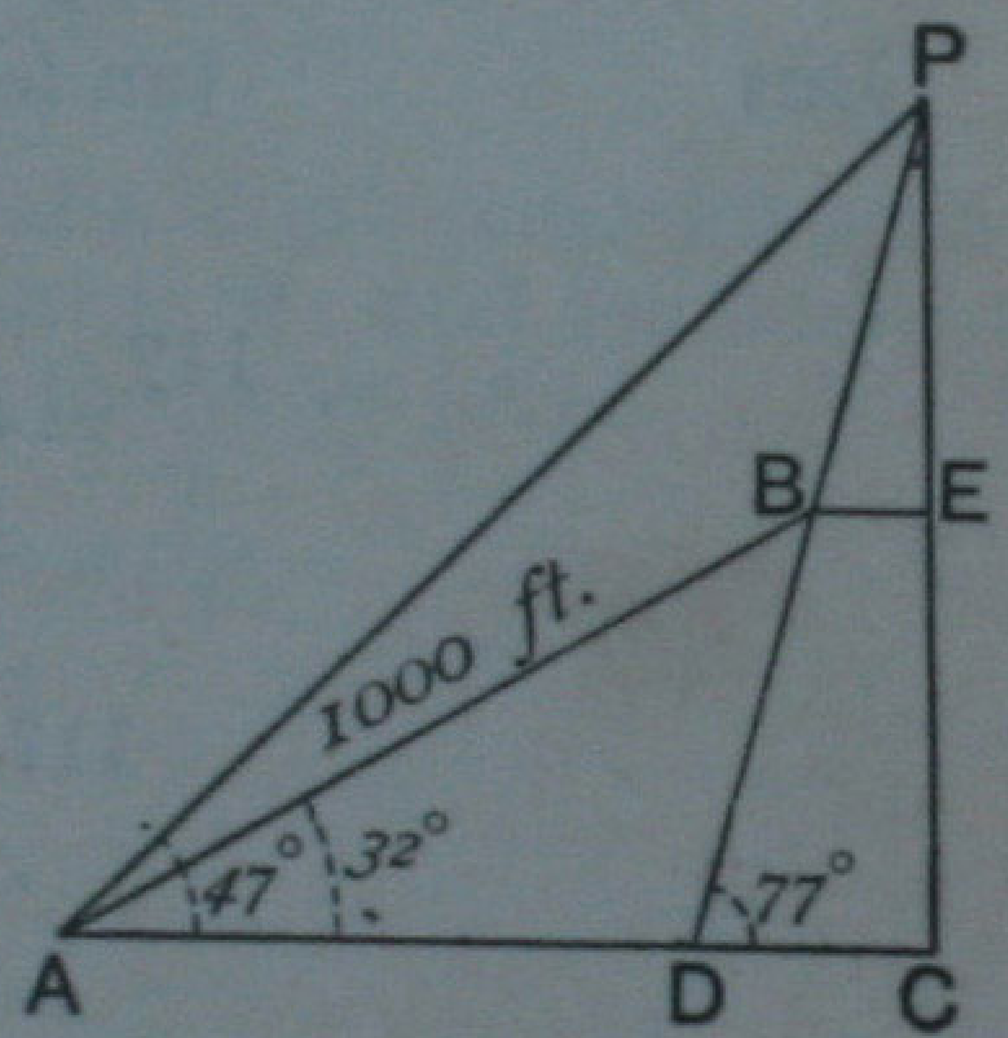
√ **11.** A tower BCD surmounted by a spire DE stands on a horizontal plane. From the extremity A of a horizontal line BA it is found that BC and DE subtend equal angles.

If $BC=9$ ft., $CD=280$ ft., and $DE=35$ ft., prove that $BA=180$ ft. nearly.

12. On the bank of a river there is a column 192 ft. high supporting a statue 24 ft. high. At a point on the opposite bank directly facing the column the statue subtends the same angle as a man 6 ft. high standing at the base of the column: find the breadth of the river.

13. A monument $ABCDE$ stands on level ground. At a point P on the ground the portions AB, AC, AD subtend angles α, β, γ respectively. Supposing that $AB=a$, $AC=b$, $AD=c$, $AP=x$, and $\alpha+\beta+\gamma=180°$, shew that $(a+b+c)\,x^2=abc$.

Example IV. The altitude of a rock is observed to be 47°; after walking 1000 ft. towards it up a slope inclined at 32° to the horizon the altitude is 77°. Find the vertical height of the rock above the first point of observation, given $\sin 47° = \cdot731$.



Let P be the top of the rock, A and B the points of observation; then in the figure $\angle PAC=47°$, $\angle BAC=32°$,

$\angle PDC=\angle PBE=77°$, $AB=1000$ ft.

Let x ft. be the height; then

$$x = PA \sin PAC = PA \sin 47°.$$

We have therefore to find PA in terms of AB.

In $\triangle PAB$, $\angle PAB=47°-32°=15°$;

$\angle APB=77°-47°=30°$;

$\therefore \angle ABP=135°$;

$$\therefore PA=\frac{AB\sin ABP}{\sin APB}$$

$$=\frac{1000\sin 135°}{\sin 30°}$$

$$=1000\sqrt{2};$$

$$\therefore x=PA\sin 47°=1000\sqrt{2}\times\cdot731$$

$$=731\sqrt{2}.$$

If we take $\sqrt{2}=1\cdot414$, we find that the height is 1034 ft. nearly.

14. From a point on the horizontal plane, the elevation of the top of a hill is 45°. After walking 500 yards towards its summit up a slope inclined at an angle of 15° to the horizon the elevation is 75°: find the height of the hill in feet.

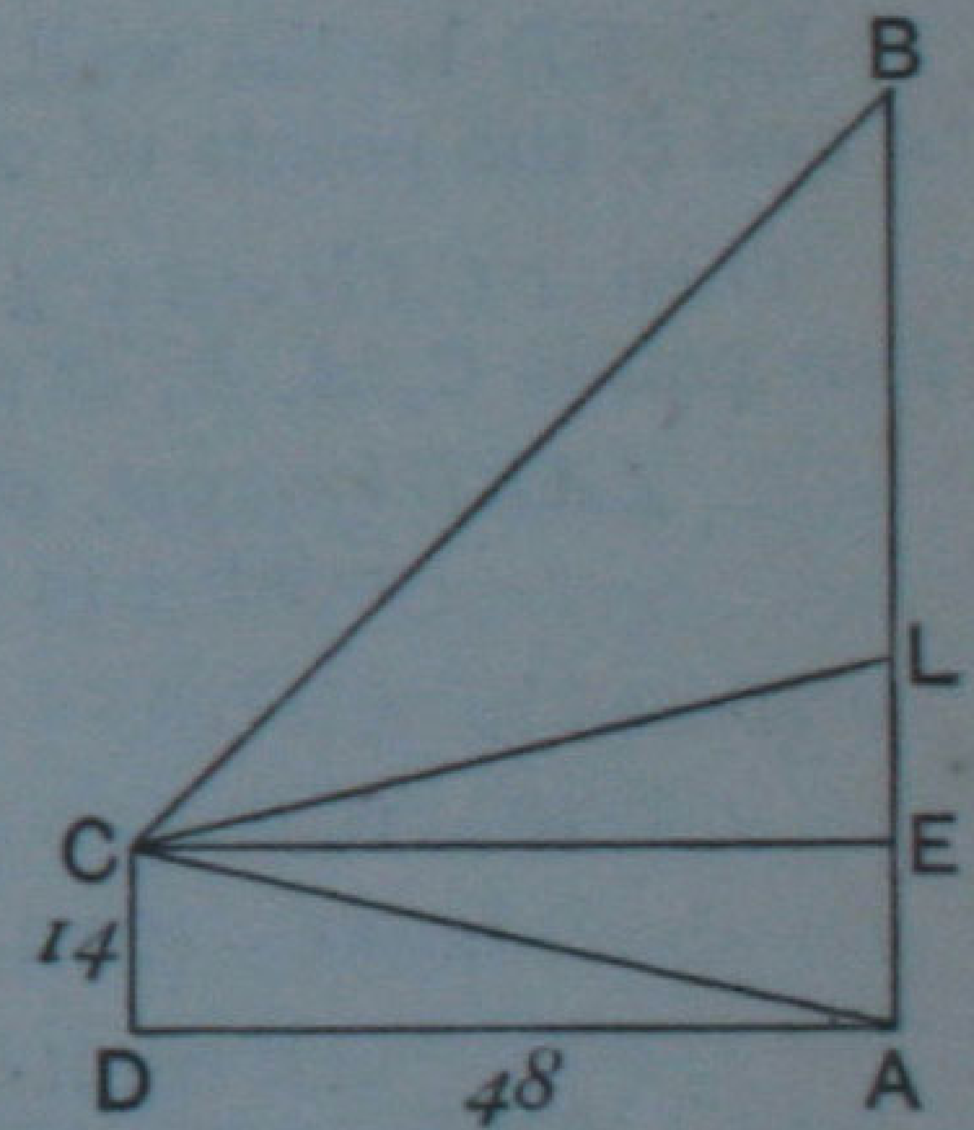
15. From a station B at the base of a mountain its summit A is seen at an elevation of 60°; after walking one mile towards the summit up a plane making an angle of 30° with the horizon to another station C, the angle BCA is observed to be 135°: find the height of the mountain in feet.

16. The elevation of the summit of a hill from a station A is α. After walking c feet towards the summit up a slope inclined at an angle β to the horizon the elevation is γ: shew that the height of the hill is $c \sin\alpha \sin(\gamma-\beta) \operatorname{cosec}(\gamma-\alpha)$ feet.

17. From a point A an observer finds that the elevation of Ben Nevis is 60°; he then walks 800 ft. on a level plane towards the foot, and then 800 ft. further up a slope of 30° and finds the elevation to be 75°: shew that the height of Ben Nevis above A is 4478 ft. approximately.

200. In many of the problems which follow, the solution depends upon the knowledge of some geometrical proposition.

Example I. A tower stands on a horizontal plane. From a mound 14 ft. above the plane and at a horizontal distance of 48 ft. from the tower an observer notices a loophole, and finds that the portions of the tower above and below the loophole subtend equal angles. If the height of the loophole is 30 ft., find the height of the tower.



Let AB be the tower, C the point of observation, L the loophole. Draw CD vertical and CE horizontal. Let $AB=x$. We have

$$CD=14,\ AD=EC=48,\ BE=x-14.$$

From $\triangle ADC$, $AC^2=(14)^2+(48)^2=2500$;

$$\therefore AC=50.$$

From $\triangle CEB$, $CB^2=(x-14)^2+(48)^2$

$$=x^2-28x+2500.$$

Now $\angle BCL=\angle ACL$;

hence by Euc. VI. 3, $\dfrac{BC}{AC}=\dfrac{BL}{AL}$;

$$\therefore \frac{\sqrt{x^2-28x+2500}}{50}=\frac{x-30}{30}.$$

By squaring, $9(x^2-28x+2500)=25(x^2-60x+900)$.

On reduction, we obtain $16x^2-1248x=0$; whence $x=78$.

Thus the tower is 78 ft. high.

EXAMPLES. XVII. b.

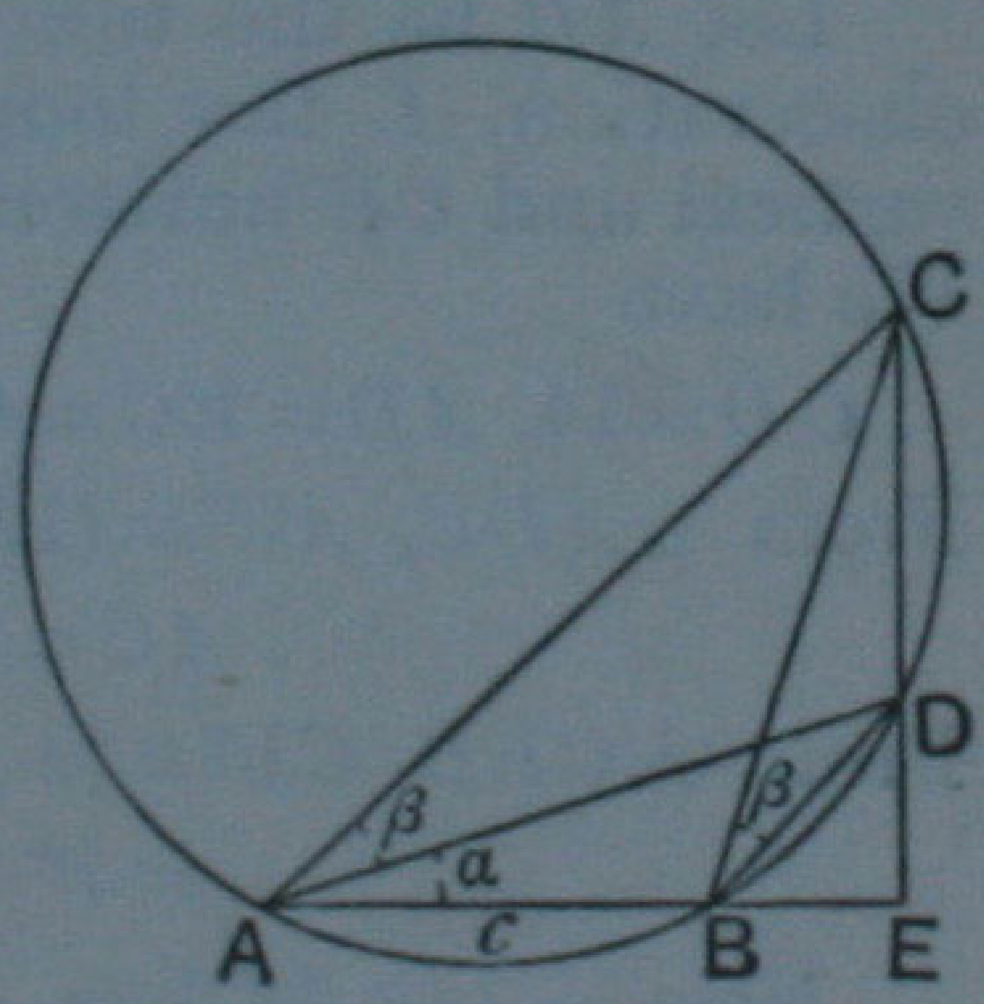
1. At one side of a road is a flagstaff 25 ft. high fixed on the top of a wall 15 ft. high. On the other side of the road, at a point on the ground directly opposite, the flagstaff and wall subtend equal angles: find the width of the road.

2. A statue a feet high stands on a column $3a$ feet high. To an observer on a level with the top of the statue, the column and statue subtend equal angles: find the distance of the observer from the top of the statue.

3. A flagstaff a feet high placed on the top of a tower b feet high subtends the same angle as the tower to an observer h feet high standing on the horizontal plane at a distance d feet from the foot of the tower: shew that

$$(a-b)\,d^2=(a+b)\,b^2-2b^2h-(a-b)\,h^2.$$

Example II. A flagstaff is fixed on the top of a wall standing upon a horizontal plane. An observer finds that the angles subtended at a point on this plane by the wall and the flagstaff are α and β. He then walks a distance c directly towards the wall and finds that the flagstaff again subtends an angle β. Find the heights of the wall and flagstaff.



Let ED be the wall, DC the flagstaff, A and B the points of observation.

Then $\angle CAD=\beta=\angle CBD$, so that the four points C, A, B, D are concyclic.

$$\therefore\ \angle ABD=\text{supp}^t.\text{ of }\angle ACE$$
$$=90^\circ+(\alpha+\beta),\text{ from }\triangle CAE.$$

Hence in $\triangle ADB$,

$$\angle ADB=180^\circ-\alpha-\{90^\circ+(\alpha+\beta)\}$$
$$=90^\circ-(2\alpha+\beta).$$

$$\therefore\ AD=\frac{AB\sin ABD}{\sin ADB}=\frac{c\cos(\alpha+\beta)}{\cos(2\alpha+\beta)}.$$

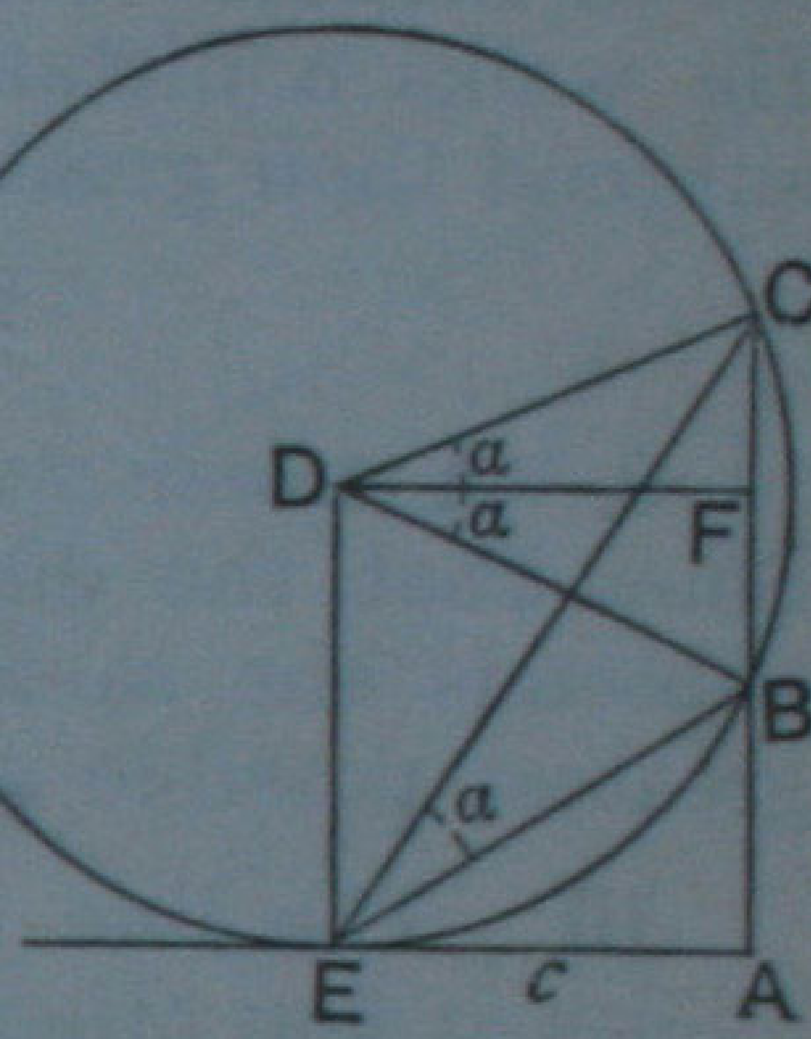
Hence in $\triangle ADE$,

$$DE=AD\sin\alpha=\frac{c\sin\alpha\cos(\alpha+\beta)}{\cos(2\alpha+\beta)}.$$

And in $\triangle CAD$,

$$CD=\frac{AD\sin CAD}{\sin ACD}=\frac{AD\sin\beta}{\cos(\alpha+\beta)}=\frac{c\sin\beta}{\cos(2\alpha+\beta)}.$$

Example III. A man walking towards a tower AB on which a flagstaff BC is fixed observes that when he is at a point E, distant c ft. from the tower, the flagstaff subtends its greatest angle. If $\angle BEC=\alpha$, prove that the heights of the tower and flagstaff are $c\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$ and $2c\tan\alpha$ ft. respectively.



Since E is the point in the horizontal line AE at which BC subtends a maximum angle, it can easily be proved that AE touches the circle passing round the triangle CBE.

[See Hall and Stevens' *Euclid*, p. 242.]

The centre D of this circle lies in the vertical line through E. Draw DF perpendicular to BC, then DF bisects BC and also $\angle CDB$.

By Euc. III. 20,

$$\angle CDB=2\angle CEB=2\alpha;$$

$$\therefore \angle CDF=\angle BDF=\alpha.$$

$$\therefore CB=2CF=2DF\tan\alpha=2c\tan\alpha.$$

Again, $\quad \angle AEB=\angle ECB$ in alternate segment

$$=\frac{1}{2}\angle EDB \text{ at centre}$$

$$=\frac{1}{2}\left(\frac{\pi}{2}-\alpha\right).$$

$$\therefore AB=c\tan AEB=c\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right).$$

4. A tower standing on a cliff subtends an angle β at each of two stations in the same horizontal line passing through the base of the cliff and at distances of a feet and b feet from the cliff. Prove that the height of the tower is $(a+b)\tan\beta$ feet.

5. A column placed on a pedestal 20 feet high subtends an angle of 45° at a point on the ground, and it also subtends an angle of 45° at a point which is 20 feet nearer the pedestal: find the height of the column.

6. A flagstaff on a tower subtends the same angle at each of two places A and B on the ground. The elevations of the top of the flagstaff as seen from A and B are α and β respectively. If $AB=a$, shew that the length of the flagstaff is

$$a\sin(\alpha+\beta-90°)\operatorname{cosec}(\alpha-\beta).$$

7. A pillar stands on a pedestal. At a distance of 60 feet from the base of the pedestal the pillar subtends its greatest angle 30°: shew that the length of the pillar is $40\sqrt{3}$ feet, and that the pedestal also subtends 30° at the point of observation.

8. A person walking along a canal observes that two objects are in the same line which is inclined at an angle a to the canal. He walks a distance c further and observes that the objects subtend their greatest angle β: shew that their distance apart is

$$2c \sin a \sin \beta / (\cos a + \cos \beta).$$

9. A tower with a flagstaff stands on a horizontal plane. Shew that the distances from the base at which the flagstaff subtends the same angle and that at which it subtends the greatest possible angle are in geometrical progression.

10. The line joining two stations A and B subtends equal angles at two other stations C and D: prove that

$$AB \sin CBD = CD \sin ADB.$$

11. Two straight lines ABC, DEC meet at C. If

$$\angle DAE = \angle DBE = a, \text{ and } \angle EAB = \beta, \ \angle EBC = \gamma,$$

shew that
$$BC = \frac{AB \sin \beta \sin (a+\beta)}{\sin (\gamma - \beta) \sin (a + \beta + \gamma)}.$$

12. Two objects P and Q subtend an angle of 30° at A. Lengths of 20 feet and 10 feet are measured from A at right angles to AP and AQ respectively to points R and S at each of which PQ subtends angles of 30°: find the length of PQ.

