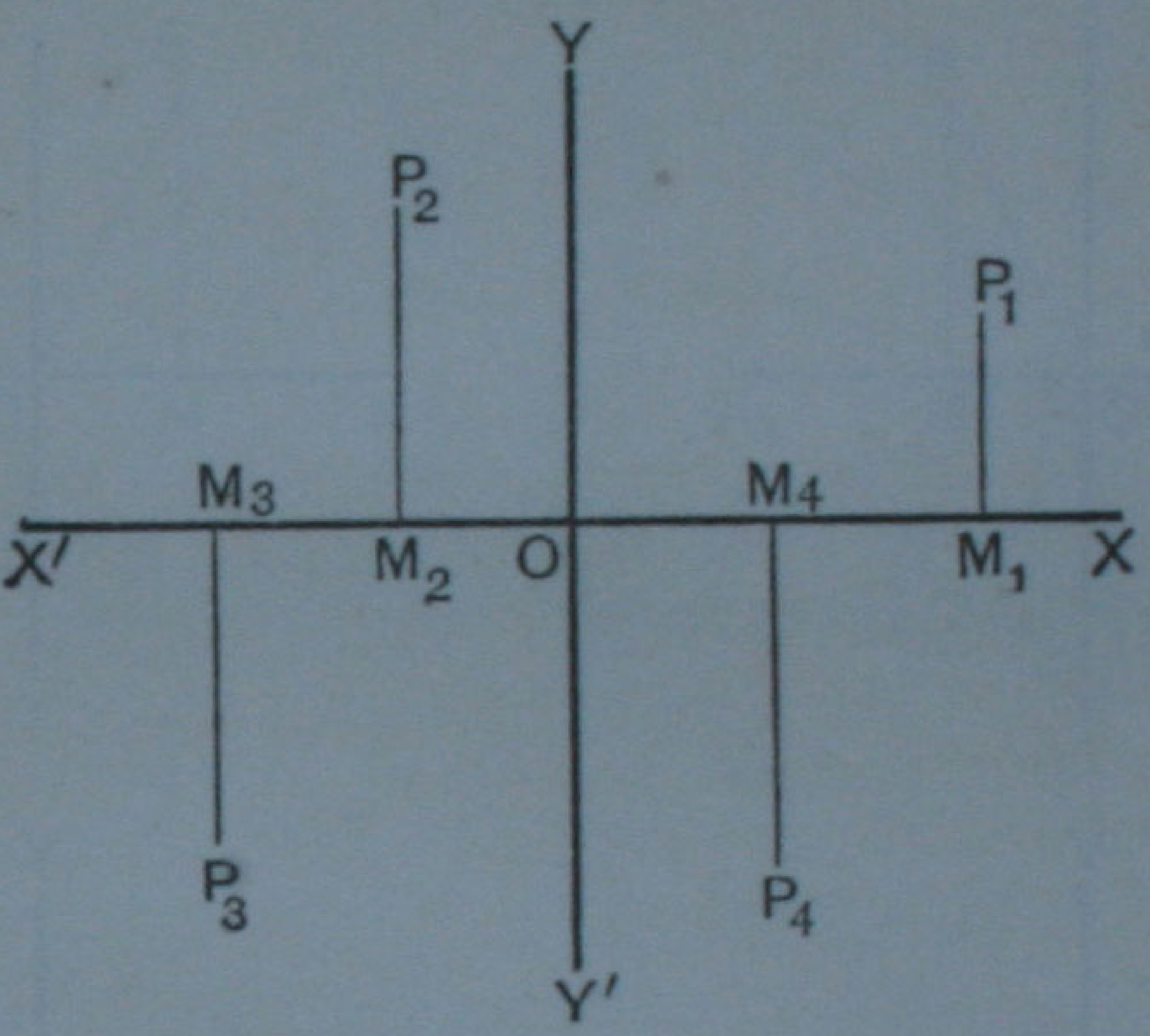
Then it is universally agreed to consider that

(1) *horizontal lines to the right of YY' are positive,*
horizontal lines to the left of YY' are negative;

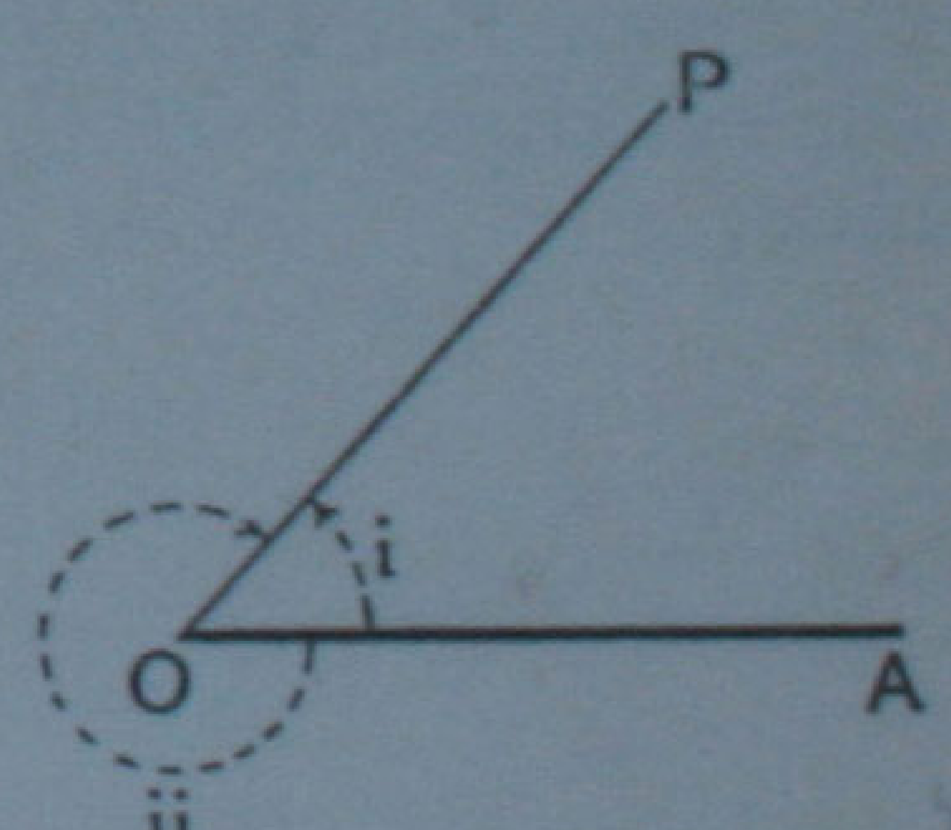
(2) *vertical lines above XX' are positive,*
vertical lines below XX' are negative.



Thus OM_1, OM_4 are positive, OM_2, OM_3 are negative;
M_1P_1, M_2P_2 are positive, M_3P_3, M_4P_4 are negative.

74. Convention of Signs for Angles. In Art. 2 an angle has been defined as the amount of revolution which the radius vector makes in passing from its initial to its final position.

In the adjoining figure the straight line OP may be supposed to have arrived at its present position from the position occupied by OA by revolution about the point O in *either* of the two directions indicated by the arrows. The angle AOP may thus be regarded in two senses according as we suppose the revolution to have been in the same direction as the hands of a clock or in the opposite direction. To distinguish between these cases we adopt the following convention:

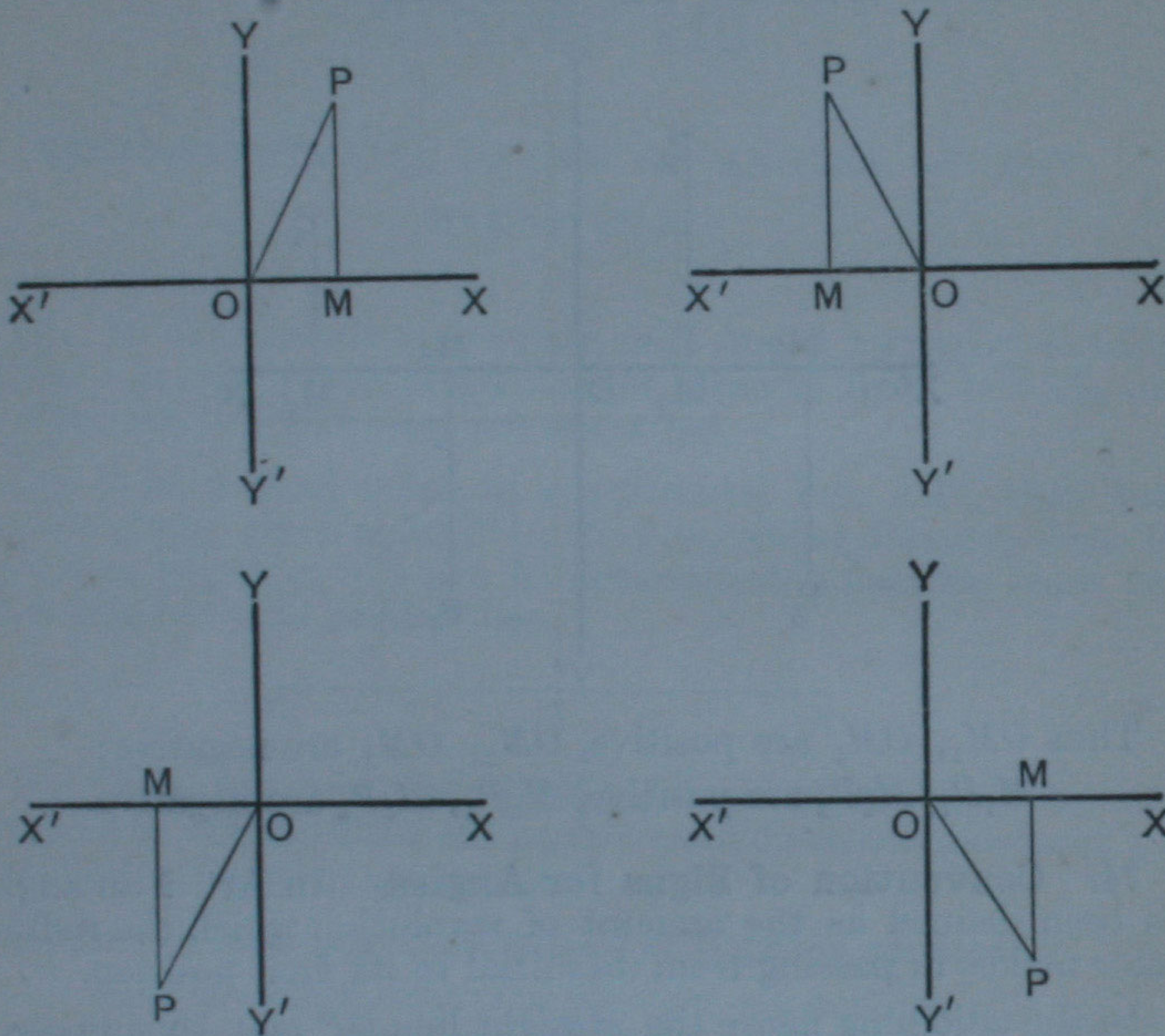


when the revolution of the radius vector is counter-clockwise the angle is positive,

when the revolution is clockwise the angle is negative.

Trigonometrical Ratios of any Angle.

75. Let XX' and YY' be two straight lines intersecting at right angles in O, and let a radius vector starting from OX revolve in either direction till it has traced out an angle A, taking up the position OP.



From P draw PM perpendicular to XX'; then in the right-angled triangle OPM, due regard being paid to the signs of the lines,

$$\sin A = \frac{MP}{OP}, \quad \operatorname{cosec} A = \frac{OP}{MP},$$

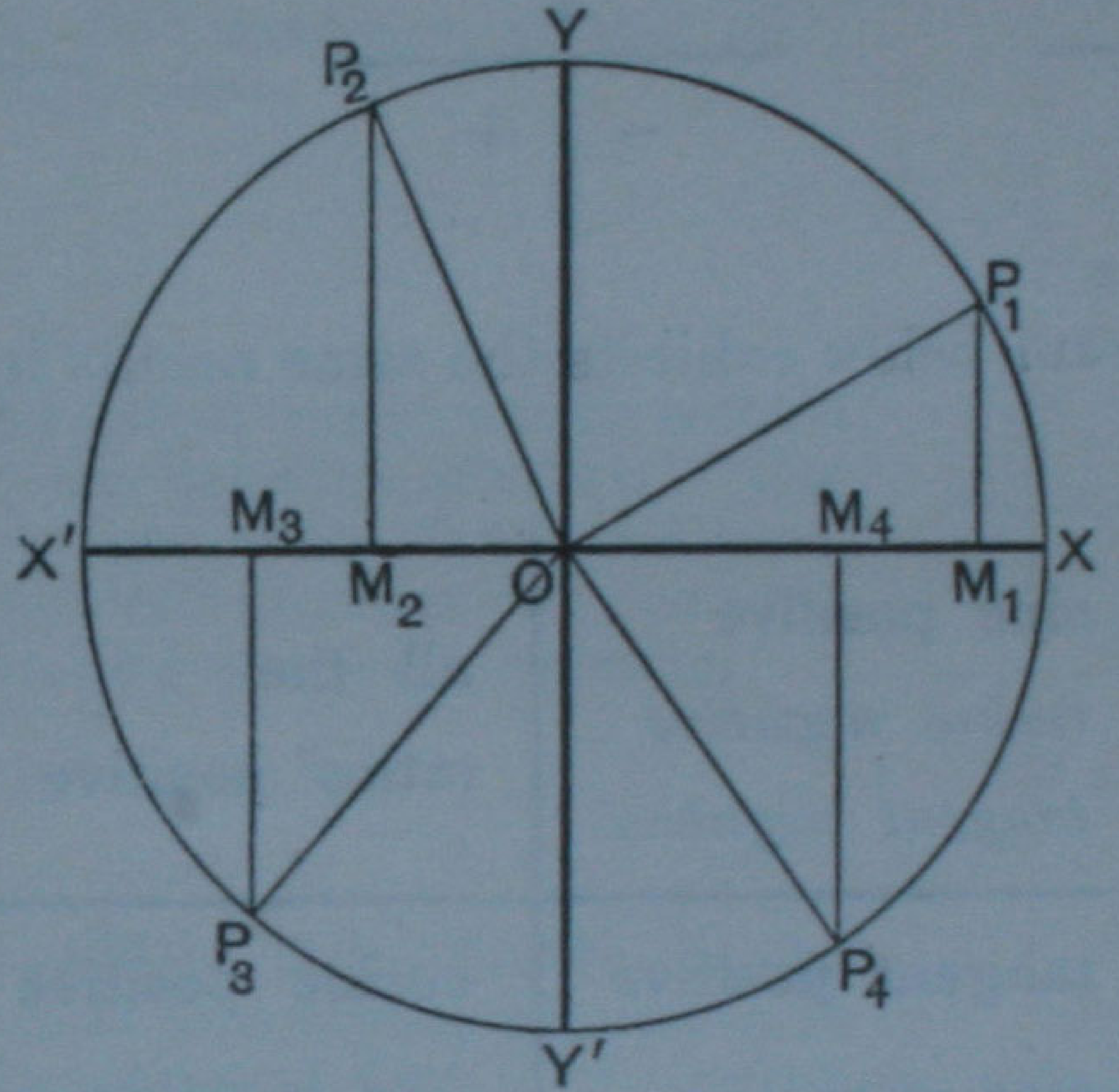
$$\cos A = \frac{OM}{OP}, \quad \sec A = \frac{OP}{OM},$$

$$\tan A = \frac{MP}{OM}, \quad \cot A = \frac{OM}{MP}.$$

The radius vector OP which only fixes the boundary of the angle is considered to be always positive.

From these definitions it will be seen that any trigonometrical function will be positive or negative according as the fraction which expresses its value has the numerator and denominator of the same sign or of opposite sign.

76. The four diagrams of the last article may be conveniently included in one.



With centre O and fixed radius let a circle be described; then the diameters XX' and YY' divide the circle into four *quadrants* XOY, YOX', $X'OY'$, $Y'OX$, named *first, second, third, fourth* respectively.

Let the positions of the radius vector in the four quadrants be denoted by OP_1, OP_2, OP_3, OP_4, and let perpendiculars P_1M_1, P_2M_2, P_3M_3, P_4M_4 be drawn to XX'; then it will be seen that in the first quadrant all the lines are positive and therefore all the functions of A are positive.

In the second quadrant, OP_2 and M_2P_2 are positive, OM_2 is negative; hence sin A is positive, cos A and tan A are negative.

In the third quadrant, OP_3 is positive, OM_3 and M_3P_3 are negative; hence tan A is positive, sin A and cos A are negative.

In the fourth quadrant, OP_4 and OM_4 are positive, M_4P_4 is negative; hence cos A is positive, sin A and tan A are negative.

77. The following diagrams shew the *signs* of the trigonometrical functions in the four quadrants. It will be sufficient to consider the three principal functions only.

sine

+	+
−	−

cosine

−	+
−	+

tangent

−	+
+	−

The diagram below exhibits the same results in another useful form.

sine positive *cosine negative* *tangent negative*	**all the ratios positive**
tangent positive *sine negative* *cosine negative*	**cosine positive** *sine negative* *tangent negative*

78. When an angle is increased or diminished by any multiple of four right angles, the radius vector is brought back again into the same position after one or more revolutions. There are thus an infinite number of angles which have the same boundary line. Such angles are called **coterminal angles.**

If n is *any* integer, all the angles coterminal with A may be represented by $n \,.\, 360^\circ + A$. Similarly, in radian measure all the angles coterminal with θ may be represented by $2n\pi + \theta$.

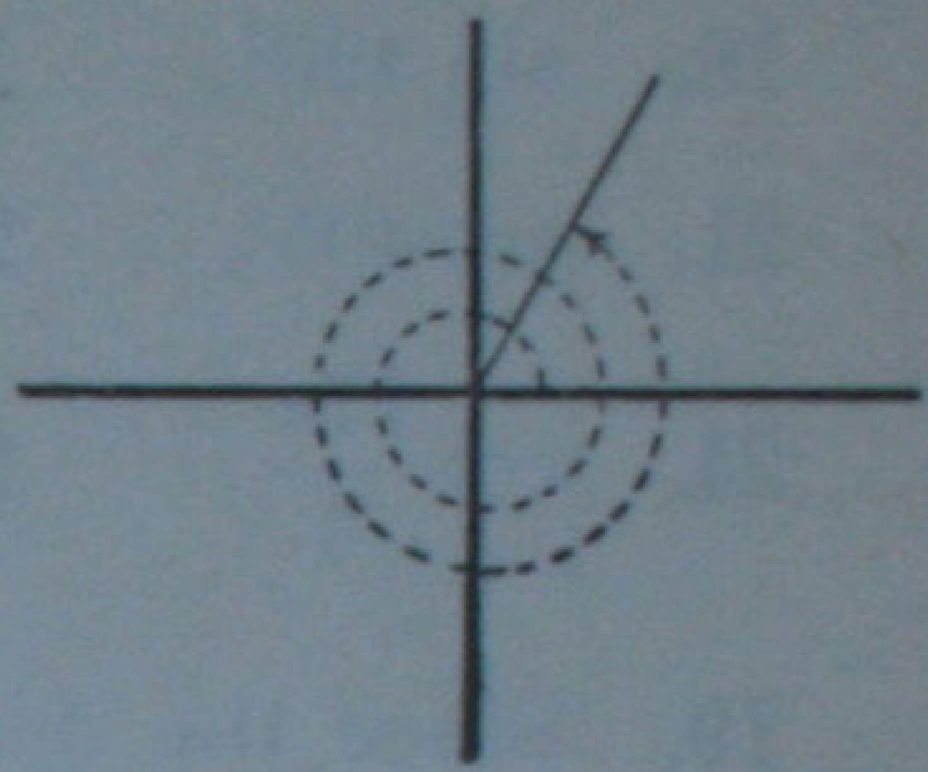
From the definitions of Art. 75, we see that the position of the boundary line is alone sufficient to determine the trigonometrical ratios of the angle; hence *all coterminal angles have the same trigonometrical ratios.*

For instance, $\sin(n \,.\, 360^\circ + 45^\circ) = \sin 45^\circ = \frac{1}{\sqrt{2}}$;

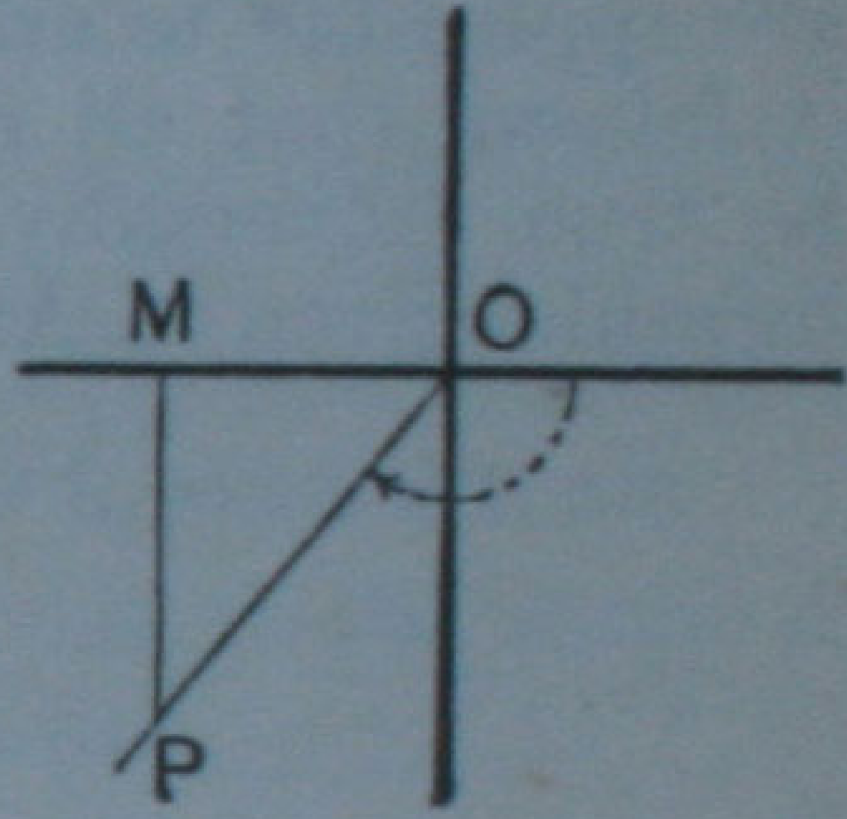
and $$\cos\left(2n\pi + \frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

Example. Draw the boundary lines of the angles 780°, $-130°$, $-400°$, and in each case state which of the trigonometrical functions are negative.

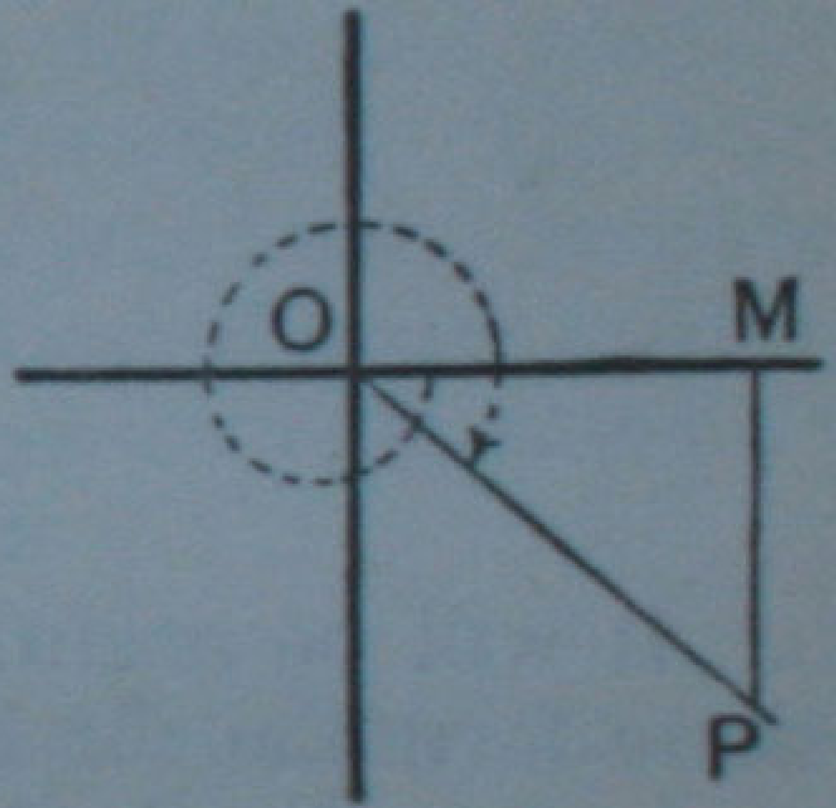
(1) Since $780 = (2 \times 360) + 60$, the radius vector has to make two complete revolutions and then turn through 60°. Thus the boundary line is in the first quadrant, so that all the functions are positive.



(2) Here the radius vector has to revolve through 130° in the negative direction. The boundary line is thus in the third quadrant, and since OM and MP are negative, the sine, cosine, cosecant, and secant are negative.



(3) Since $-400 = -(360 + 40)$, the radius vector has to make one complete revolution in the negative direction and then turn through 40°. The boundary line is thus in the fourth quadrant, and since MP is negative, the sine, tangent, cosecant, and cotangent are negative.



EXAMPLES. VIII. a.

State the quadrant in which the radius vector lies after describing the following angles:

1. 135°. 2. 265°. 3. $-315°$. 4. $-120°$.

5. $\frac{2\pi}{3}$. 6. $\frac{5\pi}{6}$. 7. $\frac{10\pi}{3}$. 8. $-\frac{11\pi}{4}$.

For each of the following angles state which of the three principal trigonometrical functions are positive.

9. 470°. 10. 330°. 11. 575°.

12. $-230°$. 13. $-620°$. 14. $-1200°$.

15. $-\frac{4\pi}{3}$. 16. $\frac{13\pi}{6}$. 17. $-\frac{13\pi}{6}$.

In each of the following cases write down the smallest positive coterminal angle, and the value of the expression.

18. $\sin 420°$. **19.** $\cos 390°$. **20.** $\tan(-315°)$.

21. $\sec 405°$. **22.** $\operatorname{cosec}(-330°)$. **23.** $\operatorname{cosec} 4380°$.

24. $\cot \frac{17\pi}{4}$. **25.** $\sec \frac{25\pi}{3}$. **26.** $\tan\left(-\frac{5\pi}{3}\right)$.

79. Since the definitions of the functions given in Art. 75 are applicable to angles of any magnitude, positive or negative, it follows that all relations derived from these definitions must be true universally. Thus we shall find that the fundamental formulæ given in Art. 29 hold in all cases; that is,

$$\sin A \times \operatorname{cosec} A = 1, \quad \cos A \times \sec A = 1, \quad \tan A \times \cot A = 1;$$

$$\tan A = \frac{\sin A}{\cos A}, \qquad \cot A = \frac{\cos A}{\sin A};$$

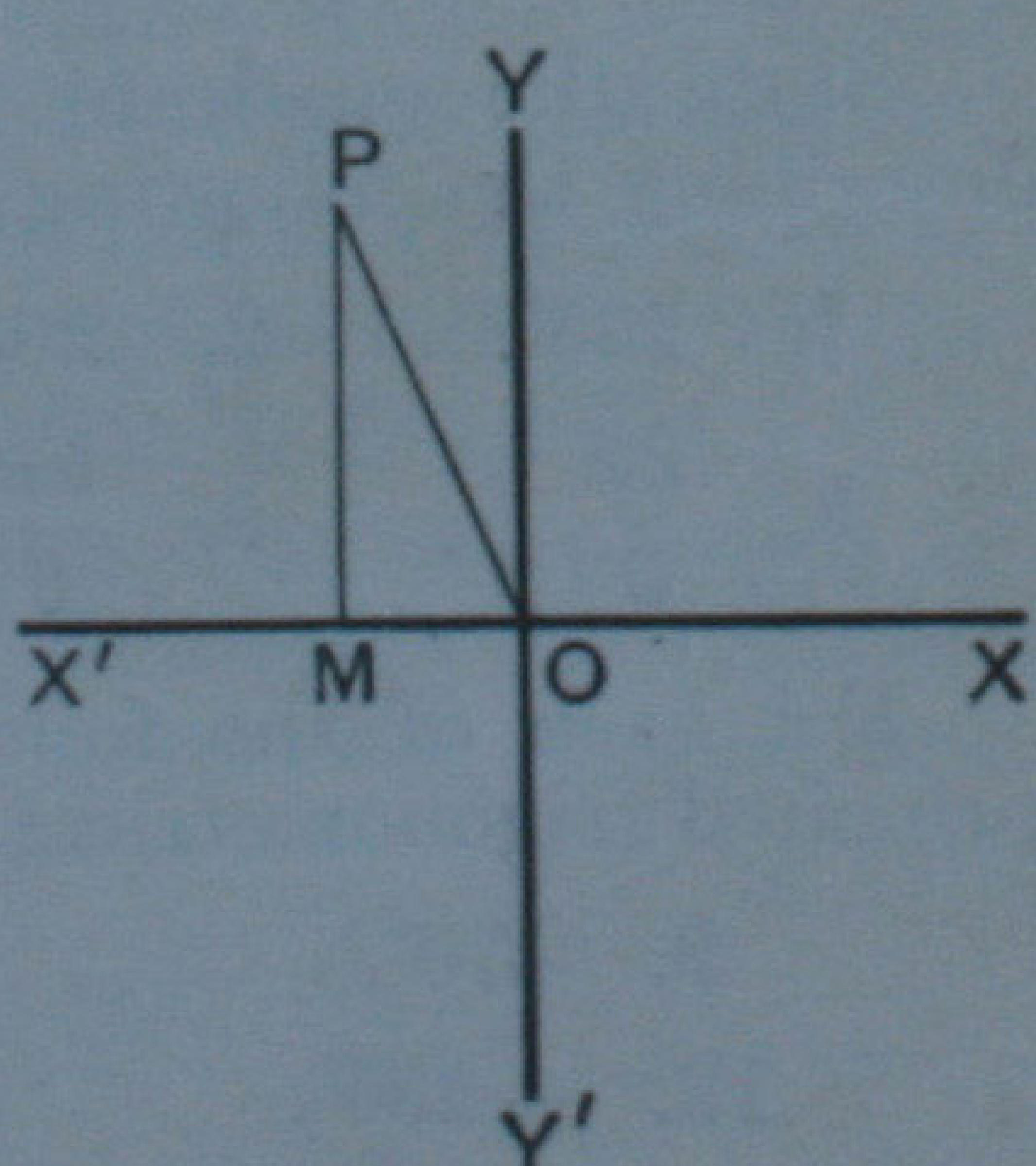
$$\sin^2 A + \cos^2 A = 1,$$

$$1 + \tan^2 A = \sec^2 A,$$

$$1 + \cot^2 A = \operatorname{cosec}^2 A.$$

It will be useful practice for the student to test the truth of these formulæ for different positions of the boundary line of the angle A. We shall give one illustration.

80. Let the radius vector revolve from its initial position OX till it has traced out an angle A and come into the position OP indicated in the figure. Draw PM perpendicular to XX'.



In the right-angled triangle OMP,

$$MP^2 + OM^2 = OP^2 \quad \text{.........(1).}$$

Divide each term by OP^2; thus

$$\left(\frac{MP}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2 = 1;$$

that is, $$\sin^2 A + \cos^2 A = 1.$$

Divide each term of (1) by OM^2; thus

$$\left(\frac{MP}{OM}\right)^2+1=\left(\frac{OP}{OM}\right)^2;$$

that is,
$$\tan^2 A+1=\sec^2 A.$$

Divide each term of (1) by MP^2; thus

$$1+\left(\frac{OM}{MP}\right)^2=\left(\frac{OP}{MP}\right)^2;$$

that is,
$$1+\cot^2 A=\text{cosec}^2 A.$$

It thus appears that the truth of these relations depends only on the statement $OP^2=MP^2+OM^2$ in the right-angled triangle OMP, and this will be the case in whatever quadrant OP lies.

NOTE. OM^2 is positive, although the line OM in the figure is negative.

81. In the statement $\cos A=\sqrt{1-\sin^2 A}$, either the positive or the negative sign may be placed before the radical. The sign of the radical hitherto has always been taken positively, because we have restricted ourselves to the consideration of acute angles. It will sometimes be necessary to examine which sign must be taken before the radical in any particular case.

Example 1. Given $\cos 126° 53'=-\frac{3}{5}$, find $\sin 126° 53'$ and $\cot 126° 53'$.

Since $\sin^2 A+\cos^2 A=1$ for angles of any magnitude, we have

$$\sin A=\pm\sqrt{1-\cos^2 A}.$$

Denote $126° 53'$ by A; then the boundary line of A lies in the second quadrant, and therefore $\sin A$ is positive. Hence the sign + must be placed before the radical;

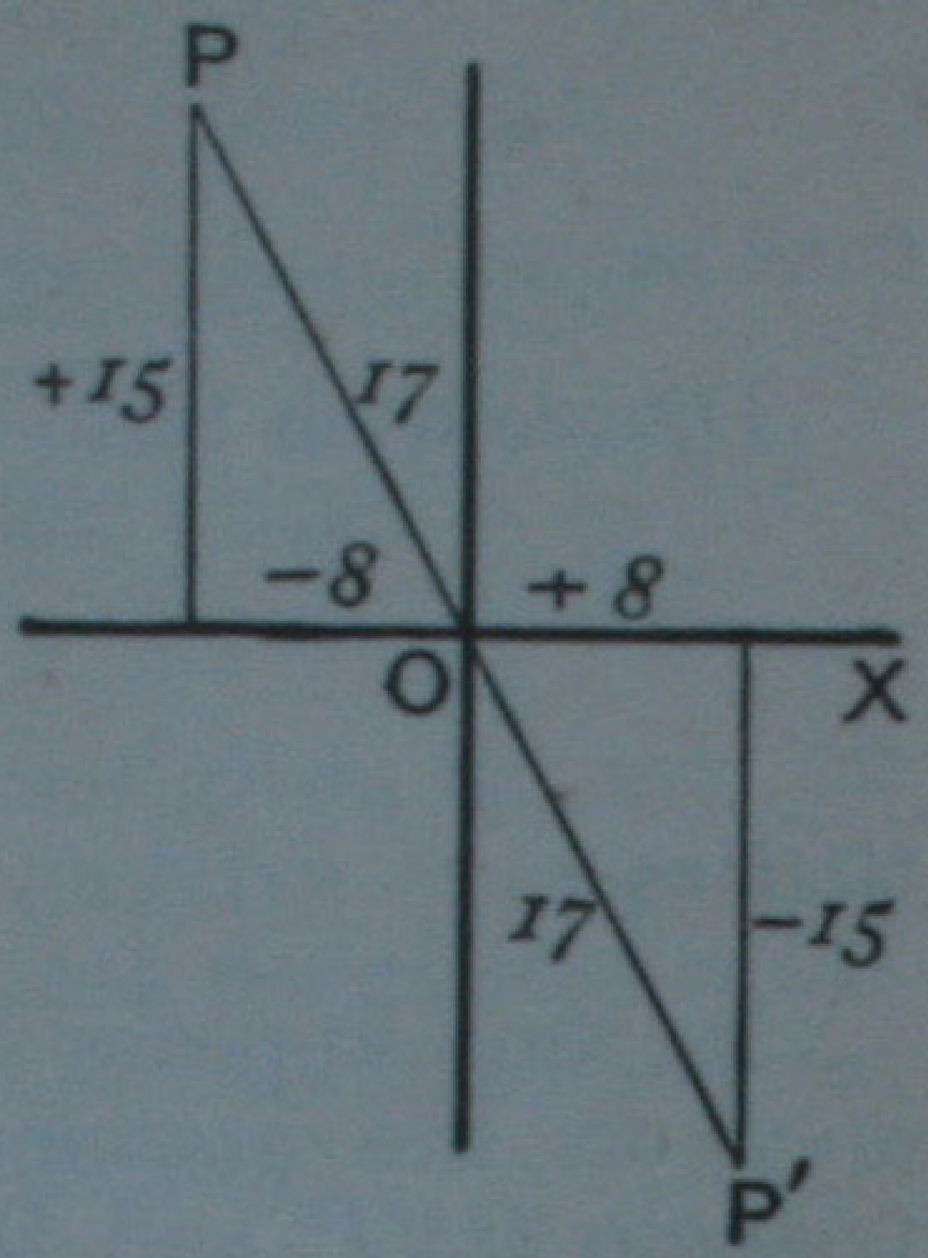
$$\therefore \sin 126° 53'=+\sqrt{1-\frac{9}{25}}=+\sqrt{\frac{16}{25}}=\frac{4}{5};$$

$$\cot 126° 53'=\frac{\cos 126° 53'}{\sin 126° 53'}=\left(-\frac{3}{5}\right)\div\left(\frac{4}{5}\right)=-\frac{3}{4}.$$

The same results may also be obtained by the method used in the following example. The appropriate signs of the lines are shewn in the figure.

Example 2. If $\tan A = -\frac{15}{8}$, find $\sin A$ and $\cos A$.

The boundary line of A will lie either in the second or in the fourth quadrant, as OP or OP'. In either position,



the radius vector $= \sqrt{(15)^2 + (8)^2}$

$= \sqrt{289} = 17.$

Hence $\sin XOP = \frac{15}{17}$, $\cos XOP = -\frac{8}{17}$;

and $\sin XOP' = -\frac{15}{17}$, $\cos XOP' = \frac{8}{17}$.

Thus corresponding to $\tan A$, there are two values of $\sin A$ and two values of $\cos A$. If however it is known in which quadrant the boundary line of A lies, $\sin A$ and $\cos A$ have each a single value.

EXAMPLES. VIII. b.

1. Given $\sin 120° = \frac{\sqrt{3}}{2}$, find $\tan 120°$.

2. Given $\tan 135° = -1$, find $\sin 135°$.

3. Find $\cos 240°$, given that $\tan 240° = \sqrt{3}$.

4. If $A = 202° 37'$ and $\sin A = -\frac{5}{13}$, find $\cos A$ and $\cot A$.

5. If $A = 143° 8'$ and $\operatorname{cosec} A = 1\frac{2}{3}$, find $\sec A$ and $\tan A$.

6. If $A = 216° 52'$ and $\cos A = -\frac{4}{5}$, find $\cot A$ and $\sin A$.

7. Given $\sec \frac{2\pi}{3} = -2$, find $\sin \frac{2\pi}{3}$ and $\cot \frac{2\pi}{3}$.

8. Given $\sin \frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$, find $\tan \frac{5\pi}{4}$ and $\sec \frac{5\pi}{4}$.

9. If $\cos A = \frac{12}{13}$, find $\sin A$ and $\tan A$.

CHAPTER IX.

VARIATIONS OF THE TRIGONOMETRICAL FUNCTIONS.

82. A CAREFUL perusal of the following remarks will render the explanations which follow more easily intelligible.

Consider the fraction $\frac{a}{x}$ in which the numerator a has a *certain fixed value* and the denominator x is a *quantity subject to change;* then it is clear that the smaller x becomes the larger does the value of the fraction $\frac{a}{x}$ become. For instance

$$\frac{a}{\frac{1}{10}}=10a, \quad \frac{a}{\frac{1}{1000}}=1000a, \quad \frac{a}{\frac{1}{10000000}}=10000000a.$$

By making the denominator x sufficiently small the value of the fraction $\frac{a}{x}$ can be made as large as we please; that is, as x approaches to the value 0, the fraction $\frac{a}{x}$ becomes infinitely great.

The symbol ∞ is used to express a quantity infinitely great, or more shortly *infinity*, and the above statement is concisely written

$$\textit{when } x=0, \textit{ the limit of } \frac{a}{x}=\infty.$$

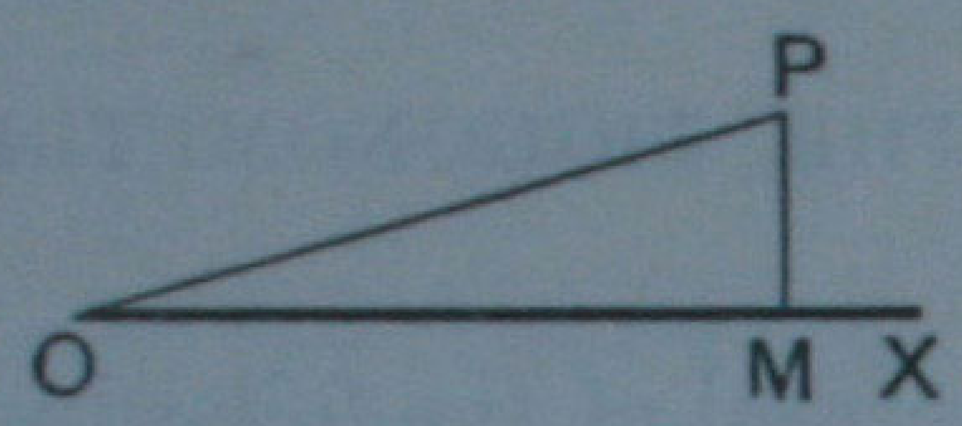
Again, if x is a quantity which gradually increases and finally becomes infinitely large, the fraction $\frac{a}{x}$ becomes infinitely small; that is,

$$\textit{when } x=\infty, \textit{ the limit of } \frac{a}{x}=0.$$

83. Definition. If y is a function of x, and if when x approaches nearer and nearer to the fixed quantity a, the value of y approaches nearer and nearer to the fixed quantity b and can be made to differ from it by as little as we please, then b is called the **limiting value** or the **limit** of y when $x=a$.

84. Trigonometrical Functions of 0°.

Let XOP be an angle traced out by a radius vector OP of fixed length.



Draw PM perpendicular to OX; then

$$\sin POM = \frac{MP}{OP}.$$

If we suppose the angle POM to be gradually decreasing, MP will also gradually decrease, and if OP ultimately come into coincidence with OM the angle POM vanishes and $MP=0$.

Hence $$\sin 0° = \frac{0}{OP} = 0.$$

Again, $\cos POM = \frac{OM}{OP}$; but when the angle POM vanishes OP becomes coincident with OM.

Hence $$\cos 0° = \frac{OM}{OM} = 1.$$

Also when the angle POM vanishes,

$$\tan 0° = \frac{0}{OM} = 0.$$

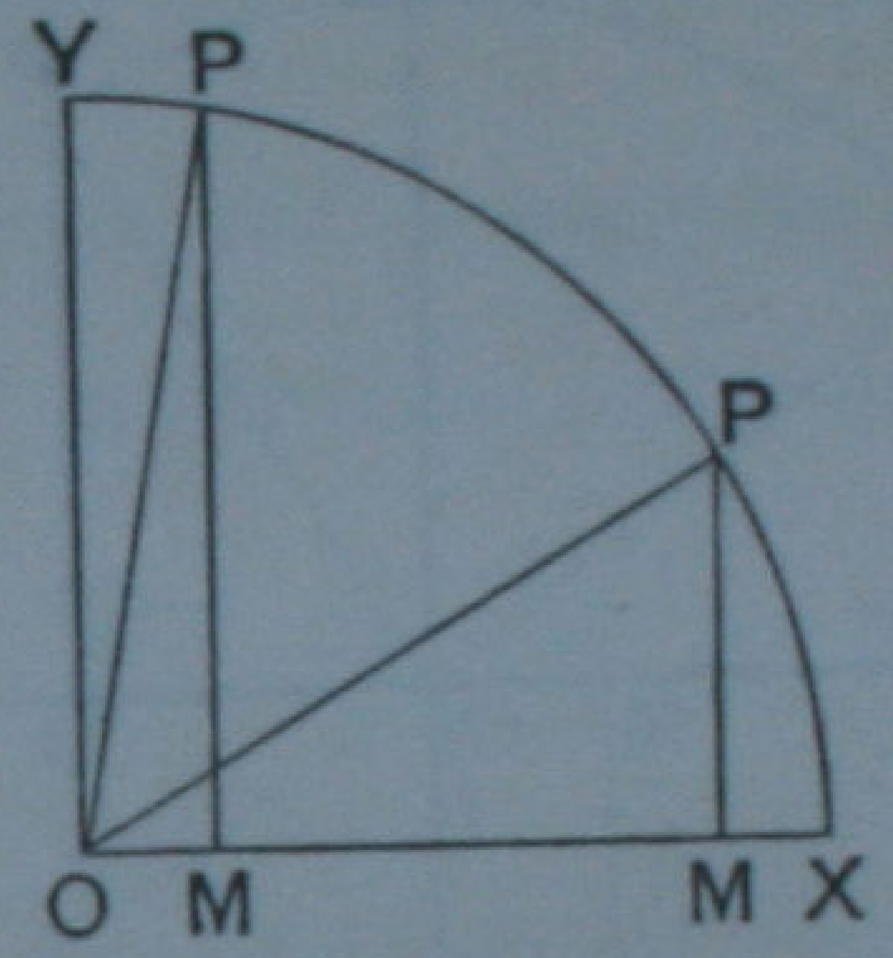
And $$\operatorname{cosec} 0° = \frac{1}{\sin 0°} = \frac{1}{0} = \infty;$$

$$\sec 0° = \frac{1}{\cos 0°} = \frac{1}{1} = 1;$$

$$\cot 0° = \frac{1}{\tan 0°} = \frac{1}{0} = \infty.$$

85. Trigonometrical Functions of 90° or $\frac{\pi}{2}$.

Let XOP be an angle traced out by a radius vector of fixed length.



Draw PM perpendicular to OX, and OY perpendicular to OX.

By definition,

$$\sin POM = \frac{MP}{OP}, \quad \cos POM = \frac{OM}{OP}, \quad \tan POM = \frac{MP}{OM}.$$

If we suppose the angle POM to be gradually increasing, MP will gradually increase and OM decrease. When OP comes into coincidence with OY the angle POM becomes equal to 90°, and OM vanishes, while MP becomes equal to OP.

Hence $$\sin 90° = \frac{OP}{OP} = 1;$$

$$\cos 90° = \frac{0}{OP} = 0;$$

$$\tan 90° = \frac{MP}{OM} = \frac{OP}{0} = \infty.$$

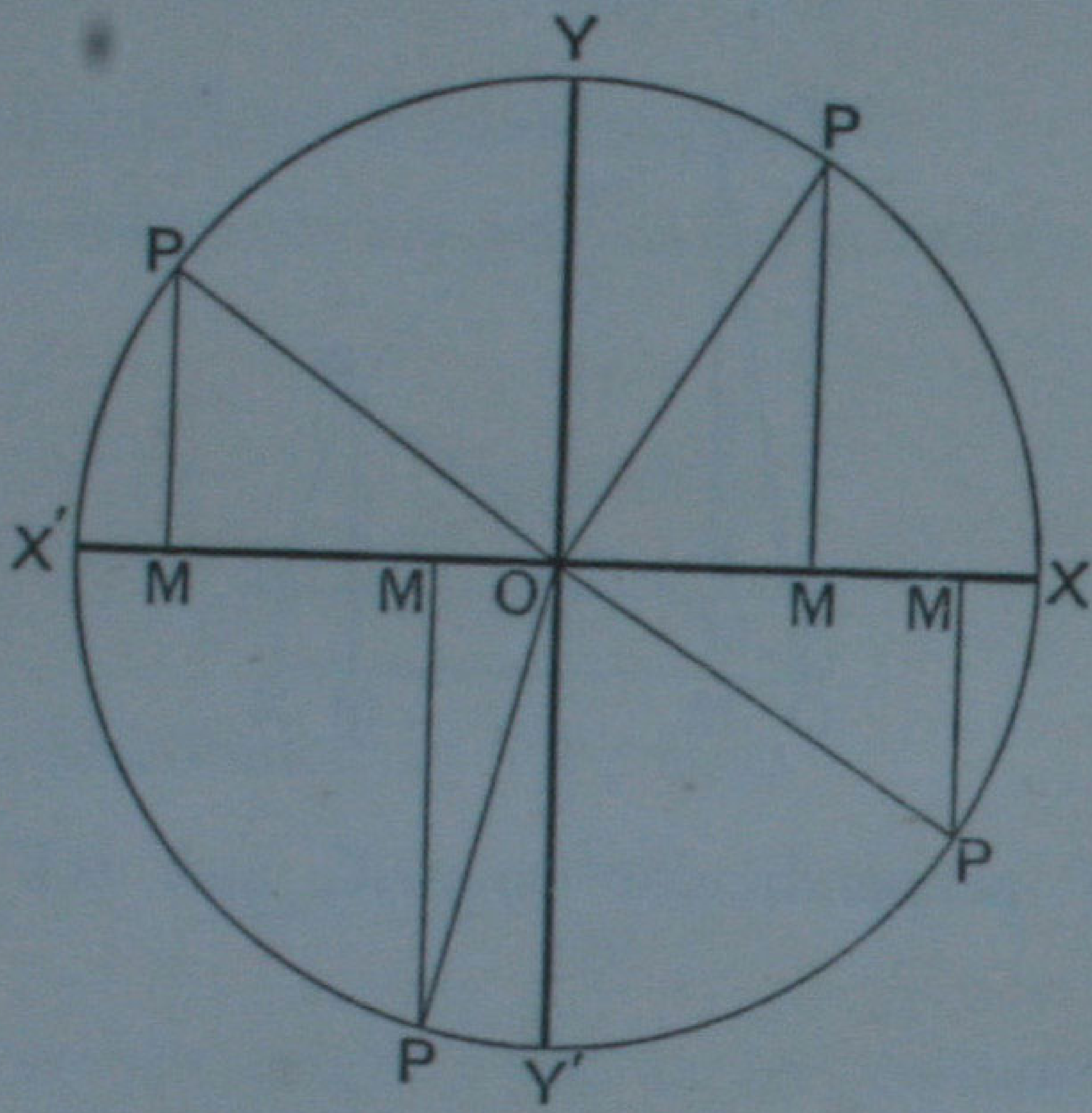
And $$\cot 90° = \frac{1}{\tan 90°} = \frac{1}{\infty} = 0;$$

$$\sec 90° = \frac{1}{\cos 90°} = \frac{1}{0} = \infty;$$

$$\operatorname{cosec} 90° = \frac{1}{\sin 90°} = 1.$$

86. *To trace the changes in sign and magnitude of* sin A *as* A *increases from* 0° *to* 360°.

Let XX' and YY' be two straight lines intersecting at right angles in O.



With centre O and any radius OP describe a circle, and suppose the angle A to be traced out by the revolution of OP through the four quadrants starting from OX.

Draw PM perpendicular to OX and let $OP=r$; then

$$\sin A = \frac{MP}{r},$$

and since r does not alter in sign or magnitude, we have only to consider the changes of MP as P moves round the circle.

When $A=0°$, $MP=0$, and $\sin 0° = \frac{0}{r} = 0$.

In the first quadrant, MP is positive and increasing;

$\therefore$ sin A is positive and increasing.

When $A=90°$, $MP=r$, and $\sin 90° = \frac{r}{r} = 1$.

In the second quadrant, MP is positive and decreasing;

$\therefore$ sin A is positive and decreasing.

When $A=180°$, $MP=0$, and $\sin 180° = \frac{0}{r} = 0$.

In the third quadrant, MP is negative and increasing;

$\therefore$ $\sin A$ is negative and increasing.

When $A=270°$, MP is equal to r, but is negative; hence

$$\sin 270° = -\frac{r}{r} = -1.$$

In the fourth quadrant, MP is negative and decreasing;

$\therefore$ $\sin A$ is negative and decreasing.

When $A=360°$, $MP=0$, and $\sin 360° = \frac{0}{r} = 0$.

87. The results of the previous article are concisely shewn in the following diagram:

sin 90° = 1

	sin A positive and decreasing	*sin A positive and increasing*	
sin 180° = 0			*sin 0° = 0*
	sin A negative and increasing	*sin A negative and decreasing*	

sin 270° = −1

88. We leave as an exercise to the student the investigation of the changes in sign and magnitude of $\cos A$ as A increases from 0° to 360°. The following diagram exhibits these changes.

cos 90° = 0

	cos A negative and increasing	*cos A positive and decreasing*	
cos 180° = −1			*cos 0° = 1*
	cos A negative and decreasing	*cos A positive and increasing*	

cos 270° = 0

89. *To trace the changes in sign and magnitude of* tan A *as* A *increases from* 0° *to* 360°.

With the figure of Art. 86, $\tan A = \frac{MP}{OM}$, and its changes will therefore depend on those of MP and OM.

When $A = 0°$, $MP = 0$, $OM = r$; $\therefore \tan 0° = \frac{0}{r} = 0$.

In the first quadrant,

MP is positive and increasing,
OM is positive and decreasing;
$\therefore$ tan A is positive and increasing.

When $A = 90°$, $MP = r$, $OM = 0$; $\therefore \tan 90° = \frac{r}{0} = \infty$.

In the second quadrant,

MP is positive and decreasing,
OM is negative and increasing;
$\therefore$ tan A is negative and decreasing.

When $A = 180°$, $MP = 0$; $\therefore \tan 180° = 0$.

In the third quadrant,

MP is negative and increasing,
OM is negative and decreasing;
$\therefore$ tan A is positive and increasing.

When $A = 270°$, $OM = 0$; $\therefore \tan 270° = \infty$.