13. A ship sailing N.E. is in a line with two beacons which are 5 miles apart, and of which one is due N. of the other. In 3 minutes and also in 21 minutes the beacons are found to subtend a right angle at the ship. Prove that the ship is sailing at the rate of 10 miles an hour, and that the beacons subtend their greatest angle at the ship at the end of $3\sqrt{7}$ minutes.

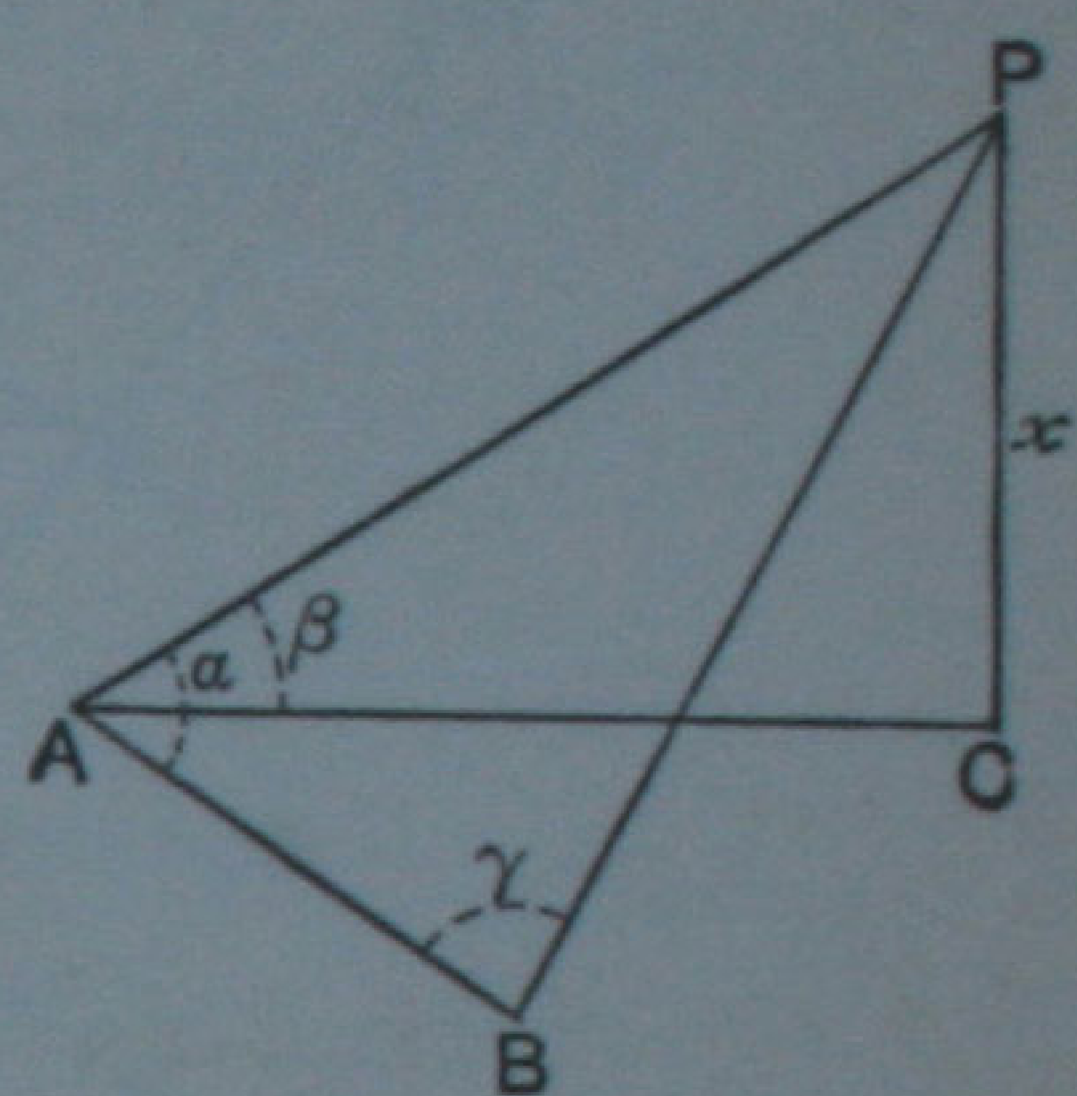
14. A flagstaff stands on the top of a tower. A man walking along a straight road towards the tower observes that the angle of elevation of the top of the flagstaff is β; after walking a distance a further along the road he notices that the flagstaff subtends its maximum angle a; shew that the height of the flagstaff is

$$\frac{2a \sin a \sin \beta}{\cos \beta + \sin (a - \beta)}.$$

Measurements in more than one plane.

201. In Art. 199 the base line AB was measured *directly towards* the object. If this is not possible we may proceed as follows.

From A measure a base line AB in any convenient direction in the horizontal plane. At A observe the two angles PAB and PAC; and at B observe the angle PBA.



Let $\angle PAB=\alpha$, $\angle PAC=\beta$,

$\angle PBA=\gamma$,

$AB=a$, $PC=x$.

From $\triangle PAC$,

$$x=PA\sin\beta.$$

From $\triangle PAB$,

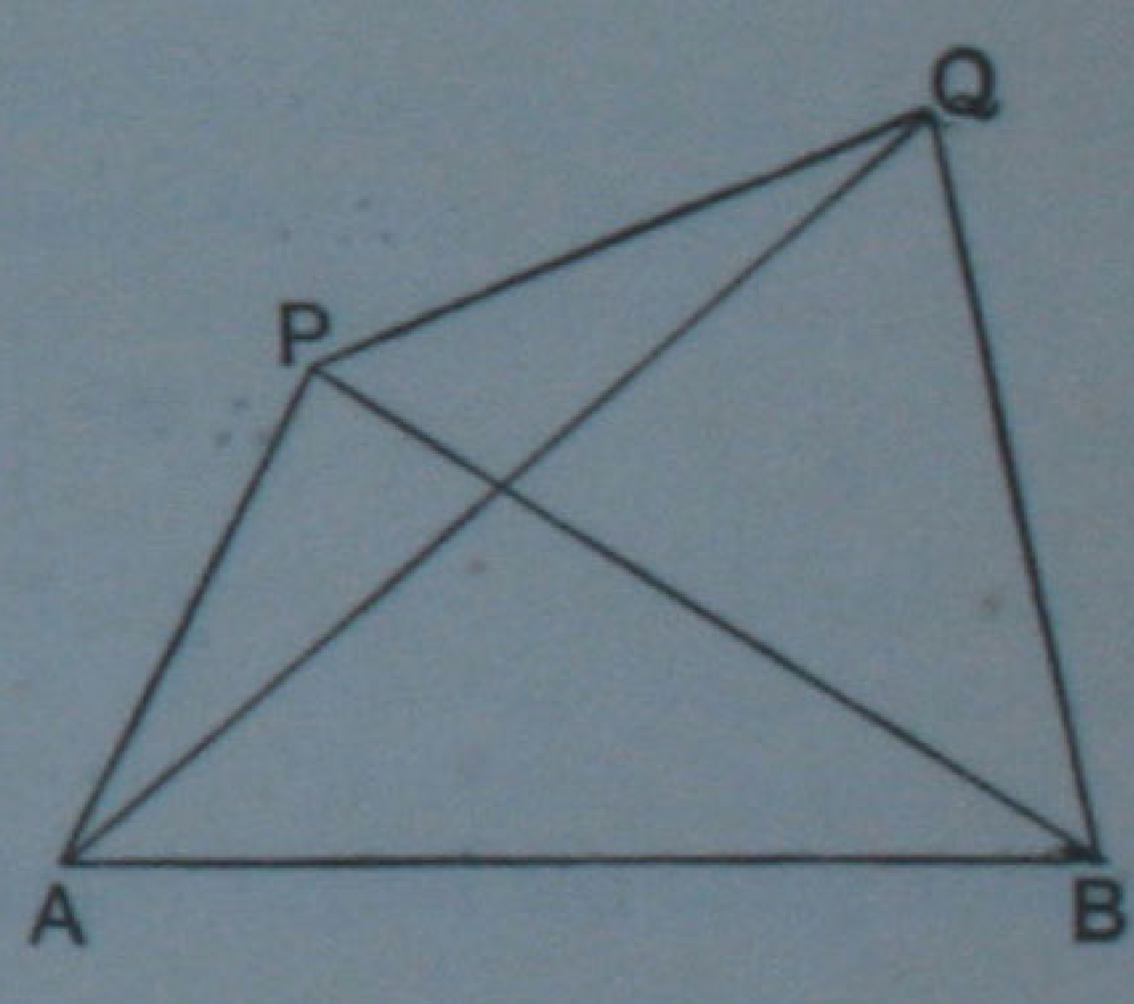
$$PA=\frac{AB\sin PBA}{\sin APB}=\frac{a\sin\gamma}{\sin(\alpha+\gamma)};$$

$$\therefore\ x=a\sin\beta\sin\gamma\operatorname{cosec}(\alpha+\gamma).$$

202. *To shew how to find the distance between two inaccessible objects.*

Let P and Q be the objects.

Take any two convenient stations A and B in the same horizontal plane, and measure the distance between them.



At A observe the angles PAQ and QAB. Also if AP, AQ, AB are not in the same plane, measure the angle PAB.

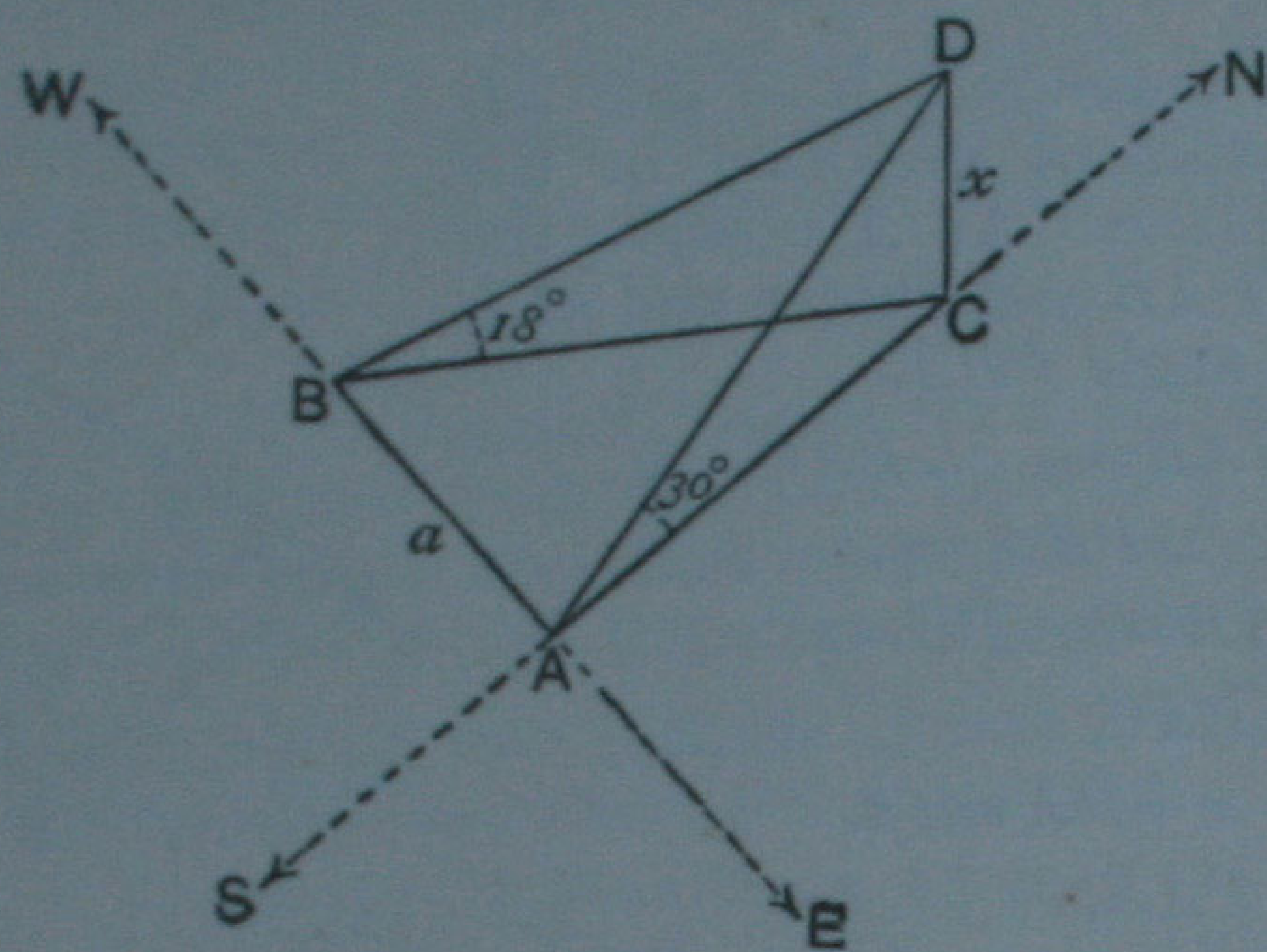
At B observe the angles ABP and ABQ.

In $\triangle PAB$, we know $\angle PAB$, $\angle PBA$, and AB; so that AP may be found.

In $\triangle QAB$, we know $\angle QAB$, $\angle QBA$, and AB; so that AQ may be found.

In $\triangle PAQ$, we know AP, AQ, and $\angle PAQ$; so that PQ may be found.

Example 1. The angular elevation of a tower CD at a place A due South of it is 30°, and at a place B due West of A the elevation is 18°. If $AB=a$, shew that the height of the tower is $\dfrac{a}{\sqrt{2+2\sqrt{5}}}$.



Let $CD=x$.

From the right-angled triangle DCA, $AC=x\cot 30°$.

From the right-angled triangle DCB, $BC=x\cot 18°$.

But $\angle BAC$ is a right angle,

$$\therefore\ BC^2-AC^2=a^2;$$

$$\therefore\ x^2(\cot^2 18°-\cot^2 30°)=a^2;$$

$$\therefore\ x^2(\operatorname{cosec}^2 18°-\operatorname{cosec}^2 30°)=a^2;$$

$$\therefore\ x^2\left\{\left(\frac{4}{\sqrt{5}-1}\right)^2-4\right\}=a^2;$$

$$\therefore\ x^2\{(\sqrt{5}+1)^2-4\}=a^2;$$

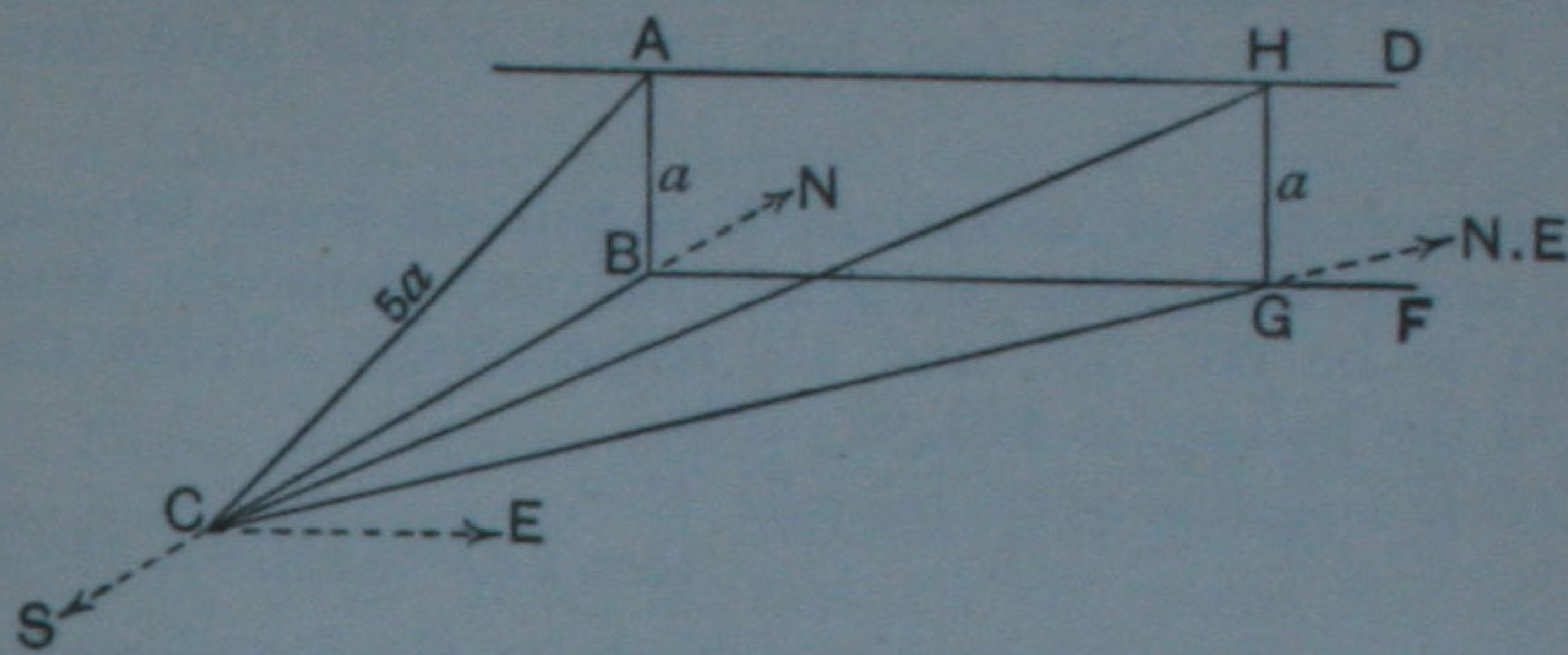
$$\therefore\ x^2(2+2\sqrt{5})=a^2,$$

which gives the height required.

Example 2. A hill of inclination 1 in 5 faces South. Shew that a road on it which takes a N.E. direction has an inclination 1 in 7.

Let AD running East and West be the ridge of the hill, and let $ABFD$ be a vertical plane through AD. Let C be a point at the foot of the hill, and ABC a section made by a vertical plane running North and South. Draw CG in a N.E. direction in the horizontal plane and let it meet BF in G; draw GH parallel to BA; then if CH is joined it will represent the direction of the road.

Since the inclination of CA is 1 in 5, we may take $AB=a$, and $AC=5a$, so that $BC^2=24a^2$.



Since CBG is a right-angled isosceles triangle,

$$CG^2=2CB^2=48a^2.$$

Hence in the right-angled triangle CGH,

$$CH^2=48a^2+a^2=49a^2;$$

$$\therefore\ CH=7a=7GH.$$

Thus the slope of the road is 1 in 7.

EXAMPLES. XVII. c.

1. The elevation of a hill at a place P due East of it is 45°, and at a place Q due South of P the elevation is 30°. If the distance from P to Q is 500 yards, find the height of the hill in feet.

2. The elevation of a spire at a point A due West of it is 60°, and at point B due South of A the elevation is 30°. If the spire is 250 feet high, find the distance between A and B.

3. A river flows due North, and a tower stands on its left bank. From a point A up-stream and on the same bank as the tower the elevation of the tower is 60°, and from a point B just opposite on the other bank the elevation is 45°. If the tower is 360 feet high, find the breadth of the river.

4. The elevation of a steeple at a place A due S. of it is 45°, and at a place B due W. of A the elevation is 15°. If $AB=2a$, shew that the height of the steeple is $a(3^{\frac{1}{4}}-3^{-\frac{1}{4}})$.

5. A person due S. of a lighthouse observes that his shadow cast by the light at the top is 24 feet long. On walking 100 yards due E. he finds his shadow to be 30 feet long. Supposing him to be 6 feet high, find the height of the light from the ground.

6. The angles of elevation of a balloon from two stations a mile apart and from a point halfway between them are observed to be 60°, 30°, and 45° respectively. Prove that the height of the balloon is $440\sqrt{6}$ yards.

[*If AD is a median of the triangle ABC, then* $2AD^2+2BD^2=AB^2+AC^2$.]

7. At each end of a base of length $2a$, the angular elevation of a mountain is θ, and at the middle point of the base the elevation is ϕ. Prove that the height of the mountain is

$$a\sin\theta\sin\phi\sqrt{\operatorname{cosec}(\phi+\theta)\operatorname{cosec}(\phi-\theta)}.$$

8. Two vertical poles, whose heights are a and b, subtend the same angle α at a point in the line joining their feet. If they subtend angles β and γ at any point in the horizontal plane at which the line joining their feet subtends a right angle, prove that

$$(a+b)^2\cot^2\alpha=a^2\cot^2\beta+b^2\cot^2\gamma.$$

9. From the top of a hill a person finds that the angles of depression of three consecutive milestones on a straight level road are α, β, γ. Shew that the height of the hill is

$$5280\sqrt{2}\,/\sqrt{\cot^2\alpha-2\cot^2\beta+\cot^2\gamma}\text{ feet.}$$

10. Two chimneys AB and CD are of equal height. A person standing between them in the line AC joining their bases observes the elevation of the one nearer to him to be 60°. After walking 80 feet in a direction at right angles to AC he observes their elevations to be 45° and 30°: find their height and distance apart.

11. Two persons who are 500 yards apart observe the bearing and angular elevation of a balloon at the same instant. One finds the elevation 60° and the bearing S.W., the other finds the elevation 45° and the bearing W. Find the height of the balloon.

12. The side of a hill faces due S. and is inclined to the horizon at an angle α. A straight railway upon it is inclined at an angle β to the horizon: if the bearing of the railway be x degrees E. of N., shew that $\cos x=\cot\alpha\tan\beta$.

EXAMPLES. XVII. d.

[*In the following examples the logarithms are to be taken from the Tables.*]

1. A man in a balloon observes that two churches which he knows to be one mile apart subtend an angle of $11° 25' 20''$ when he is exactly over the middle point between them: find the height of the balloon in miles.

2. There are three points A, B, C in a straight line on a level piece of ground. A vertical pole erected at C has an elevation of $5° 30'$ from A and $10° 45'$ from B. If AB is 100 yards, find the height of the pole and the distance BC.

3. The angular altitude of a lighthouse seen from a point on the shore is $12° 31' 46''$, and from a point 500 feet nearer the altitude is $26° 33' 55''$: find its height above the sea-level.

4. From a boat the angles of elevation of the highest and lowest points of a flagstaff 30 ft. high on the edge of a cliff are $46° 12'$ and $44° 13'$: find the height and distance of the cliff.

5. From the top of a hill the angles of depression of two successive milestones on level ground, and in the same vertical plane as the observer, are $5°$ and $10°$. Find the height of the hill in feet and the distance of the nearer milestone in miles.

6. An observer whose eye is 15 feet above the roadway finds that the angle of elevation of the top of a telegraph post is $17° 18' 35''$, and that the angle of depression of the foot of the post is $8° 32' 15''$: find the height of the post and its distance from the observer.

7. Two straight railroads are inclined at an angle of $20° 16'$. At the same instant two engines start from the point of intersection, one along each line; one travels at the rate of 20 miles an hour: at what rate must the other travel so that after 3 hours the distance between them shall be 30 miles?

8. An observer finds that from the doorstep of his house the elevation of the top of a spire is $5a$, and that from the roof above the doorstep it is $4a$. If h be the height of the roof above the doorstep, prove that the height of the spire above the doorstep and the horizontal distance of the spire from the house are respectively

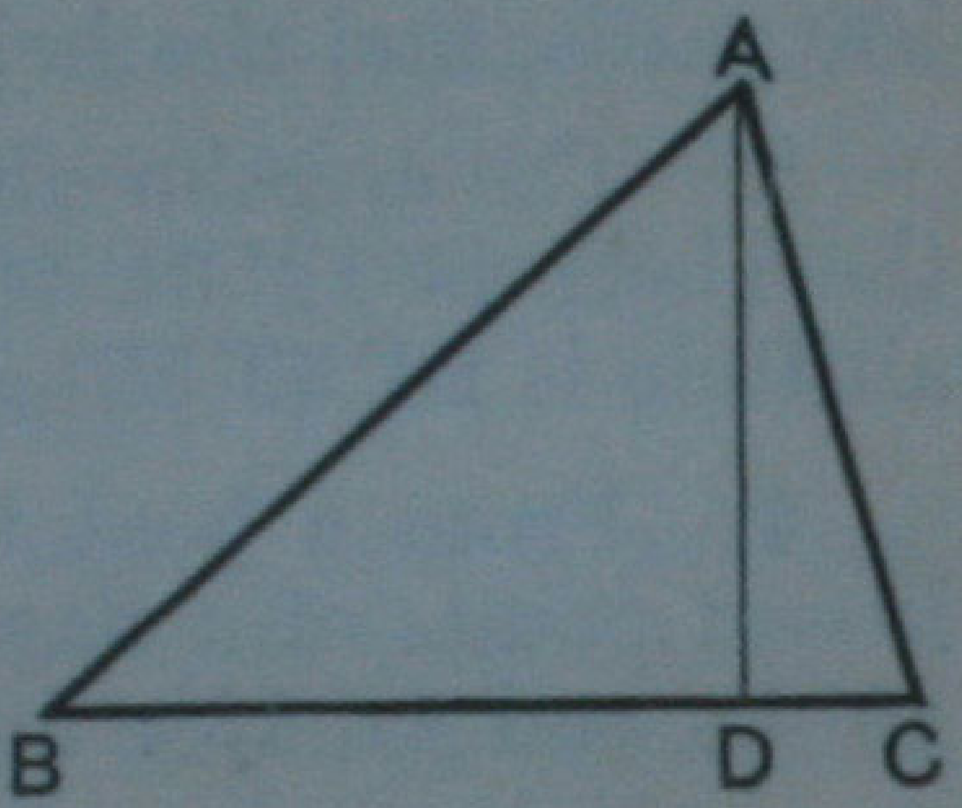
$$h \operatorname{cosec} a \cos 4a \sin 5a \text{ and } h \operatorname{cosec} a \cos 4a \cos 5a.$$

If $h = 39$ feet, and $a = 7° 17' 39''$, calculate the height and the distance.

CHAPTER XVIII.

PROPERTIES OF TRIANGLES AND POLYGONS.

203. *To find the area of a triangle.*



Let Δ denote the area of the triangle ABC. Draw AD perpendicular to BC.

By Euc. I. 41, the area of a triangle is half the area of a rectangle on the same base and of the same altitude.

$$\therefore \quad \Delta = \frac{1}{2}(\text{base} \times \text{altitude})$$

$$= \frac{1}{2} BC \, . \, AD = \frac{1}{2} BC \, . \, AB \sin B$$

$$= \frac{1}{2} ca \sin B.$$

Similarly, it may be proved that

$$\Delta = \frac{1}{2} ab \sin C, \text{ and } \Delta = \frac{1}{2} bc \sin A.$$

These three expressions for the area are comprised in the single statement

$$\Delta = \frac{1}{2} (\textit{product of two sides}) \times (\textit{sine of included angle}).$$

Again, $\quad \Delta = \frac{1}{2} bc \sin A = bc \sin \frac{A}{2} \cos \frac{A}{2}$

$$= bc \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{s(s-a)(s-b)(s-c)},$$

which gives the area in terms of the sides.

Again, $\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} \sin A \cdot \frac{a \sin B}{\sin A} \cdot \frac{a \sin C}{\sin A}$

$$= \frac{a^2 \sin B \sin C}{2 \sin A}$$

$$= \frac{a^2 \sin B \sin C}{2 \sin (B + C)},$$

which gives the area in terms of one side and the functions of the adjacent angles.

NOTE. Many writers use the symbol S for the area of a triangle, but to avoid confusion between S and s in manuscript work the symbol Δ is preferable.

Example 1. The sides of a triangle are 17, 25, 28: find the lengths of the perpendiculars from the angles upon the opposite sides.

From the formula $\Delta = \frac{1}{2}$ (base × altitude),

it is evident that the three perpendiculars are found by dividing 2Δ by the three sides in turn.

Now $\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{35 \times 18 \times 10 \times 7}$

$$= 5 \times 7 \times 6 = 210.$$

Thus the perpendiculars are $\frac{420}{17}, \frac{420}{25}, \frac{420}{28}$, or $\frac{420}{17}, \frac{84}{5}$, 15.

Example 2. Two angles of a triangular field are 22·5° and 45°, and the length of the side opposite to the latter is one furlong: find the area.

Let $A = 22\frac{1}{2}°$, $B = 45°$, then $b = 220$ yds., and $C = 112\frac{1}{2}°$.

From the formula $\Delta = \frac{b^2 \sin A \sin C}{2 \sin B}$,

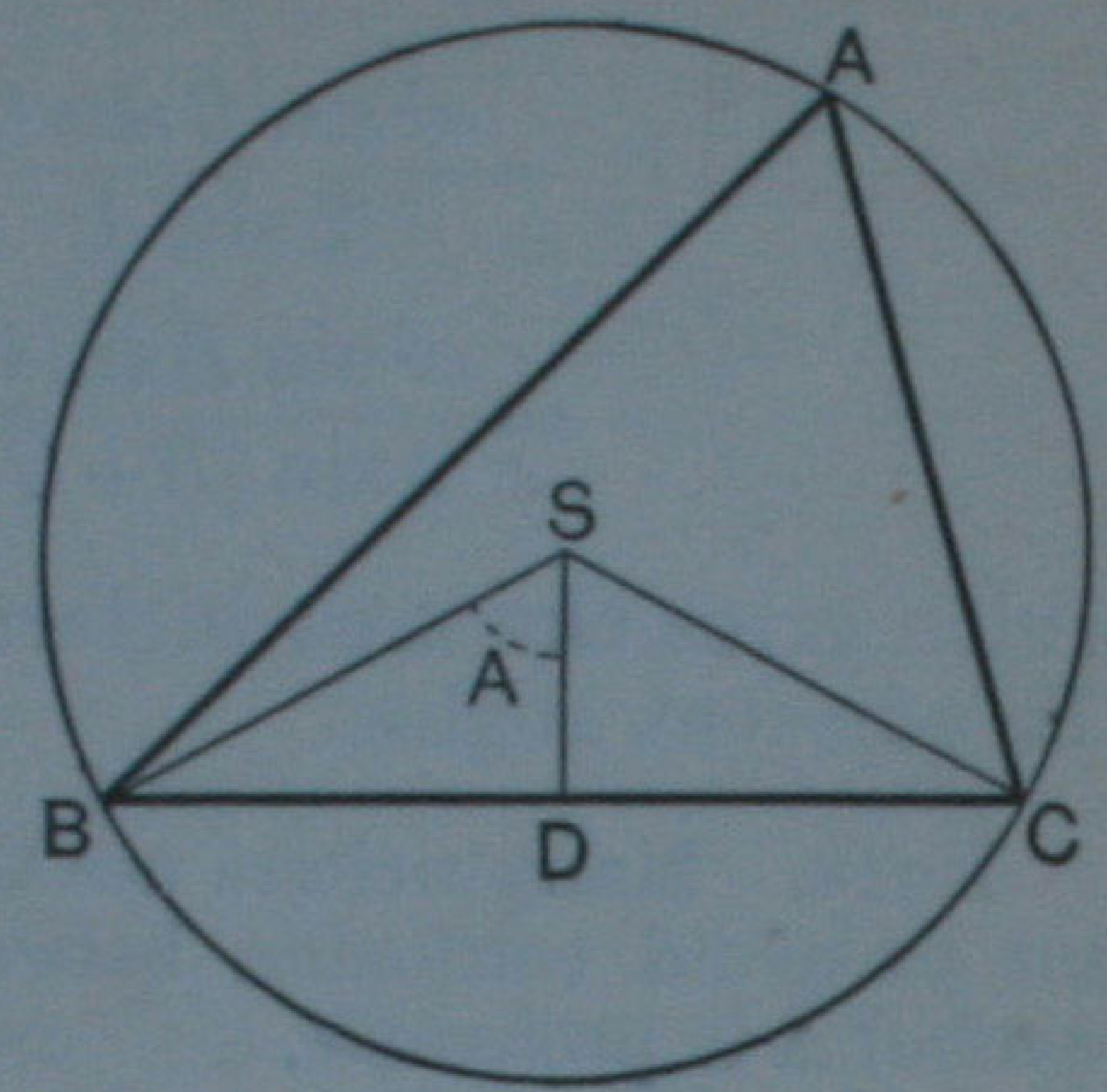
the area in sq. yds. $= \frac{220 \times 220 \times \sin 22\frac{1}{2}° \times \sin 112\frac{1}{2}°}{2 \sin 45°}$

$$= \frac{220 \times 220 \times \sin 22\frac{1}{2}° \times \cos 22\frac{1}{2}°}{2 \times 2 \sin 22\frac{1}{2}° \cos 22\frac{1}{2}°}$$

$$= 110 \times 110.$$

Expressed in acres, the area $= \frac{110 \times 110}{4840} = 2\frac{1}{2}$.

204. *To find the radius of the circle circumscribing a triangle.*



Let S be the centre of the circle circumscribing the triangle ABC, and R its radius.

Bisect $\angle BSC$ by SD, which will also bisect BC at right angles.

Now by Euc. III. 20,

$\angle BSC$ at centre

$$= \text{twice } \angle BAC$$
$$= 2A;$$

and
$$\frac{a}{2} = BD = BS \sin BSD = R \sin A;$$

$$\therefore\ R = \frac{a}{2 \sin A}.$$

Thus
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

or
$$a = 2R \sin A, \quad b = 2R \sin B, \quad c = 2R \sin C.$$

Example. Shew that $2R^2 \sin A \sin B \sin C = \Delta$.

The first side $= \frac{1}{2} \cdot 2R \sin A \cdot 2R \sin B \cdot \sin C$

$$= \frac{1}{2} ab \sin C$$
$$= \Delta.$$

205. From the result of the last article we deduce the following important theorem:

If a chord of length l *subtend an angle* θ *at the circumference of a circle whose radius is* R, *then* $l = 2R \sin \theta$.

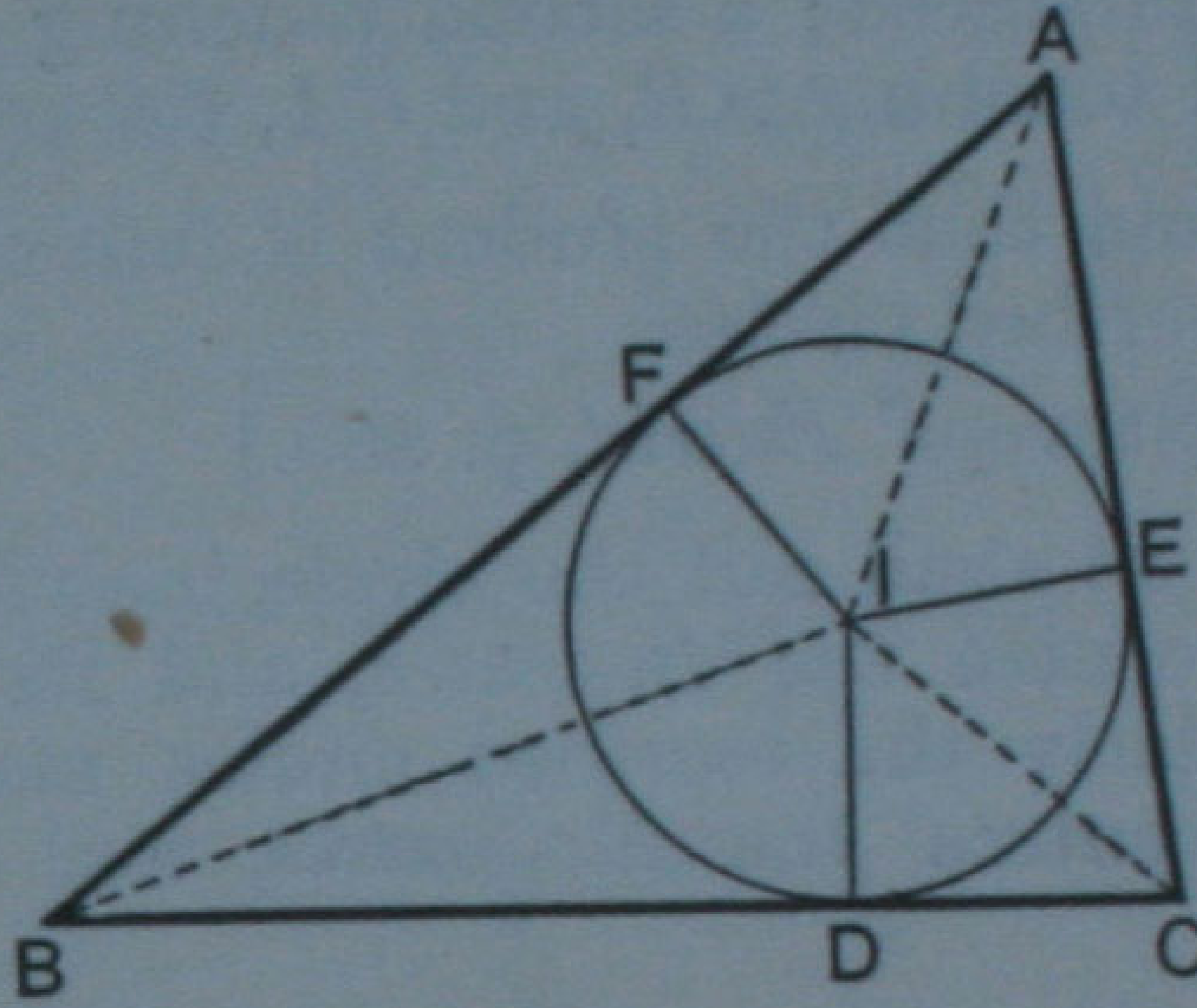
206. For shortness, the circle circumscribing a triangle may be called the *Circum-circle*, its centre the *Circum-centre*, and its radius the *Circum-radius*.

The circum-radius may be expressed in a form not involving the angles, for

$$R = \frac{a}{2 \sin A} = \frac{abc}{2bc \sin A} = \frac{abc}{4\Delta}.$$

207. *To find the radius of the circle inscribed in a triangle.*

Let I be the centre of the circle inscribed in the triangle ABC, and D, E, F the points of contact; then ID, IE, IF are perpendicular to the sides.



Now Δ = sum of the areas of the triangles BIC, CIA, AIB

$$=\frac{1}{2}ar+\frac{1}{2}br+\frac{1}{2}cr=\frac{1}{2}(a+b+c)r$$

$$=sr;$$

whence
$$r=\frac{\Delta}{s}.$$

208. *To express the radius of the inscribed circle in terms of one side and the functions of the half-angles.*

In the figure of the previous article, we know from Euc. IV. 4 that I is the point of intersection of the lines bisecting the angles, so that

$$\angle IBD=\frac{B}{2}, \quad \angle ICD=\frac{C}{2}.$$

$$\therefore \ BD=r\cot\frac{B}{2}, \quad CD=r\cot\frac{C}{2}.$$

$$\therefore \ r\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)=a;$$

$$\therefore \ r\sin\frac{B+C}{2}=a\sin\frac{B}{2}\sin\frac{C}{2};$$

$$\therefore \ r=\frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}.$$

209. Definition. A circle which touches one side of a triangle and the other two sides produced is said to be an **escribed circle** of the triangle.

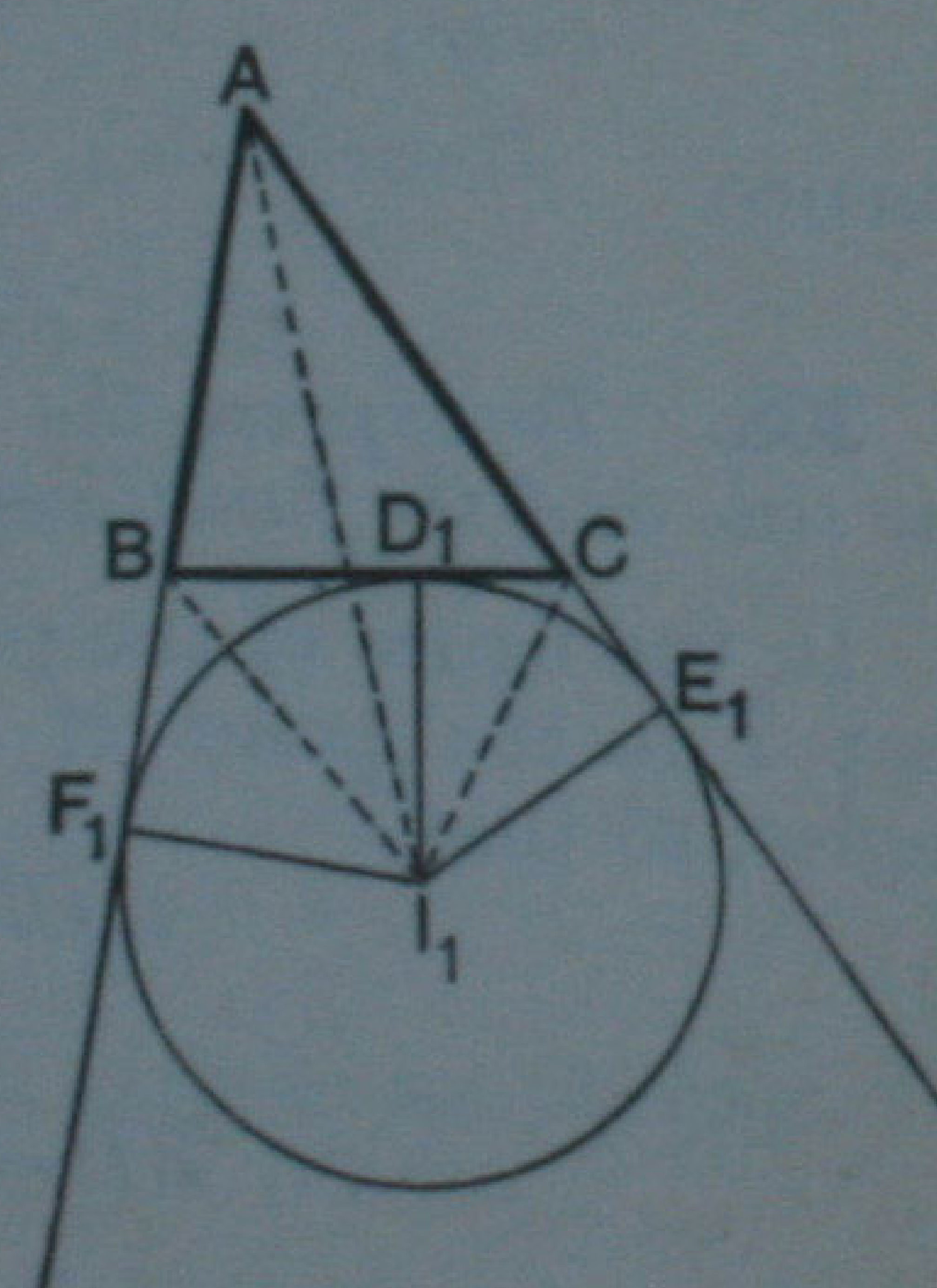
Thus the triangle ABC has *three* escribed circles, one touching BC, and AB, AC produced; a second touching CA, and BC, BA produced; a third touching AB, and CA, CB produced.

We shall assume that the student is familiar with the construction of the escribed circles.

[See Hall and Stevens' *Euclid*, p. 255.]

For shortness, we shall call the circle inscribed in a triangle the *In-circle*, its centre the *In-centre*, and its radius the *In-radius*; and similarly the escribed circles may be called the *Ex-circles*, their centres the *Ex-centres*, and their radii the *Ex-radii*.

210. *To find the radius of an escribed circle of a triangle.*



Let I_1 be the centre of the circle touching the side BC and the two sides AB and AC produced. Let D_1, E_1, F_1 be the points of contact; then the lines joining I_1 to these points are perpendicular to the sides.