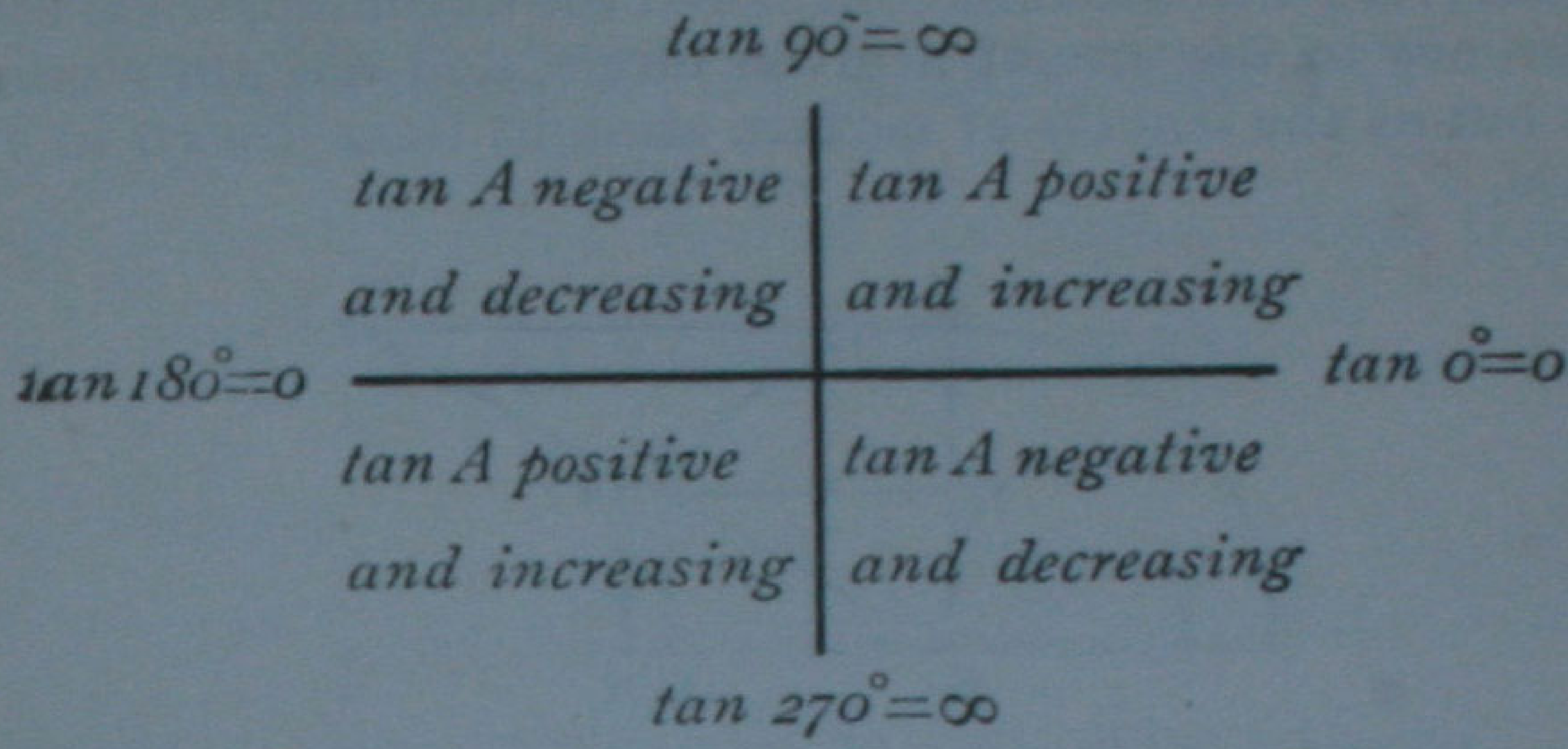
In the fourth quadrant,

MP is negative and decreasing,
OM is positive and increasing;
$\therefore$ tan A is negative and decreasing.

When $A = 360°$, $MP = 0$; $\therefore \tan 360° = 0$.

NOTE. When the numerator of a fraction changes continually from a small positive to a small negative quantity the fraction changes sign by passing through the value 0. When the denominator changes continually from a small positive to a small negative quantity the fraction changes sign by passing through the value ∞. For instance, as A passes through the value 90°, OM changes from a small positive to a small negative quantity, hence $\frac{OM}{OP}$, that is $\cos A$, changes sign by passing through the value 0, while $\frac{PM}{OM}$, that is $\tan A$, changes sign by passing through the value ∞.

90. The results of Art. 89 are shewn in the following diagram :



The student will now have no difficulty in tracing the variations in sign and magnitude of the other functions.

91. In Arts. 86 and 89 we have seen that the variations of the trigonometrical functions of the angle XOP depend on the position of P as P moves round the circumference of the circle. On this account the trigonometrical functions of an angle are called **circular functions.** This name is one that we shall use frequently.

EXAMPLES. IX.

Trace the changes in sign and magnitude of

1. $\cot A$, between $0°$ and $360°$.

2. $\operatorname{cosec}\theta$, between 0 and π.

3. $\cos\theta$, between π and 2π.

4. $\tan A$, between $-90°$ and $-270°$.

5. $\sec\theta$, between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

Find the value of

6. $\cos 0° \sin^2 270° - 2\cos 180° \tan 45°$.

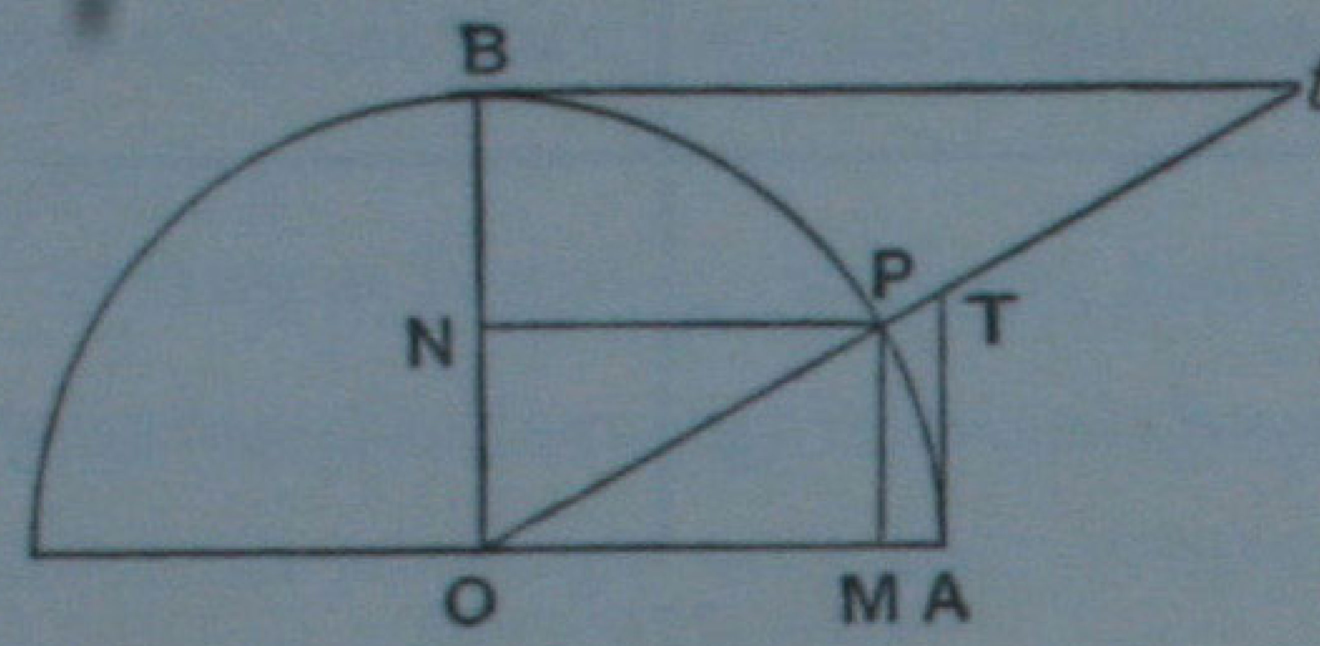
7. $3\sin 0° \sec 180° + 2\operatorname{cosec} 90° - \cos 360°$.

8. $2\sec^2\pi \cos 0 + 3\sin^3\frac{3\pi}{2} - \operatorname{cosec}\frac{\pi}{2}$.

9. $\tan\pi \cos\frac{3\pi}{2} + \sec 2\pi - \operatorname{cosec}\frac{3\pi}{2}$.

Note on the old definitions of the Trigonometrical Functions.

Formerly, Mathematicians considered the trigonometrical functions with reference to the *arc* of a given circle, and did not regard them as *ratios* but as the *lengths* of certain straight lines drawn in relation to this arc.



Let OA and OB be two radii of a circle at right angles, and let P be any point on the circumference. Draw PM and PN perpendicular to OA and OB respectively, and let the tangents at A and B meet OP produced in T and t respectively.

The *lines* PM, AT, OT, AM were named respectively the sine, tangent, secant, versed-sine of the arc AP, and PN, Bt, Ot, BN, which are the sine, tangent, secant, versed-sine of the complementary arc BP, were named respectively the cosine, cotangent, cosecant, coversed-sine of the arc AP.

As thus defined each trigonometrical function of the *arc* is equal to the corresponding function of the *angle*, which it subtends at the centre of the circle, multiplied by the radius. Thus

$$\frac{AT}{OA} = \tan POA; \text{ that is, } AT = OA \times \tan POA;$$

and

$$\frac{Ot}{OB} = \sec BOP = \operatorname{cosec} POA; \text{ that is, } Ot = OB \times \operatorname{cosec} POA.$$

The values of the functions of the arc therefore depended on the length of the radius of the circle as well as on the angle subtended by the arc at the centre of the circle, so that in Tables of the functions it was necessary to state the magnitude of the radius.

The names of the trigonometrical functions and the abbreviations for them now in use were introduced by different Mathematicians chiefly towards the end of the sixteenth and during the seventeenth century, but were not generally employed until their re-introduction by Euler. The development of the science of Trigonometry may be considered to date from the publication in 1748 of Euler's *Introductio in analysin Infinitorum.*

The reader will find some interesting information regarding the progress of Trigonometry in Ball's *Short History of Mathematics.*

MISCELLANEOUS EXAMPLES. C.

1. Draw the boundary lines of the angles whose tangent is equal to $-\frac{3}{4}$, and find the cosine of these angles.

2. Shew that

$$\cos A\,(2\sec A + \tan A)(\sec A - 2\tan A) = 2\cos A - 3\tan A.$$

3. Given $C=90^\circ$, $b=10{\cdot}5$, $c=21$, solve the triangle.

4. If $\sec A = -\frac{25}{7}$, and A lies between 180° and 270°, find $\cot A$.

5. The latitude of Bombay is 19° N.: find its distance from the equator, taking the diameter of the earth to be 7920 miles.

6. From the top of a cliff 200 ft. high, the angles of depression of two boats due east of the observer are $34^\circ\,30'$ and $18^\circ\,40'$: find their distance apart, given

$$\cot 34^\circ\,30' = 1{\cdot}455, \qquad \cot 18^\circ\,40' = 2{\cdot}96.$$

7. If A lies between 180° and 270°, and $3\tan A = 4$, find the value of $2\cot A - 5\cos A + \sin A$.

8. Find, correct to three decimal places, the radius of a circle in which an arc 15 inches long subtends at the centre an angle of $71^\circ\,36'\,3{\cdot}6''$.

9. Shew that

$$\frac{\tan^3\theta}{1+\tan^2\theta} + \frac{\cot^3\theta}{1+\cot^2\theta} = \frac{1-2\sin^2\theta\cos^2\theta}{\sin\theta\cos\theta}.$$

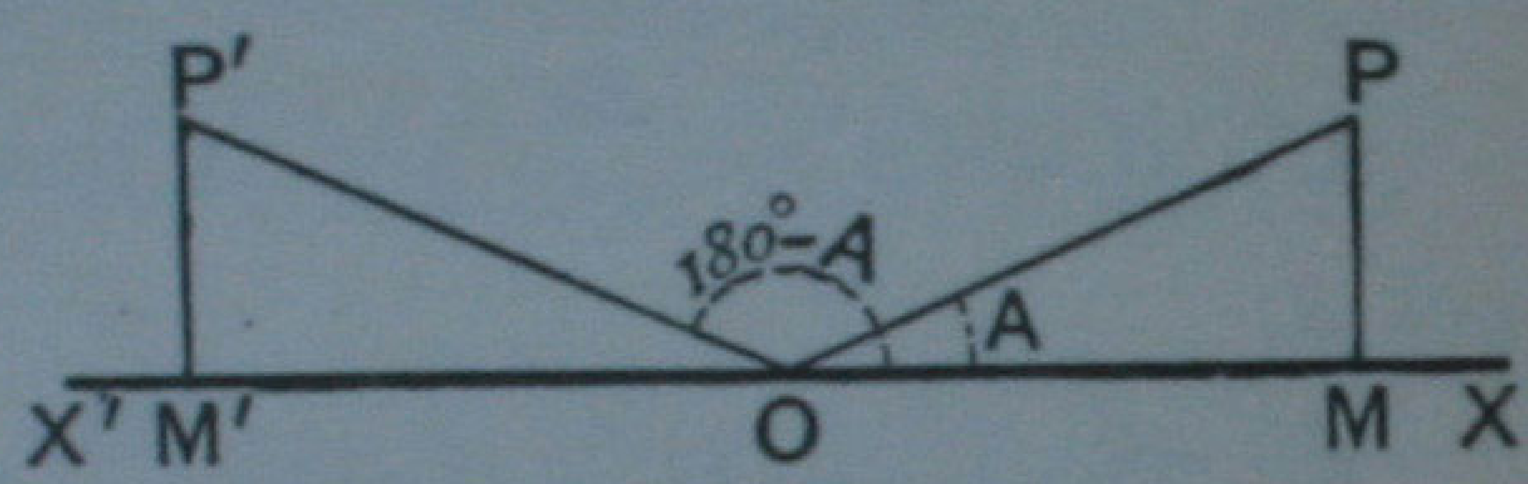
10. The angle of elevation of the top of a tower is $68^\circ\,11'$, and a flagstaff 24 ft. high on the summit of the tower subtends an angle of $2^\circ\,10'$ at the observer's eye. Find the height of the tower, given

$$\tan 70^\circ\,21' = 2{\cdot}8, \qquad \cot 68^\circ\,11' = {\cdot}4.$$

CHAPTER X.

CIRCULAR FUNCTIONS OF CERTAIN ALLIED ANGLES.

92. Circular Functions of 180° – A.



Take any straight line XOX', and let a radius vector starting from OX revolve until it has traced the angle A, taking up the position OP.

Again, let the radius vector starting from OX revolve through 180° into the position OX' and then *back again* through an angle A taking up the final position OP'. Thus XOP' is the angle $180°-A$.

From P and P' draw PM and $P'M'$ perpendicular to XX'; then by Euc. I. 26 the triangles OPM and $OP'M'$ are geometrically equal.

By definition,

$$\sin(180°-A)=\frac{M'P'}{OP'};$$

but $M'P'$ is equal to MP in magnitude and is of the same sign;

$$\therefore\ \sin(180°-A)=\frac{MP}{OP}=\sin A.$$

Again, $$\cos(180°-A)=\frac{OM'}{OP'};$$

and OM' is equal to OM in magnitude, but is of opposite sign;

$$\therefore\ \cos(180°-A)=\frac{-OM}{OP}=-\frac{OM}{OP}=-\cos A.$$

Also $$\tan(180°-A)=\frac{M'P'}{OM'}=\frac{MP}{-OM}=-\frac{MP}{OM}=-\tan A.$$

93. In the last article, for the sake of simplicity we have supposed the angle A to be less than a right angle, but all the formulæ of this chapter may be shewn to be true for angles of any magnitude. A general proof of one case is given in Art. 102, and the same method may be applied to all the other cases.

94. If the angles are expressed in radian measure, the formulæ of Art. 92 become

$$\sin(\pi-\theta)=\sin\theta,$$

$$\cos(\pi-\theta)=-\cos\theta,$$

$$\tan(\pi-\theta)=-\tan\theta.$$

Example 1. Find the sine and cosine of 120°.

$$\sin 120^\circ=\sin(180^\circ-60^\circ)=\sin 60^\circ=\frac{\sqrt{3}}{2}.$$

$$\cos 120^\circ=\cos(180^\circ-60^\circ)=-\cos 60^\circ=-\frac{1}{2}.$$

Example 2. Find the cosine and cotangent of $\frac{5\pi}{6}$.

$$\cos\frac{5\pi}{6}=\cos\left(\pi-\frac{\pi}{6}\right)=-\cos\frac{\pi}{6}=-\frac{\sqrt{3}}{2}.$$

$$\cot\frac{5\pi}{6}=\cot\left(\pi-\frac{\pi}{6}\right)=-\cot\frac{\pi}{6}=-\sqrt{3}.$$

95. DEFINITION. When the sum of two angles is equal to two right angles each is said to be the **supplement** of the other and the angles are said to be **supplementary**. Thus if A is any angle its supplement is $180^\circ-A$.

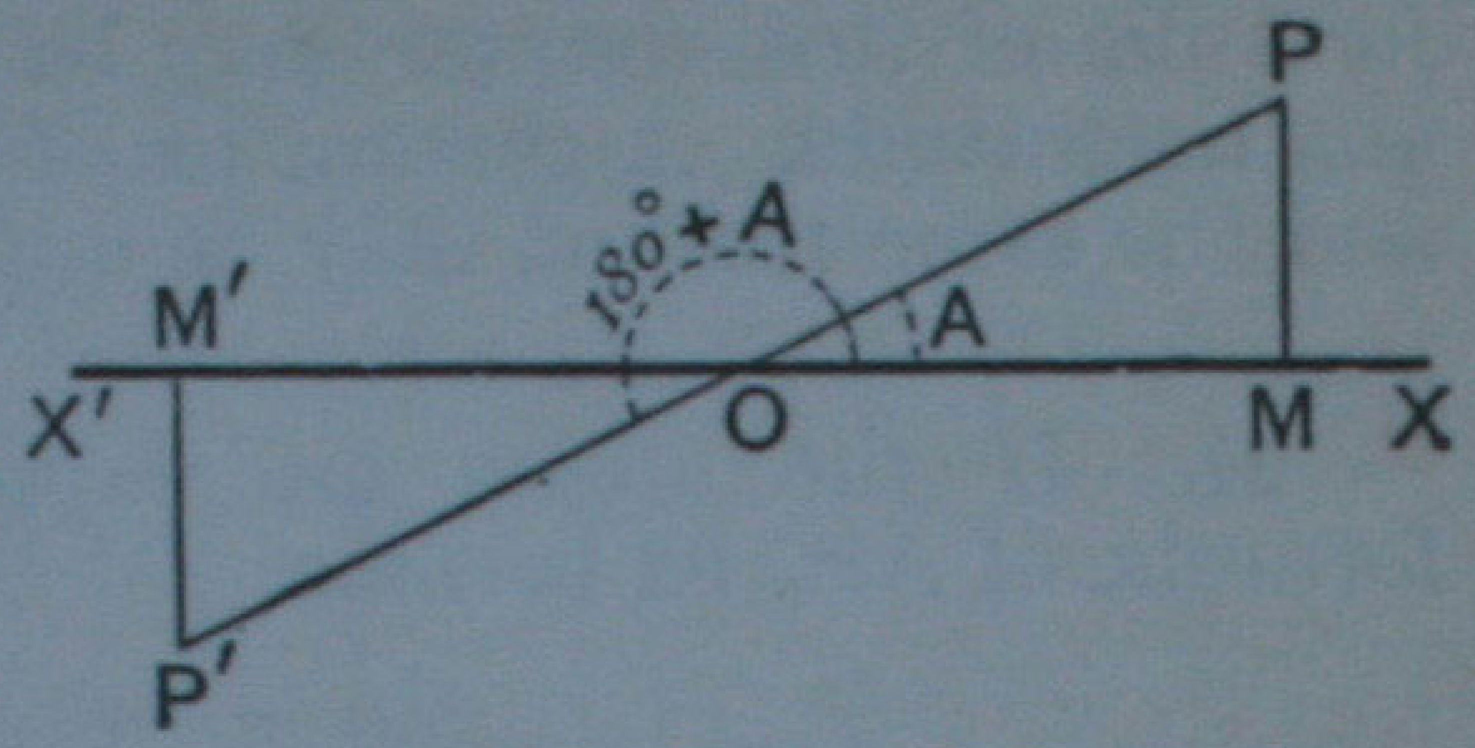
96. The results of Art. 92 are so important in a later part of the subject that it is desirable to emphasize them. We therefore repeat them in a verbal form:

the sines of supplementary angles are equal in magnitude and are of the same sign;

the cosines of supplementary angles are equal in magnitude but are of opposite sign;

the tangents of supplementary angles are equal in magnitude but are of opposite sign.

97. Circular Functions of 180°+A.



Take any straight line XOX' and let a radius vector starting from OX revolve until it has traced the angle A, taking up the position OP.

Again, let the radius vector starting from OX revolve through 180° into the position OX', and then further through an angle A, taking up the final position OP'. Thus XOP' is the angle $180°+A$.

From P and P' draw PM and $P'M'$ perpendicular to XX'; then OP and OP' are in the same straight line, and by Euc. I. 26 the triangles OPM and $OP'M'$ are geometrically equal.

By definition,

$$\sin(180°+A)=\frac{M'P'}{OP'};$$

and $M'P'$ is equal to MP in magnitude but is of opposite sign;

$$\therefore \sin(180°+A)=\frac{-MP}{OP}=-\frac{MP}{OP}=-\sin A.$$

Again, $$\cos(180°+A)=\frac{OM'}{OP'};$$

and OM' is equal to OM in magnitude but is of opposite sign;

$$\therefore \cos(180°+A)=\frac{-OM}{OP}=-\frac{OM}{OP}=-\cos A.$$

Also $$\tan(180°+A)=\frac{M'P'}{OM'}=\frac{-MP}{-OM}=\frac{MP}{OM}=\tan A.$$

Expressed in radian measure, the above formulæ are written

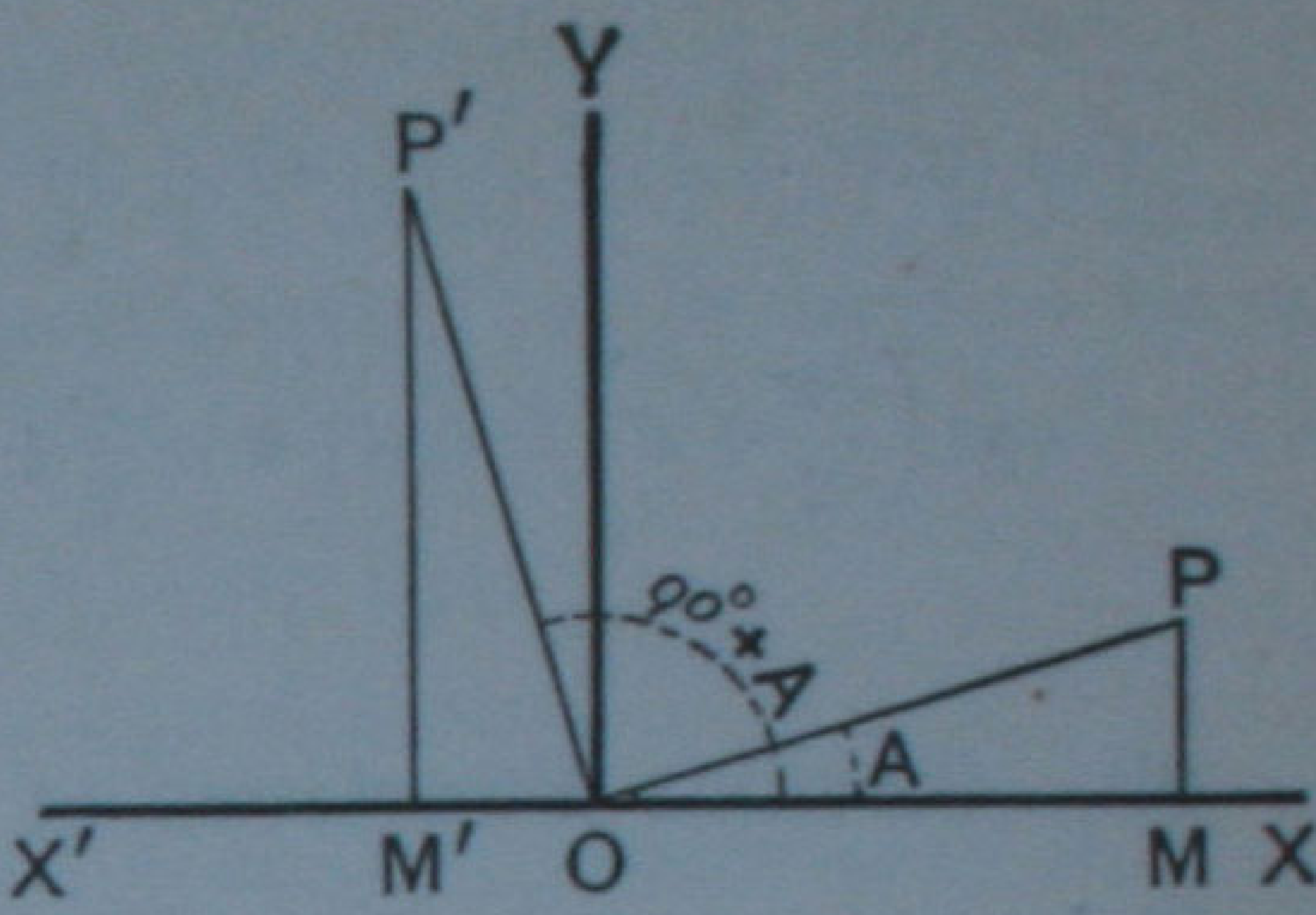
$$\sin(\pi+\theta)=-\sin\theta,\quad \cos(\pi+\theta)=-\cos\theta,\quad \tan(\pi+\theta)=\tan\theta.$$

In these results we may draw especial attention to the fact that an angle may be increased or diminished by two right angles as often as we please without altering the value of the tangent.

Example. Find the value of cot 210°.

$$\cot 210°=\cot(180°+30°)=\cot 30°=\sqrt{3}.$$

98. Circular Functions of 90°+A.



Take any straight line XOX', and let a radius vector starting from OX revolve until it has traced the angle A, taking up the position OP.

Again, let the radius vector starting from OX revolve through 90° into the position OY, and then further through an angle A, taking up the final position OP'. Thus XOP' is the angle $90° + A$.

From P and P' draw PM and $P'M'$ perpendicular to XX'; then $\angle M'P'O = \angle P'OY = A = \angle POM$.

By Euc. I. 26, the triangles OPM and $OP'M'$ are geometrically equal; hence

$M'P'$ is equal to OM in magnitude and is of the same sign,

and OM' is equal to MP in magnitude but is of opposite sign.