Let r_1 be the radius; then

$$\begin{aligned}\Delta &= \text{area } ABC \\ &= \text{area } ABI_1C - \text{area } BI_1C \\ &= \text{area } BI_1A + \text{area } CI_1A - \text{area } BI_1C \\ &= \frac{1}{2}cr_1 + \frac{1}{2}br_1 - \frac{1}{2}ar_1 \\ &= \frac{1}{2}(c+b-a)\,r_1 \\ &= (s-a)\,r_1;\end{aligned}$$

$$\therefore\ r_1 = \frac{\Delta}{s-a}.$$

Similarly, if r_2, r_3 be the radii of the escribed circles opposite to the angles B and C respectively,

$$r_2 = \frac{\Delta}{s-b}, \qquad r_3 = \frac{\Delta}{s-c}.$$

211. *To find the radii of the escribed circles in terms of one side and the functions of the half-angles.*

In the figure of the last article, I_1 is the point of intersection of the lines bisecting the angles B and C externally; so that

$$\angle I_1BD_1 = 90^\circ - \frac{B}{2}, \quad \angle I_1CD_1 = 90^\circ - \frac{C}{2}.$$

$$\therefore \quad BD_1 = r_1 \cot\left(90^\circ - \frac{B}{2}\right) = r_1 \tan\frac{B}{2},$$

$$CD_1 = r_1 \cot\left(90^\circ - \frac{C}{2}\right) = r_1 \tan\frac{C}{2};$$

$$\therefore \quad r_1\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = a;$$

$$\therefore \quad r_1 \sin\frac{B+C}{2} = a\cos\frac{B}{2}\cos\frac{C}{2};$$

$$\therefore \quad r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}.$$

Similarly,

$$r_2 = \frac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}, \quad r_3 = \frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}.$$

212. By substituting

$$a = 2R\sin A, \quad b = 2R\sin B, \quad c = 2R\sin C,$$

in the formulæ of Art. 208 and Art. 211, we have

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

$$r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

$$r_2 = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2},$$

$$r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

Example 1. Shew that $\dfrac{r_1 - r}{a} + \dfrac{r_2 - r}{b} = \dfrac{c}{r_3}$.

$$\text{The first side} = \frac{1}{a}\left(\frac{\Delta}{s-a} - \frac{\Delta}{s}\right) + \frac{1}{b}\left(\frac{\Delta}{s-b} - \frac{\Delta}{s}\right)$$

$$= \frac{\Delta}{s(s-a)} + \frac{\Delta}{s(s-b)} = \frac{\Delta(2s-a-b)}{s(s-a)(s-b)}$$

$$= \frac{\Delta c}{s(s-a)(s-b)} = \frac{\Delta c(s-c)}{s(s-a)(s-b)(s-c)}$$

$$= \frac{\Delta c(s-c)}{\Delta^2} = \frac{c(s-c)}{\Delta}$$

$$= \frac{c}{r_3}.$$

Example 2. If $r_1 = r_2 + r_3 + r$, prove that the triangle is right-angled.

By transposition, $\qquad r_1 - r = r_2 + r_3;$

$$\therefore\ 4R \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} - 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} + 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2};$$

$$\therefore\ \sin\frac{A}{2}\left(\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= \cos\frac{A}{2}\left(\sin\frac{B}{2}\cos\frac{C}{2} + \cos\frac{B}{2}\sin\frac{C}{2}\right);$$

$$\therefore\ \sin\frac{A}{2}\cos\frac{B+C}{2} = \cos\frac{A}{2}\sin\frac{B+C}{2};$$

$$\therefore\ \sin^2\frac{A}{2} = \cos^2\frac{A}{2};$$

whence $\dfrac{A}{2} = 45°$, and $A = 90°$.

213. Many important relations connecting a triangle and its circles may be established by elementary geometry.

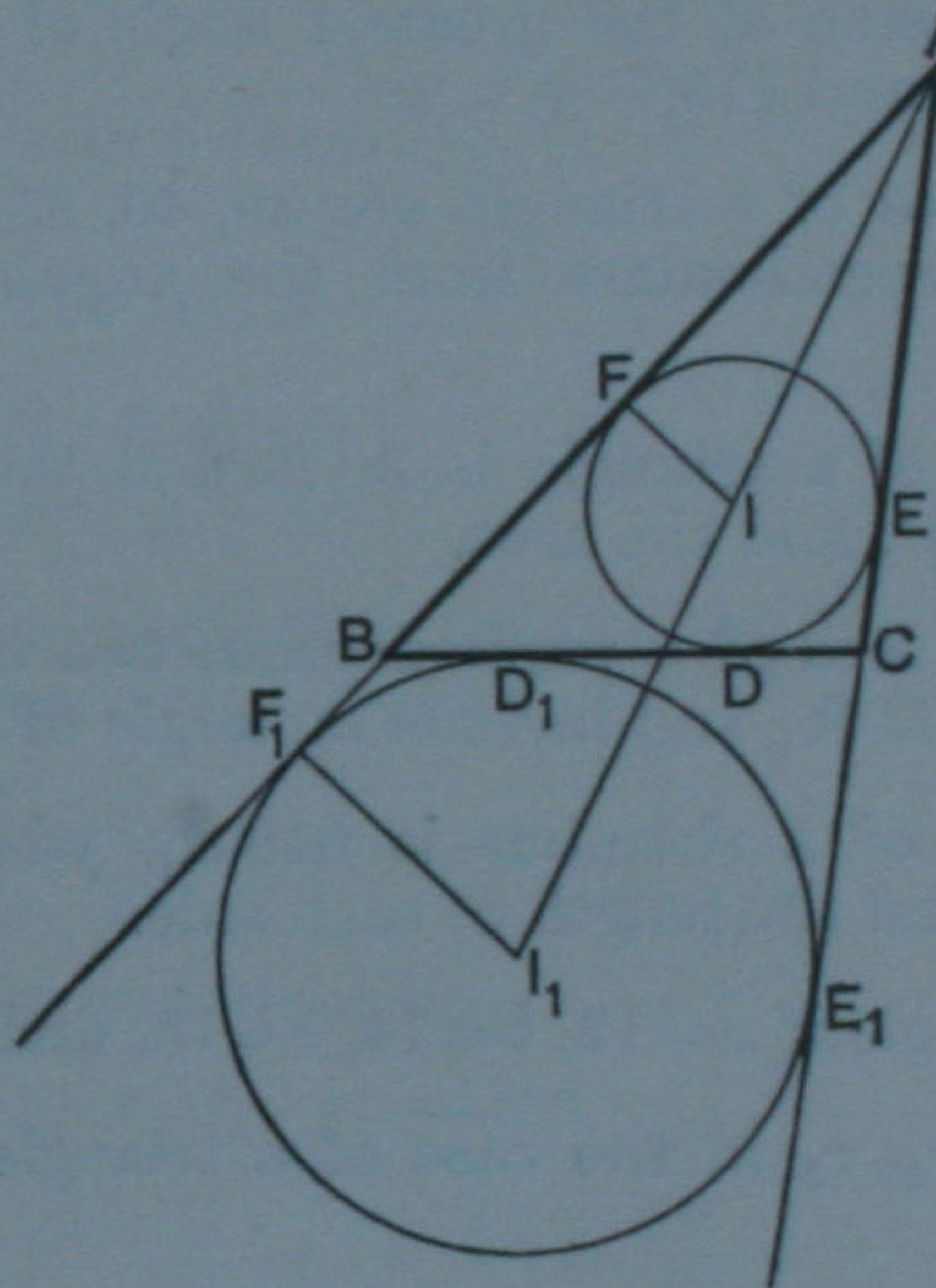
With the notation of previous articles, since tangents to a circle from the same point are equal, we have $AF = AE, \quad BD = BF, \quad CD = CE;$

$$\therefore \quad AF + (BD + CD) = \text{half the sum of the sides};$$

$$\therefore \quad AF + a = s.$$

$$\therefore \quad AF = s - a = AE.$$

Similarly, $BD = BF = s - b, \quad CD = CE = s - c$



Also $$r = AF \tan \frac{A}{2} = (s - a) \tan \frac{A}{2}.$$

Similarly, $r = (s - b) \tan \dfrac{B}{2}, \quad r = (s - c) \tan \dfrac{C}{2}.$

Again, $AF_1 = AE_1, \quad BF_1 = BD_1, \quad CE_1 = CD_1;$

$$\therefore \quad 2AF_1 = AF_1 + AE_1 = (AB + BD_1) + (AC + CD_1)$$
$$= \text{sum of the sides};$$

$$\therefore \quad AF_1 = s = AE_1.$$

$$\therefore \quad BD_1 = BF_1 = s - c, \quad CD_1 = CE_1 = s - b.$$

Also $$r_1 = AF_1 \tan \frac{A}{2} = s \tan \frac{A}{2}.$$

Similarly, $r_2 = s \tan \dfrac{B}{2}, \quad r_3 = s \tan \dfrac{C}{2}.$

EXAMPLES. XVIII. a.

1. Two sides of a triangle are 300 ft. and 120 ft., and the included angle is 150°; find the area.

2. Find the area of the triangle whose sides are 171, 204, 195.

3. Find the sine of the greatest angle of a triangle whose sides are 70, 147, and 119.

4. If the sides of a triangle are 39, 40, 25, find the lengths of the three perpendiculars from the angular points on the opposite sides.

5. One side of a triangle is 30 ft. and the adjacent angles are $22\frac{1}{2}°$ and $112\frac{1}{2}°$, find the area.

6. Find the area of a parallelogram two of whose adjacent sides are 42 and 32 ft., and include an angle of 30°.

7. The area of a rhombus is 648 sq. yds. and one of the angles is 150°: find the length of each side.

8. In a triangle if $a=13$, $b=14$, $c=15$, find r and R.

9. Find r_1, r_2, r_3 in the case of a triangle whose sides are 17, 10, 21.

10. If the area of a triangle is 96, and the radii of the escribed circles are 8, 12, 24, find the sides.

Prove the following formulæ:

11. $\sqrt{rr_1r_2r_3}=\Delta$.

12. $s(s-a)\tan\frac{A}{2}=\Delta$.

13. $rr_1\cot\frac{A}{2}=\Delta$.

14. $4Rrs=abc$.

15. $r_1r_2r_3=rs^2$.

16. $r\cot\frac{B}{2}\cot\frac{C}{2}=r_1$.

17. $Rr(\sin A+\sin B+\sin C)=\Delta$.

18. $r_1r_2+rr_3=ab$.

19. $\cos\frac{A}{2}\sqrt{bc(s-b)(s-c)}=\Delta$.

20. $r_1+r_2=c\cot\frac{C}{2}$.

21. $(r_1-r)(r_2+r_3)=a^2$.

22. $r_1 \cot \frac{A}{2} = r_2 \cot \frac{B}{2} = r_3 \cot \frac{C}{2} = r \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$.

23. $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$.

24. $r_2 r_3 + r_3 r_1 + r_1 r_2 = s^2$.

25. $r_1 + r_2 + r_3 - r = 4R$.

26. $r + r_1 + r_2 - r_3 = 4R \cos C$.

27. $b^2 \sin 2C + c^2 \sin 2B = 4\Delta$.

28. $4R \cos \frac{C}{2} = (a+b) \sec \frac{A-B}{2}$.

29. $a^2 - b^2 = 2Rc \sin (A-B)$.

30. $\frac{a^2-b^2}{2} \cdot \frac{\sin A \sin B}{\sin (A-B)} = \Delta$.

31. If the perpendiculars from A, B, C to the opposite sides are p_1, p_2, p_3 respectively, prove that

(1) $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r}$; (2) $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = \frac{1}{r_3}$.

Prove the following identities:

32. $(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$.

33. $\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_3}\right) = \frac{4R}{r^2 s^2}$.

34. $4\Delta (\cot A + \cot B + \cot C) = a^2 + b^2 + c^2$.

35. $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} = 0$.

36. $a^2 b^2 c^2 (\sin 2A + \sin 2B + \sin 2C) = 32\Delta^3$.

37. $a \cos A + b \cos B + c \cos C = 4R \sin A \sin B \sin C$.

38. $a \cot A + b \cot B + c \cot C = 2(R + r)$.

39. $(b+c) \tan \frac{A}{2} + (c+a) \tan \frac{B}{2} + (a+b) \tan \frac{C}{2}$
$= 4R (\cos A + \cos B + \cos C)$.

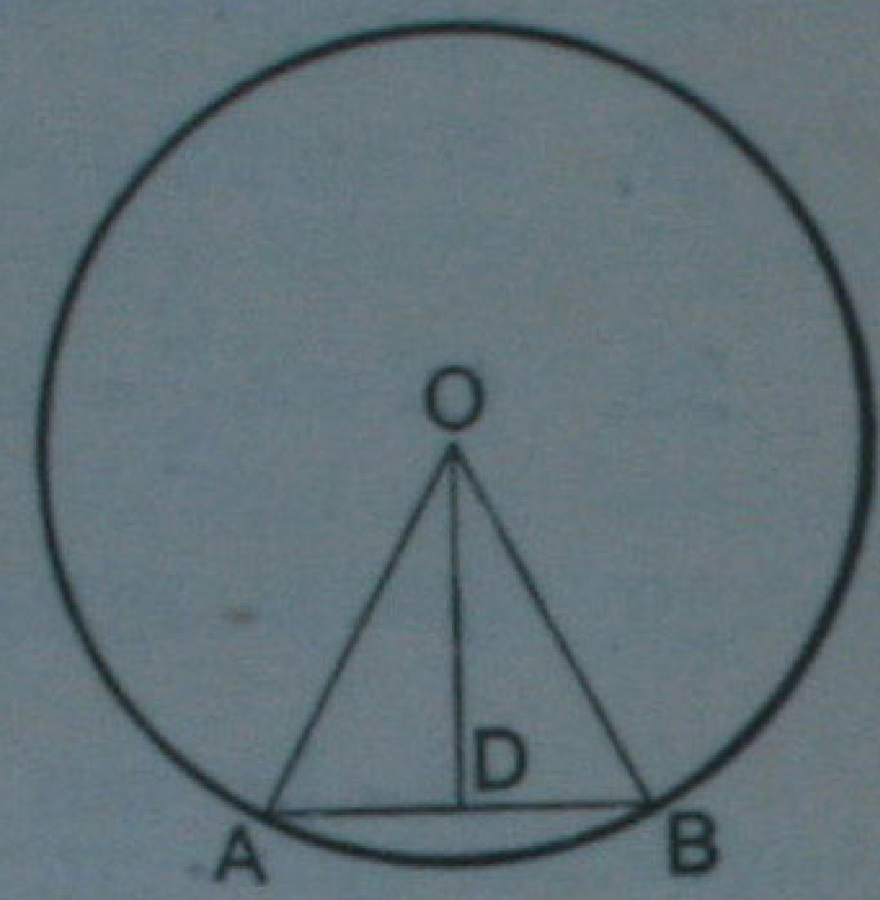
40. $r (\sin A + \sin B + \sin C) = 2R \sin A \sin B \sin C$.

41. $\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = 2 + \frac{r}{2R}$.

Inscribed and circumscribed Polygons.

214. *To find the perimeter and area of a regular polygon of* n *sides inscribed in a circle.*

Let r be the radius of the circle, and AB a side of the polygon.



Join OA, OB, and draw OD bisecting $\angle AOB$; then AB is bisected at right angles in D.

And $\angle AOB = \frac{1}{n}$ (four right angles)

$$= \frac{2\pi}{n}.$$

$$\text{Perimeter of polygon} = nAB = 2nAD = 2nOA \sin AOD$$

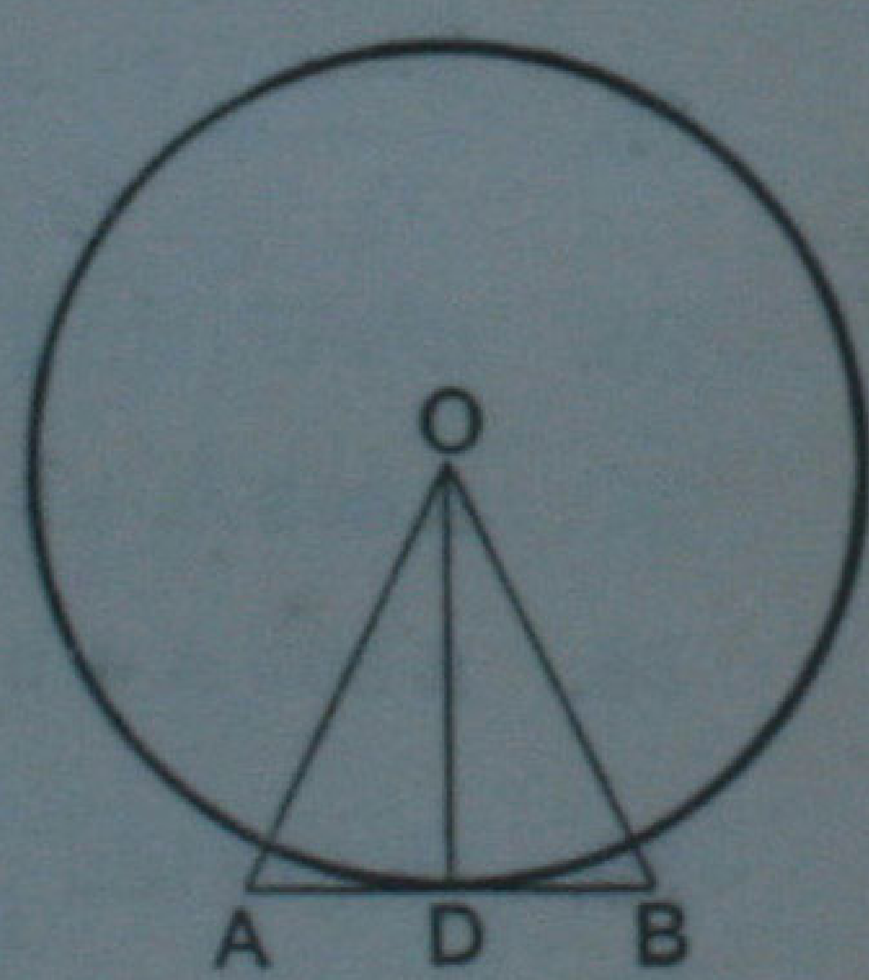
$$= 2nr \sin \frac{\pi}{n}.$$

$$\text{Area of polygon} = n\,(\text{area of triangle } AOB)$$

$$= \frac{1}{2} nr^2 \sin \frac{2\pi}{n}.$$

215. *To find the perimeter and area of a regular polygon of* n *sides circumscribed about a given circle.*

Let r be the radius of the circle, and AB a side of the polygon. Let AB touch the circle at D. Join OA, OB, OD; then OD bisects AB at right angles, and also bisects $\angle AOB$.



Perimeter of polygon

$$= nAB = 2nAD = 2nOD \tan AOD$$

$$= 2nr \tan \frac{\pi}{n}.$$

$$\text{Area of polygon} = n\,(\text{area of triangle } AOB)$$

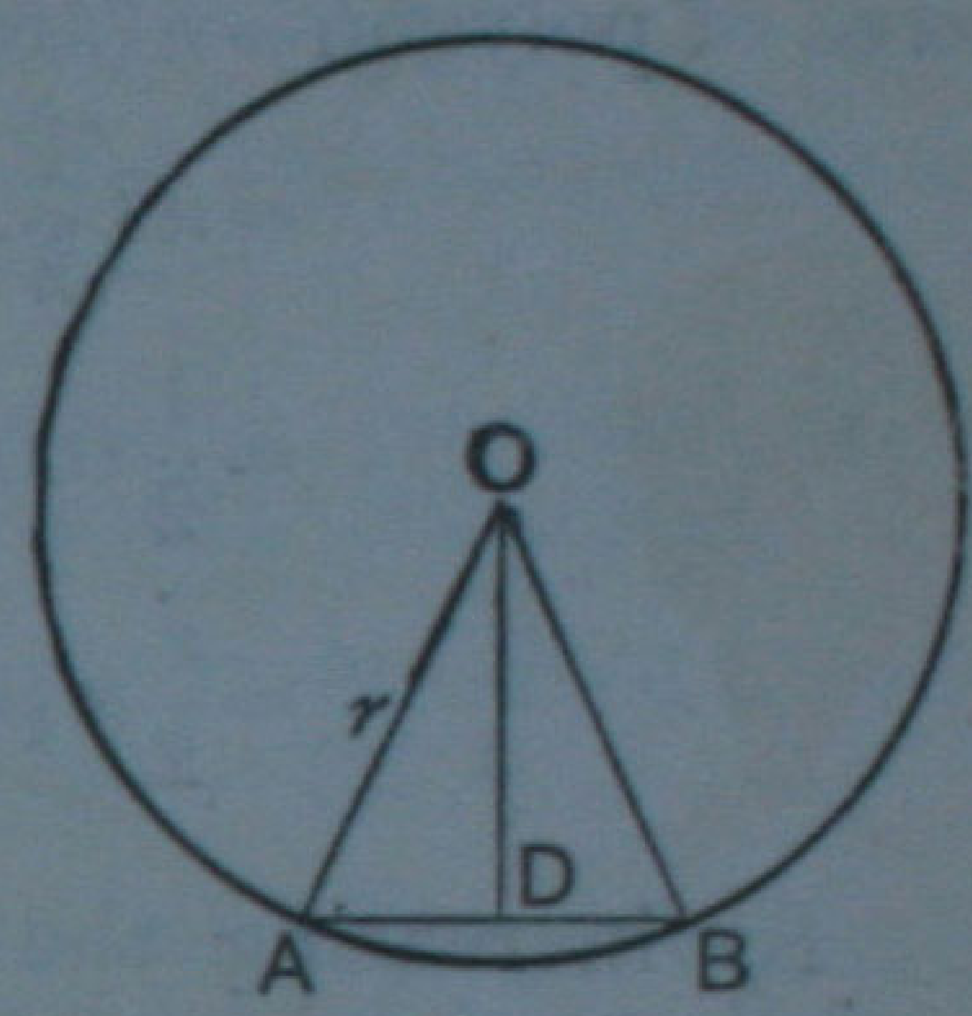
$$= nOD \,.\, AD$$

$$= nr^2 \tan \frac{\pi}{n}.$$

216. There is no need to burden the memory with the formulæ of the last two articles, as in any particular instance they are very readily obtained.

Example 1. The side of a regular dodecagon is 2 ft., find the radius of the circumscribed circle.

Let r be the required radius. In the adjoining figure we have



$$AB=2, \quad \angle AOB=\frac{2\pi}{12}.$$

$$AB=2AD=2r\sin\frac{\pi}{12};$$

$$\therefore\ 2r\sin 15^\circ=2;$$

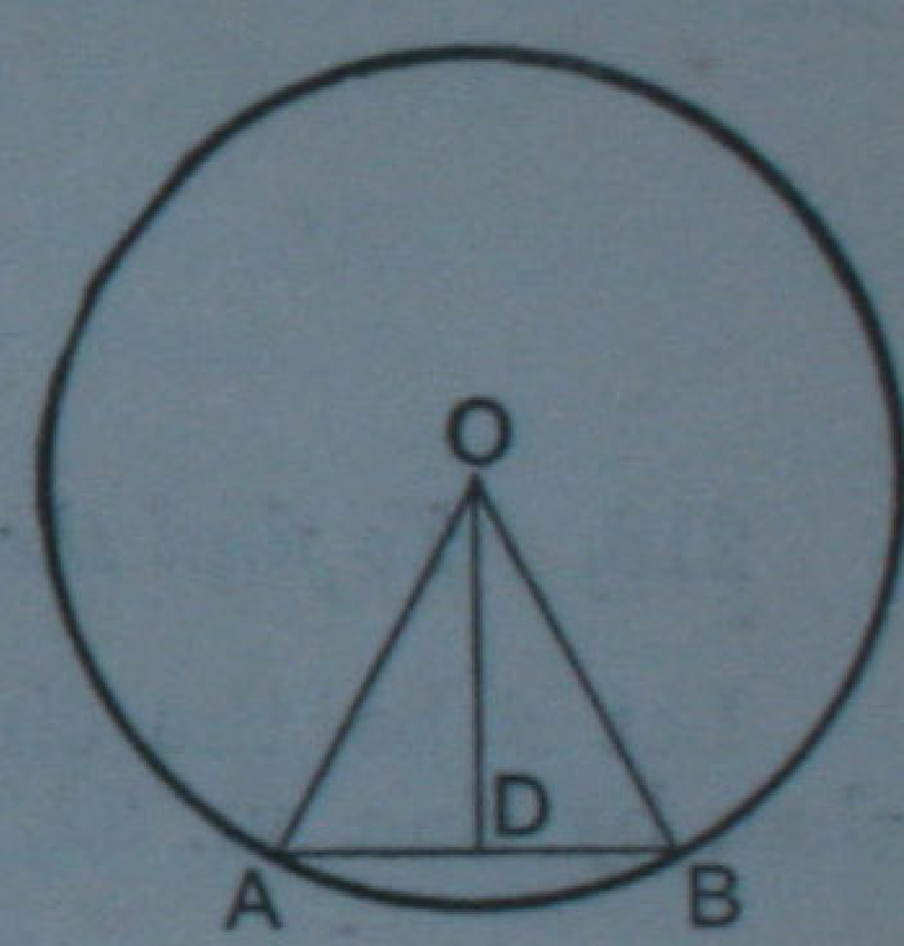
$$\therefore\ r=\frac{1}{\sin 15^\circ}=\frac{2\sqrt{2}}{\sqrt{3}-1}=\sqrt{2}\,(\sqrt{3}+1).$$

Thus the radius is $\sqrt{6}+\sqrt{2}$ feet.

Example 2. A regular pentagon and a regular decagon have the same perimeter, prove that their areas are as 2 to $\sqrt{5}$.

Let AB be one of the n sides of a regular polygon, O the centre of the circumscribed circle, OD perpendicular to AB.

Then if $AB=a$,



$$\text{area of polygon}=nAD\,.\,OD$$

$$=nAD\,.\,AD\cot\frac{\pi}{n}$$

$$=\frac{na^2}{4}\cot\frac{\pi}{n}.$$

Denote the perimeter of the pentagon and decagon by $10c$. Then each side of the pentagon is $2c$, and its area is $5c^2\cot\frac{\pi}{5}$.

Each side of the decagon is c, and its area is $\frac{5}{2}c^2\cot\frac{\pi}{10}$.

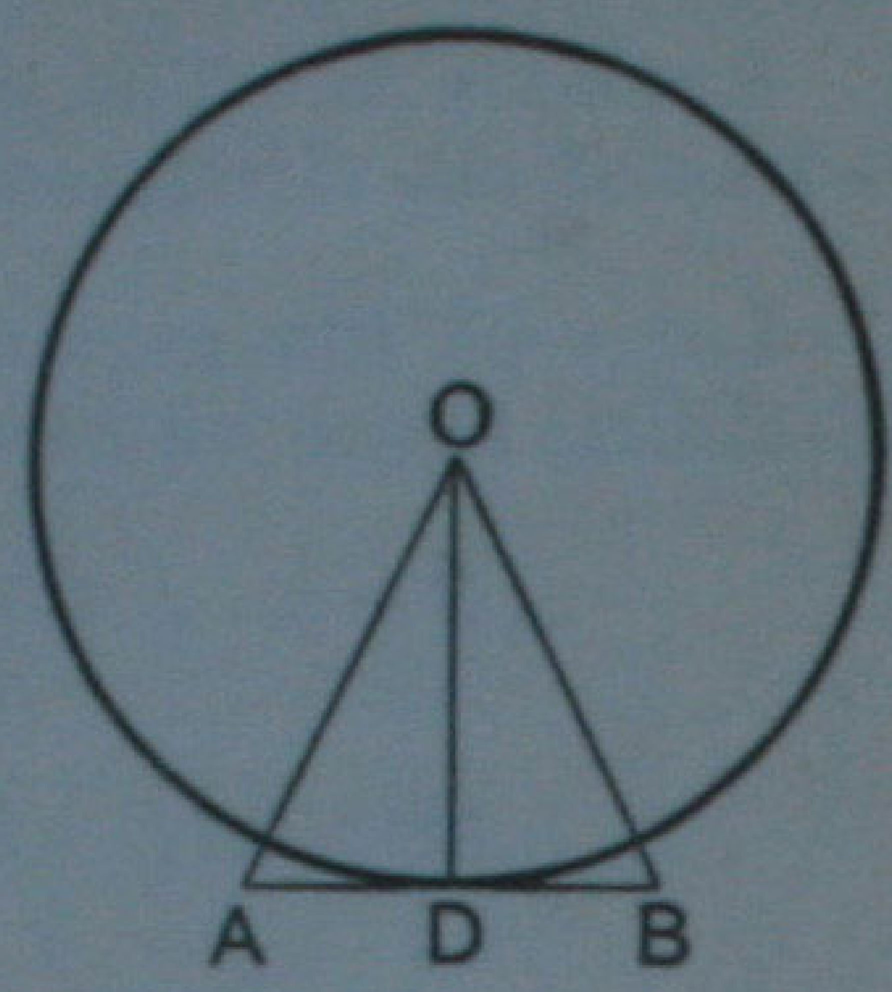
$$\therefore\ \frac{\text{Area of pentagon}}{\text{Area of decagon}}=\frac{2\cot 36^\circ}{\cot 18^\circ}=\frac{2\cos 36^\circ\sin 18^\circ}{\sin 36^\circ\cos 18^\circ}=\frac{2\cos 36^\circ}{2\cos^2 18^\circ}$$

$$=\frac{2\cos 36^\circ}{1+\cos 36^\circ}=\frac{2\,(\sqrt{5}+1)}{4}\div\left(1+\frac{\sqrt{5}+1}{4}\right)$$

$$=\frac{2\,(\sqrt{5}+1)}{5+\sqrt{5}}=\frac{2}{\sqrt{5}}.$$

217. *To find the area of a circle.*

Let r be the radius of the circle, and let a regular polygon of n sides be described about it. Then from the adjoining figure, we have



area of polygon $= n$ (area of triangle AOB)

$$= n\left(\frac{1}{2}AB \, . \, OD\right)$$

$$= \frac{1}{2}OD \, . \, nAB$$

$$= \frac{r}{2} \times \text{perimeter of polygon.}$$

By increasing the number of sides without limit, the area and the perimeter of the polygon may be made to differ as little as we please from the area and the circumference of the circle. Hence

$$\text{area of a circle} = \frac{r}{2} \times \text{circumference}$$

$$= \frac{r}{2} \times 2\pi r \quad \text{[Art. 59.]}$$

$$= \pi r^2.$$

218. *To find the area of the sector of a circle.*

Let θ be the circular measure of the angle of the sector; then by Euc. VI. 33,

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{2\pi};$$

$$\therefore \text{ area of sector} = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \theta.$$

EXAMPLES. XVIII. b.

$$\left[\textit{In this Exercise take } \pi = \frac{22}{7}.\right]$$

1. Find the area of a regular decagon inscribed in a circle whose radius is 3 feet; given sin 36° = ·588.

2. Find the perimeter and area of a regular quindecagon described about a circle whose diameter is 3 yards; given

$$\tan 12^\circ = \cdot 213.$$

3. Shew that the areas of the inscribed and circumscribed circles of a regular hexagon are in the ratio of 3 to 4.

4. Find the area of a circle inscribed in a regular pentagon whose area is 250 sq. ft.; given $\cot 36^\circ = 1 \cdot 376$.

5. Find the perimeter of a regular octagon inscribed in a circle whose area is 1386 sq. inches; given $\sin 22^\circ\, 30' = \cdot 382$.

6. Find the perimeter of a regular pentagon described about a circle whose area is 616 sq. ft.; given $\tan 36^\circ = \cdot 727$.

7. Find the diameter of the circle circumscribing a regular quindecagon, whose inscribed circle has an area of 2464 sq. ft.; given $\sec 12^\circ = 1 \cdot 022$.

8. Find the area of a regular dodecagon in a circle about a regular pentagon 50 sq. ft. in area; given $\operatorname{cosec} 72^\circ = 1 \cdot 0515$.

9. A regular pentagon and a regular decagon have the same area, prove that the ratio of their perimeters is $\sqrt[4]{5} : \sqrt{2}$.

10. Two regular polygons of n sides and $2n$ sides have the same perimeter; shew that the ratio of their areas is

$$2 \cos \frac{\pi}{n} : 1 + \cos \frac{\pi}{n}.$$

11. If $2a$ be the side of a regular polygon of n sides, R and r the radii of the circumscribed and inscribed circles, prove that

$$R + r = a \cot \frac{\pi}{2n}.$$

12. Prove that the square of the side of a regular pentagon inscribed in a circle is equal to the sum of the squares of the sides of a regular hexagon and decagon inscribed in the same circle.

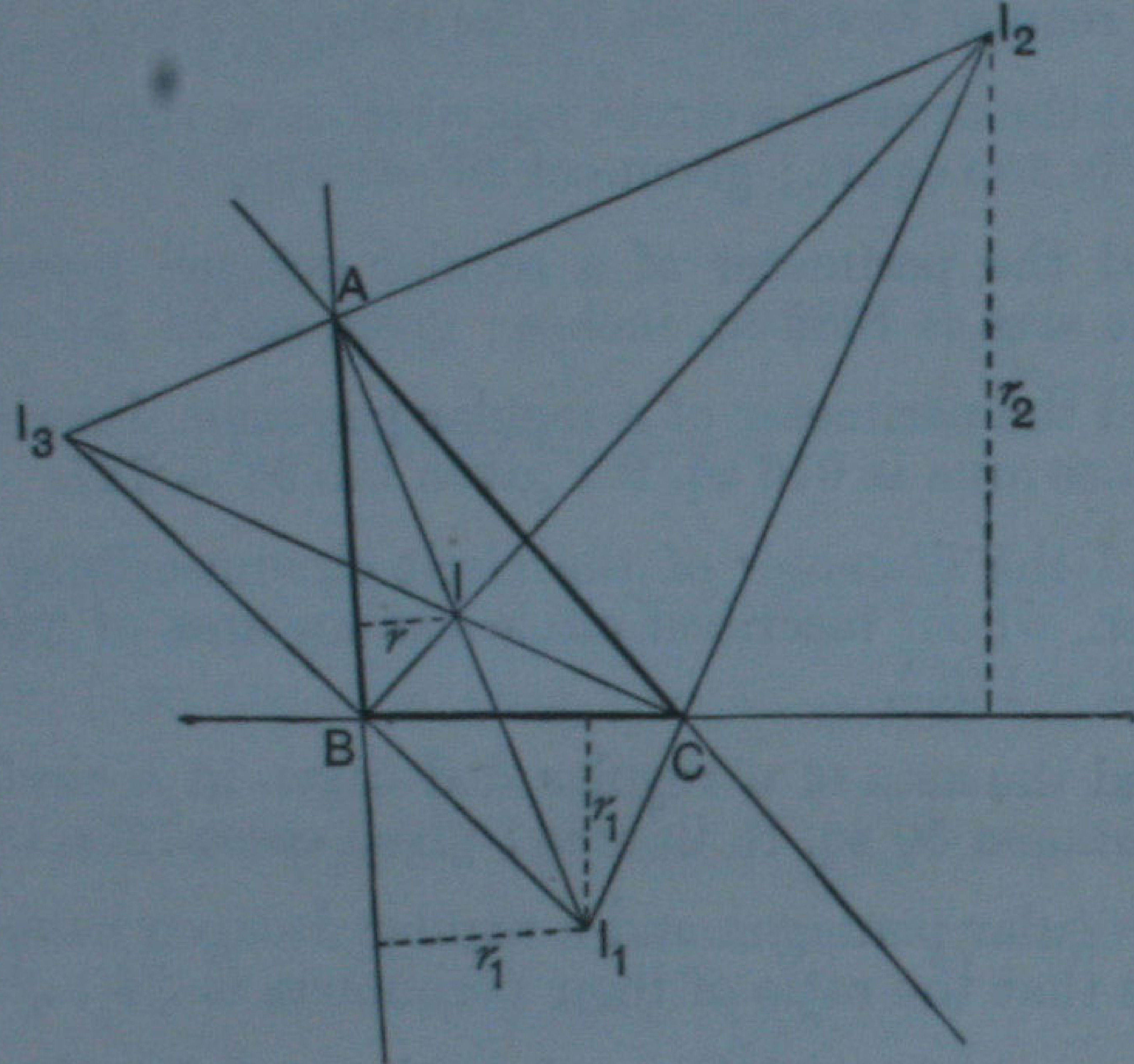
13. With reference to a given circle, A_1 and B_1 are the areas of the inscribed and circumscribed regular polygons of n sides, A_2 and B_2 are corresponding quantities for regular polygons of $2n$ sides: prove that

(1) A_2 is a geometric mean between A_1 and B_1;

(2) B_2 is a harmonic mean between A_2 and B_1.

The Ex-central Triangle.

***219.** Let ABC be a triangle, I_1, I_2, I_3 its ex-centres; then $I_1I_2I_3$ is called the **Ex-central triangle** of ABC.



Let I be the in-centre; then from the construction for finding the positions of the in-centre and ex-centres, it follows that:

(i) The points I, I_1 lie on the line bisecting the angle BAC; the points I, I_2 lie on the line bisecting the angle ABC; the points I, I_3 lie on the line bisecting the angle ACB.

(ii) The points I_2, I_3 lie on the line bisecting the angle BAC externally; the points I_3, I_1 lie on the line bisecting the angle ABC externally; the points I_1, I_2 lie on the line bisecting the angle ACB externally.

(iii) The line AI_1 is perpendicular to I_2I_3; the line BI_2 is perpendicular to I_3I_1; the line CI_3 is perpendicular to I_1I_2. Thus the triangle ABC is the *Pedal triangle* of its ex-central triangle $I_1I_2I_3$. [See Art. 223.]

(iv) The angles IBI_1 and ICI_1 are right angles; hence the points B, I, C, I_1 are concyclic. Similarly, the points C, I, A, I_2, and the points A, I, B, I_3 are concyclic.

(v) The lines AI_1, BI_2, CI_3 meet at the in-centre I, which is therefore the *Orthocentre* of the ex-central triangle $I_1I_2I_3$.

(vi) Each of the four points I, I_1, I_2, I_3 is the orthocentre of the triangle formed by joining the other three points.

***220.** *To find the sides and angles of the ex-central triangle.*

With the figure of the last article,

$$\angle BI_1C = \angle BI_1I + \angle CI_1I$$

$$= \angle BCI + \angle CBI \qquad \text{[Euc. III. 21]}$$

$$= \frac{C}{2} + \frac{B}{2} = 90° - \frac{A}{2}.$$

Thus the angles are

$$90° - \frac{A}{2}, \quad 90° - \frac{B}{2}, \quad 90° - \frac{C}{2}.$$

Again, the points B, I_3, I_2, C are concyclic;

$$\therefore \angle I_1I_2I_3 = \text{supplement of } \angle I_3BC = \angle I_1BC;$$

$\therefore$ the triangles $I_1I_2I_3$, I_1BC are similar;

$$\therefore \frac{I_2I_3}{BC} = \frac{I_3I_1}{I_1C} = \sec\left(90° - \frac{A}{2}\right) = \operatorname{cosec}\frac{A}{2};$$

$$\therefore I_2I_3 = a \operatorname{cosec}\frac{A}{2} = 4R\cos\frac{A}{2}.$$

Thus the sides are

$$4R\cos\frac{A}{2}, \quad 4R\cos\frac{B}{2}, \quad 4R\cos\frac{C}{2}.$$

***221.** *To find the area and circum-radius of the ex-central triangle.*

$$\text{The area} = \frac{1}{2}\,(\text{product of two sides}) \times (\text{sine of included angle})$$

$$= \frac{1}{2} \times 4R\cos\frac{B}{2} \times 4R\cos\frac{C}{2} \times \sin\left(90° - \frac{A}{2}\right)$$

$$= 8R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

$$\text{The circum-radius} = \frac{I_2I_3}{2\sin I_2I_1I_3} = \frac{4R\cos\frac{A}{2}}{2\sin\left(90° - \frac{A}{2}\right)} = 2R.$$

***222.** *To find the distances between the in-centre and ex-centres.*

With the figure of Art. 219,

the $\angle$s IBI_1, ICI_1 are right angles;

$\therefore$ II_1 is the diameter of the circum-circle of the triangle BCI_1;

$$\therefore \; II_1 = \frac{BC}{\sin BI_1C} = \frac{a}{\cos \frac{A}{2}}$$

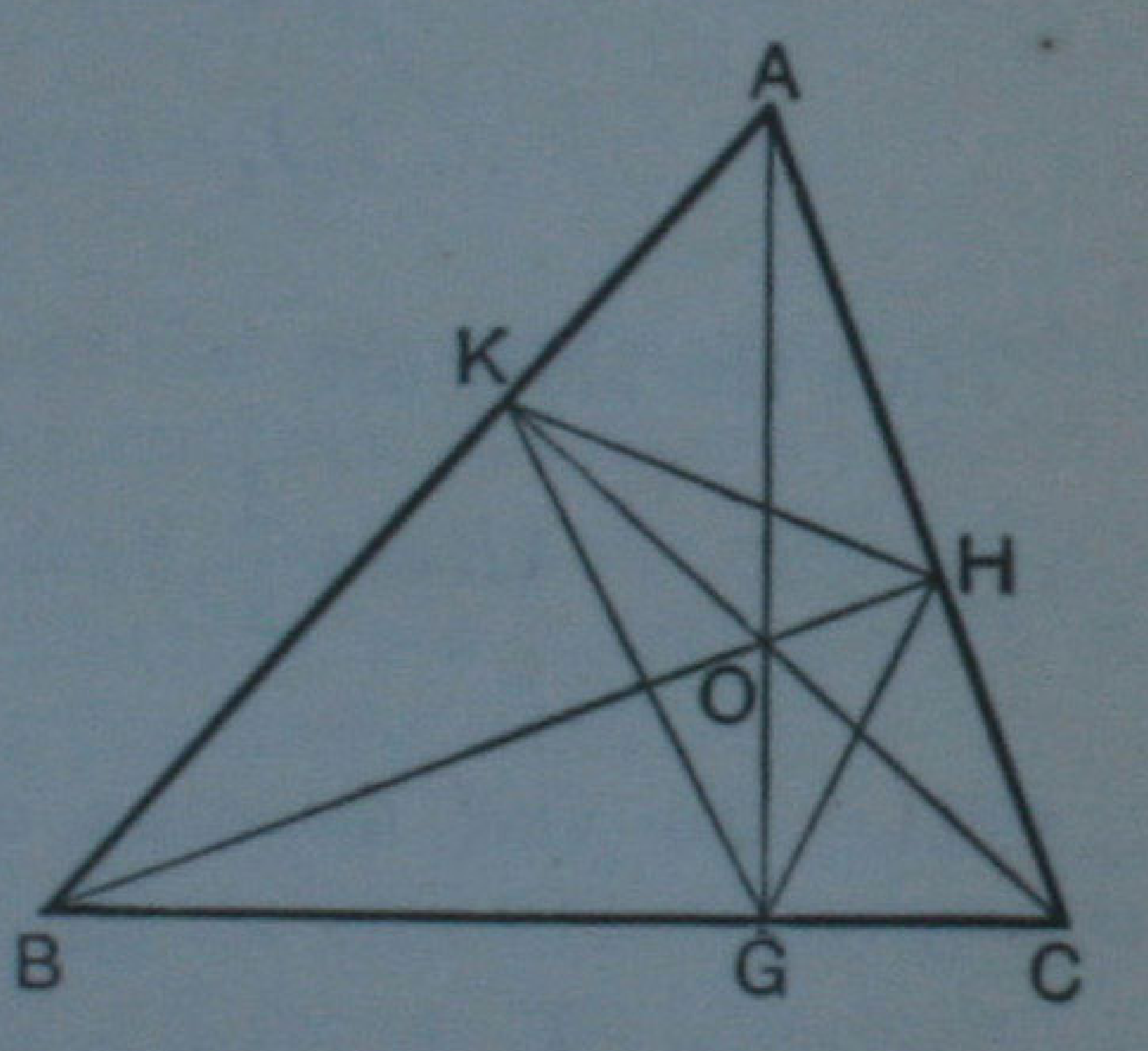
$$= 4R \sin \frac{A}{2}.$$

Thus the distances are

$$4R \sin \frac{A}{2}, \quad 4R \sin \frac{B}{2}, \quad 4R \sin \frac{C}{2}.$$

The Pedal Triangle.