By definition,

$$\sin(90° + A) = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos A;$$

$$\cos(90° + A) = \frac{OM'}{OP'} = \frac{-MP}{OP} = -\frac{MP}{OP} = -\sin A;$$

$$\tan(90° + A) = \frac{M'P'}{OM'} = \frac{OM}{-MP} = -\frac{OM}{MP} = -\cot A.$$

Expressed in radian measure the above formulæ become

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta, \quad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta, \quad \tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta.$$

Example 1. Find the value of sin 120°.

$$\sin 120° = \sin(90° + 30°) = \cos 30° = \frac{\sqrt{3}}{2}.$$

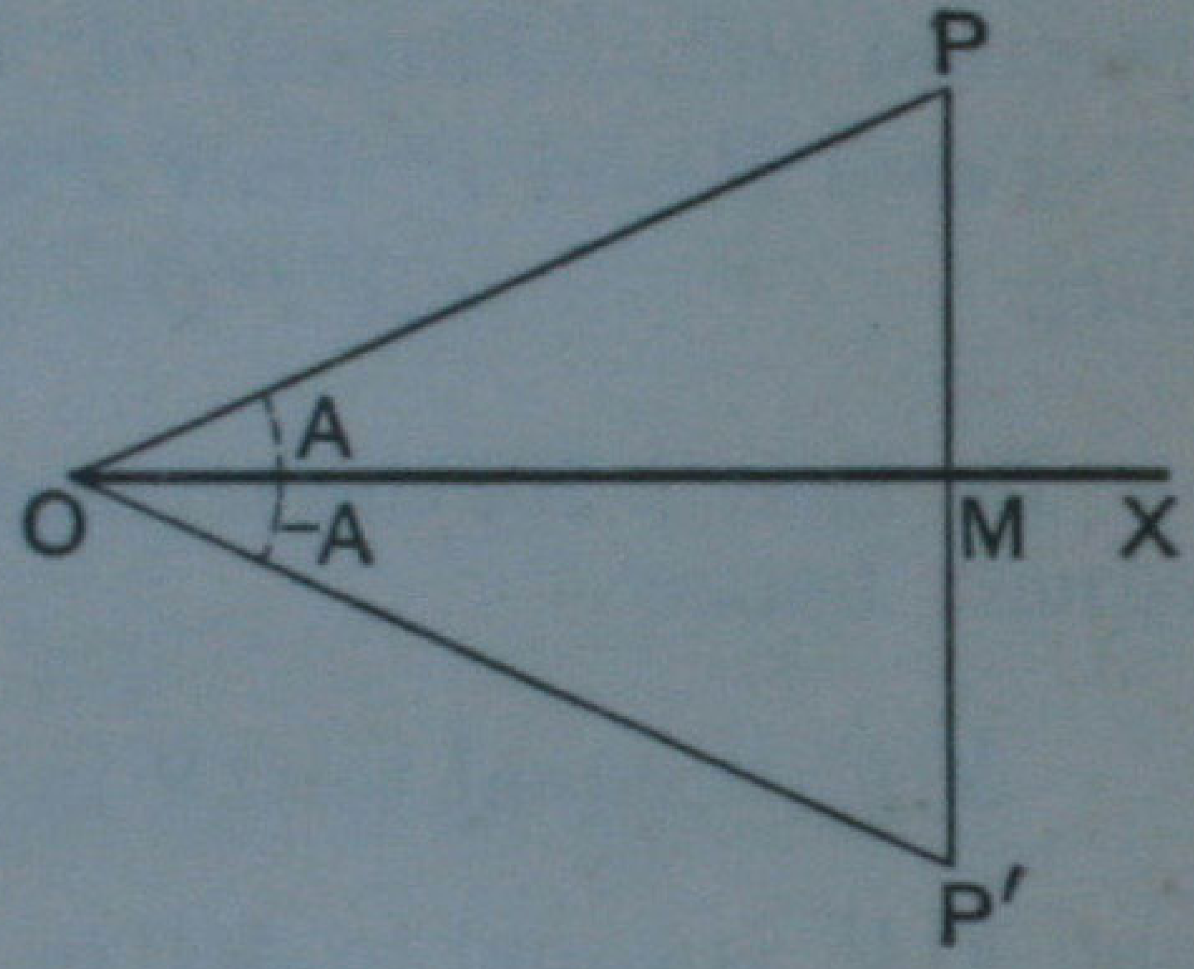
Example 2. Find the values of $\tan(270°+A)$ and $\cos\left(\frac{3\pi}{2}+\theta\right)$.

$$\tan(270°+A)=\tan(180°+\overline{90°+A})=\tan(90°+A)=-\cot A;$$

$$\cos\left(\frac{3\pi}{2}+\theta\right)=\cos\left(\pi+\overline{\frac{\pi}{2}+\theta}\right)=-\cos\left(\frac{\pi}{2}+\theta\right)=\sin\theta.$$

99. Circular Functions of $-A$.

Take any straight line OX and let a radius vector starting from OX revolve until it has traced the angle A, taking up the position OP.



Again, let the radius vector starting from OX revolve in the *opposite* direction until it has traced the angle A, taking up the position OP'. Join PP'; then MP' is equal to MP in magnitude, and the angles at M are right angles. [Euc. I. 4.]

By definition,

$$\sin(-A)=\frac{MP'}{OP'}=\frac{-MP}{OP}=-\sin A;$$

$$\cos(-A)=\frac{OM}{OP'}=\frac{OM}{OP}=\cos A;$$

$$\tan(-A)=\frac{MP'}{OM}=\frac{-MP}{OM}=-\tan A.$$

It is especially worthy of notice that *we may change the sign of an angle without altering the value of its cosine.*

Example. Find the values of

$$\operatorname{cosec}(-210°) \text{ and } \cos(A-270°).$$

$$\operatorname{cosec}(-210°)=-\operatorname{cosec}210°=-\operatorname{cosec}(180°+30°)=\operatorname{cosec}30°=2.$$

$$\cos(A-270°)=\cos(270°-A)=\cos(180°+\overline{90°-A})$$

$$=-\cos(90°-A)=-\sin A.$$

100. If $f(A)$ denotes a function of A which is unaltered in magnitude and sign when $-A$ is written for A, then $f(A)$ is said to be an **even function** of A. In this case $f(-A)=f(A)$.

If when $-A$ is written for A, the sign of $f(A)$ is changed while the magnitude remains unaltered, $f(A)$ is said to be an **odd function** of A, and in this case $f(-A)=-f(A)$.

From the last article it will be seen that

$\cos A$ *and* $\sec A$ *are even functions of* A,

$\sin A$, $\operatorname{cosec} A$, $\tan A$, $\cot A$ *are odd functions of* A.

EXAMPLES. X. a.

Find the numerical value of

1. $\cos 135°$. **2.** $\sin 150°$. **3.** $\tan 240°$.

4. $\operatorname{cosec} 225°$. **5.** $\sin(-120°)$. **6.** $\cot(-135°)$.

7. $\cot 315°$. **8.** $\cos(-240°)$. **9.** $\sec(-300°)$.

10. $\tan \frac{3\pi}{4}$. **11.** $\sin \frac{4\pi}{3}$. **12.** $\sec \frac{2\pi}{3}$.

13. $\operatorname{cosec}\left(-\frac{\pi}{6}\right)$. **14.** $\cos\left(-\frac{3\pi}{4}\right)$. **15.** $\cot\left(-\frac{5\pi}{6}\right)$.

Express as functions of A:

16. $\cos(270°+A)$. **17.** $\cot(270°-A)$. **18.** $\sin(A-90°)$.

19. $\sec(A-180°)$. **20.** $\sin(270°-A)$. **21.** $\cot(A-90°)$.

Express as functions of θ:

22. $\sin\left(\theta-\frac{\pi}{2}\right)$. **23.** $\tan(\theta-\pi)$. **24.** $\sec\left(\frac{3\pi}{2}-\theta\right)$.

Express in the simplest form:

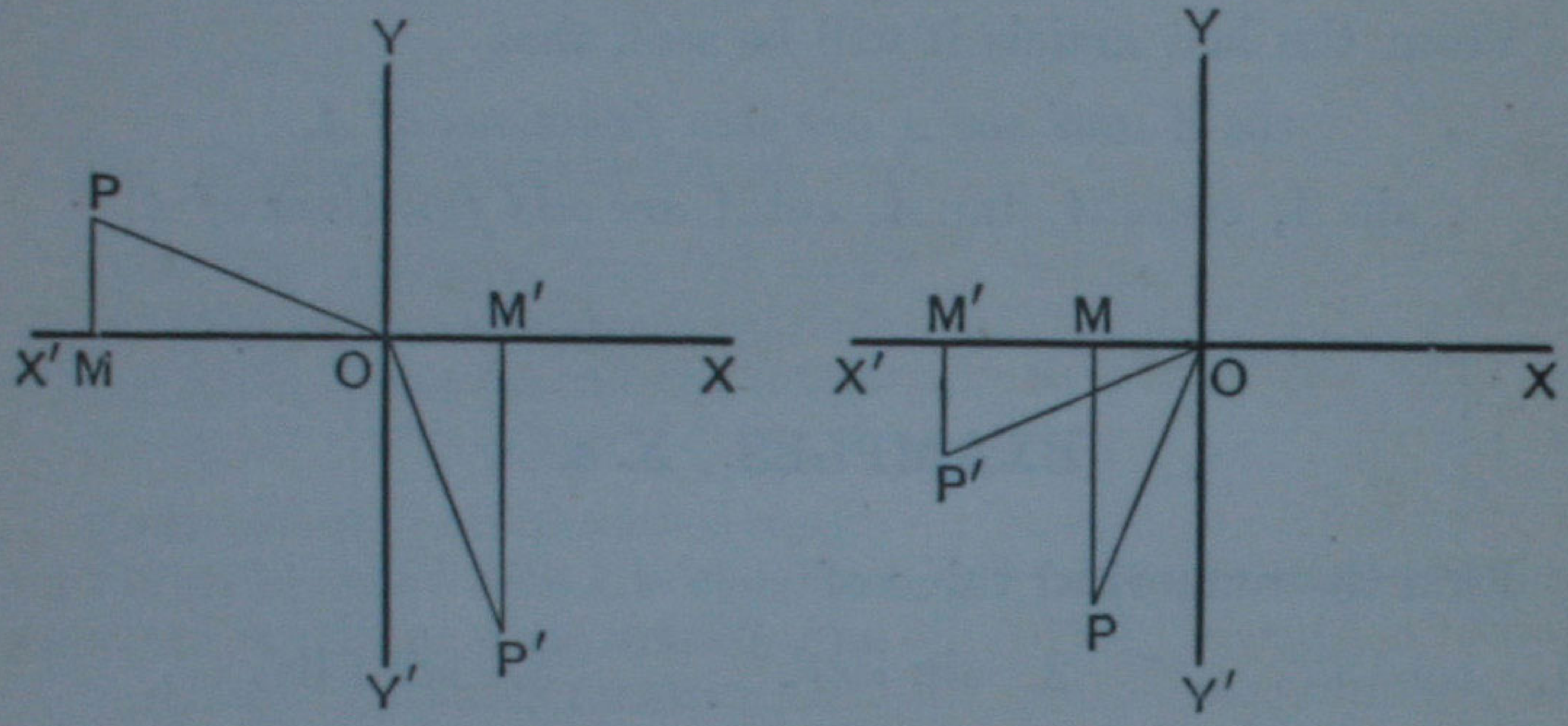
25. $\tan(180°+A)\sin(90°+A)\sec(90°-A)$.

26. $\cos(90°+A)+\sin(180°-A)-\sin(180°+A)-\sin(-A)$.

27. $\sec(180°+A)\sec(180°-A)+\cot(90°+A)\tan(180°+A)$.

101. In Art. 38 we have established the relations which subsist between the trigonometrical ratios of $90° - A$ and those of A, when A is an acute angle. We shall now give a general proof which is applicable whatever be the magnitude of A.

102. Circular Functions of 90° – A for any value of A.



Let a radius vector starting from OX revolve until it has traced the angle A, taking up the position OP in each of the two figures.

Again, let the radius vector starting from OX revolve through 90° into the position OY and then *back again* through an angle A, taking up the final position OP' in each of the two figures.

Draw PM and $P'M'$ perpendicular to XX'; then whatever be the value of A, it will be found that $\angle OP'M' = \angle POM$, so that the triangles OMP and $OM'P'$ are geometrically equal, having MP equal to OM', and OM equal to $M'P'$, *in magnitude*.

When P is above XX', P' is to the right of YY',
and when P is below XX', P' is to the left of YY'.

When P' is above XX', P is to the right of YY',
and when P' is below XX', P is to the left of YY'.

Hence MP is equal to OM' in magnitude and is always of the same sign as OM';

and $M'P'$ is equal to OM in magnitude and is always of the same sign as OM.

By definition,

$$\sin(90^\circ - A) = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos A;$$

$$\cos(90^\circ - A) = \frac{OM'}{OP'} = \frac{MP}{OP} = \sin A;$$

$$\tan(90^\circ - A) = \frac{M'P'}{OM'} = \frac{OM}{MP} = \cot A.$$

A general method similar to the above may be applied to all the other cases of this chapter.

103. Circular Functions of n . 360° + A.

If n is any integer, $n \,.\, 360^\circ$ represents n complete revolutions of the radius vector, and therefore the boundary line of the angle $n \,.\, 360^\circ + A$ is coincident with that of A. The value of each function of the angle $n \,.\, 360^\circ + A$ is thus the same as the value of the corresponding function of A both in magnitude and in sign.

104. Since the functions of all coterminal angles are equal, there is a *recurrence* of the values of the functions each time the boundary line completes its revolution and comes round into its original position. This is otherwise expressed by saying that *the circular functions are periodic*, and 360° is said to be *the amplitude of the period.*

In radian measure, the amplitude of the period is 2π.

NOTE. In the case of the tangent and cotangent the amplitude of the period is half that of the other circular functions, being 180° or π radians. [Art. 97.]

105. Circular Functions of n . 360° – A.

If n is any integer, the boundary line of $n \,.\, 360^\circ - A$ is coincident with that of $-A$. The value of each function of $n \,.\, 360^\circ - A$ is thus the same as the value of the corresponding function of $-A$ both in magnitude and in sign; hence

$$\sin(n \,.\, 360^\circ - A) = \sin(-A) = -\sin A;$$

$$\cos(n \,.\, 360^\circ - A) = \cos(-A) = \cos A;$$

$$\tan(n \,.\, 360^\circ - A) = \tan(-A) = -\tan A.$$

106. We can always express the functions of any angle in terms of the functions of some positive acute angle. In the arrangement of the work it is advisable to follow a uniform plan.

(1) If the angle is negative, use the relations connecting the functions of $-A$ and A. [Art. 99.]

Thus $$\sin(-30^\circ) = -\sin 30^\circ = -\frac{1}{2};$$

$$\cos(-845^\circ) = \cos 845^\circ.$$

(2) If the angle is greater than 360°, by taking off multiples of 360° the angle may be replaced by a coterminal angle less than 360°. [Art. 103.]

Thus $$\tan 735^\circ = \tan(2\times 360^\circ + 15^\circ) = \tan 15^\circ.$$

(3) If the angle is still greater than 180°, use the relations connecting the functions of $180^\circ + A$ and A. [Art. 97.]

Thus $$\cot 585^\circ = \cot(360^\circ + 225^\circ) = \cot 225^\circ$$
$$= \cot(180^\circ + 45^\circ) = \cot 45^\circ = 1.$$

(4) If the angle is still greater than 90°, use the relations connecting the functions of $180^\circ - A$ and A. [Art. 92.]

Thus $$\cos 675^\circ = \cos(360^\circ + 315^\circ) = \cos 315^\circ$$
$$= \cos(180^\circ + 135^\circ) = -\cos 135^\circ$$
$$= -\cos(180^\circ - 45^\circ) = \cos 45^\circ = \frac{1}{\sqrt{2}}.$$

Example. Express $\sin(-1190^\circ)$, $\tan 1000^\circ$, $\cos(-3860^\circ)$ as functions of positive acute angles.

$$\sin(-1190^\circ) = -\sin 1190^\circ = -\sin(3\times 360^\circ + 110^\circ) = -\sin 110^\circ$$
$$= -\sin(180^\circ - 70^\circ) = -\sin 70^\circ.$$

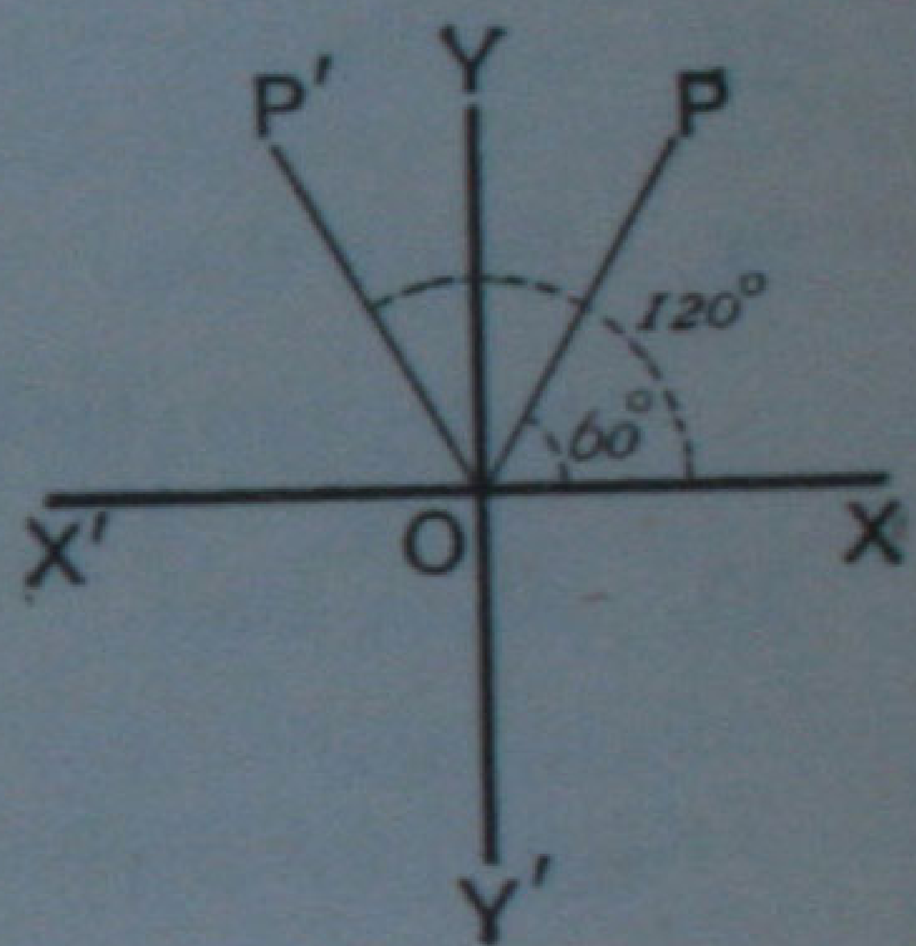
$$\tan 1000^\circ = \tan(2\times 360^\circ + 280^\circ) = \tan 280^\circ$$
$$= \tan(180^\circ + 100^\circ) = \tan 100^\circ$$
$$= \tan(180^\circ - 80^\circ) = -\tan 80^\circ.$$

$$\cos(-3860^\circ) = \cos 3860^\circ = \cos(10\times 360^\circ + 260^\circ) = \cos 260^\circ$$
$$= \cos(180^\circ + 80^\circ) = -\cos 80^\circ.$$

107. From the investigations of this chapter we see that the number of angles which have the same circular function is unlimited. Thus if $\tan\theta=1$, θ may be any one of the angles coterminal with 45° or 225°.

Example. Draw the boundary lines of A when $\sin A=\frac{\sqrt{3}}{2}$, and write down all the angles numerically less than 360° which satisfy the equation.

Since $\sin 60°=\frac{\sqrt{3}}{2}$, if we draw OP making $\angle XOP=60°$, then OP is one position of the boundary line.



Again, $\sin 60°=\sin(180°-60°)=\sin 120°$, so that another position of the boundary line will be found by making $XOP'=120°$.

There will be no position of the boundary line in the third or fourth quadrant, since in these quadrants the sine is negative.

Thus in one complete revolution OP and OP' are the only two positions of the boundary line of the angle A.

Hence the positive angles are 60° and 120°;

and the negative angles are $-(360°-120°)$ and $-(360°-60°)$; that is, $-240°$ and $-300°$.

EXAMPLES. X. b.

Find the numerical value of

1. $\cos 480°$.	**2.** $\sin 960°$.	**3.** $\cos(-780°)$.
4. $\sin(-870°)$.	**5.** $\sec 900°$.	**6.** $\tan(-855°)$.
7. $\operatorname{cosec}(-660°)$.	**8.** $\cot 840°$.	**9.** $\operatorname{cosec}(-765°)$.
10. $\cos 4005°$.	**11.** $\cot 990°$.	**12.** $\sin 3015°$.
13. $\sec 2745°$.	**14.** $\cos 2400°$.	**15.** $\sec(-5895°)$.
16. $\sin\frac{15\pi}{4}$.	**17.** $\cot\frac{23\pi}{4}$.	**18.** $\sec\frac{7\pi}{3}$.
19. $\cot\frac{16\pi}{3}$.	**20.** $\sec\left(\frac{3\pi}{2}+\frac{\pi}{3}\right)$.	

Find all the angles numerically less than 360° which satisfy the equations :

21. $\cos\theta = \frac{\sqrt{3}}{2}$. **22.** $\sin\theta = -\frac{1}{2}$.

23. $\tan\theta = -\sqrt{3}$. **24.** $\cot\theta = -1$.

If A is less than 90°, prove geometrically

25. $\sec(A - 180°) = -\sec A$.

26. $\tan(270° + A) = -\cot A$.

27. $\cos(A - 90°) = \sin A$.

28. Prove that

$$\tan A + \tan(180° - A) + \cot(90° + A) = \tan(360° - A).$$

29. Shew that

$$\frac{\sin(180° - A)}{\tan(180° + A)} \cdot \frac{\cot(90° - A)}{\tan(90° + A)} \cdot \frac{\cos(360° - A)}{\sin(-A)} = \sin A.$$

Express in the simplest form

30. $\frac{\sin(-A)}{\sin(180° + A)} - \frac{\tan(90° + A)}{\cot A} + \frac{\cos A}{\sin(90° + A)}$.

31. $\frac{\operatorname{cosec}(180° - A)}{\sec(180° + A)} \cdot \frac{\cos(-A)}{\cos(90° + A)}$.

32. $\frac{\cos(90° + A)\sec(-A)\tan(180° - A)}{\sec(360° + A)\sin(180° + A)\cot(90° - A)}$.

33. Prove that $\sin\left(\frac{\pi}{2} + \theta\right)\cos(\pi - \theta)\cot\left(\frac{3\pi}{2} + \theta\right)$

$$= \sin\left(\frac{\pi}{2} - \theta\right)\sin\left(\frac{3\pi}{2} - \theta\right)\cot\left(\frac{\pi}{2} + \theta\right).$$

34. When $a = \frac{11\pi}{4}$, find the numerical value of

$$\sin^2 a - \cos^2 a + 2\tan a - \sec^2 a.$$

CHAPTER XI.

FUNCTIONS OF COMPOUND ANGLES.

108. WHEN an angle is made up by the algebraical sum of two or more angles it is called a **compound angle**; thus $A+B$, $A-B$, and $A+B-C$ are compound angles.

109. Hitherto we have only discussed the properties of the functions of single angles, such as A, B, α, θ. In the present chapter we shall prove some fundamental properties relating to the functions of compound angles. We shall begin by finding expressions for the sine, cosine, and tangent of $A+B$ and $A-B$ in terms of the functions of A and B.

It may be useful to caution the student against the prevalent mistake of supposing that a function of $A+B$ is equal to the sum of the corresponding functions of A and B, and a function of $A-B$ to the difference of the corresponding functions.

Thus $\sin(A+B)$ is not equal to $\sin A+\sin B$,

and $\qquad \cos(A-B)$ is not equal to $\cos A-\cos B$.

A numerical instance will illustrate this.

Thus if $A=60°$, $B=30°$, then $A+B=90°$,

so that $$\cos(A+B)=\cos 90°=0;$$

but $$\cos A+\cos B=\cos 60°+\cos 30°=\frac{1}{2}+\frac{\sqrt{3}}{2}.$$

Hence $\cos(A+B)$ is not equal to $\cos A+\cos B$.

In like manner, $\sin(A+A)$ is not equal to $\sin A+\sin A$;

that is, $\qquad \sin 2A$ is not equal to $2\sin A$.

Similarly $\qquad \tan 3A$ is not equal to $3\tan A$.

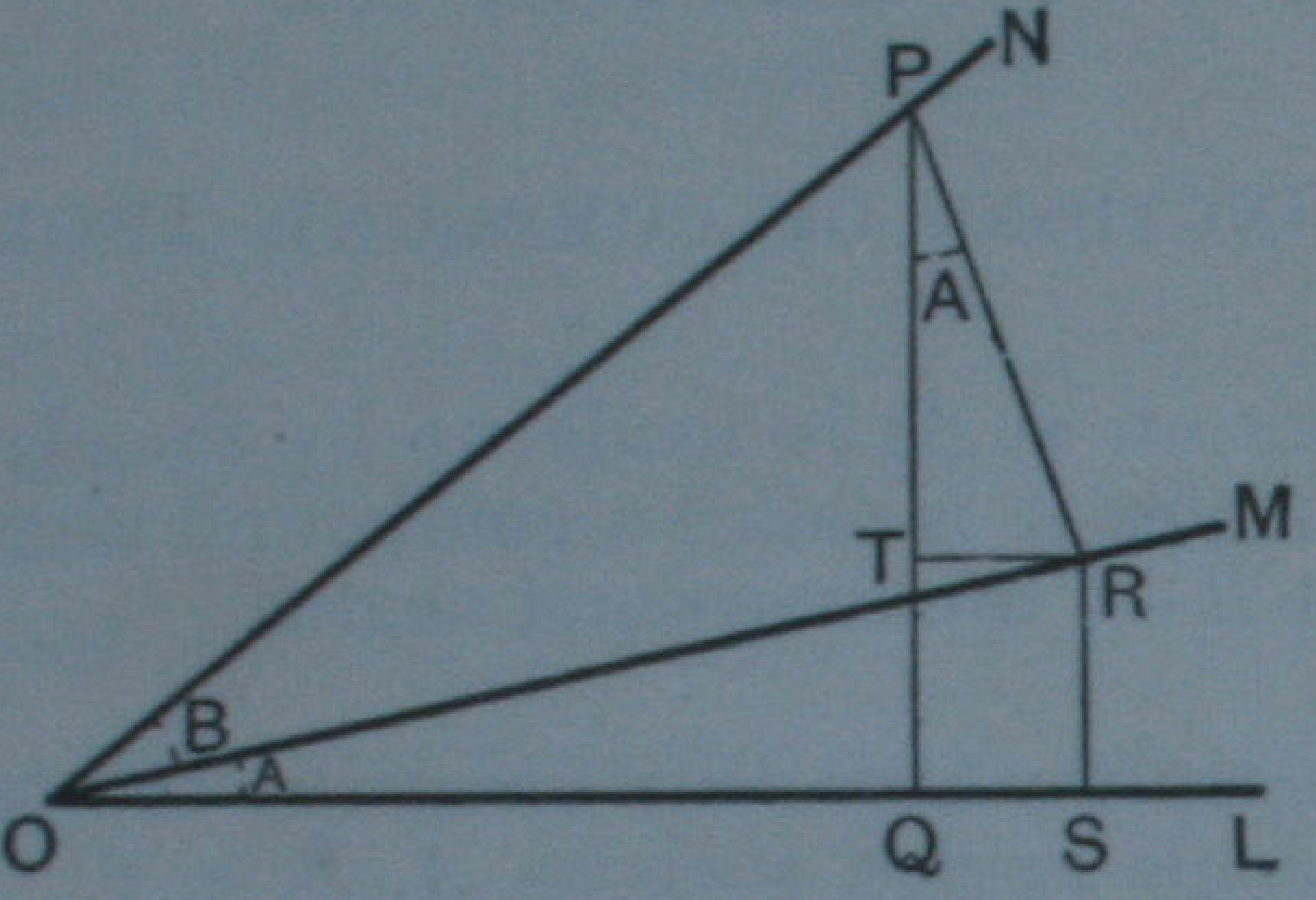
The Addition

110. *To prove the formulæ*

$$\sin(A+B)=\sin A\cos B+\cos A\sin B,$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B.$$

Let $\angle LOM=A$, and $\angle MON=B$; then $\angle LON=A+B$.