***223.** Let G, H, K be the feet of the perpendiculars from the angular points on the opposite sides of the triangle ABC; then GHK is called the **Pedal triangle** of ABC.



The three perpendiculars AG, BH, CK meet in a point O which is called the **Orthocentre** of the triangle ABC.

***224.** *To find the sides and angles of the pedal triangle.*

In the figure of the last article, the points K, O, G, B are concyclic;

$$\therefore \; \angle OGK = \angle OBK = 90° - A.$$

Also the points H, O, G, C are concyclic;

$$\therefore \; \angle OGH = \angle OCH = 90° - A;$$

$$\therefore \; \angle KGH = 180° - 2A.$$

Thus the angles of the pedal triangle are

$$180° - 2A, \quad 180° - 2B, \quad 180° - 2C.$$

Again, the triangles AKH, ABC are similar;

$$\therefore \frac{HK}{BC}=\frac{AK}{AC}=\cos A;$$

$$\therefore HK=a\cos A.$$

Thus the sides of the pedal triangle are

$$a\cos A, \quad b\cos B, \quad c\cos C.$$

In terms of R, the equivalent forms become

$$R\sin 2A, \quad R\sin 2B, \quad R\sin 2C.$$

If the angle ACB of the given triangle is obtuse, the expressions $180^\circ-2C$ and $c\cos C$ are both negative, and the values we have obtained require some modification. We leave the student to shew that in this case the angles are $2A, 2B, 2C-180^\circ$, and the sides $\quad a\cos A, \quad b\cos B, \quad -c\cos C$.

***225.** *To find the area and circum-radius of the pedal triangle.*

The area $=\frac{1}{2}$ (product of two sides) $\times$ (sine of included angle)

$$=\frac{1}{2}R\sin 2B \,.\, R\sin 2C \,.\, \sin(180-2A)$$

$$=\frac{1}{2}R^2\sin 2A\sin 2B\sin 2C.$$

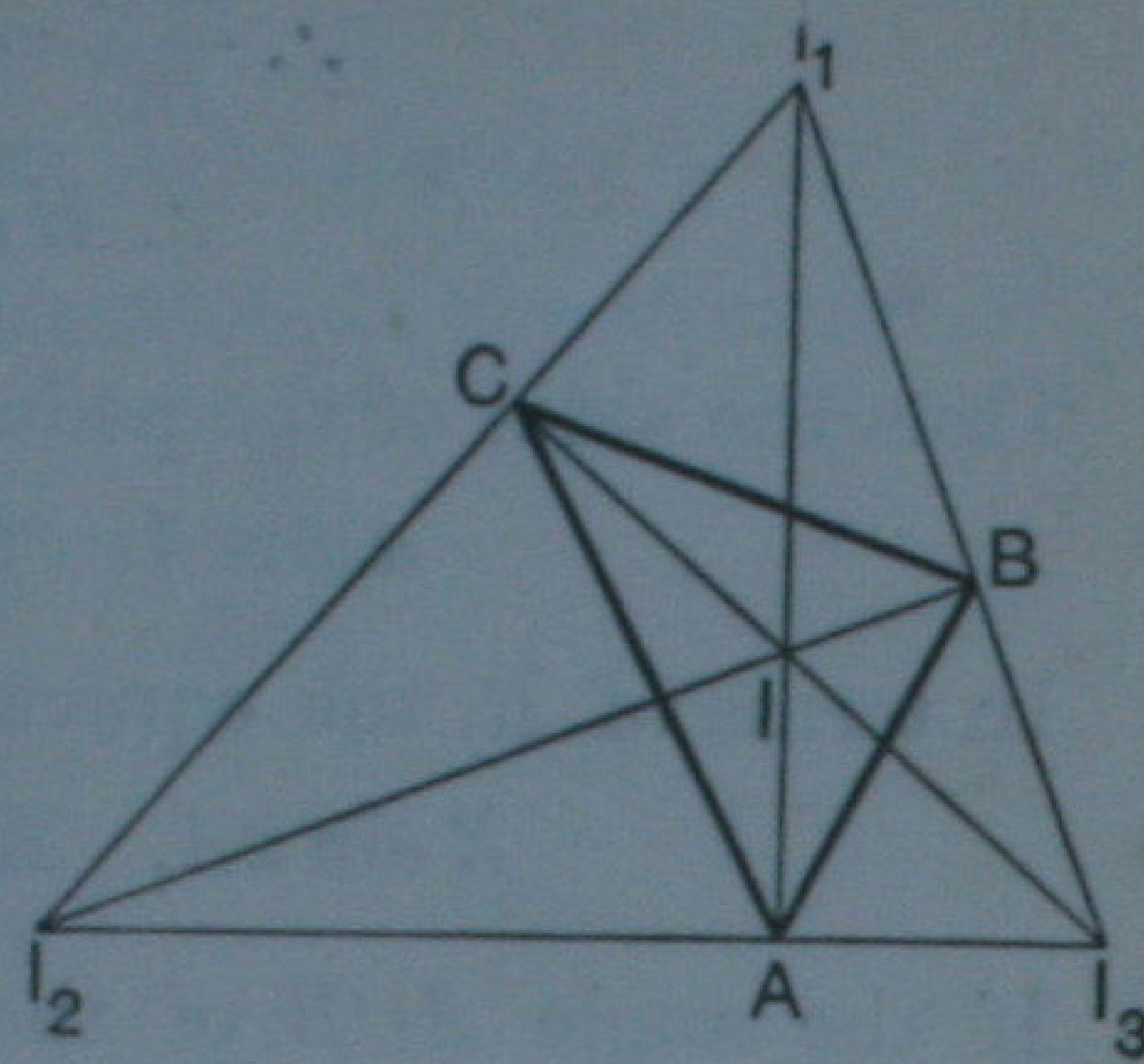
The circum-radius $=\dfrac{HK}{2\sin HGK}=\dfrac{R\sin 2A}{2\sin(180^\circ-2A)}=\dfrac{R}{2}$.

NOTE. The circum-circle of the pedal triangle is the nine points circle of the triangle ABC. Thus the radius of the nine points circle of the triangle ABC is $\frac{R}{2}$. [See Hall and Stevens' *Euclid*, p. 281.]

***226.** In Art. 224, we have proved that OG, OH, OK bisect the angles HGK, KHG, GKH respectively, so that O is the in-centre of the triangle GHK. Thus the orthocentre of a triangle is the in-centre of the pedal triangle.

Again, the line CGB which is at right angles to OG bisects $\angle HGK$ externally. Similarly the lines AHC and BKA bisect $\angle KHG$ and $\angle GKH$ externally, so that ABC is the ex-central triangle of its pedal triangle GHK.

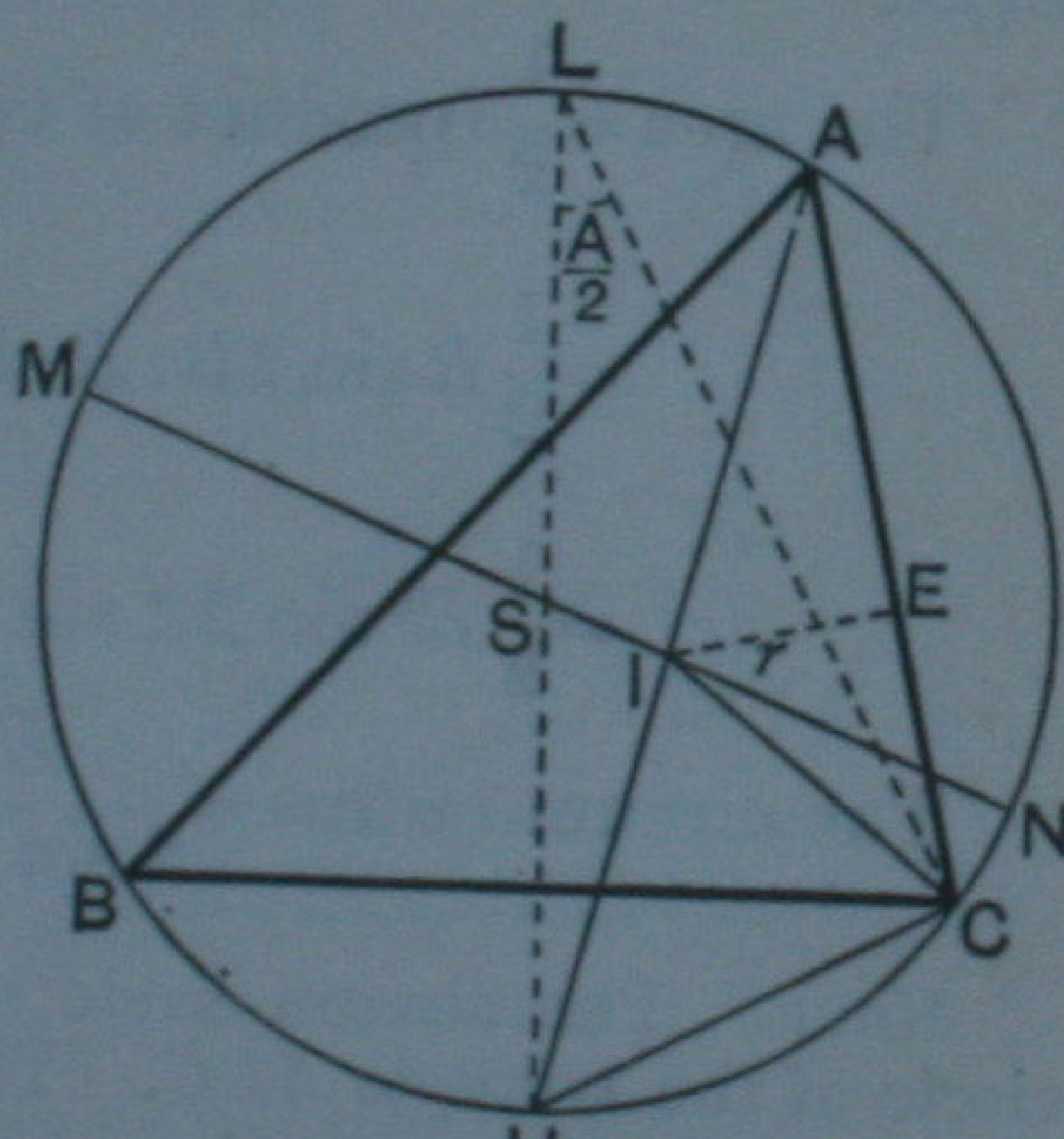
***227.** In Art. 219, we have seen that ABC is the pedal triangle of its ex-central triangle $I_1I_2I_3$. Certain theorems depending on this connection are more evident from the adjoining figure, in which the fact that ABC is the pedal triangle of $I_1I_2I_3$ is brought more prominently into view. For instance, the circum-circle of the triangle ABC is the nine points circle of the triangle $I_1I_2I_3$, and passes through the middle points of II_1, II_2, II_3 and of I_2I_3, I_3I_1, I_1I_2.



***228.** *To find the distance between the in-centre and circum-centre.*

Let S be the circum-centre and I the in-centre. Produce AI to meet the circum-circle in H; join CH and CI.

Draw IE perpendicular to AC. Produce HS to meet the circumference in L, and join CL. Then



$$\angle HIC = \angle IAC + \angle ICA$$
$$= \frac{A}{2} + \frac{C}{2};$$
$$\angle HCI = \angle ICB + \angle BCH$$
$$= \frac{C}{2} + \angle BAH$$
$$= \frac{C}{2} + \frac{A}{2};$$

$$\therefore \angle HCI = \angle HIC;$$

$$\therefore HI = HC = 2R \sin\frac{A}{2}.$$

Also
$$AI = IE \operatorname{cosec}\frac{A}{2} = r \operatorname{cosec}\frac{A}{2};$$

$$\therefore AI \,.\, IH = 2Rr.$$

Produce SI to meet the circumference in M and N.

By Euc. III. 35,

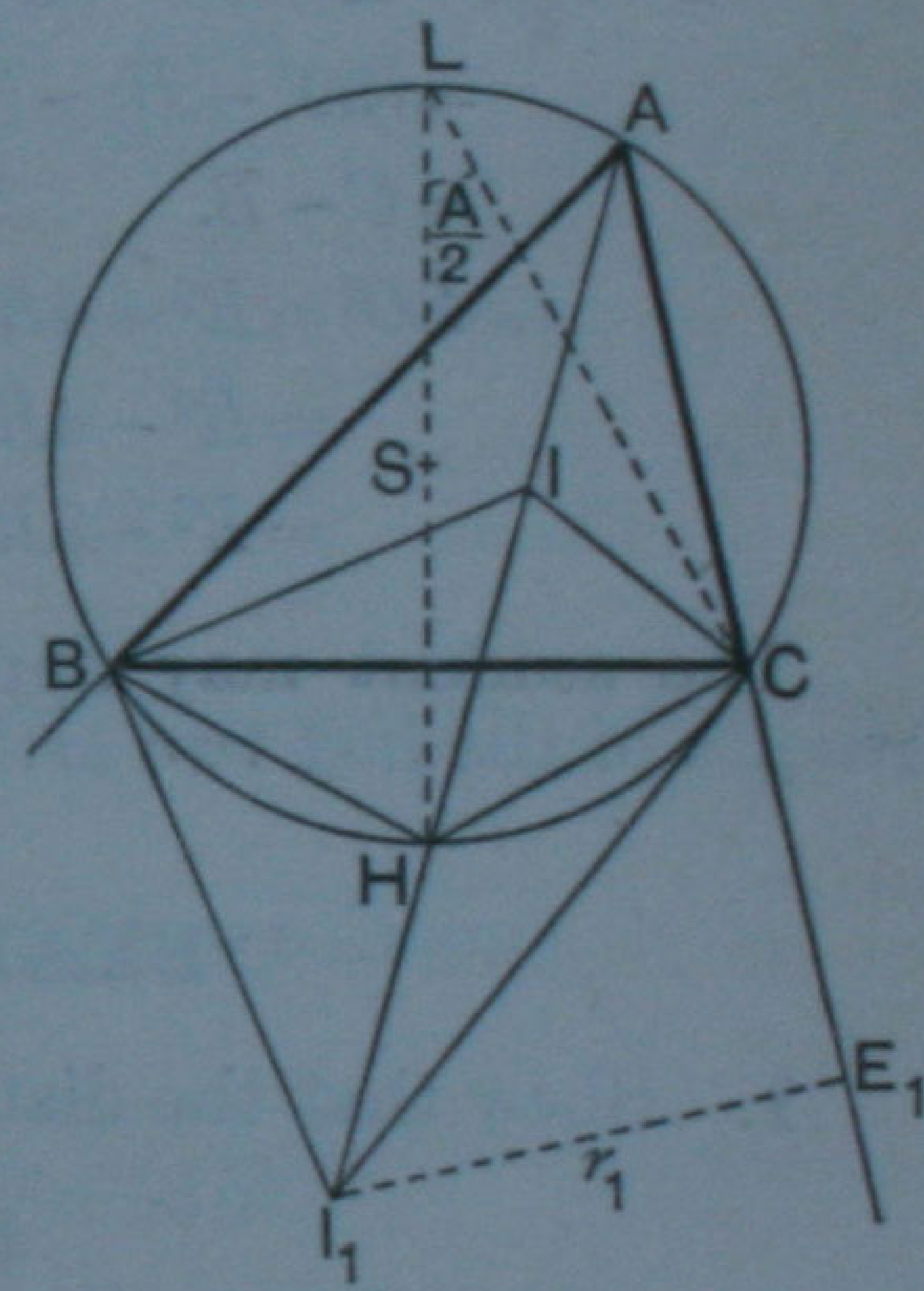
$$AI \,.\, IH = MI \,.\, IN = (R + SI)(R - SI);$$

$$\therefore\ 2Rr = R^2 - SI^2;$$

that is, $$SI^2 = R^2 - 2Rr.$$

***229.** *To find the distance of an ex-centre from the circum-centre.*

Let S be the circum-centre, and I the in-centre; then AI produced passes through the ex-centre I_1.



Let AI_1 meet the circum-circle in H; join CI, BI, CH, BH, CI_1, BI_1. Draw I_1E_1 perpendicular to AC.

Produce HS to meet the circumference in L, and join CL.

The angles IBI_1 and ICI_1 are right angles; hence the circle on II_1 as diameter passes through B and C.

The chords BH and CH of the circum-circle subtend equal angles at A, and are therefore equal.

But from the last article, $HC = HI$;

$$\therefore\ HB = HC = HI;$$

hence H is the centre of the circle round IBI_1C.

$$\therefore\ HI_1 = HC = 2R \sin\frac{A}{2}.$$

Now $$SI_1^2 - R^2 = \text{square of tangent from } I_1$$

$$= I_1H \,.\, I_1A$$

$$= 2R\sin\frac{A}{2} \,.\, r_1 \operatorname{cosec}\frac{A}{2}$$

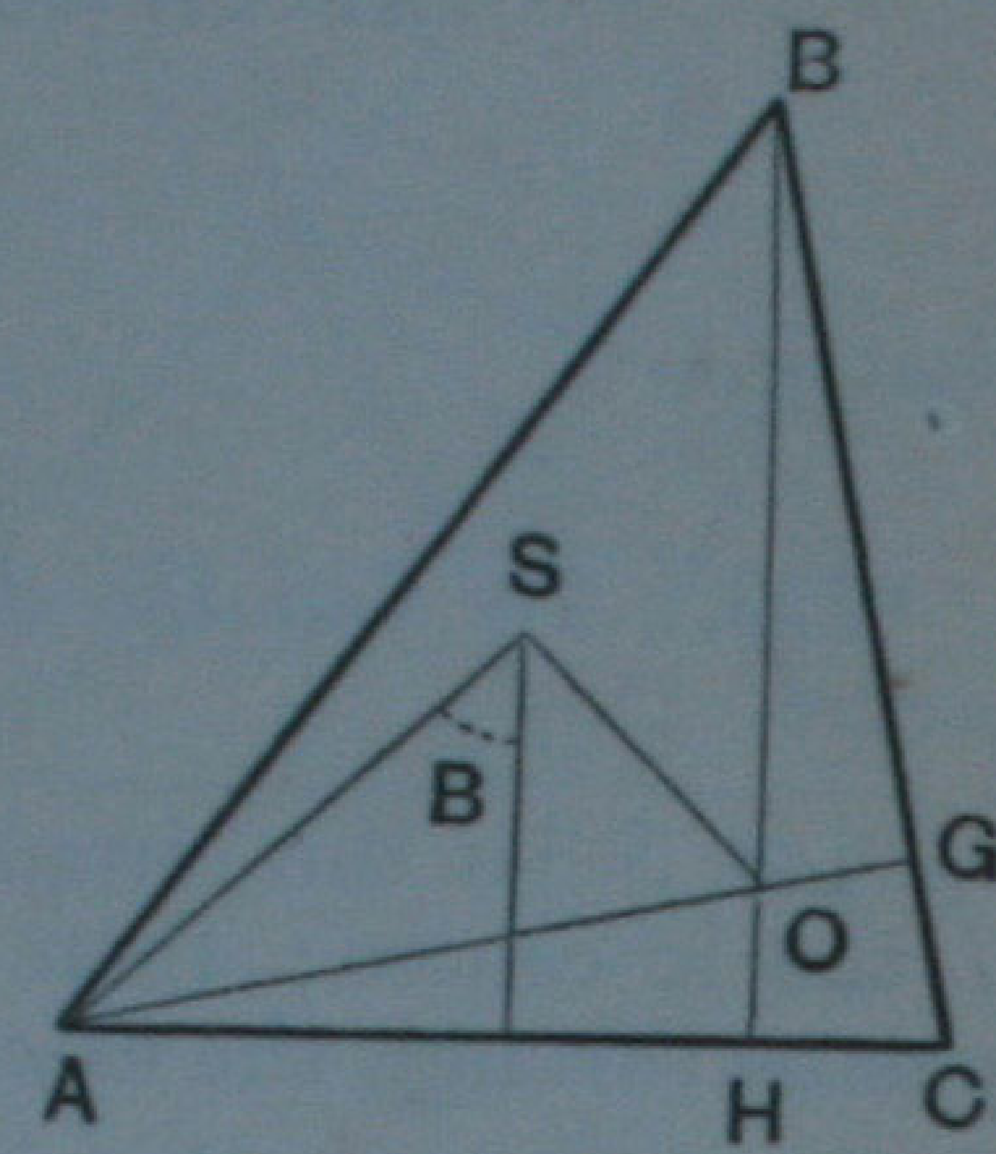
$$= 2Rr_1.$$

$$\therefore\ SI_1^2 = R^2 + 2Rr_1.$$

***230.** *To find the distance of the orthocentre from the circumcentre.*

With the usual notation, we have

$$SO^2 = SA^2 + AO^2 - 2SA \cdot AO \cos SAO.$$



Now $AS = R$;

$$\begin{aligned} AO &= AH \operatorname{cosec} C \\ &= c \cos A \operatorname{cosec} C \\ &= 2R \sin C \cos A \operatorname{cosec} C \\ &= 2R \cos A; \end{aligned}$$

$$\begin{aligned} \angle SAO &= \angle SAC - \angle OAC \\ &= (90^\circ - B) - (90^\circ - C) \\ &= C - B. \end{aligned}$$

$$\begin{aligned} \therefore\ SO^2 &= R^2 + 4R^2\cos^2 A - 4R^2 \cos A \cos(C - B) \\ &= R^2 - 4R^2 \cos A \{\cos(B + C) + \cos(C - B)\} \\ &= R^2 - 8R^2 \cos A \cos B \cos C. \end{aligned}$$

The student may apply a similar method to establish the results of the last two articles.

*EXAMPLES. XVIII. c.

1. Shew that the distance of the in-centre from A is

$$4R \sin\frac{B}{2} \sin\frac{C}{2}.$$

2. Shew that the distances of the ex-centre I_1 from the angular points A, B, C are

$$4R\cos\frac{B}{2}\cos\frac{C}{2}, \quad 4R\sin\frac{A}{2}\cos\frac{C}{2}, \quad 4R\sin\frac{A}{2}\cos\frac{B}{2}.$$

3. Prove that the area of the ex-central triangle is equal to

(1) $2Rs$; (2) $\frac{1}{2}\Delta \operatorname{cosec}\frac{A}{2}\operatorname{cosec}\frac{B}{2}\operatorname{cosec}\frac{C}{2}$.

4. Shew that

$$r \cdot II_1 \cdot II_2 \cdot II_3 = 4R \cdot IA \cdot IB \cdot IC.$$

5. Shew that the perimeter and in-radius of the pedal triangle are respectively

$$4R \sin A \sin B \sin C \quad \text{and} \quad 2R \cos A \cos B \cos C.$$

6. If g, h, k denote the sides of the pedal triangle, prove that

(1) $$\frac{g}{a^2}+\frac{h}{b^2}+\frac{k}{c^2}=\frac{a^2+b^2+c^2}{2abc};$$

(2) $$\frac{(b^2-c^2)\,g}{a^2}+\frac{(c^2-a^2)\,h}{b^2}+\frac{(a^2-b^2)\,k}{c^2}=0.$$

7. Prove that the ex-radii of the pedal triangle are
$2R\cos A\sin B\sin C$, $2R\sin A\cos B\sin C$, $2R\sin A\sin B\cos C$.

8. Prove that any formula which connects the sides and angles of a triangle holds if we replace

(1) a, b, c by $a\cos A$, $b\cos B$, $c\cos C$,
and A, B, C by $180°-2A$, $180°-2B$, $180°-2C$;

(2) a, b, c by $a\operatorname{cosec}\frac{A}{2}$, $b\operatorname{cosec}\frac{B}{2}$, $c\operatorname{cosec}\frac{C}{2}$,

and A, B, C by $90°-\frac{A}{2}$, $90°-\frac{B}{2}$, $90°-\frac{C}{2}$.

9. Prove that the radius of the circum-circle is never less than the diameter of the in-circle.

10. If $R=2r$, shew that the triangle is equilateral.

11. Prove that

$$SI^2+SI_1^2+SI_2^2+SI_3^2=12R^2.$$

12. Prove that

(1) $a\,.\,AI^2+b\,.\,BI^2+c\,.\,CI^2=abc$;

(2) $a\,.\,AI_1^2-b\,.\,BI_1^2-c\,.\,CI_1^2=abc$.

13. If GHK be the pedal triangle, and O the orthocentre, prove that

(1) $$\frac{OG}{AG}+\frac{OH}{BH}+\frac{OK}{CK}=1;$$

(2) $$\frac{OG}{OG+a\cot A}+\frac{OH}{OH+b\cot B}+\frac{OK}{OK+c\cot C}=1.$$

14. If GHK be the pedal triangle, shew that the sum of the circum-radii of the triangles AHK, BKG, CGH is equal to $R+r$.

15. If $A_1B_1C_1$ is the ex-central triangle of ABC, and $A_2B_2C_2$ the ex-central triangle of $A_1B_1C_1$, and $A_3B_3C_3$ the ex-central triangle of $A_2B_2C_2$, and so on: find the angles of the triangle $A_nB_nC_n$, and prove that when n is indefinitely increased the triangle becomes equilateral.

16. Prove that

(1) $OS^2=9R^2-a^2-b^2-c^2$;

(2) $OI^2=2r^2-4R^2\cos A\cos B\cos C$;

(3) $OI_1^2=2r_1^2-4R^2\cos A\cos B\cos C$.

17. If f, g, h denote the distances of the circum-centre of the pedal triangle from the angular points of the original triangle, shew that

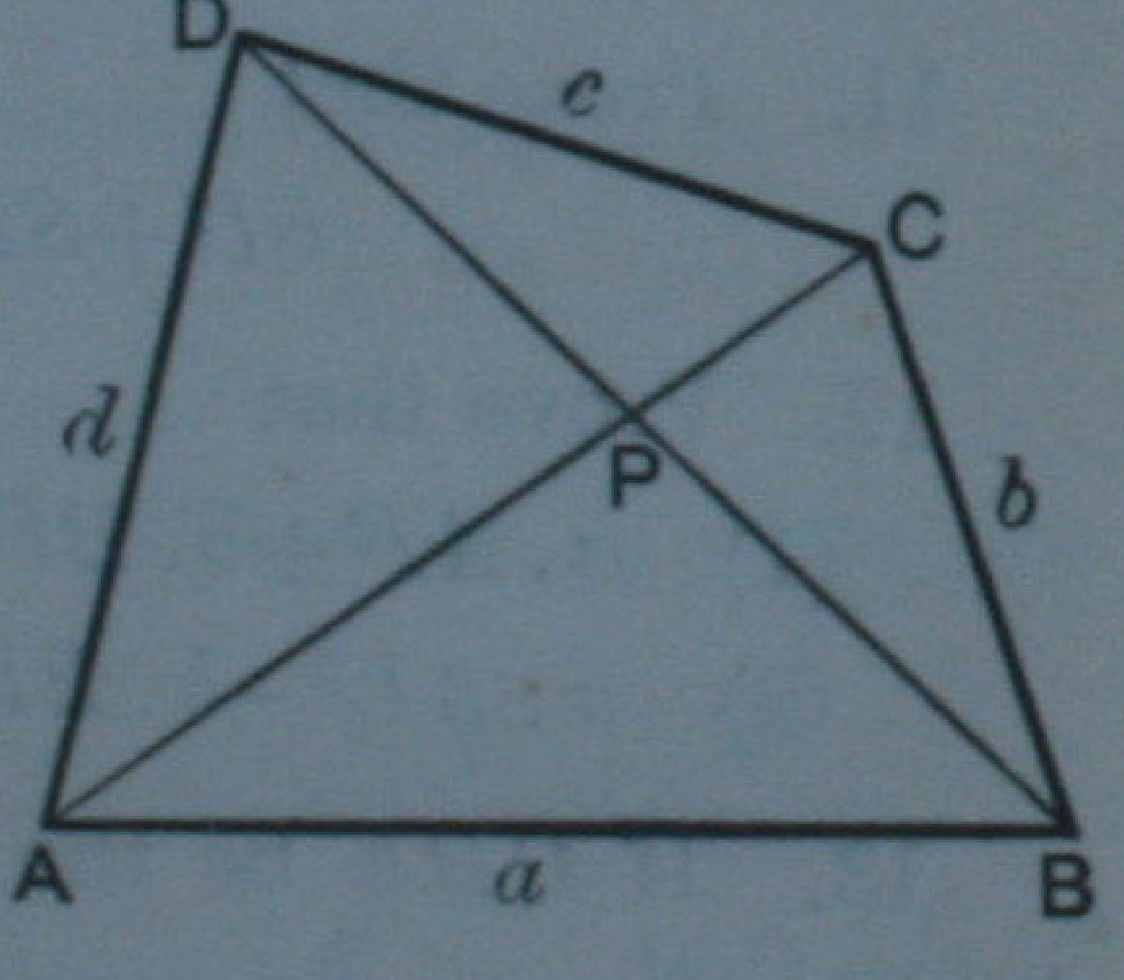
$$4(f^2+g^2+h^2)=11R^2+8R^2\cos A\cos B\cos C.$$

Quadrilaterals.

***231.** *To prove that the area of a quadrilateral is equal to*

$$\frac{1}{2}\,(\textit{product of the diagonals})\times(\textit{sine of included angle}).$$

Let the diagonals AC, BD intersect at P, and let $\angle DPA=\alpha$, and let S denote the area of the quadrilateral.



$$\Delta DAC=\Delta APD+\Delta CPD$$

$$=\frac{1}{2}DP\,.\,AP\sin\alpha$$

$$+\frac{1}{2}DP\,.\,PC\sin(\pi-\alpha)$$

$$=\frac{1}{2}DP(AP+PC)\sin\alpha$$

$$=\frac{1}{2}DP\,.\,AC\sin\alpha.$$

Similarly $$\Delta ABC=\frac{1}{2}BP\,.\,AC\sin\alpha.$$

$$\therefore\ S=\frac{1}{2}(DP+BP)\,AC\sin\alpha$$

$$=\frac{1}{2}DB\,.\,AC\sin\alpha.$$

***232.** *To find the area of a quadrilateral in terms of the sides and the sum of two opposite angles.*

Let $ABCD$ be the quadrilateral, and let a, b, c, d be the lengths of its sides, S its area.

By equating the two values of BD^2 found from the triangles BAD, BCD, we have

$$a^2+d^2-2ad\cos A=b^2+c^2-2bc\cos C;$$

$$\therefore\ a^2+d^2-b^2-c^2=2ad\cos A-2bc\cos C \quad\text{.........(1)}.$$

Also $\quad S=$ sum of areas of triangles BAD, BCD

$$=\frac{1}{2}ad\sin A+\frac{1}{2}bc\sin C;$$

$$\therefore\quad 4S=2ad\sin A+2bc\sin C \quad\text{...............(2)}.$$

Square (2) and add to the square of (1);

$$\therefore\quad 16S^2+(a^2+d^2-b^2-c^2)^2=4a^2d^2+4b^2c^2-8abcd\cos(A+C).$$

Let $A+C=2\alpha$; then

$$\cos(A+C)=\cos 2\alpha=2\cos^2\alpha-1;$$

$$\therefore\quad 16S^2=4(ad+bc)^2-(a^2+d^2-b^2-c^2)^2-16abcd\cos^2\alpha.$$

But the first two terms on the right

$$=(2ad+2bc+a^2+d^2-b^2-c^2)(2ad+2bc-a^2-d^2+b^2+c^2)$$

$$=\{(a+d)^2-(b-c)^2\}\{(b+c)^2-(a-d)^2\}$$

$$=(a+d+b-c)(a+d-b+c)(b+c+a-d)(b+c-a+d)$$

$$=(2\sigma-2c)(2\sigma-2b)(2\sigma-2d)(2\sigma-2a),$$

where $a+b+c+d=2\sigma$,

$$=16(\sigma-a)(\sigma-b)(\sigma-c)(\sigma-d).$$

Thus $\quad S^2=(\sigma-a)(\sigma-b)(\sigma-c)(\sigma-d)-abcd\cos^2\alpha$, where 2σ denotes the sum of the sides, 2α the sum of either pair of opposite angles.

***233.** In the case of a *cyclic quadrilateral*, $A+C=180°$, so that $\alpha=90°$; hence

$$S=\sqrt{(\sigma-a)(\sigma-b)(\sigma-c)(\sigma-d)}.$$

This formula may be obtained directly as in the last article by making use of the condition $A+C=180°$ during the course of the work. In this case $\cos C=-\cos A$, and $\sin C=\sin A$, so that the expressions (1) and (2) become

$$a^2+d^2-b^2-c^2=2(ad+bc)\cos A,$$

and

$$4S=2(ad+bc)\sin A;$$

whence by eliminating A we obtain

$$16S^2+(a^2+d^2-b^2-c^2)^2=4(ad+bc)^2.$$

***234.** *To find the diagonals and the circum-radius of a cyclic quadrilateral.*

If $ABCD$ is a cyclic quadrilateral, we have just proved that

$$2(ad+bc)\cos A=a^2+d^2-b^2-c^2.$$

Now $BD^2=a^2+d^2-2ad\cos A$

$$=a^2+d^2-\frac{ad(a^2+d^2-b^2-c^2)}{ad+bc}$$

$$=\frac{bc(a^2+d^2)+ad(b^2+c^2)}{ad+bc}$$

$$=\frac{(ab+cd)(ac+bd)}{ad+bc}.$$

Similarly, we may prove that

$$AC^2=\frac{(ad+bc)(ac+bd)}{ab+cd}.$$

Thus

$$AC\,.\,BD=ac+bd, \quad \text{[Compare Euc. VI. D.]}$$

and

$$\frac{AC}{BD}=\frac{ad+bc}{ab+cd}.$$

The circle passing round the quadrilateral circumscribes the triangle ABD; hence

the circum-radius $=\dfrac{BD}{2\sin A}$

$$=\frac{(ad+bc)BD}{2(ad+bc)\sin A}=\frac{(ad+bc)BD}{4S}$$

$$=\frac{1}{4S}\sqrt{(ab+cd)(ac+bd)(ad+bc)}.$$

Example. A quadrilateral $ABCD$ is such that one circle can be inscribed in it and another circle circumscribed about it; shew that $\tan^2 \frac{A}{2} = \frac{bc}{ad}$.

If a circle can be inscribed in a quadrilateral, the sum of one pair of the opposite sides is equal to that of the other pair;

$$\therefore a+c=b+d.$$

Since the quadrilateral is cyclic,

$$\cos A = \frac{a^2+d^2-b^2-c^2}{2(ad+bc)}. \qquad \text{[Art. 233.]}$$

But $a-d=b-c$, so that $a^2-2ad+d^2=b^2-2bc+c^2$;

$$\therefore a^2+d^2-b^2-c^2=2(ad-bc);$$

$$\therefore \cos A = \frac{ad-bc}{ad+bc};$$

$$\therefore \tan^2 \frac{A}{2} = \frac{1-\cos A}{1+\cos A} = \frac{bc}{ad}.$$

*EXAMPLES. XVIII. d.

1. If a circle can be inscribed in a quadrilateral, shew that its radius is S/σ where S is the area and 2σ the sum of the sides of the quadrilateral.

2. If the sides of a cyclic quadrilateral be 3, 3, 4, 4, shew that a circle can be inscribed in it, and find the radii of the inscribed and circumscribed circles.

3. If the sides of a cyclic quadrilateral be 1, 2, 4, 3, shew that the cosine of the angle between the two greatest sides is $\frac{5}{7}$, and that the radius of the inscribed circle is ·98 nearly.

4. The sides of a cyclic quadrilateral are 60, 25, 52, 39: shew that two of the angles are right angles, and find the diagonals and the area.

5. The sides of a quadrilateral are 4, 5, 8, 9, and one diagonal is 9: find the area.

6. If a circle can be inscribed in a cyclic quadrilateral, shew that the area of the quadrilateral is $\sqrt{abcd}$, and that the radius of the circle is

$$2\sqrt{abcd}/(a+b+c+d).$$

7. If the sides of a quadrilateral are given, shew that the area is a maximum when the quadrilateral can be inscribed in a circle.

8. If the sides of a quadrilateral are 23, 29, 37, 41 inches, prove that the maximum area is 7 sq. ft.

9. If $ABCD$ is a cyclic quadrilateral, prove that

$$\tan^2\frac{B}{2}=\frac{(\sigma-a)(\sigma-b)}{(\sigma-c)(\sigma-d)}.$$

10. If f, g denote the diagonals of a quadrilateral and β the angle between them, prove that

$$2fg\cos\beta=(a^2+c^2)\sim(b^2+d^2).$$

11. If β is the angle between the diagonals of any quadrilateral, prove that the area is

$$\frac{1}{4}\{(a^2+c^2)\sim(b^2+d^2)\}\tan\beta.$$

12. Prove that the area of a quadrilateral in which a circle can be inscribed is

$$\sqrt{abcd}\sin\frac{A+C}{2}.$$

13. If a circle can be inscribed in a quadrilateral whose diagonals are f and g, prove that

$$4S^2=f^2g^2-(ac-bd)^2.$$

14. If β is the angle between the diagonals of a cyclic quadrilateral, prove that

(1) $(ac+bd)\sin\beta=(ad+bc)\sin A$;

(2) $\cos\beta=\dfrac{(a^2+c^2)\sim(b^2+d^2)}{2(ac+bd)}$;

(3) $\tan^2\dfrac{\beta}{2}=\dfrac{(\sigma-b)(\sigma-d)}{(\sigma-a)(\sigma-c)}$ or $\dfrac{(\sigma-a)(\sigma-c)}{(\sigma-b)(\sigma-d)}$.

15. If f, g are the diagonals of a quadrilateral, shew that

$$S=\frac{1}{4}\sqrt{4f^2g^2-(a^2+c^2-b^2-d^2)^2}.$$

16. In a cyclic quadrilateral, prove that the product of the segments of a diagonal is

$$abcd(ac+bd)/(ab+cd)(ad+bc).$$

235. The following exercise consists of miscellaneous questions involving the properties of triangles.

EXAMPLES. XVIII. e.

1. If the sides of a triangle are 242, 1212, 1450 yards, shew that the area is 6 acres.

2. One of the sides of a triangle is 200 yards and the adjacent angles are 22·5° and 67·5° : find the area.

3. If $r_1=2r_2=2r_3$, shew that $3a=4b$.

4. If a, b, c are in A. P., shew that r_1, r_2, r_3 are in H. P.

5. Find the area of a triangle whose sides are

$$\frac{y}{z}+\frac{z}{x},\quad \frac{z}{x}+\frac{x}{y},\quad \frac{x}{y}+\frac{y}{z}.$$

6. If $\sin A : \sin C=\sin(A-B) : \sin(B-C)$, shew that a^2, b^2, c^2 are in A. P.

Prove that

7. $$\frac{a\sin A+b\sin B+c\sin C}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{a^2+b^2+c^2}{2s}.$$

8. $$\left(\frac{a^2}{\sin A}+\frac{b^2}{\sin B}+\frac{c^2}{\sin C}\right)\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\Delta.$$

9. $$(r_2+r_3)(r_3+r_1)(r_1+r_2)=4R(r_2r_3+r_3r_1+r_1r_2).$$

10. $$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{r_1+r_2+r_3}{(r_2r_3+r_3r_1+r_1r_2)^{\frac{1}{2}}}.$$

11. $$bc\cot\frac{A}{2}+ca\cot\frac{B}{2}+ab\cot\frac{C}{2}=4Rs^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{3}{s}\right).$$

12. $$\left(\frac{1}{r}+\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)^2=\frac{4}{r}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right).$$

13. The perimeter of a right-angled triangle is 70, and the in-radius is 6 : find the sides.

14. If f, g, h are the perpendiculars from the circum-centre on the sides, prove that

$$\frac{a}{f}+\frac{b}{g}+\frac{c}{h}=\frac{abc}{4fgh}.$$

15. An equilateral triangle and a regular hexagon have the same perimeter: shew that the areas of their inscribed circles are as 4 to 9.

***16.** Shew that the perimeter of the pedal triangle is equal to

$$abc/2R^2.$$

***17.** Shew that the area of the ex-central triangle is equal to

$$abc\,(a+b+c)/4\Delta.$$

18. In the ambiguous case, if A, a, b are the given parts, and c_1, c_2 the two values of the third side, shew that the distance between the circum-centres of the two triangles is $\dfrac{c_1 \sim c_2}{2\sin A}$.

***19.** If β be the angle between the diagonals of a cyclic quadrilateral, shew that

$$\sin\beta = \frac{2S}{ac+bd}.$$

***20.** Shew that

$$r^3 . II_1 . II_2 . II_3 = IA^2 . IB^2 . IC^2.$$

***21.** Shew that the sum of the squares of the sides of the ex-central triangle is equal to $8R\,(4R+r)$.

***22.** If circles can be inscribed in and circumscribed about a quadrilateral, and if β be the angle between the diagonals, shew that

$$\cos\beta = (ac \sim bd)/(ac+bd).$$

23. If l, m, n are the lengths of the medians of a triangle, prove that

(1) $4\,(l^2+m^2+n^2) = 3\,(a^2+b^2+c^2)$;

(2) $(b^2-c^2)\,l^2 + (c^2-a^2)\,m^2 + (a^2-b^2)\,n^2 = 0$;

(3) $16\,(l^4+m^4+n^4) = 9\,(a^4+b^4+c^4)$.

24. Shew that the radii of the escribed circles are the roots of the equation

$$x^3 - (4R+r)\,x^2 + s^2x - s^2r = 0.$$

25. If Δ_1, Δ_2, Δ_3 be the areas of the triangles cut off by tangents to the in-circle parallel to the sides of a triangle, prove that

$$\frac{\Delta_1}{(s-a)^2} = \frac{\Delta_2}{(s-b)^2} = \frac{\Delta_3}{(s-c)^2} = \frac{\Delta}{s^2}.$$

***26.** The triangle LMN is formed by joining the points of contact of the in-circle; shew that it is similar to the ex-central triangle, and that their areas are as r^2 to $4R^2$.

27. In the triangle PQR formed by drawing tangents at A, B, C to the circum-circle, prove that the angles and sides are

$$180° - 2A, \quad 180° - 2B, \quad 180° - 2C;$$

and
$$\frac{a}{2\cos B\cos C}, \quad \frac{b}{2\cos C\cos A}, \quad \frac{c}{2\cos A\cos B}.$$

28. If p, q, r be the lengths of the bisectors of the angles of a triangle, prove that

(1) $\frac{1}{p}\cos\frac{A}{2}+\frac{1}{q}\cos\frac{B}{2}+\frac{1}{r}\cos\frac{C}{2}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$;

(2) $\frac{pqr}{4\Delta}=\frac{abc\,(a+b+c)}{(b+c)(c+a)(a+b)}$.

29. If the perpendiculars AG, BH, CK are produced to meet the circum-circle in L, M, N, prove that

(1) area of triangle $LMN = 8\Delta\cos A\cos B\cos C$;

(2) $AL\sin A + BM\sin B + CN\sin C = 8R\sin A\sin B\sin C$.

30. If r_a, r_b, r_c be the radii of the circles inscribed between the in-circle and the sides containing the angles A, B, C respectively, shew that

(1) $r_a = r\tan^2\frac{\pi - A}{4}$; (2) $\sqrt{r_b r_c}+\sqrt{r_c r_a}+\sqrt{r_a r_b}=r$.