In ON, *the boundary line of the compound angle* $A+B$, take any point P, and draw PQ and PR perpendicular to OL and OM respectively; also draw RS and RT perpendicular to OL and PQ respectively.

By definition,

$$\sin(A+B)=\frac{PQ}{OP}=\frac{RS+PT}{OP}=\frac{RS}{OP}+\frac{PT}{OP}$$

$$=\frac{RS}{OR}\cdot\frac{OR}{OP}+\frac{PT}{PR}\cdot\frac{PR}{OP}$$

$$=\sin A\,.\cos B+\cos TPR\,.\sin B.$$

But $\angle TPR=90^\circ-\angle TRP=\angle TRO=\angle ROS=A$;

$$\therefore\ \sin(A+B)=\sin A\cos B+\cos A\sin B.$$

Also $$\cos(A+B)=\frac{OQ}{OP}=\frac{OS-TR}{OP}=\frac{OS}{OP}-\frac{TR}{OP}$$

$$=\frac{OS}{OR}\cdot\frac{OR}{OP}-\frac{TR}{PR}\cdot\frac{PR}{OP}$$

$$=\cos A\,.\cos B-\sin TPR\,.\sin B$$

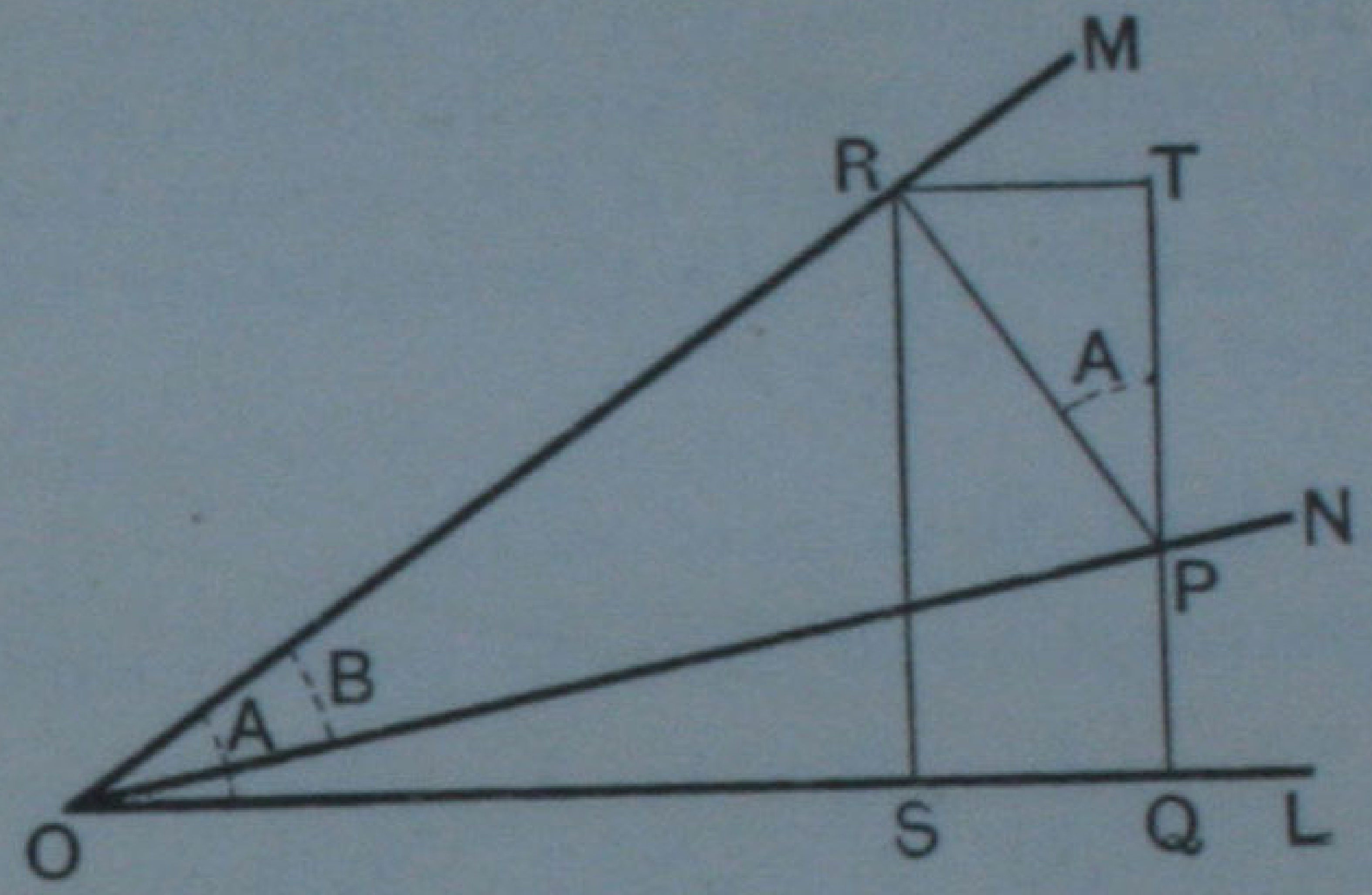
$$=\cos A\cos B-\sin A\sin B.$$

Formulae.

111. *To prove the formulæ*

$$\sin(A-B)=\sin A\cos B-\cos A\sin B,$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B.$$

Let $\angle LOM=A$, and $\angle MON=B$; then $\angle LON=A-B$.



In ON, *the boundary line of the compound angle* $A-B$, take any point P, and draw PQ and PR perpendicular to OL and OM respectively; also draw RS and RT perpendicular to OL and QP respectively.

By definition,

$$\sin(A-B)=\frac{PQ}{OP}=\frac{RS-PT}{OP}=\frac{RS}{OP}-\frac{PT}{OP}$$
$$=\frac{RS}{OR}\cdot\frac{OR}{OP}-\frac{PT}{PR}\cdot\frac{PR}{OP}$$
$$=\sin A\,.\,\cos B-\cos TPR\,.\,\sin B.$$

But $\angle TPR=90°-\angle TRP=\angle MRT=\angle MOL=A$;

$$\therefore\ \sin(A-B)=\sin A\cos B-\cos A\sin B.$$

Also $$\cos(A-B)=\frac{OQ}{OP}=\frac{OS+RT}{OP}=\frac{OS}{OP}+\frac{RT}{OP}$$
$$=\frac{OS}{OR}\cdot\frac{OR}{OP}+\frac{RT}{RP}\cdot\frac{RP}{OP}$$
$$=\cos A\,.\,\cos B+\sin TPR\,.\,\sin B$$
$$=\cos A\cos B+\sin A\sin B.$$

112. The *expansions* of $\sin(A \pm B)$ and $\cos(A \pm B)$ are frequently called the "Addition Formulæ." We shall sometimes refer to them as the "$A+B$" and "$A-B$" formulæ.

113. In the foregoing geometrical proofs we have supposed that the angles A, B, $A+B$ are all less than a right angle, and that $A-B$ is positive. If the angles are not so restricted some modification of the figures will be required. It is however unnecessary to consider these cases in detail, as in Chap. XXII. we shall shew by the Method of Projections that the Addition Formulæ hold universally. In the meantime the student may assume that they are always true.

Example 1. Find the value of $\cos 75°$.

$$\cos 75° = \cos(45° + 30°) = \cos 45° \cos 30° - \sin 45° \sin 30°$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Example 2. If $\sin A = \frac{4}{5}$ and $\sin B = \frac{5}{13}$, find $\sin(A-B)$.

$$\sin(A-B) = \sin A \cos B - \cos A \sin B.$$

But
$$\cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\frac{16}{25}} = \frac{3}{5};$$

and
$$\cos B = \sqrt{1-\sin^2 B} = \sqrt{1-\frac{25}{169}} = \frac{12}{13};$$

$$\therefore \sin(A-B) = \frac{4}{5} \cdot \frac{12}{13} - \frac{3}{5} \cdot \frac{5}{13} = \frac{33}{65}.$$

Note. Strictly speaking $\cos A = \pm \frac{3}{5}$ and $\cos B = \pm \frac{12}{13}$, so that $\sin(A-B)$ has *four* values. We shall however suppose that in similar cases only the positive value of the square root is taken.

114. *To prove that* $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$.

The first side

$$= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B)$$
$$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$$
$$= \sin^2 A\,(1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B$$
$$= \sin^2 A - \sin^2 B.$$

EXAMPLES. XI. a.

[*The examples printed in more prominent type are important, and should be regarded as standard formulæ.*]

Prove that

1. $\sin(A+45^\circ)=\frac{1}{\sqrt{2}}(\sin A+\cos A)$.

2. $\cos(A+45^\circ)=\frac{1}{\sqrt{2}}(\cos A-\sin A)$.

3. $2\sin(30^\circ-A)=\cos A-\sqrt{3}\sin A$.

4. If $\cos A=\frac{4}{5}$, $\cos B=\frac{3}{5}$, find $\sin(A+B)$ and $\cos(A-B)$.

5. If $\sin A=\frac{3}{5}$, $\cos B=\frac{12}{13}$, find $\cos(A+B)$ and $\sin(A-B)$.

6. If $\sec A=\frac{17}{8}$, $\operatorname{cosec} B=\frac{5}{4}$, find $\sec(A+B)$.

Prove that

7. $\mathbf{\sin 75^\circ=\cos 15^\circ=\frac{\sqrt{3}+1}{2\sqrt{2}}}$.

8. $\mathbf{\sin 15^\circ=\cos 75^\circ=\frac{\sqrt{3}-1}{2\sqrt{2}}}$.

9. $\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}=\tan\alpha+\tan\beta$.

10. $\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta}=\cot\beta-\cot\alpha$.

11. $\frac{\cos(\alpha-\beta)}{\cos\alpha\sin\beta}=\cot\beta+\tan\alpha$.

12. $\mathbf{\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B}$.

13. $\mathbf{\sin(A+B)\sin(A-B)=\cos^2 B-\cos^2 A}$.

14. $\cos(45^\circ-A)-\sin(45^\circ+A)=0$.

15. $\cos(45^\circ+A)+\sin(A-45^\circ)=0$.

16. $\cos(A-B)-\sin(A+B)=(\cos A-\sin A)(\cos B-\sin B)$.

17. $\cos(A+B)+\sin(A-B)=(\cos A+\sin A)(\cos B-\sin B)$.

Prove the following identities:

18. $2\sin(A+45^\circ)\sin(A-45^\circ)=\sin^2 A-\cos^2 A.$

19. $2\cos\left(\frac{\pi}{4}+\alpha\right)\cos\left(\frac{\pi}{4}-\alpha\right)=\cos^2\alpha-\sin^2\alpha.$

20. $2\sin\left(\frac{\pi}{4}+\alpha\right)\cos\left(\frac{\pi}{4}+\beta\right)=\cos(\alpha+\beta)+\sin(\alpha-\beta).$

21. $\frac{\sin(\beta-\gamma)}{\cos\beta\cos\gamma}+\frac{\sin(\gamma-\alpha)}{\cos\gamma\cos\alpha}+\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}=0.$

115. *To expand* $\tan(A+B)$ *in terms of* $\tan A$ *and* $\tan B$.

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}.$$

To express this fraction in terms of *tangents*, divide each term of numerator and denominator by $\cos A\cos B$;

$$\therefore\ \tan(A+B)=\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{1-\dfrac{\sin A}{\cos A}\cdot\dfrac{\sin B}{\cos B}};$$

that is,
$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

A geometrical proof of this result is given in Chap. XXII.

Similarly, we may prove that

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

Example. Find the value of tan 75°.

$$\tan 75^\circ=\tan(45^\circ+30^\circ)=\frac{\tan 45^\circ+\tan 30^\circ}{1-\tan 45^\circ\tan 30^\circ}$$

$$=\frac{1+\dfrac{1}{\sqrt{3}}}{1-\dfrac{1}{\sqrt{3}}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$=\frac{(\sqrt{3}+1)(\sqrt{3}+1)}{3-1}=\frac{4+2\sqrt{3}}{2}$$

$$=2+\sqrt{3}.$$

116. *To expand* cot $(A+B)$ *in terms of* cot A *and* cot B.

$$\cot(A+B)=\frac{\cos(A+B)}{\sin(A+B)}=\frac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}.$$

To express this fraction in terms of *cotangents*, divide each term of numerator and denominator by $\sin A\sin B$;

$$\therefore\ \cot(A+B)=\frac{\dfrac{\cos A\cos B}{\sin A\sin B}-1}{\dfrac{\cos B}{\sin B}+\dfrac{\cos A}{\sin A}}=\frac{\cot A\cot B-1}{\cot B+\cot A}.$$

Similarly, we may prove that

$$\cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A}.$$

117. *To find the expansion of* $\sin(A+B+C)$.

$$\begin{aligned}\sin(A+B+C)&=\sin\{(A+B)+C\}\\&=\sin(A+B)\cos C+\cos(A+B)\sin C\\&=(\sin A\cos B+\cos A\sin B)\cos C\\&\qquad+(\cos A\cos B-\sin A\sin B)\sin C\\&=\sin A\cos B\cos C+\cos A\sin B\cos C\\&\qquad+\cos A\cos B\sin C-\sin A\sin B\sin C.\end{aligned}$$

118. *To find the expansion of* $\tan(A+B+C)$.

$$\tan(A+B+C)=\tan\{(A+B)+C\}=\frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}$$

$$=\frac{\dfrac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\dfrac{\tan A+\tan B}{1-\tan A\tan B}\cdot\tan C}$$

$$=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}.$$

COR. If $A+B+C=180°$, then $\tan(A+B+C)=0$; hence the numerator of the above expression must be zero.

$$\therefore\ \tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

EXAMPLES. XI. b.

[*The examples printed in more prominent type are important, and should be regarded as standard formulæ.*]

1. Find $\tan(A+B)$ when $\tan A=\frac{1}{2}$, $\tan B=\frac{1}{3}$.

2. If $\tan A=\frac{4}{3}$, and $B=45°$, find $\tan(A-B)$.

3. If $\cot A=\frac{5}{7}$, $\cot B=\frac{7}{5}$, find $\cot(A+B)$ and $\tan(A-B)$.

4. If $\cot A=\frac{11}{2}$, $\tan B=\frac{7}{24}$, find $\cot(A-B)$ and $\tan(A+B)$.

5. $\mathbf{\tan(45°+A)=\frac{1+\tan A}{1-\tan A}}$.

6. $\mathbf{\tan(45°-A)=\frac{1-\tan A}{1+\tan A}}$.

7. $\cot\left(\frac{\pi}{4}-\theta\right)=\frac{\cot\theta+1}{\cot\theta-1}$.

8. $\cot\left(\frac{\pi}{4}+\theta\right)=\frac{\cot\theta-1}{\cot\theta+1}$.

9. $\mathbf{\tan 15°=2-\sqrt{3}}$.

10. $\mathbf{\cot 15°=2+\sqrt{3}}$.

11. Find the expansions of

$$\cos(A+B+C) \text{ and } \sin(A-B+C).$$

12. Express $\tan(A-B-C)$ in terms of $\tan A$, $\tan B$, $\tan C$.

13. Express $\cot(A+B+C)$ in terms of $\cot A$, $\cot B$, $\cot C$.

119. Beginners not unfrequently find a difficulty in the converse use of the $A+B$ and $A-B$ formulæ; that is, they fail to recognise when an expression is merely an expansion belonging to one of the standard forms.

Example 1. Simplify $\cos(\alpha-\beta)\cos(\alpha+\beta)-\sin(\alpha-\beta)\sin(\alpha+\beta)$.

This expression is the expansion of the cosine of the compound angle $(\alpha+\beta)+(\alpha-\beta)$, and is therefore equal to $\cos\{(\alpha+\beta)+(\alpha-\beta)\}$; that is, to $\cos 2\alpha$.

Example 2. Shew that $\frac{\tan A+\tan 2A}{1-\tan A\tan 2A}=\tan 3A$.

By Art. 115, the first side is the expansion of $\tan(A+2A)$, and is therefore equal to $\tan 3A$.

Example 3. Prove that $\cot 2A + \tan A = \operatorname{cosec} 2A$.

$$\text{The first side} = \frac{\cos 2A}{\sin 2A} + \frac{\sin A}{\cos A} = \frac{\cos 2A \cos A + \sin 2A \sin A}{\sin 2A \cos A}$$

$$= \frac{\cos (2A - A)}{\sin 2A \cos A} = \frac{\cos A}{\sin 2A \cos A}$$

$$= \frac{1}{\sin 2A} = \operatorname{cosec} 2A.$$

Example 4. Prove that

$$\cos 4\theta \cos \theta + \sin 4\theta \sin \theta = \cos 2\theta \cos \theta - \sin 2\theta \sin \theta.$$

$$\text{The first side} = \cos (4\theta - \theta) = \cos 3\theta = \cos (2\theta + \theta)$$

$$= \cos 2\theta \cos \theta - \sin 2\theta \sin \theta.$$

EXAMPLES. XI. c.

Prove the following identities :

1. $\cos (A + B) \cos B + \sin (A + B) \sin B = \cos A.$
2. $\sin 3A \cos A - \cos 3A \sin A = \sin 2A.$
3. $\cos 2a \cos a + \sin 2a \sin a = \cos a.$
4. $\cos (30° + A) \cos (30° - A) - \sin (30° + A) \sin (30° - A) = \frac{1}{2}.$
5. $\sin (60° - A) \cos (30° + A) + \cos (60° - A) \sin (30° + A) = 1.$
6. $\frac{\cos 2a}{\sec a} - \frac{\sin 2a}{\operatorname{cosec} a} = \cos 3a.$
7. $\frac{\tan (a - \beta) + \tan \beta}{1 - \tan (a - \beta) \tan \beta} = \tan a.$
8. $\frac{\cot (a + \beta) \cot a + 1}{\cot a - \cot (a + \beta)} = \cot \beta.$
9. $\frac{\tan 4A - \tan 3A}{1 + \tan 4A \tan 3A} = \tan A.$
10. $\cot \theta - \cot 2\theta = \operatorname{cosec} 2\theta.$
11. $1 + \tan 2\theta \tan \theta = \sec 2\theta.$
12. $1 + \cot 2\theta \cot \theta = \operatorname{cosec} 2\theta \cot \theta.$
13. $\sin 2\theta \cos \theta + \cos 2\theta \sin \theta = \sin 4\theta \cos \theta - \cos 4\theta \sin \theta.$
14. $\cos 4a \cos a - \sin 4a \sin a = \cos 3a \cos 2a - \sin 3a \sin 2a.$

Functions of Multiple Angles.

120. *To express* $\sin 2A$ *in terms of* $\sin A$ *and* $\cos A$.

$$\sin 2A = \sin(A+A) = \sin A \cos A + \cos A \sin A;$$

that is, $$\sin 2A = 2\sin A \cos A.$$

Since A may have any value, this is a perfectly general formula for the sine of an angle in terms of the sine and cosine of the half angle. Thus if $2A$ be replaced by θ, we have

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.$$

Similarly, $$\sin 4A = 2\sin 2A\cos 2A$$
$$= 4\sin A\cos A\cos 2A.$$

121. *To express* $\cos 2A$ *in terms of* $\cos A$ *and* $\sin A$.

$$\cos 2A = \cos(A+A) = \cos A\cos A - \sin A\sin A;$$

that is, $$\cos 2A = \cos^2 A - \sin^2 A \quad \text{.....................(1)}.$$

There are two other useful forms in which $\cos 2A$ may be expressed, one involving $\cos A$ only, the other $\sin A$ only.

Thus from (1),

$$\cos 2A = \cos^2 A - (1 - \cos^2 A);$$

that is, $$\cos 2A = 2\cos^2 A - 1 \quad \text{.....................(2)}.$$

Again, from (1),

$$\cos 2A = (1 - \sin^2 A) - \sin^2 A;$$

that is, $$\cos 2A = 1 - 2\sin^2 A \quad \text{.....................(3)}.$$

From formulæ (2) and (3), we obtain by transposition

$$1 + \cos 2A = 2\cos^2 A \quad \text{.....................(4)},$$

and $$1 - \cos 2A = 2\sin^2 A \quad \text{.....................(5)}.$$

By division, $$\frac{1-\cos 2A}{1+\cos 2A} = \tan^2 A \quad \text{.....................(6)}.$$

Example. Express $\cos 4\alpha$ in terms of $\sin\alpha$.

From (3), $\cos 4\alpha = 1 - 2\sin^2 2\alpha = 1 - 2(4\sin^2\alpha\cos^2\alpha)$
$= 1 - 8\sin^2\alpha(1 - \sin^2\alpha)$
$= 1 - 8\sin^2\alpha + 8\sin^4\alpha.$

122. The six formulæ of the last article deserve special attention. They are universally true so long as one of the angles involved is double of the other. For instance,

$$\cos a = \cos^2\frac{a}{2} - \sin^2\frac{a}{2},$$

$$\cos a = 2\cos^2\frac{a}{2} - 1, \qquad \cos a = 1 - 2\sin^2\frac{a}{2},$$

$$1 + \cos\theta = 2\cos^2\frac{\theta}{2}, \qquad 1 - \cos\theta = 2\sin^2\frac{\theta}{2}.$$

Example. If $\cos\theta = \cdot 28$, find the value of $\tan\frac{\theta}{2}$.

$$\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{1-\cdot 28}{1+\cdot 28} = \frac{\cdot 72}{1\cdot 28} = \frac{72}{128} = \frac{9}{16};$$

$$\therefore \tan\frac{\theta}{2} = \pm\frac{3}{4}.$$

123. *To express* $\tan 2A$ *in terms of* $\tan A$.

$$\tan 2A = \tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A};$$

that is,
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}.$$

124. *To express* $\sin 2A$ *and* $\cos 2A$ *in terms of* $\tan A$.

$$\sin 2A = 2\sin A\cos A = 2\frac{\sin A}{\cos A}\cos^2 A = 2\tan A\cos^2 A;$$

$$\therefore \sin 2A = \frac{2\tan A}{\sec^2 A} = \frac{2\tan A}{1+\tan^2 A}.$$

Again,

$$\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A\,(1 - \tan^2 A);$$

$$\therefore \cos 2A = \frac{1-\tan^2 A}{\sec^2 A} = \frac{1-\tan^2 A}{1+\tan^2 A}.$$

Example. Shew that $\dfrac{1-\tan^2(45^\circ - A)}{1+\tan^2(45^\circ - A)} = \sin 2A.$

The first side $= \cos 2(45^\circ - A) = \cos(90^\circ - 2A) = \sin 2A$.

EXAMPLES. XI. d.

[*The examples printed in more prominent type are important, and should be regarded as standard formulæ.*]

1. If $\cos A = \frac{1}{3}$, find $\cos 2A$.

2. Find $\cos 2A$ when $\sin A = \frac{2}{5}$.

3. If $\sin A = \frac{3}{5}$, find $\sin 2A$.

4. If $\tan \theta = \frac{1}{3}$, find $\tan 2\theta$.

5. If $\tan \theta = \frac{1}{7}$, find $\sin 2\theta$ and $\cos 2\theta$.

6. If $\cos \alpha = \frac{4}{5}$, find $\tan \frac{\alpha}{2}$.

7. Find $\tan A$ when $\cos 2A = \cdot 96$.

Prove the following identities:

8. $\mathbf{\frac{\sin 2A}{1+\cos 2A} = \tan A}$.

9. $\mathbf{\frac{\sin 2A}{1-\cos 2A} = \cot A}$.

10. $\mathbf{\frac{1-\cos A}{\sin A} = \tan \frac{A}{2}}$.

11. $\mathbf{\frac{1+\cos A}{\sin A} = \cot \frac{A}{2}}$.

12. $2 \operatorname{cosec} 2\alpha = \sec \alpha \operatorname{cosec} \alpha$.

13. $\tan \alpha + \cot \alpha = 2 \operatorname{cosec} 2\alpha$.

14. $\cos^4 \alpha - \sin^4 \alpha = \cos 2\alpha$.

15. $\mathbf{\cot \alpha - \tan \alpha = 2 \cot 2\alpha}$.

16. $\mathbf{\cot 2A = \frac{\cot^2 A - 1}{2 \cot A}}$.

17. $\frac{\cot A - \tan A}{\cot A + \tan A} = \cos 2A$.

18. $\frac{1+\cot^2 A}{2 \cot A} = \operatorname{cosec} 2A$.

19. $\frac{\cot^2 A + 1}{\cot^2 A - 1} = \sec 2A$.

20. $\frac{1+\sec \theta}{\sec \theta} = 2 \cos^2 \frac{\theta}{2}$.

21. $\frac{\sec \theta - 1}{\sec \theta} = 2 \sin^2 \frac{\theta}{2}$.

22. $\frac{2-\sec^2 \theta}{\sec^2 \theta} = \cos 2\theta$.

23. $\frac{\operatorname{cosec}^2 \theta - 2}{\operatorname{cosec}^2 \theta} = \cos 2\theta$.

Prove the following identities:

24. $\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)^2=1+\sin A.$

25. $\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)^2=1-\sin A.$

26. $\frac{\cos 2a}{1+\sin 2a}=\tan(45^\circ-a).$

27. $\frac{\cos 2a}{1-\sin 2a}=\cot(45^\circ-a).$

28. $\sin 8A=8\sin A\cos A\cos 2A\cos 4A.$

29. $\cos 4A=8\cos^4 A-8\cos^2 A+1.$

30. $\sin A=1-2\sin^2\left(45^\circ-\frac{A}{2}\right).$

31. $\cos^2\left(\frac{\pi}{4}-a\right)-\sin^2\left(\frac{\pi}{4}-a\right)=\sin 2a.$

32. $\tan(45^\circ+A)-\tan(45^\circ-A)=2\tan 2A.$

33. $\tan(45^\circ+A)+\tan(45^\circ-A)=2\sec 2A.$

125. Functions of 3A.

$$\begin{aligned}\sin 3A=\sin(2A+A)&=\sin 2A\cos A+\cos 2A\sin A\\&=2\sin A\cos^2 A+(1-2\sin^2 A)\sin A\\&=2\sin A(1-\sin^2 A)+(1-2\sin^2 A)\sin A;\\&=3\sin A-4\sin^3 A.\end{aligned}$$

Similarly it may be proved that

$$\cos 3A=4\cos^3 A-3\cos A.$$

Again, $\tan 3A=\tan(2A+A)=\frac{\tan 2A+\tan A}{1-\tan 2A\tan A}$;

by putting $\tan 2A=\frac{2\tan A}{1-\tan^2 A}$,

we obtain on reduction

$$\tan 3A=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}.$$

These formulæ are perfectly general and may be applied to cases of any two angles, one of which is three times the other; thus

$$\cos 6a = 4\cos^3 2a - 3\cos 2a;$$

$$\sin 9A = 3\sin 3A - 4\sin^3 3A.$$

126. *To find the value of* sin 18°.