***31.** Lines drawn through the angular points of a triangle ABC parallel to the sides of the pedal triangle form a triangle XYZ: shew that the perimeter and area of XYZ are respectively

$$2R\tan A\tan B\tan C \text{ and } R^2\tan A\tan B\tan C.$$

***32.** A straight line cuts three concentric circles in A, B, C and passes at a distance p from their centre: shew that the area of the triangle formed by the tangents at A, B, C is

$$\frac{BC\,.\,CA\,.\,AB}{2p}.$$

MISCELLANEOUS EXAMPLES. F.

1. If $\alpha+\beta+\gamma+\delta=180°$, shew that

$$\cos\alpha\cos\beta+\cos\gamma\cos\delta=\sin\alpha\sin\beta+\sin\gamma\sin\delta.$$

2. Prove that

$$\cos(15°-A)\sec 15°-\sin(15°-A)\operatorname{cosec}15°=4\sin A.$$

3. Shew that in a triangle

$$\cot A+\sin A\operatorname{cosec}B\operatorname{cosec}C$$

retains the same value if any two of the angles A, B, C are interchanged.

4. If $a=2$, $b=\sqrt{8}$, $A=30°$, solve the triangle.

5. Shew that

(1) $\cot 18°=\sqrt{5}\cot 36°$;

(2) $16\sin 36°\sin 72°\sin 108°\sin 144°=5$.

6. Find the number of ciphers before the first significant digit in $(\cdot 0396)^{90}$, given

$$\log 2=\cdot 30103,\ \log 3=\cdot 47712,\ \log 11=1\cdot 04139.$$

7. An observer finds that the angle subtended by the line joining two points A and B on the horizontal plane is 30°. On walking 50 yards directly towards A the angle increases to 75°: find his distance from B at each observation.

8. Prove that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2(\alpha+\beta+\gamma)$

$$=2+2\cos(\beta+\gamma)\cos(\gamma+\alpha)\cos(\alpha+\beta).$$

9. Shew that

(1) $\tan 40°+\cot 40°=2\sec 10°$;

(2) $\tan 70°+\tan 20°=2\operatorname{cosec}40°$.

10. Prove that

(1) $2\sin 4a-\sin 10a+\sin 2a=16\sin a\cos a\cos 2a\sin^2 3a$;

(2) $\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}-\sin\frac{6\pi}{7}=4\sin\frac{\pi}{7}\sin\frac{3\pi}{7}\sin\frac{5\pi}{7}$.

11. If $B=30°$, $b=3\sqrt{2}-\sqrt{6}$, $c=6-2\sqrt{3}$, solve the triangle.

12. From a ship which is sailing N.E., the bearing of a rock is N.N.W. After the ship has sailed 10 miles the rock bears due W.: find the distance of the ship from the rock at each observation.

13. Shew that in any triangle

$$\frac{b^2-c^2}{\cos B+\cos C}+\frac{c^2-a^2}{\cos C+\cos A}+\frac{a^2-b^2}{\cos A+\cos B}=0.$$

14. If $\cos(\theta-a)$, $\cos\theta$, $\cos(\theta+a)$ are in harmonical progression, shew that

$$\cos\theta=\sqrt{2}\cos\frac{a}{2}.$$

15. If $\sin\beta$ be the geometric mean between $\sin a$ and $\cos a$, prove that $$\cos 2\beta=2\cos^2\left(\frac{\pi}{4}+a\right).$$

16. Shew that the distances of the orthocentre from the sides are $2R\cos B\cos C$, $2R\cos C\cos A$, $2R\cos A\cos B$.

17. If $$\cos\theta=\frac{\cos u-e}{1-e\cos u},$$
prove that $$\tan\frac{\theta}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}.$$

18. If the sides of a right-angled triangle are

$$2(1+\sin\theta)+\cos\theta \quad \text{and} \quad 2(1+\cos\theta)+\sin\theta,$$

prove that the hypotenuse is

$$3+2(\cos\theta+\sin\theta).$$

*19. Prove that the distances of the in-centre of the ex-central triangle $I_1I_2I_3$ from its ex-centres are

$$8R\sin\frac{B+C}{4},\quad 8R\sin\frac{C+A}{4},\quad 8R\sin\frac{A+B}{4}.$$

*20. Prove that the distances between the ex-centres of the ex-central triangle $I_1I_2I_3$ are

$$8R\cos\frac{B+C}{4},\quad 8R\cos\frac{C+A}{4},\quad 8R\cos\frac{A+B}{4}.$$

21. If

$$(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)=(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma),$$

shew that each expression is equal to $\pm\sin\alpha\sin\beta\sin\gamma$.

22. If the sum of four angles is 180°, shew that the sum of the products of their sines taken two together is equal to the sum of the products of their cosines taken two together.

***23.** In a triangle, shew that

(1) $II_1 \,.\, II_2 \,.\, II_3 = 16R^2 r$; (2) $II_1{}^2 + I_2I_3{}^2 = 16R^2$.

24. Find the angles of a triangle whose sides are proportional to

(1) $\cos\dfrac{A}{2}$, $\cos\dfrac{B}{2}$, $\cos\dfrac{C}{2}$;

(2) $\sin 2A$, $\sin 2B$, $\sin 2C$.

25. Prove that the expression

$$\sin^2(\theta+\alpha)+\sin^2(\theta+\beta)-2\cos(\alpha-\beta)\sin(\theta+\alpha)\sin(\theta+\beta)$$

is independent of θ.

***26.** If a, b, c, d are the sides of a quadrilateral described about a circle, prove that

$$ad\sin^2\frac{A}{2}=bc\sin^2\frac{C}{2}.$$

27. Tangents parallel to the three sides are drawn to the in-circle. If p, q, r be the lengths of the parts of the tangents within the triangle, prove that $\dfrac{p}{a}+\dfrac{q}{b}+\dfrac{r}{c}=1$.

[*The Tables will be required for Examples* 28 *and* 29.]

28. From the top of a cliff 1566 ft. in height a train, which is travelling at a uniform speed in a straight line to a tunnel immediately below the observer, is seen to pass two consecutive stations at an interval of 3 minutes. The angles of depression of the two stations are 13° 14′ 12″ and 56° 24′ 36″ respectively; how fast is the train travelling?

29. A harbour lies in a direction 46° 8′ 8·6″ South of West from a fort, and at a distance of 27·23 miles from it. A ship sets out from the harbour at noon and sails due East at 10 miles an hour; when will the ship be 20 miles from the fort?

CHAPTER XIX.

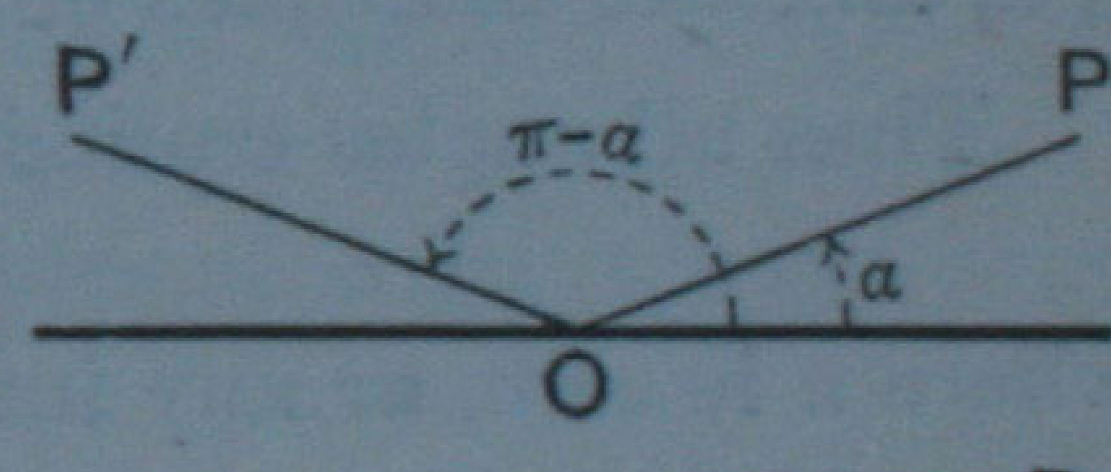
GENERAL VALUES AND INVERSE FUNCTIONS.

236. THE equation $\sin\theta=\frac{1}{2}$ is satisfied by $\theta=\frac{\pi}{6}$, and by $\theta=\pi-\frac{\pi}{6}$, and all angles coterminal with these will have the same sine. This example shews that there are an infinite number of angles whose sine is equal to a given quantity. Similar remarks apply to the other functions.

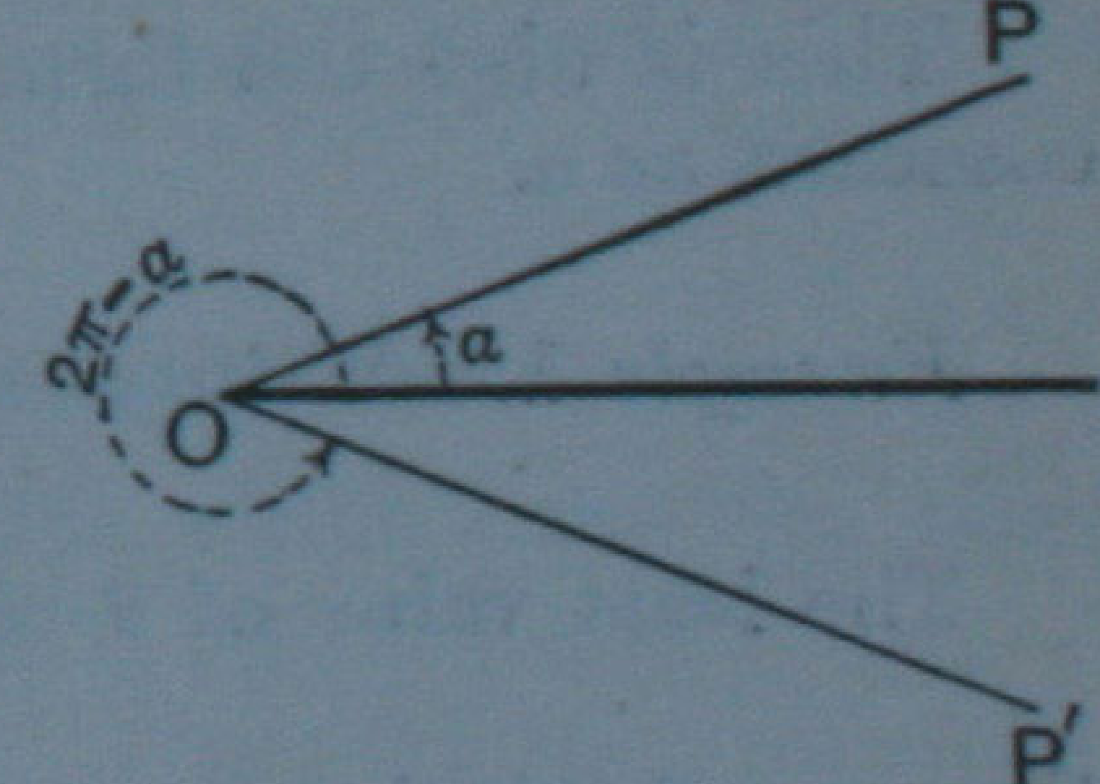
We proceed to shew how to express by a single formula all angles which have a given sine, cosine, or tangent.

237. From the results proved in Chap. IX., it is easily seen that in going once through the four quadrants, there are two and only two positions of the boundary line which give angles with the same sine, cosine, or tangent.

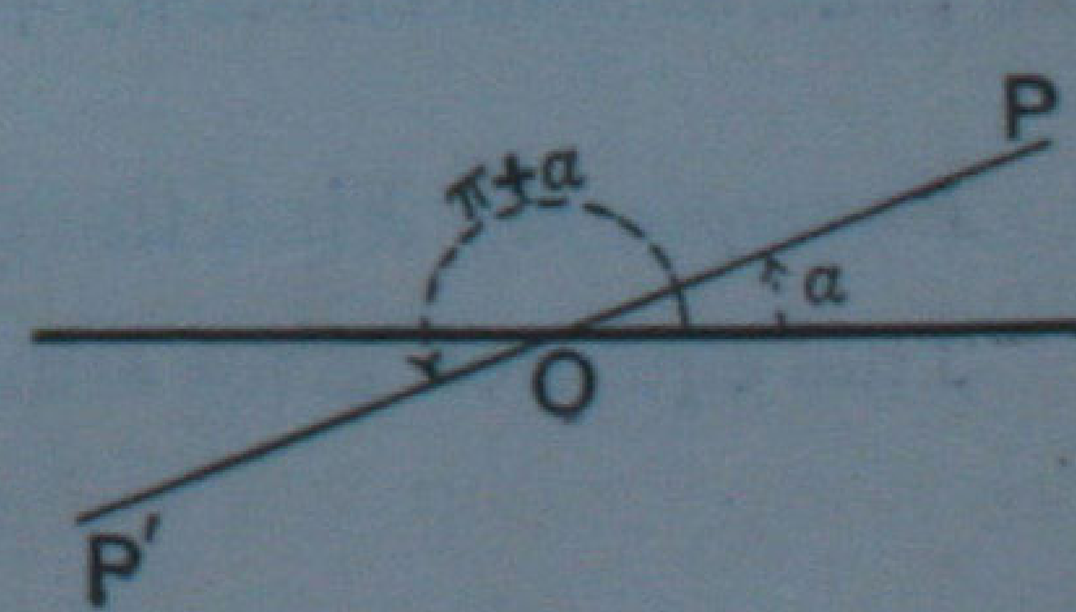
Thus if $\sin\alpha$ has a given value, the positions of the radius vector are OP and OP' bounding the angles α and $\pi-\alpha$. [Art. 92.]



If $\cos\alpha$ has a given value, the positions of the radius vector are OP and OP' bounding the angles α and $2\pi-\alpha$. [Art. 105.]

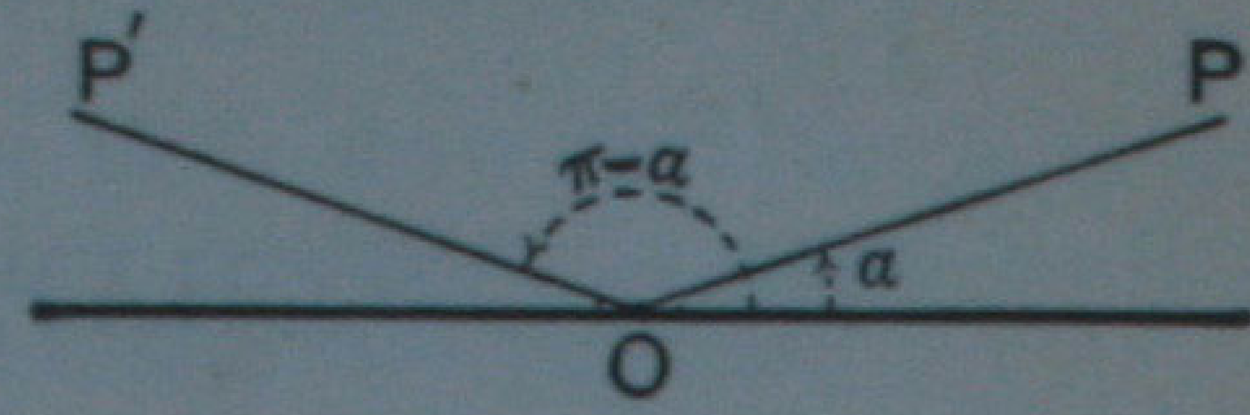


If $\tan\alpha$ has a given value, the positions of the radius vector are OP and OP' bounding the angles α and $\pi+\alpha$. [Art. 97.]



238. *To find a formula for all the angles which have a given sine.*

Let a be the smallest positive angle which has a given sine. Draw OP and OP' bounding the angles a and $\pi - a$; then the required angles are those coterminal with OP and OP'.



The positive angles are

$$2p\pi + a \quad \text{and} \quad 2p\pi + (\pi - a),$$

where p is zero, or any positive integer.

The negative angles are

$$-(\pi + a) \quad \text{and} \quad -(2\pi - a),$$

and those which may be obtained from them by the addition of any negative multiple of 2π; that is, angles denoted by

$$2q\pi - (\pi + a) \quad \text{and} \quad 2q\pi - (2\pi - a),$$

where q is zero, or any negative integer.

These angles may be grouped as follows:

$$\left.\begin{matrix} 2p\pi + a, \\ (2q-2)\pi + a, \end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix} (2p+1)\pi - a, \\ (2q-1)\pi - a, \end{matrix}\right.$$

and it will be noticed that even multiples of π are followed by $+a$, and odd multiples of π by $-a$.

Thus all angles equi-sinal with a are included in the formula

$$n\pi + (-1)^n a,$$

where n is zero, or any integer positive or negative.

This is also the formula for all angles which have the same cosecant as a.

Example 1. Write down the general solution of $\sin\theta = \frac{\sqrt{3}}{2}$.

The least value of θ which satisfies the equation is $\frac{\pi}{3}$; therefore the general solution is $n\pi + (-1)^n \frac{\pi}{3}$.

Example 2. Find the general solution of $\sin^2\theta = \sin^2 a$.

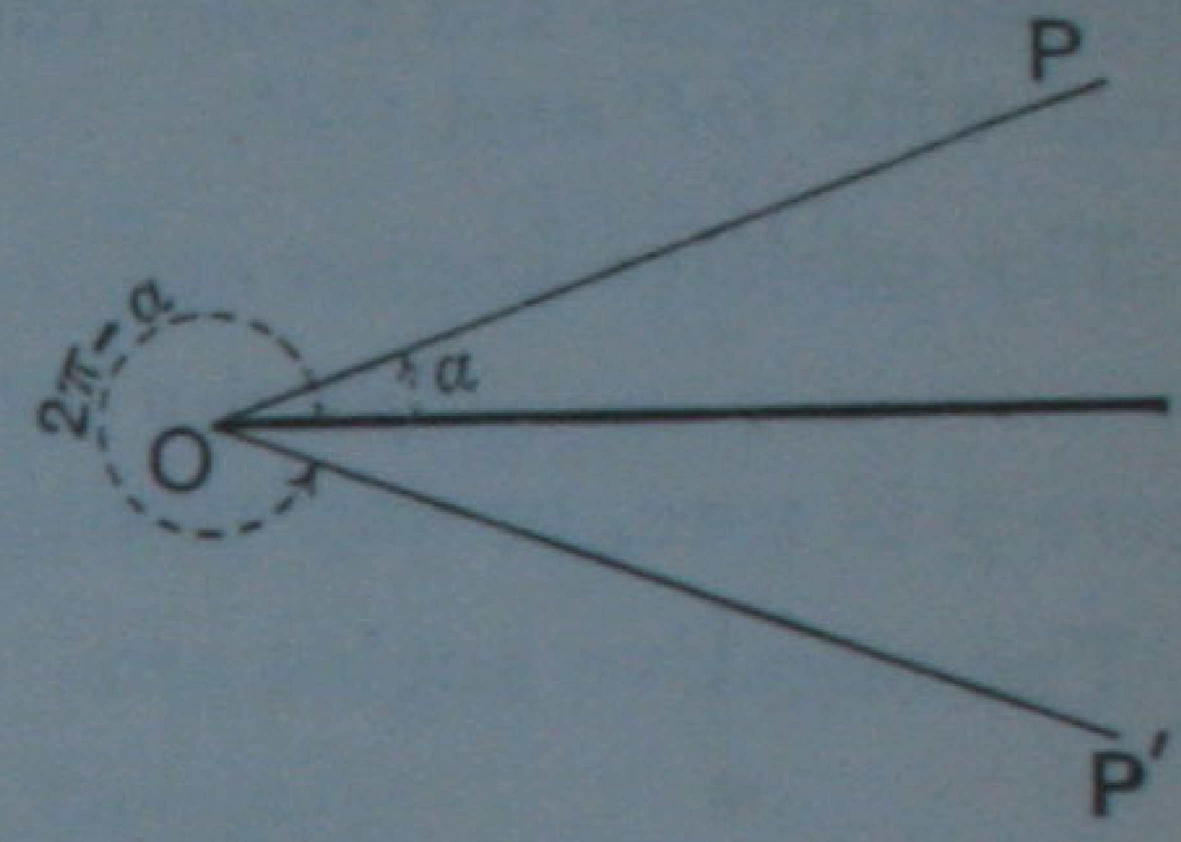
This equation gives either $\sin\theta = +\sin a$..........................(1),

or $\sin\theta = -\sin a = \sin(-a)$ ....(2).

From (1), $\theta = n\pi + (-1)^n \alpha$;
and from (2), $\theta = n\pi + (-1)^n (-\alpha)$.

Both values are included in the formula $\theta = n\pi \pm \alpha$.

239. *To find a formula for all the angles which have a given cosine.*



Let α be the smallest positive angle which has a given cosine. Draw OP and OP' bounding the angles α and $2\pi - \alpha$; then the required angles are those coterminal with OP and OP'.

The positive angles are

$$2p\pi + \alpha \text{ and } 2p\pi + (2\pi - \alpha),$$

where p is zero, or any positive integer.

The negative angles are

$$-\alpha \text{ and } -(2\pi - \alpha),$$

and those which may be obtained from them by the addition of any negative multiple of 2π; that is, angles denoted by

$$2q\pi - \alpha \text{ and } 2q\pi - (2\pi - \alpha),$$

where q is zero, or any negative integer.

The angles may be grouped as follows :

$$\left.\begin{array}{l} 2p\pi + \alpha, \\ 2q\pi - \alpha, \end{array}\right\} \text{ and } \left\{\begin{array}{l} (2p+2)\pi - \alpha, \\ (2q-2)\pi + \alpha, \end{array}\right.$$

and it will be noticed that the multiples of π are always even, but may be followed by $+\alpha$ or by $-\alpha$.

Thus all angles equi-cosinal with α are included in the formula

$$\boxed{2n\pi \pm \alpha,}$$

where n is zero, or any integer positive or negative.

This is also the formula for all angles which have the same secant as α.

Example 1. Find the general solution of $\cos\theta = -\frac{1}{2}$.

The least value of θ is $\pi - \frac{\pi}{3}$, or $\frac{2\pi}{3}$; hence the general solution is $2n\pi \pm \frac{2\pi}{3}$.