Let $A = 18°$, then $5A = 90°$, so that $2A = 90° - 3A$.

$$\therefore \sin 2A = \sin(90° - 3A) = \cos 3A;$$

$$\therefore 2\sin A\cos A = 4\cos^3 A - 3\cos A.$$

Divide by $\cos A$ (which is not equal to zero);

$$\therefore 2\sin A = 4\cos^2 A - 3 = 4(1 - \sin^2 A) - 3;$$

$$\therefore 4\sin^2 A + 2\sin A - 1 = 0;$$

$$\therefore \sin A = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-1 \pm \sqrt{5}}{4}.$$

Since 18° is an acute angle, we take the positive sign;

$$\therefore \sin 18° = \frac{\sqrt{5}-1}{4}.$$

Example. Find cos 18° and sin 54°.

$$\cos 18° = \sqrt{1 - \sin^2 18°} = \sqrt{1 - \frac{6 - 2\sqrt{5}}{16}} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}.$$

Since 54° and 36° are complementary, $\sin 54° = \cos 36°$.

Now $\cos 36° = 1 - 2\sin^2 18° = 1 - \frac{2(6 - 2\sqrt{5})}{16} = \frac{\sqrt{5}+1}{4};$

$$\therefore \sin 54° = \frac{\sqrt{5}+1}{4}.$$

EXAMPLES. XI. e.

1. If $\cos A = \frac{1}{3}$, find $\cos 3A$.

2. Find $\sin 3A$ when $\sin A = \frac{3}{5}$.

3. Given $\tan A = 3$, find $\tan 3A$.

Prove the following identities:

4. $\dfrac{\sin 3A}{\sin A} - \dfrac{\cos 3A}{\cos A} = 2.$

5. $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}.$

6. $\dfrac{3\cos a + \cos 3a}{3\sin a - \sin 3a} = \cot^3 a.$

7. $\dfrac{\sin 3a + \sin^3 a}{\cos^3 a - \cos 3a} = \cot a.$

8. $\dfrac{\cos^3 a - \cos 3a}{\cos a} + \dfrac{\sin^3 a + \sin 3a}{\sin a} = 3.$

9. $\sin 18^\circ + \sin 30^\circ = \sin 54^\circ.$

10. $\cos 36^\circ - \sin 18^\circ = \dfrac{1}{2}.$

11. $\cos^2 36^\circ + \sin^2 18^\circ = \dfrac{3}{4}.$

12. $4\sin 18^\circ \cos 36^\circ = 1.$

127. The following examples further illustrate the formulæ proved in this chapter.

Example 1. Shew that $\cos^6\alpha + \sin^6\alpha = 1 - \frac{3}{4}\sin^2 2\alpha.$

$$\text{The first side} = (\cos^2\alpha + \sin^2\alpha)(\cos^4\alpha + \sin^4\alpha - \cos^2\alpha\sin^2\alpha)$$
$$= (\cos^2\alpha + \sin^2\alpha)^2 - 3\cos^2\alpha\sin^2\alpha$$
$$= 1 - \frac{3}{4}(4\cos^2\alpha\sin^2\alpha)$$
$$= 1 - \frac{3}{4}\sin^2 2\alpha.$$

Example 2. Prove that $\dfrac{\cos A - \sin A}{\cos A + \sin A} = \sec 2A - \tan 2A.$

$$\text{The right side} = \frac{1}{\cos 2A} - \frac{\sin 2A}{\cos 2A} = \frac{1 - \sin 2A}{\cos 2A},$$

and since $\cos 2A = \cos^2 A - \sin^2 A = (\cos A + \sin A)(\cos A - \sin A)$, this suggests that we should multiply the numerator and denominator of the left side by $\cos A - \sin A$; thus

$$\text{the first side} = \frac{(\cos A - \sin A)(\cos A - \sin A)}{(\cos A + \sin A)(\cos A - \sin A)}$$
$$= \frac{\cos^2 A + \sin^2 A - 2\cos A\sin A}{\cos^2 A - \sin^2 A}$$
$$= \frac{1 - \sin 2A}{\cos 2A} = \sec 2A - \tan 2A.$$

Example 3. Shew that $\dfrac{1}{\tan 3A - \tan A} - \dfrac{1}{\cot 3A - \cot A} = \cot 2A.$

$$\text{The first side} = \frac{1}{\dfrac{\sin 3A}{\cos 3A} - \dfrac{\sin A}{\cos A}} - \frac{1}{\dfrac{\cos 3A}{\sin 3A} - \dfrac{\cos A}{\sin A}}$$

$$= \frac{\cos 3A \cos A}{\sin 3A \cos A - \cos 3A \sin A} - \frac{\sin 3A \sin A}{\cos 3A \sin A - \sin 3A \cos A}$$

$$= \frac{\cos 3A \cos A + \sin 3A \sin A}{\sin 3A \cos A - \cos 3A \sin A}$$

$$= \frac{\cos (3A - A)}{\sin (3A - A)} = \frac{\cos 2A}{\sin 2A} = \cot 2A.$$

Note. This example has been given to emphasize the fact that in identities involving the functions of $2A$ and $3A$ it is sometimes best not to substitute their equivalents in terms of functions of A.

EXAMPLES. XI. f.

Prove the following identities:

1. $\tan 2A - \sec A \sin A = \tan A \sec 2A.$

2. $\tan 2A + \cos A \operatorname{cosec} A = \cot A \sec 2A.$

3. $\dfrac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} = \tan \theta.$

4. $\dfrac{1 + \cos \theta + \cos \dfrac{\theta}{2}}{\sin \theta + \sin \dfrac{\theta}{2}} = \cot \dfrac{\theta}{2}.$

5. $\cos^6 \alpha - \sin^6 \alpha = \cos 2\alpha \left(1 - \dfrac{1}{4} \sin^2 2\alpha\right).$

6. $4(\cos^6 \theta + \sin^6 \theta) = 1 + 3 \cos^2 2\theta.$

7. $\dfrac{\cos 3\alpha + \sin 3\alpha}{\cos \alpha - \sin \alpha} = 1 + 2 \sin 2\alpha.$

8. $\dfrac{\cos 3\alpha - \sin 3\alpha}{\cos \alpha + \sin \alpha} = 1 - 2 \sin 2\alpha.$

9. $\dfrac{\cos \alpha + \sin \alpha}{\cos \alpha - \sin \alpha} = \tan 2\alpha + \sec 2\alpha.$

Prove the following identities:

10. $\dfrac{\cot a-1}{\cot a+1}=\dfrac{1-\sin 2a}{\cos 2a}$.

11. $\dfrac{1+\sin\theta}{\cos\theta}=\dfrac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$.

12. $\dfrac{\cos\theta}{1-\sin\theta}=\dfrac{\cot\frac{\theta}{2}+1}{\cot\frac{\theta}{2}-1}$.

13. $\sec A-\tan A=\tan\left(45^\circ-\dfrac{A}{2}\right)$.

14. $\tan A+\sec A=\cot\left(45^\circ-\dfrac{A}{2}\right)$.

15. $\dfrac{1+\sin\theta}{1-\sin\theta}=\tan^2\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)$.

16. $(2\cos A+1)(2\cos A-1)=2\cos 2A+1$.

17. $\dfrac{\sin 2A}{1+\cos 2A}\cdot\dfrac{\cos A}{1+\cos A}=\tan\dfrac{A}{2}$.

18. $\dfrac{\sin 2A}{1-\cos 2A}\cdot\dfrac{1-\cos A}{\cos A}=\tan\dfrac{A}{2}$.

19. $4\sin^3 a\cos 3a+4\cos^3 a\sin 3a=3\sin 4a$.

[*Put* $4\sin^3 a=3\sin a-\sin 3a$ *and* $4\cos^3 a=3\cos a+\cos 3a$.]

20. $\cos^3 a\cos 3a+\sin^3 a\sin 3a=\cos^3 2a$.

21. $4(\cos^3 20^\circ+\cos^3 40^\circ)=3(\cos 20^\circ+\cos 40^\circ)$.

22. $4(\cos^3 10^\circ+\sin^3 20^\circ)=3(\cos 10^\circ+\sin 20^\circ)$.

23. $\tan 3A-\tan 2A-\tan A=\tan 3A\tan 2A\tan A$.

[*Use* $\tan 3A=\tan(2A+A)$.]

24. $\dfrac{\cot\theta}{\cot\theta-\cot 3\theta}+\dfrac{\tan\theta}{\tan\theta-\tan 3\theta}=1$.

25. $\dfrac{1}{\tan 3\theta+\tan\theta}-\dfrac{1}{\cot 3\theta+\cot\theta}=\cot 4\theta$.

CHAPTER XII.

TRANSFORMATION OF PRODUCTS AND SUMS.

Transformation of products into sums or differences.

128. In the last chapter we have proved that

$$\sin A \cos B + \cos A \sin B = \sin (A+B),$$

and

$$\sin A \cos B - \cos A \sin B = \sin (A-B).$$

By addition,

$$2 \sin A \cos B = \sin (A+B) + \sin (A-B) \quad \text{.........(1)};$$

by subtraction

$$2 \cos A \sin B = \sin (A+B) - \sin (A-B) \quad \text{.........(2)}.$$

These formulæ enable us to express the product of a sine and cosine as the sum or difference of two sines.

Again, $\cos A \cos B - \sin A \sin B = \cos (A+B),$

and $\cos A \cos B + \sin A \sin B = \cos (A-B).$

By addition,

$$2 \cos A \cos B = \cos (A+B) + \cos (A-B) \quad \text{.........(3)};$$

by subtraction,

$$2 \sin A \sin B = \cos (A-B) - \cos (A+B) \quad \text{.........(4)}.$$

These formulæ enable us to express

(i) the product of two cosines as the sum of two cosines;

(ii) the product of two sines as the difference of two cosines.

129. In each of the four formulæ of the previous article it should be noticed that on the left side we have any two angles A and B, and on the right side the sum and difference of these angles.

For practical purposes the following verbal statements of the results are more useful.

$$2 \sin A \cos B = \sin(\mathit{sum}) + \sin(\mathit{difference});$$
$$2 \cos A \sin B = \sin(\mathit{sum}) - \sin(\mathit{difference});$$
$$2 \cos A \cos B = \cos(\mathit{sum}) + \cos(\mathit{difference});$$
$$2 \sin A \sin B = \cos(\mathit{difference}) - \cos(\mathit{sum}).$$

N.B. In the last of these formulæ, *the difference precedes the sum.*

Example 1. $2 \sin 7A \cos 4A = \sin(\mathit{sum}) + \sin(\mathit{difference})$

$$= \sin 11A + \sin 3A.$$

Example 2. $2 \cos 3\theta \sin 6\theta = \sin(3\theta + 6\theta) - \sin(3\theta - 6\theta)$

$$= \sin 9\theta - \sin(-3\theta)$$
$$= \sin 9\theta + \sin 3\theta.$$

Example 3. $\cos\frac{3A}{2}\cos\frac{5A}{2} = \frac{1}{2}\left\{\cos\left(\frac{3A}{2}+\frac{5A}{2}\right) + \cos\left(\frac{3A}{2}-\frac{5A}{2}\right)\right\}$

$$= \frac{1}{2}\{\cos 4A + \cos(-A)\}$$
$$= \frac{1}{2}(\cos 4A + \cos A).$$

Example 4. $2 \sin 75^\circ \sin 15^\circ = \cos(75^\circ - 15^\circ) - \cos(75^\circ + 15^\circ)$

$$= \cos 60^\circ - \cos 90^\circ$$
$$= \frac{1}{2} - 0$$
$$= \frac{1}{2}.$$

130. After a little practice the student will be able to omit some of the steps and find the equivalent very rapidly.

Example 1. $2\cos\left(\frac{\pi}{4}+\theta\right)\cos\left(\frac{\pi}{4}-\theta\right) = \cos\frac{\pi}{2} + \cos 2\theta = \cos 2\theta.$

Example 2. $\sin(\alpha - 2\beta)\cos(\alpha + 2\beta) = \frac{1}{2}\{\sin 2\alpha + \sin(-4\beta)\}$

$$= \frac{1}{2}(\sin 2\alpha - \sin 4\beta).$$

EXAMPLES. XII. a.

Express in the form of a sum or difference

1. $2\sin 3\theta\cos\theta$.
2. $2\cos 6\theta\sin 3\theta$.
3. $2\cos 7A\cos 5A$.
4. $2\sin 3A\sin 2A$.
5. $2\cos 5\theta\sin 4\theta$.
6. $2\sin 4\theta\cos 8\theta$.
7. $2\sin 9\theta\sin 3\theta$.
8. $2\cos 9\theta\sin 7\theta$.
9. $2\cos 2a\cos 11a$.
10. $2\sin 5a\sin 10a$.
11. $\sin 4a\cos 7a$.
12. $\sin 3a\sin a$.
13. $\cos\frac{A}{2}\sin\frac{3A}{2}$.
14. $\sin\frac{5A}{2}\cos\frac{7A}{2}$.
15. $2\cos\frac{2\theta}{3}\cos\frac{5\theta}{3}$.
16. $\sin\frac{\theta}{4}\sin\frac{3\theta}{4}$.
17. $2\cos 2\beta\cos(a-\beta)$.
18. $2\sin 3a\sin(a+\beta)$.
19. $2\sin(2\theta+\phi)\cos(\theta-2\phi)$.
20. $2\cos(3\theta+\phi)\sin(\theta-2\phi)$.
21. $\cos(60^\circ+a)\sin(60^\circ-a)$.

Transformation of sums or differences into products.

131. Since $\sin(A+B)=\sin A\cos B+\cos A\sin B$,

and $\sin(A-B)=\sin A\cos B-\cos A\sin B$;

by addition,

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B \quad \text{.........(1)};$$

by subtraction,

$$\sin(A+B)-\sin(A-B)=2\cos A\sin B \quad \text{.........(2)}.$$

Again, $\cos(A+B)=\cos A\cos B-\sin A\sin B$,

and $\cos(A-B)=\cos A\cos B+\sin A\sin B$.

By addition,

$$\cos(A+B)+\cos(A-B)=2\cos A\cos B \quad \text{.........(3)};$$

by subtraction,

$$\cos(A+B)-\cos(A-B)=-2\sin A\sin B$$

$$=2\sin A\sin(-B) \quad \text{......(4)}.$$

Let $$A+B=C, \text{ and } A-B=D;$$

then $$A=\frac{C+D}{2}, \text{ and } B=\frac{C-D}{2}.$$

By substituting for A and B in the formulæ (1), (2), (3), (4), we obtain

$$\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2},$$

$$\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2},$$

$$\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2},$$

$$\cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}.$$

132. In practice, it is more convenient to quote the formulæ we have just obtained verbally as follows:

sum of two sines = 2 sin (*half-sum*) cos (*half-difference*);

difference of two sines = 2 cos (*half-sum*) sin (*half-difference*);

sum of two cosines = 2 cos (*half-sum*) cos (*half-difference*);

difference of two cosines

= 2 sin (*half-sum*) sin (*half-difference reversed*)

Example 1. $\sin 14\theta+\sin 6\theta=2\sin\frac{14\theta+6\theta}{2}\cos\frac{14\theta-6\theta}{2}$

$=2\sin 10\theta\cos 4\theta.$

Example 2. $\sin 9A-\sin 7A=2\cos\frac{9A+7A}{2}\sin\frac{9A-7A}{2}$

$=2\cos 8A\sin A.$

Example 3. $\cos A+\cos 8A=2\cos\frac{9A}{2}\cos\left(-\frac{7A}{2}\right)$

$=2\cos\frac{9A}{2}\cos\frac{7A}{2}.$

Example 4. $\cos 70^\circ-\cos 10^\circ=2\sin 40^\circ\sin(-30^\circ)$

$=-2\sin 40^\circ\sin 30^\circ=-\sin 40^\circ.$

EXAMPLES. XII. b.

Express in the form of a product

1. $\sin 8\theta + \sin 4\theta$.
2. $\sin 5\theta - \sin \theta$.
3. $\cos 7\theta + \cos 3\theta$.
4. $\cos 9\theta - \cos 11\theta$.
5. $\sin 7a - \sin 5a$.
6. $\cos 3a + \cos 8a$.
7. $\sin 3a + \sin 13a$.
8. $\cos 5a - \cos a$.
9. $\cos 2A + \cos 9A$.
10. $\sin 3A - \sin 11A$.
11. $\cos 10° - \cos 50°$.
12. $\sin 70° + \sin 50°$.

Prove that

13. $\dfrac{\cos a - \cos 3a}{\sin 3a - \sin a} = \tan 2a$.
14. $\dfrac{\sin 2a + \sin 3a}{\cos 2a - \cos 3a} = \cot \dfrac{a}{2}$.
15. $\dfrac{\cos 4\theta - \cos \theta}{\sin \theta - \sin 4\theta} = \tan \dfrac{5\theta}{2}$.
16. $\dfrac{\cos 2\theta - \cos 12\theta}{\sin 12\theta + \sin 2\theta} = \tan 5\theta$.
17. $\sin(60° + A) - \sin(60° - A) = \sin A$.
18. $\cos(30° - A) + \cos(30° + A) = \sqrt{3} \cos A$.
19. $\cos\left(\dfrac{\pi}{4} + a\right) - \cos\left(\dfrac{\pi}{4} - a\right) = -\sqrt{2} \sin a$.
20. $\dfrac{\cos(2a - 3\beta) + \cos 3\beta}{\sin(2a - 3\beta) + \sin 3\beta} = \cot a$.
21. $\dfrac{\cos(\theta - 3\phi) - \cos(3\theta + \phi)}{\sin(3\theta + \phi) + \sin(\theta - 3\phi)} = \tan(\theta + 2\phi)$.
22. $\dfrac{\sin(a + \beta) - \sin 4\beta}{\cos(a + \beta) + \cos 4\beta} = \tan \dfrac{a - 3\beta}{2}$.

133. The eight formulæ proved in this chapter are of the utmost importance and very little further progress can be made until they have been thoroughly learnt. In the first group, the transformation is from products to sums and differences; in the second group, there is the converse transformation from sums and differences to products.

Many examples admit of solution by applying either of these transformations, but it is absolutely necessary that the student should master all the formulæ and apply them with equal readiness.

134. The following examples should be studied with great care.

Example 1. Prove that

$$\sin 5A + \sin 2A - \sin A = \sin 2A\,(2\cos 3A + 1).$$

The first side $= (\sin 5A - \sin A) + \sin 2A$

$$= 2\cos 3A \sin 2A + \sin 2A$$

$$= \sin 2A\,(2\cos 3A + 1).$$

Example 2. Prove that

$$\cos 2\theta \cos\theta - \sin 4\theta \sin\theta = \cos 3\theta \cos 2\theta.$$

The first side $= \frac{1}{2}(\cos 3\theta + \cos\theta) - \frac{1}{2}(\cos 3\theta - \cos 5\theta)$

$$= \frac{1}{2}(\cos\theta + \cos 5\theta)$$

$$= \cos 3\theta \cos 2\theta.$$

Example 3. Find the value of

$$\cos 20° + \cos 100° + \cos 140°.$$

The expression $= \cos 20° + (\cos 100° + \cos 140°)$

$$= \cos 20° + 2\cos 120° \cos 20°$$

$$= \cos 20° + 2\left(-\frac{1}{2}\right)\cos 20°$$

$$= \cos 20° - \cos 20° = 0.$$

Example 4. Express as the product of three sines

$$\sin(\beta+\gamma-\alpha) + \sin(\gamma+\alpha-\beta) + \sin(\alpha+\beta-\gamma) - \sin(\alpha+\beta+\gamma).$$

The expression $= 2\sin\gamma\cos(\beta-\alpha) + 2\cos(\alpha+\beta)\sin(-\gamma)$

$$= 2\sin\gamma\,\{\cos(\beta-\alpha) - \cos(\alpha+\beta)\}$$

$$= 2\sin\gamma\,(2\sin\beta\sin\alpha)$$

$$= 4\sin\alpha\sin\beta\sin\gamma.$$

Example 5. Express $4\cos\alpha\cos\beta\cos\gamma$ as the sum of four cosines.

The expression $= 2\cos\alpha\,\{\cos(\beta+\gamma) + \cos(\beta-\gamma)\}$

$$= 2\cos\alpha\cos(\beta+\gamma) + 2\cos\alpha\cos(\beta-\gamma)$$

$$= \cos(\alpha+\beta+\gamma) + \cos(\alpha-\beta-\gamma) + \cos(\alpha+\beta-\gamma) + \cos(\alpha-\beta+\gamma)$$

$$= \cos(\alpha+\beta+\gamma) + \cos(\beta+\gamma-\alpha) + \cos(\gamma+\alpha-\beta) + \cos(\alpha+\beta-\gamma).$$

Example 6. Prove that $\sin^2 5x - \sin^2 3x = \sin 8x \sin 2x$.

First solution.

$$\sin^2 5x - \sin^2 3x = (\sin 5x + \sin 3x)(\sin 5x - \sin 3x)$$
$$= (2\sin 4x \cos x)(2\cos 4x \sin x)$$
$$= (2\sin 4x \cos 4x)(2\sin x \cos x)$$
$$= \sin 8x \sin 2x.$$

Second solution.

$$\sin 8x \sin 2x = \frac{1}{2}(\cos 6x - \cos 10x)$$
$$= \frac{1}{2}\{1 - 2\sin^2 3x - (1 - 2\sin^2 5x)\}$$
$$= \sin^2 5x - \sin^2 3x.$$

Third solution.

By using the formula of Art. 114 we have at once

$$\sin^2 5x - \sin^2 3x = \sin(5x + 3x)\sin(5x - 3x) = \sin 8x \sin 2x.$$

EXAMPLES. XII. c.

Prove the following identities:

1. $\cos 3A + \sin 2A - \sin 4A = \cos 3A(1 - 2\sin A)$.
2. $\sin 3\theta - \sin\theta - \sin 5\theta = \sin 3\theta(1 - 2\cos 2\theta)$.
3. $\cos\theta + \cos 2\theta + \cos 5\theta = \cos 2\theta(1 + 2\cos 3\theta)$.
4. $\sin\alpha - \sin 2\alpha + \sin 3\alpha = 4\sin\frac{\alpha}{2}\cos\alpha\cos\frac{3\alpha}{2}$.
5. $\sin 3\alpha + \sin 7\alpha + \sin 10\alpha = 4\sin 5\alpha\cos\frac{7\alpha}{2}\cos\frac{3\alpha}{2}$.
6. $\sin A + 2\sin 3A + \sin 5A = 4\sin 3A\cos^2 A$.
7. $\dfrac{\sin 2\alpha + \sin 5\alpha - \sin\alpha}{\cos 2\alpha + \cos 5\alpha + \cos\alpha} = \tan 2\alpha$.
8. $\dfrac{\sin\alpha + \sin 2\alpha + \sin 4\alpha + \sin 5\alpha}{\cos\alpha + \cos 2\alpha + \cos 4\alpha + \cos 5\alpha} = \tan 3\alpha$.
9. $\dfrac{\cos 7\theta + \cos 3\theta - \cos 5\theta - \cos\theta}{\sin 7\theta - \sin 3\theta - \sin 5\theta + \sin\theta} = \cot 2\theta$.
10. $\cos 3A \sin 2A - \cos 4A \sin A = \cos 2A \sin A$.

Prove the following identities:

11. $\cos 5A \cos 2A - \cos 4A \cos 3A = -\sin 2A \sin A$.

12. $\sin 4\theta \cos \theta - \sin 3\theta \cos 2\theta = \sin \theta \cos 2\theta$.

13. $\cos 5° - \sin 25° = \sin 35°$.

[*Use* $\sin 25° = \cos 65°$.]

14. $\sin 65° + \cos 65° = \sqrt{2} \cos 20°$.

15. $\cos 80° + \cos 40° - \cos 20° = 0$.

16. $\sin 78° - \sin 18° + \cos 132° = 0$.

17. $\sin^2 5A - \sin^2 2A = \sin 7A \sin 3A$.

18. $\cos 2A \cos 5A = \cos^2 \frac{7A}{2} - \sin^2 \frac{3A}{2}$.

19. $\sin(\alpha+\beta+\gamma) + \sin(\alpha-\beta-\gamma) + \sin(\alpha+\beta-\gamma)$
$+ \sin(\alpha-\beta+\gamma) = 4 \sin \alpha \cos \beta \cos \gamma$.

20. $\cos(\beta+\gamma-\alpha) - \cos(\gamma+\alpha-\beta) + \cos(\alpha+\beta-\gamma)$
$- \cos(\alpha+\beta+\gamma) = 4 \sin \alpha \cos \beta \sin \gamma$.

21. $\sin 2\alpha + \sin 2\beta + \sin 2\gamma - \sin 2(\alpha+\beta+\gamma)$
$= 4 \sin(\beta+\gamma) \sin(\gamma+\alpha) \sin(\alpha+\beta)$.

22. $\cos \alpha + \cos \beta + \cos \gamma + \cos(\alpha+\beta+\gamma)$
$= 4 \cos \frac{\beta+\gamma}{2} \cos \frac{\gamma+\alpha}{2} \cos \frac{\alpha+\beta}{2}$.

23. $4 \sin A \sin(60° + A) \sin(60° - A) = \sin 3A$.

24. $4 \cos \theta \cos\left(\frac{2\pi}{3} + \theta\right) \cos\left(\frac{2\pi}{3} - \theta\right) = \cos 3\theta$.

25. $\cos \theta + \cos\left(\frac{2\pi}{3} - \theta\right) + \cos\left(\frac{2\pi}{3} + \theta\right) = 0$.

26. $\cos^2 A + \cos^2(60° + A) + \cos^2(60° - A) = \frac{3}{2}$.

[*Put* $2 \cos^2 A = 1 + \cos 2A$.]

27. $\sin^2 A + \sin^2(120° + A) + \sin^2(120° - A) = \frac{3}{2}$.

28. $\cos 20° \cos 40° \cos 80° = \frac{1}{8}$.

29. $\sin 20° \sin 40° \sin 80° = \frac{1}{8}\sqrt{3}$.

135. Many interesting identities can be established connecting the functions of the three angles A, B, C, which satisfy the relation $A+B+C=180°$. In proving these it will be necessary to keep clearly in view the properties of complementary and supplementary angles. [Arts. 39 and 96.]

From the given relation, the sum of any two of the angles is the supplement of the third; thus

$$\sin(B+C)=\sin A, \qquad \cos(A+B)=-\cos C,$$
$$\tan(C+A)=-\tan B, \qquad \cos B=-\cos(C+A),$$
$$\sin C=\sin(A+B), \qquad \cot A=-\cot(B+C).$$

Again, $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=90°$, so that each half angle is the complement of the sum of the other two; thus

$$\cos\frac{A+B}{2}=\sin\frac{C}{2}, \quad \sin\frac{C+A}{2}=\cos\frac{B}{2}, \quad \tan\frac{B+C}{2}=\cot\frac{A}{2},$$

$$\cos\frac{C}{2}=\sin\frac{A+B}{2}, \quad \sin\frac{A}{2}=\cos\frac{B+C}{2}, \quad \tan\frac{B}{2}=\cot\frac{C+A}{2}.$$

Example 1. If $A+B+C=180°$, prove that

$$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C.$$

$$\begin{aligned}\text{The first side}&=2\sin(A+B)\cos(A-B)+2\sin C\cos C\\&=2\sin C\cos(A-B)+2\sin C\cos C\\&=2\sin C\{\cos(A-B)+\cos C\}\\&=2\sin C\{\cos(A-B)-\cos(A+B)\}\\&=2\sin C\times 2\sin A\sin B\\&=4\sin A\sin B\sin C.\end{aligned}$$

Example 2. If $A+B+C=180°$, prove that

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

Since $A+B$ is the supplement of C, we have

$$\tan(A+B)=-\tan C;$$

$$\therefore\ \frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C;$$

whence by multiplying up and rearranging,

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

Example 3. If $A+B+C=180°$, prove that

$$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

The first side $=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+\cos C$

$$=2\sin\frac{C}{2}\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2}$$

$$=1+2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\sin\frac{C}{2}\right)$$

$$=1+2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)$$

$$=1+2\sin\frac{C}{2}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)$$

$$=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

EXAMPLES. XII. d.

If $A+B+C=180°$, prove that

1. $\sin 2A-\sin 2B+\sin 2C=4\cos A\sin B\cos C.$

2. $\sin 2A-\sin 2B-\sin 2C=-4\sin A\cos B\cos C.$

3. $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$

4. $\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$

5. $\cos A-\cos B+\cos C=4\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}-1.$

6. $\dfrac{\sin B+\sin C-\sin A}{\sin A+\sin B+\sin C}=\tan\frac{B}{2}\tan\frac{C}{2}.$

7. $\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{A}{2}\tan\frac{B}{2}=1.$

[*Use* $\tan\frac{A+B}{2}=\cot\frac{C}{2}$, *and therefore* $\tan\frac{A+B}{2}\tan\frac{C}{2}=1.$]

If $A+B+C=180°$, prove that

8. $\dfrac{1+\cos A-\cos B+\cos C}{1+\cos A+\cos B-\cos C}=\tan\dfrac{B}{2}\cot\dfrac{C}{2}$.

9. $\cos 2A+\cos 2B+\cos 2C+4\cos A\cos B\cos C+1=0$.

10. $\cot B\cot C+\cot C\cot A+\cot A\cot B=1$.

11. $(\cot B+\cot C)(\cot C+\cot A)(\cot A+\cot B)$
$=\operatorname{cosec} A\operatorname{cosec} B\operatorname{cosec} C$.

12. $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1$.
[*Use* $2\cos^2 A=1+\cos 2A$.]

13. $\sin^2\dfrac{A}{2}+\sin^2\dfrac{B}{2}+\sin^2\dfrac{C}{2}=1-2\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$.

14. $\cos^2 2A+\cos^2 2B+\cos^2 2C=1+2\cos 2A\cos 2B\cos 2C$.

15. $\dfrac{\cot B+\cot C}{\tan B+\tan C}+\dfrac{\cot C+\cot A}{\tan C+\tan A}+\dfrac{\cot A+\cot B}{\tan A+\tan B}=1$.

16. $\dfrac{\tan A+\tan B+\tan C}{(\sin A+\sin B+\sin C)^2}=\dfrac{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}{2\cos A\cos B\cos C}$.

136. The following examples further illustrate the formulæ proved in this and the preceding chapter.

Example 1. Prove that $\cot(A+15°)-\tan(A-15°)=\dfrac{4\cos 2A}{2\sin 2A+1}$.

$$\text{The first side}=\frac{\cos(A+15°)}{\sin(A+15°)}-\frac{\sin(A-15°)}{\cos(A-15°)}$$

$$=\frac{\cos(A+15°)\cos(A-15°)-\sin(A+15°)\sin(A-15°)}{\sin(A+15°)\cos(A-15°)}$$

$$=\frac{\cos\{(A+15°)+(A-15°)\}}{\sin(A+15°)\cos(A-15°)}$$

$$=\frac{2\cos 2A}{2\sin(A+15°)\cos(A-15°)}=\frac{2\cos 2A}{\sin 2A+\sin 30°}$$

$$=\frac{4\cos 2A}{2\sin 2A+1}.$$

NOTE. In dealing with expressions which involve numerical angles it is usually advisable to effect some simplification before substituting the known values of the functions of the angles, especially if these contain surds.

Example 2. If $A+B+C=\pi$, prove that

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}.$$

$$\text{The second side}=2\cos\frac{\pi-A}{4}\left[\cos\frac{2\pi-(B+C)}{4}+\cos\frac{B-C}{4}\right]$$

$$=2\cos\frac{\pi-A}{4}\cos\frac{\pi+A}{4}+2\cos\frac{\pi-A}{4}\cos\frac{B-C}{4}$$

$$=\left(\cos\frac{\pi}{2}+\cos\frac{A}{2}\right)+2\cos\frac{B+C}{4}\cos\frac{B-C}{4}$$

$$=\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}.$$

EXAMPLES. XII. e.

Prove the following identities :

1. $\cos(\alpha+\beta)\sin(\alpha-\beta)+\cos(\beta+\gamma)\sin(\beta-\gamma)$
$+\cos(\gamma+\delta)\sin(\gamma-\delta)+\cos(\delta+\alpha)\sin(\delta-\alpha)=0.$

2. $\dfrac{\sin(\beta-\gamma)}{\sin\beta\sin\gamma}+\dfrac{\sin(\gamma-\alpha)}{\sin\gamma\sin\alpha}+\dfrac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta}=0.$

3. $\dfrac{\sin\alpha+\sin\beta+\sin(\alpha+\beta)}{\sin\alpha+\sin\beta-\sin(\alpha+\beta)}=\cot\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}.$

4. $\sin\alpha\cos(\beta+\gamma)-\sin\beta\cos(\alpha+\gamma)=\cos\gamma\sin(\alpha-\beta).$

5. $\cos\alpha\cos(\beta+\gamma)-\cos\beta\cos(\alpha+\gamma)=\sin\gamma\sin(\alpha-\beta).$

6. $(\cos A-\sin A)(\cos 2A-\sin 2A)=\cos A-\sin 3A.$

7. If $\tan\theta=\dfrac{b}{a}$, prove that $a\cos 2\theta+b\sin 2\theta=a$.

[*See Art.* 124.]

8. Prove that $\sin 2A+\cos 2A=\dfrac{(1+\tan A)^2-2\tan^2 A}{1+\tan^2 A}.$

9. Prove that $\sin 4A=\dfrac{4\tan A(1-\tan^2 A)}{(1+\tan^2 A)^2}.$

10. If $A+B=45^\circ$, prove that
$(1+\tan A)(1+\tan B)=2.$

Prove the following identities :

11. $\cot(15^\circ - A) + \tan(15^\circ + A) = \dfrac{4\cos 2A}{1 - 2\sin 2A}$.

12. $\cot(15^\circ + A) + \tan(15^\circ + A) = \dfrac{4}{\cos 2A + \sqrt{3}\sin 2A}$.

13. $\tan(A + 30^\circ)\tan(A - 30^\circ) = \dfrac{1 - 2\cos 2A}{1 + 2\cos 2A}$.

14. $(2\cos A + 1)(2\cos A - 1)(2\cos 2A - 1) = 2\cos 4A + 1$.

15. $\tan(\beta - \gamma) + \tan(\gamma - \alpha) + \tan(\alpha - \beta)$
$= \tan(\beta - \gamma)\tan(\gamma - \alpha)\tan(\alpha - \beta)$.

16. $\sin(\beta - \gamma) + \sin(\gamma - \alpha) + \sin(\alpha - \beta)$
$+ 4\sin\dfrac{\beta - \gamma}{2}\sin\dfrac{\gamma - \alpha}{2}\sin\dfrac{\alpha - \beta}{2} = 0$.

17. $\cos^2(\beta - \gamma) + \cos^2(\gamma - \alpha) + \cos^2(\alpha - \beta)$
$= 1 + 2\cos(\beta - \gamma)\cos(\gamma - \alpha)\cos(\alpha - \beta)$.

18. $\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos(\alpha + \beta) = \sin^2(\alpha + \beta)$.

19. $\sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta\cos(\alpha + \beta) = \sin^2(\alpha + \beta)$.

20. $\cos 12^\circ + \cos 60^\circ + \cos 84^\circ = \cos 24^\circ + \cos 48^\circ$.

If $A + B + C = 180^\circ$, shew that

21. $\cos\dfrac{A}{2} + \cos\dfrac{B}{2} + \cos\dfrac{C}{2} = 4\cos\dfrac{B + C}{4}\cos\dfrac{C + A}{4}\cos\dfrac{A + B}{4}$.

22. $\cos\dfrac{A}{2} - \cos\dfrac{B}{2} + \cos\dfrac{C}{2} = 4\cos\dfrac{\pi + A}{4}\cos\dfrac{\pi - B}{4}\cos\dfrac{\pi + C}{4}$.

23. $\sin\dfrac{A}{2} + \sin\dfrac{B}{2} + \sin\dfrac{C}{2} = 1 + 4\sin\dfrac{\pi - A}{4}\sin\dfrac{\pi - B}{4}\sin\dfrac{\pi - C}{4}$.

If $\alpha + \beta + \gamma = \dfrac{\pi}{2}$, shew that

24. $\dfrac{\sin 2\alpha + \sin 2\beta + \sin 2\gamma}{\sin 2\alpha + \sin 2\beta - \sin 2\gamma} = \cot\alpha\cot\beta$.

25. $\tan\beta\tan\gamma + \tan\gamma\tan\alpha + \tan\alpha\tan\beta = 1$.

CHAPTER XIII.

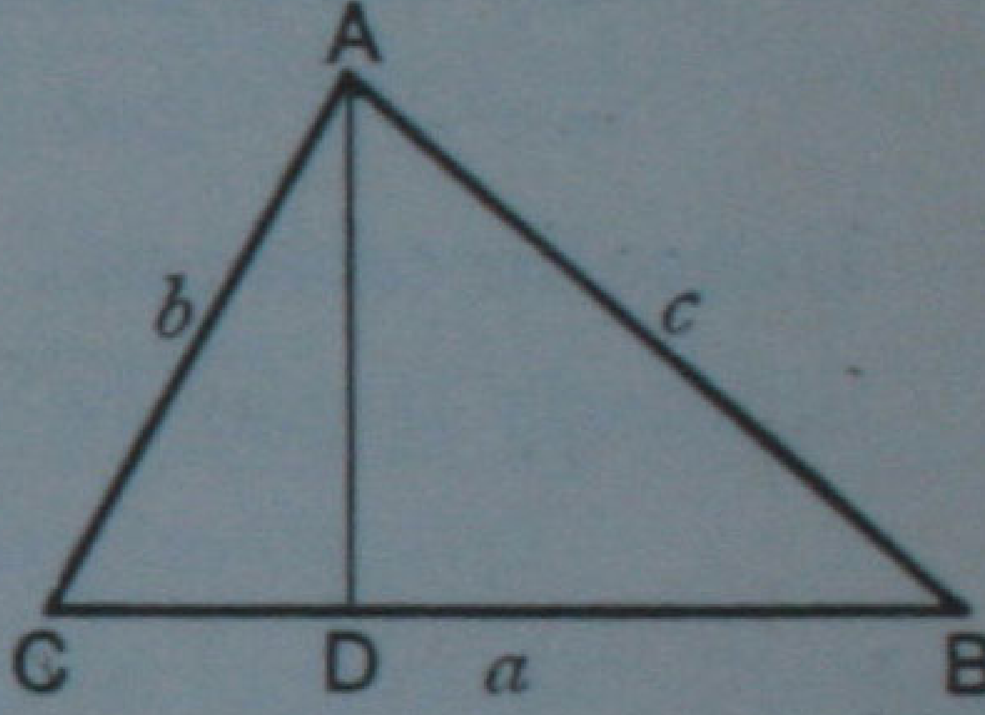
RELATIONS BETWEEN THE SIDES AND ANGLES OF A TRIANGLE.

137. *In any triangle the sides are proportional to the sines of the opposite angles; that is,*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

(1) Let the triangle ABC be acute-angled.

From A draw AD perpendicular to the opposite side; then



$$AD = AB \sin ABD = c \sin B,$$

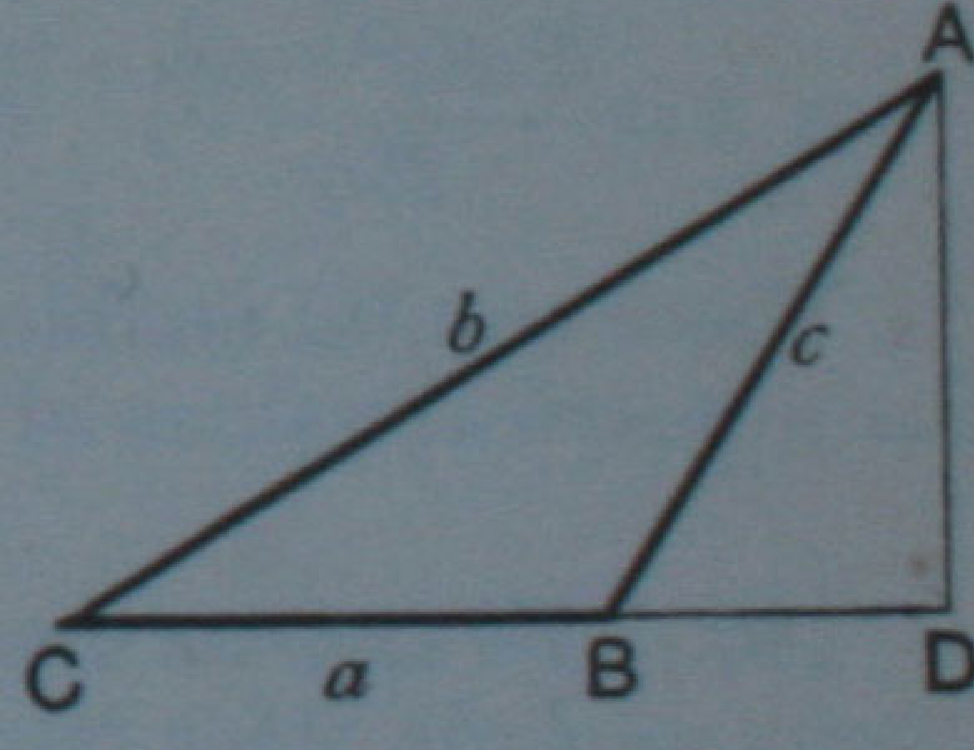
and $$AD = AC \sin ACD = b \sin C;$$

$$\therefore\ b \sin C = c \sin B,$$

that is, $$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

(2) Let the triangle ABC have an obtuse angle B.

Draw AD perpendicular to CB produced; then



$$AD = AC \sin ACD = b \sin C,$$

and $$AD = AB \sin ABD$$

$$= c \sin(180° - B) = c \sin B;$$

$$\therefore\ b \sin C = c \sin B;$$

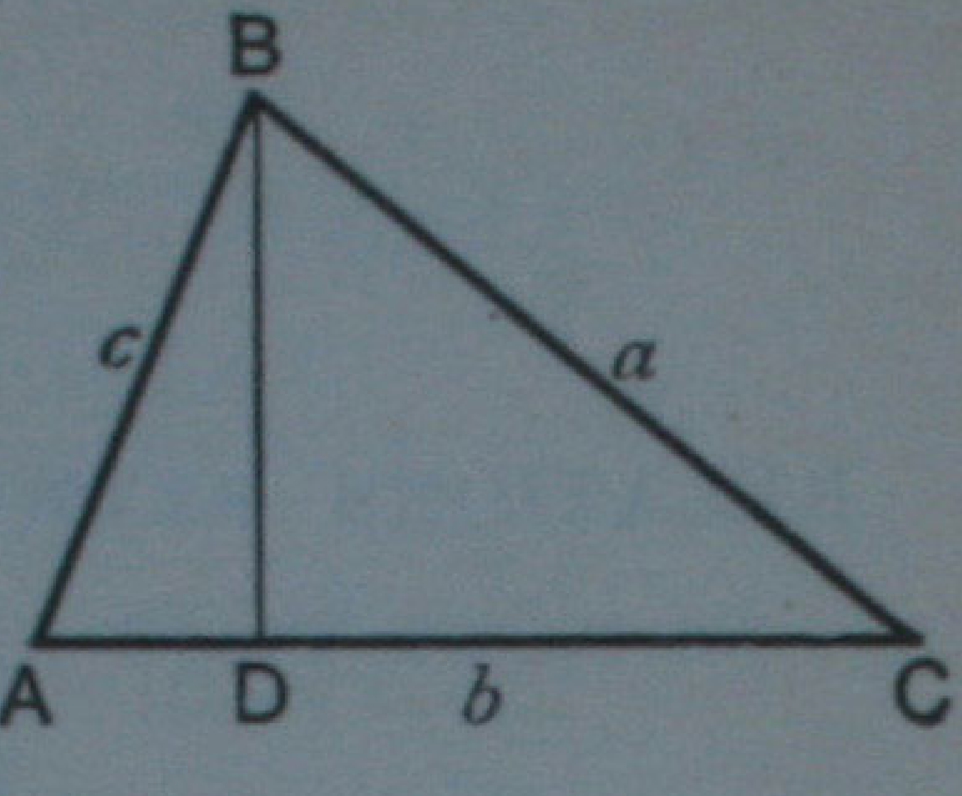
that is, $$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

In like manner it may be proved that either of these ratios is equal to $\frac{a}{\sin A}$.

Thus $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

138. *To find an expression for one side of a triangle in terms of the other two sides and the included angle.*

(1) Let ABC be an acute-angled triangle.



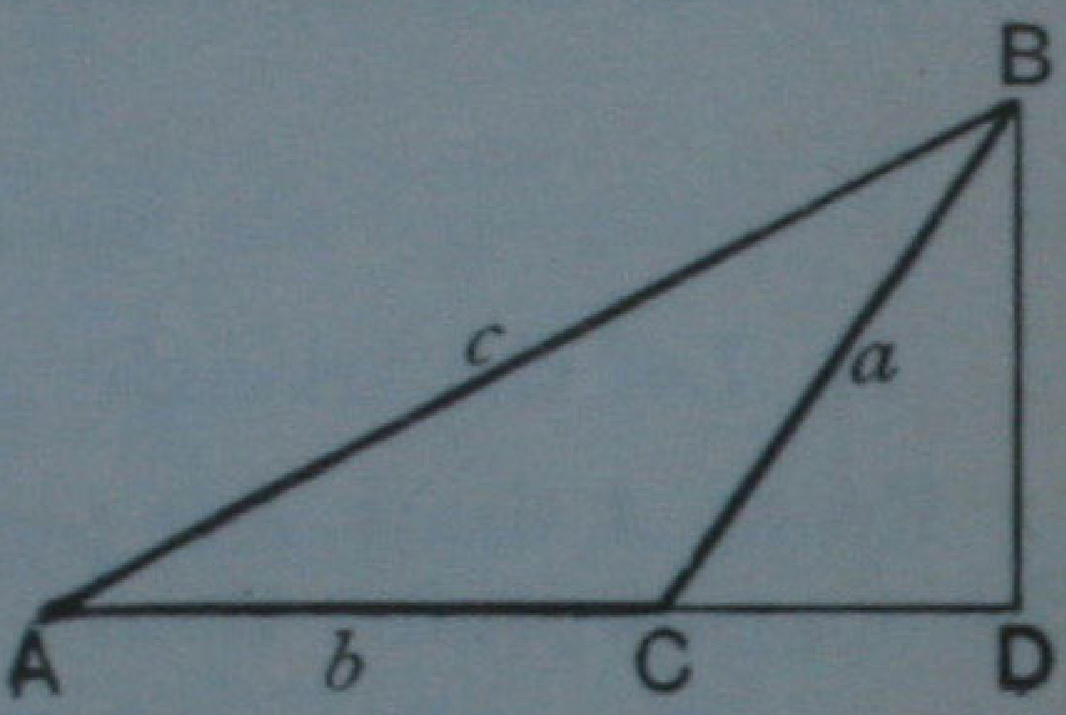
Draw BD perpendicular to AC; then by Euc. II. 13,

$$AB^2 = BC^2 + CA^2 - 2AC \,.\, CD;$$

$$\therefore\ c^2 = a^2 + b^2 - 2b \,.\, a \cos C$$

$$= a^2 + b^2 - 2ab \cos C.$$

(2) Let the triangle ABC have an obtuse angle C.



Draw BD perpendicular to AC produced; then by Euc. II. 12,

$$AB^2 = BC^2 + CA^2 + 2AC \,.\, CD;$$

$$\therefore\ c^2 = a^2 + b^2 + 2b \,.\, a \cos BCD$$

$$= a^2 + b^2 + 2ab \cos (180° - C)$$

$$= a^2 + b^2 - 2ab \cos C.$$

Hence in each case, $c^2 = a^2 + b^2 - 2ab \cos C$.

Similarly it may be shewn that

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

and $$b^2 = c^2 + a^2 - 2ca \cos B.$$

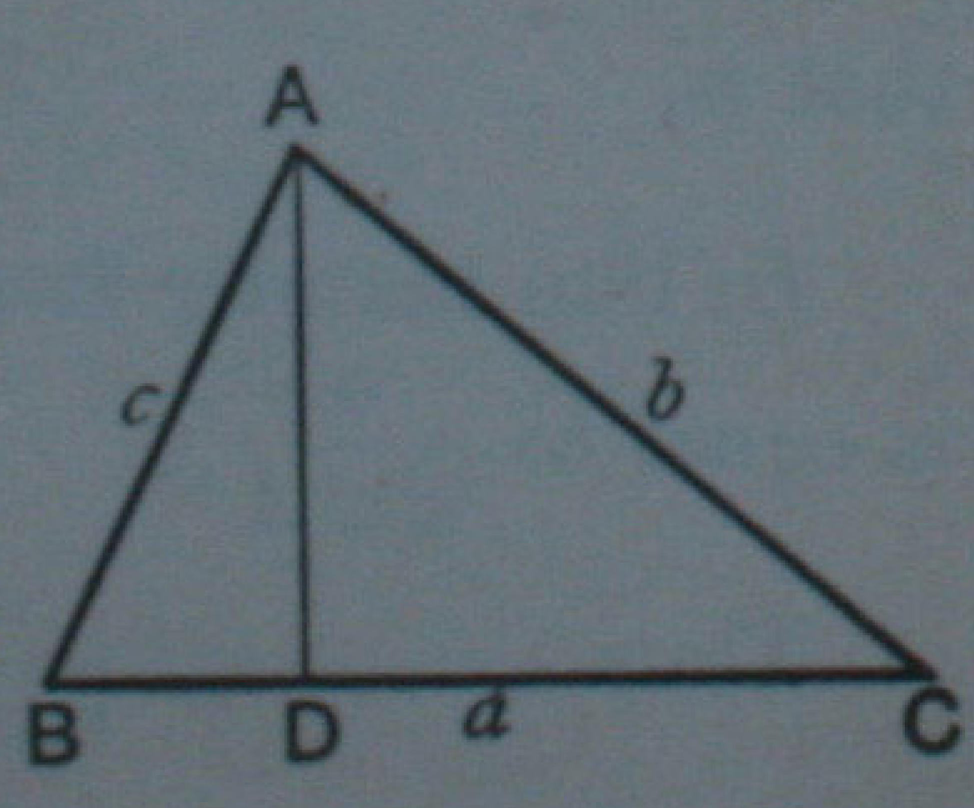
139. From the formulæ of the last article, we obtain

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}; \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}; \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

These results enable us to find the cosines of the angles when the numerical values of the sides are given.

140. *To express one side of a triangle in terms of the adjacent angles and the other two sides.*

(1) Let ABC be an acute-angled triangle.



Draw AD perpendicular to BC; then

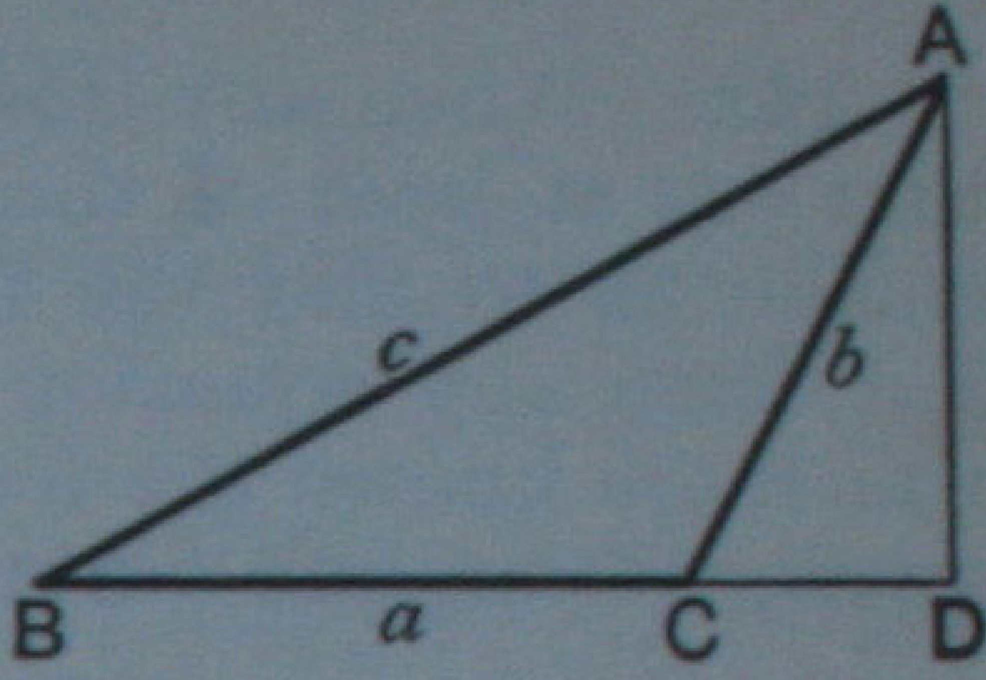
$$BC = BD + CD$$

$$= AB \cos ABD + AC \cos ACD;$$

that is, $$a = c \cos B + b \cos C.$$

(2) Let the triangle ABC have an obtuse angle C.

Draw AD perpendicular to BC produced; then



$$BC = BD - CD$$
$$= AB \cos ABD - AC \cos ACD;$$
$$\therefore\ a = c \cos B - b \cos (180° - C)$$
$$= c \cos B + b \cos C.$$

Thus in each case $a = b \cos C + c \cos B$.

Similarly it may be shewn that

$$b = c \cos A + a \cos C, \text{ and } c = a \cos B + b \cos A.$$

NOTE. The formulæ we have proved in this chapter are quite general and may be regarded as the fundamental relations subsisting between the sides and angles of a triangle. The modified forms which they assume in the case of right-angled triangles have already been considered in Chap. V.; it will therefore be unnecessary in the present chapter to make any direct reference to right-angled triangles.

141. The sets of formulæ in Arts. 137, 138, and 140 have been established independently of one another; they are however not independent, for from any one set the other two may be derived by the help of the relation $A + B + C = 180°$.

For instance, suppose we have proved as in Art. 137 that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C};$$

then since $\sin A = \sin (B + C) = \sin B \cos C + \sin C \cos B$;

$$\therefore\ 1 = \frac{\sin B}{\sin A} \cos C + \frac{\sin C}{\sin A} \cos B;$$

$$\therefore\ 1 = \frac{b}{a} \cos C + \frac{c}{a} \cos B;$$

$$\therefore\ a = b \cos C + c \cos B.$$

Similarly, we may prove that

$$b = c \cos A + a \cos C, \text{ and } c = a \cos B + b \cos A.$$

Multiplying these last three equations by a, b, $-c$ respectively and adding, we have

$$a^2 + b^2 - c^2 = 2ab \cos C;$$
$$\therefore\ c^2 = a^2 + b^2 - 2ab \cos C.$$

Similarly the other relations of Art. 138 may be deduced.

Solution of Triangles.

142. When any three parts of a triangle are given, provided that one at least of these is a side, the relations we have proved enable us to find the numerical values of the unknown parts. For from any equation which connects four quantities three of which are known the fourth may be found. Thus if c, a, B are given, we can find b from the formula

$$b^2=c^2+a^2-2ca\cos B;$$

and if B, C, b are given, we find c from the formula

$$\frac{c}{\sin C}=\frac{b}{\sin B}.$$

We may remark that if the three angles alone are given, the formula

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

enables us to find the *ratios* of the sides but not their actual *lengths*, and thus the triangle cannot be completely solved. In such a case there may be an infinite number of equiangular triangles all satisfying the data of the question. [See Euc. VI. 4.]

143. CASE I. *To solve a triangle having given the three sides.*

The angles A and B may be found from the formulæ

$$\cos A=\frac{b^2+c^2-a^2}{2bc}, \text{ and } \cos B=\frac{c^2+a^2-b^2}{2ca};$$

then the angle C is known from the equation $C=180°-A-B$.

Example 1. If $a=7$, $b=5$, $c=8$, find the angles A and B, having given that $\cos 38° 11'=\frac{11}{14}$.

$$\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{5^2+8^2-7^2}{2\times5\times8}=\frac{40}{2\times5\times8}=\frac{1}{2};$$

$$\therefore A=60°.$$

$$\cos B=\frac{c^2+a^2-b^2}{2ca}=\frac{8^2+7^2-5^2}{2\times8\times7}=\frac{88}{2\times8\times7}=\frac{11}{14};$$

$$\therefore B=38° 11'.$$

Example 2. Find the greatest angle of the triangle whose sides are 6, 13, 11, having given that $\cos 84^\circ 47' = \frac{1}{11}$.

Let $a=6$, $b=13$, $c=11$. Since the greatest angle is opposite to the greatest side, the required angle is B.

And $$\cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{11^2+6^2-13^2}{2\times 11\times 6} = \frac{-12}{2\times 11\times 6};$$

$$\therefore \cos B = -\frac{1}{11} = -\cos 84^\circ 47';$$

$$\therefore B = 180^\circ - 84^\circ 47' = 95^\circ 13'.$$

Thus the required angle is $95^\circ 13'$.

144. CASE II. *To solve a triangle having given two sides and the included angle.*

Let b, c, A be given; then a can be found from the formula

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

We may now obtain B from either of the formulæ

$$\cos B = \frac{c^2+a^2-b^2}{2ca}, \text{ or } \sin B = \frac{b \sin A}{a};$$

then C is known from the equation $C = 180^\circ - A - B$.

Example. If $a=3$, $b=7$, $C=98^\circ 13'$, solve the triangle, having given $\cos 81^\circ 47' = \frac{1}{7}$.

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$= 9 + 49 - 2\times 3\times 7 \cos 98^\circ 13'.$$

But $98^\circ 13'$ is the supplement of $81^\circ 47'$;

$$\therefore c^2 = 58 + (2\times 3\times 7 \cos 81^\circ 47')$$

$$= 58 + \left(2\times 3\times 7\times \frac{1}{7}\right) = 58 + 6 = 64;$$

$$\therefore c = 8.$$

$$\cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{64+9-49}{2\times 8\times 3} = \frac{24}{2\times 8\times 3} = \frac{1}{2};$$

$$\therefore B = 60^\circ.$$

$$C = 180^\circ - 60^\circ - 98^\circ 13' = 21^\circ 47'.$$

145. Case III. *To solve a triangle having given two angles and a side.*

Let B, C, a be given.

The angle A is found from $A=180°-B-C$; and the sides b and c from

$$b=\frac{a\sin B}{\sin A} \text{ and } c=\frac{a\sin C}{\sin A}.$$

Example. If $A=105°$, $C=60°$, $b=4$, solve the triangle.

$$B=180°-105°-60°=15°.$$

$$\therefore c=\frac{b\sin C}{\sin B}=\frac{4\sin 60°}{\sin 15°}=\frac{4\sqrt{3}}{2}\cdot\frac{2\sqrt{2}}{\sqrt{3}-1}=\frac{4\sqrt{6}}{\sqrt{3}-1}$$

$$=\frac{4\sqrt{6}(\sqrt{3}+1)}{3-1}=2\sqrt{6}(\sqrt{3}+1);$$

$$\therefore c=6\sqrt{2}+2\sqrt{6}.$$

$$a=\frac{b\sin A}{\sin B}=\frac{4\sin 105°}{\sin 15°}=\frac{4\sin 75°}{\sin 15°}$$

$$=4\times\frac{\sqrt{3}+1}{2\sqrt{2}}\times\frac{2\sqrt{2}}{\sqrt{3}-1}=\frac{4(\sqrt{3}+1)}{\sqrt{3}-1};$$

$$\therefore a=4(2+\sqrt{3}).$$

EXAMPLES. XIII. a.

1. If $a=15$, $b=7$, $c=13$, find C.
2. If $a=7$, $b=3$, $c=5$, find A.
3. If $a=5$, $b=5\sqrt{3}$, $c=5$, find the angles.
4. If $a=25$, $b=31$, $c=7\sqrt{2}$, find A.
5. The sides of a triangle are 2, $2\frac{2}{3}$, $3\frac{1}{3}$, find the greatest angle.
6. Solve the triangle when $a=\sqrt{3}+1$, $b=2$, $c=\sqrt{6}$.
7. Solve the triangle when $a=\sqrt{2}$, $b=2$, $c=\sqrt{3}-1$.
8. If $a=8$, $b=5$, $c=\sqrt{19}$, find C; given $\cos 28° 56'=\frac{7}{8}$.
9. If the sides are as 4 : 7 : 5, find the greatest angle;

   given $\cos 78° 27'=\frac{1}{5}$.

10. If $a=2$, $b=\sqrt{3}+1$, $C=60^\circ$, find c.

11. Given $a=3$, $c=5$, $B=120^\circ$, find b.

12. Given $b=7$, $c=6$, $A=75^\circ\,31'$, find a; given $\cos 75^\circ\,31'=\cdot 25$.

13. If $b=8$, $c=11$, $A=93^\circ\,35'$, find a; given $\cos 86^\circ\,25'=\cdot 0625$.

14. If $a=7$, $c=3$, $B=123^\circ\,12'$, find b; given $\cos 56^\circ\,48'=\frac{23}{42}$.

15. Solve the triangle when $a=2\sqrt{6}$, $c=6-2\sqrt{3}$, $B=75^\circ$.

16. Solve the triangle when $A=72^\circ$, $b=2$, $c=\sqrt{5}+1$.

17. Given $A=75^\circ$, $B=30^\circ$, $b=\sqrt{8}$, solve the triangle.

18. If $B=60^\circ$, $C=15^\circ$, $b=\sqrt{6}$, solve the triangle.

19. If $A=45^\circ$, $B=105^\circ$, $c=\sqrt{2}$, solve the triangle.

20. Given $A=45^\circ$, $B=60^\circ$, shew that $c:a=\sqrt{3}+1:2$.

21. If $C=120^\circ$, $c=2\sqrt{3}$, $a=2$, find b.

22. If $B=60^\circ$, $a=3$, $b=3\sqrt{3}$, find c.

23. Given $(a+b+c)(b+c-a)=3bc$, find A.

24. Find the angles of the triangle whose sides are

$$3+\sqrt{3},\quad 2\sqrt{3},\quad \sqrt{6}.$$

25. Find the angles of the triangle whose sides are

$$\frac{\sqrt{3}+1}{2\sqrt{2}},\quad \frac{\sqrt{3}-1}{2\sqrt{2}},\quad \frac{\sqrt{3}}{2}.$$

26. Two sides of a triangle are $\frac{1}{\sqrt{6}-\sqrt{2}}$ and $\frac{1}{\sqrt{6}+\sqrt{2}}$, and the included angle is 60°: solve the triangle.

146. When an angle of a triangle is obtained through the medium of the sine there may be ambiguity, for the sines of supplementary angles are equal in magnitude and are of the same sign, so that there are two angles less than 180° which have the same sine. When an angle is obtained through the medium of the cosine there is no ambiguity, for there is only one angle less than 180° whose cosine is equal to a given quantity.

Thus if $\qquad \sin A=\frac{1}{2}$, then $A=30^\circ$ or 150°;

if $\qquad \cos A=\frac{1}{2}$, then $A=60^\circ$.

Example. If $C=60°$, $b=2\sqrt{3}$, $c=3\sqrt{2}$, find A.

From the equation $$\sin B=\frac{b\sin C}{c},$$

we have $$\sin B=\frac{2\sqrt{3}}{3\sqrt{2}}\cdot\frac{\sqrt{3}}{2}=\frac{1}{\sqrt{2}};$$

$$\therefore B=45° \text{ or } 135°.$$

The value $B=135°$ is inadmissible, for in this case the sum of B and C would be greater than 180°.

Thus $$A=180°-60°-45°=75°.$$

147. CASE IV. *To solve a triangle having given two sides and an angle opposite to one of them.*

Let a, b, A be given; then B is to be found from the equation

$$\sin B=\frac{b}{a}\sin A.$$

(i) If $a<b\sin A$, then $\frac{b\sin A}{a}>1$, so that $\sin B>1$, which is impossible. Thus there is no solution.

(ii) If $a=b\sin A$, then $\frac{b\sin A}{a}=1$, so that $\sin B=1$, and B has only the value 90°.

(iii) If $a>b\sin A$, then $\frac{b\sin A}{a}<1$, and two values for B may be found from $\sin B=\frac{b\sin A}{a}$. These values are supplementary, so that one angle is acute, the other obtuse.

(1) If $a<b$, then $A<B$, and therefore B may either be acute or obtuse, so that both values are admissible. This is known as **the ambiguous case.**

(2) If $a=b$, then $A=B$; and if $a>b$, then $A>B$; in either case B cannot be obtuse, and therefore only the smaller value of B is admissible.

When B is found, C is determined from $C=180°-A-B$. Finally, c may be found from the equation $c=\frac{a\sin C}{\sin A}$.

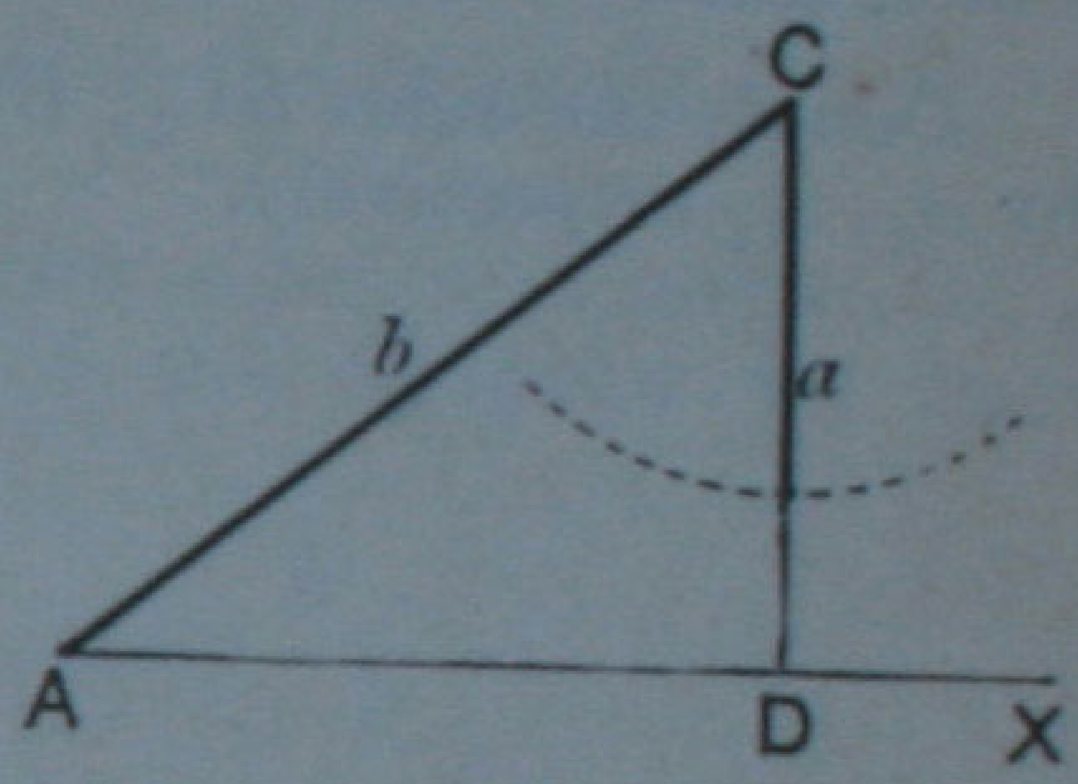
From the foregoing investigation it appears that *the only case in which an ambiguous solution can arise is when the smaller of the two given sides is opposite to the given angle.*

148. *To discuss the Ambiguous Case geometrically.*

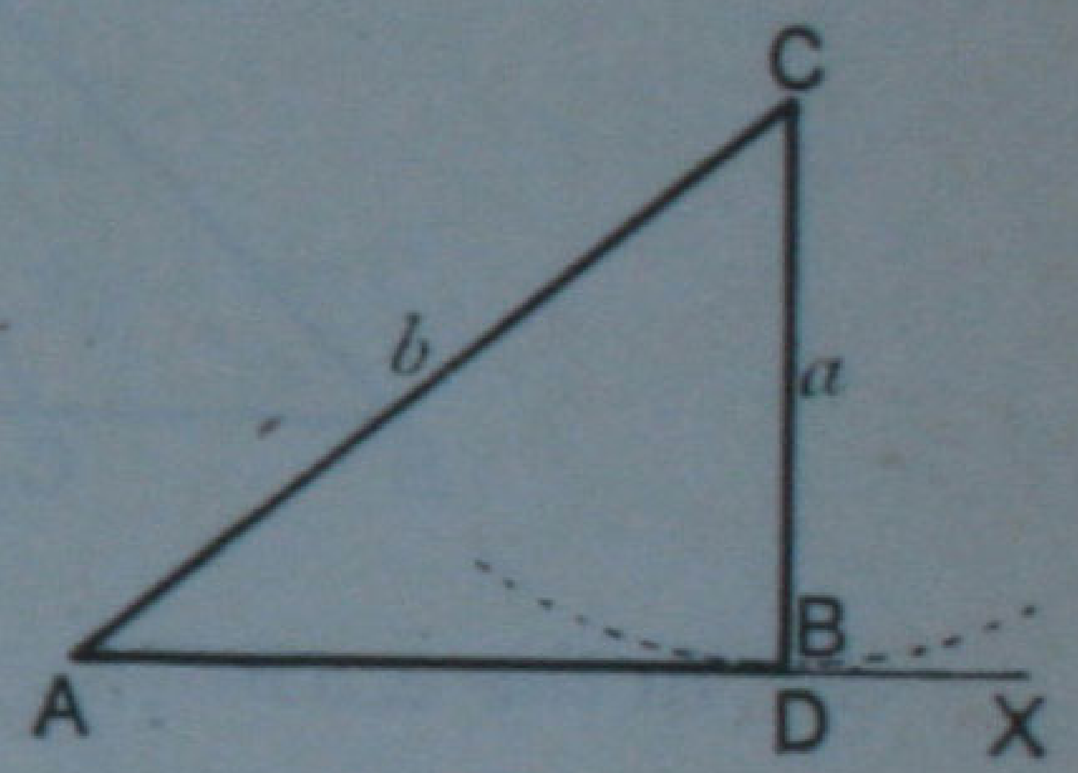
Let a, b, A be the given parts. Take a line AX unlimited towards X; make $\angle XAC$ equal to A, and AC equal to b. Draw CD perpendicular to AX, then $CD = b \sin A$.

With centre C and radius equal to a describe a circle.

(i) If $a < b \sin A$, the circle will not meet AX; thus no triangle can be constructed with the given parts.



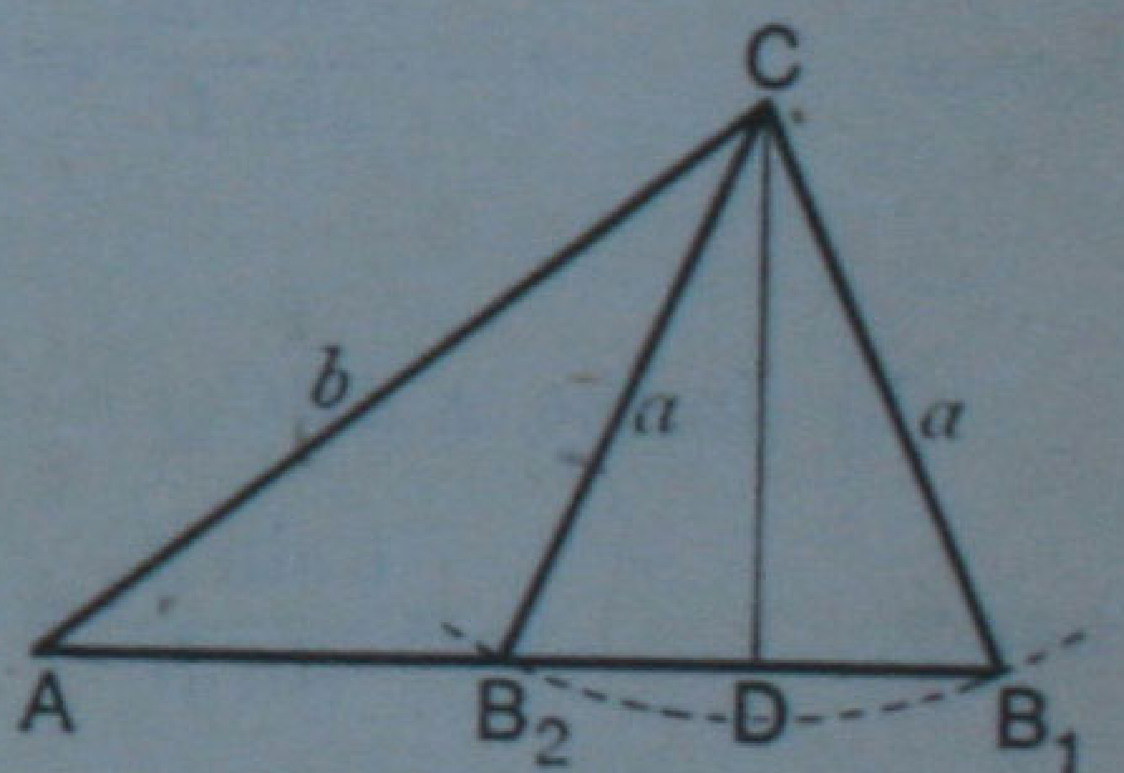
(ii) If $a = b \sin A$, the circle will *touch* AX at D; thus there is a right-angled triangle with the given parts.



(iii) If $a > b \sin A$, the circle will cut AX in two points B_1, B_2.

(1) These points will be both on the same side of A, when $a < b$, in which case there are two solutions, namely the triangles

$$AB_1C, \quad AB_2C.$$

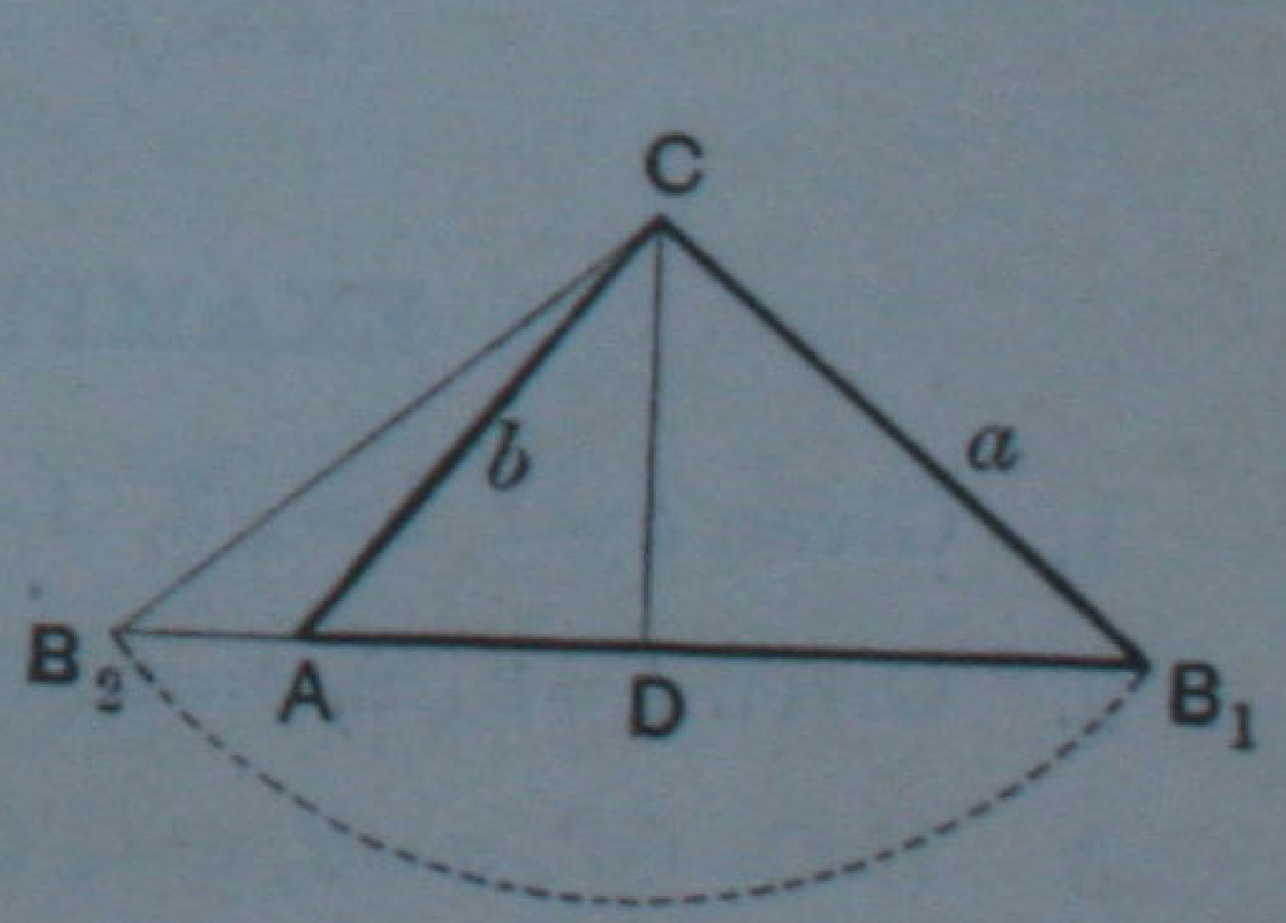


This is *the Ambiguous Case.*

(2) The points B_1, B_2 will be on opposite sides of A when $a > b$.

In this case there is only one solution, for the angle CAB_2 is *the supplement of the given angle*, and thus the triangle AB_2C does not satisfy the data.

(3) If $a = b$, the point B_2 coincides with A, so that there is only one solution.



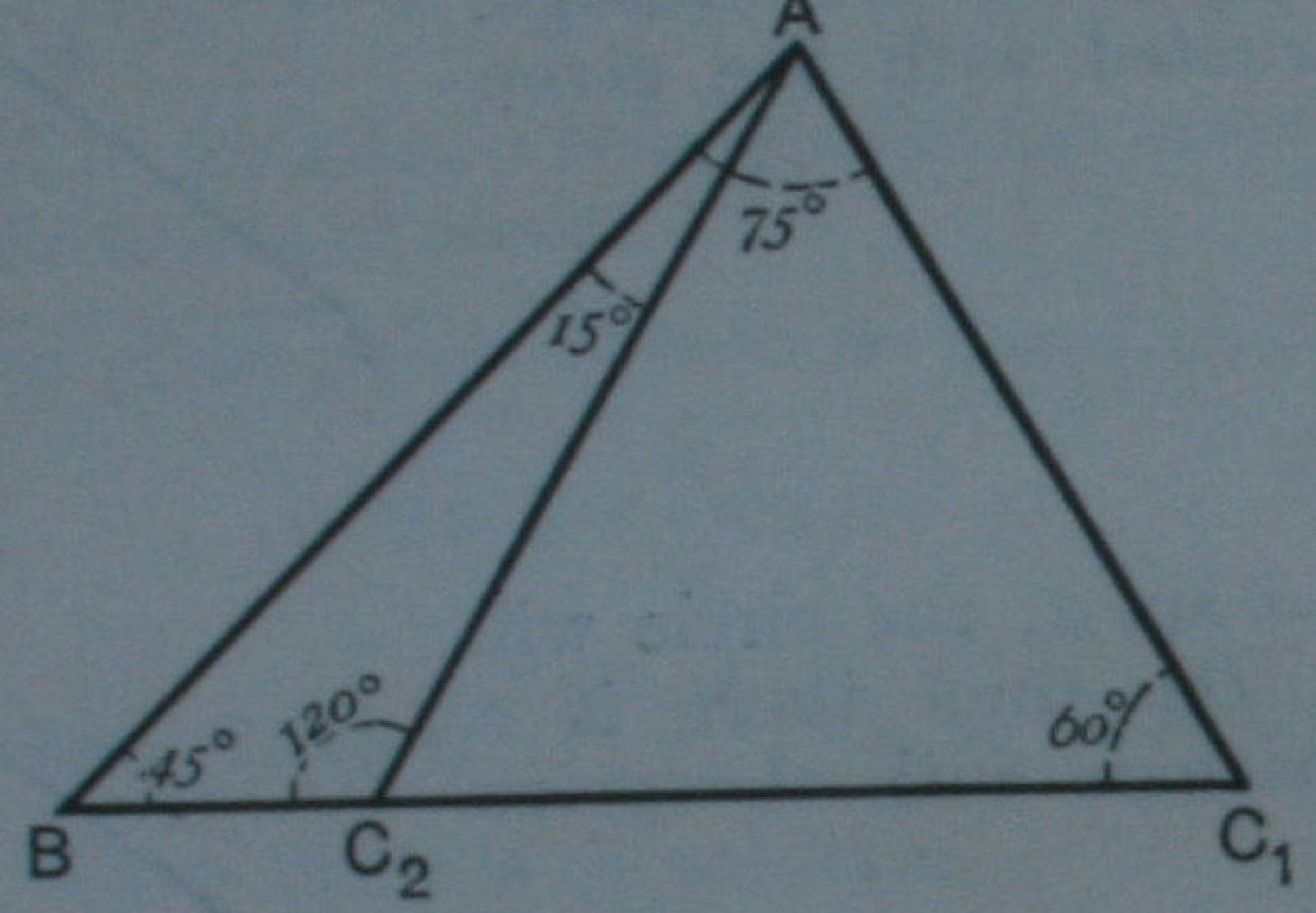
Example. Given $B=45^\circ$, $c=\sqrt{12}$, $b=\sqrt{8}$, solve the triangle.

We have $$\sin C=\frac{c\sin B}{b}=\frac{2\sqrt{3}}{2\sqrt{2}}\cdot\frac{1}{\sqrt{2}}=\frac{\sqrt{3}}{2}.$$

$$\therefore\ C=60^\circ \text{ or } 120^\circ,$$

and since $b<c$, both these values are admissible. The two triangles which satisfy the data are shewn in the figure.

Denote the sides BC_1, BC_2 by a_1, a_2, and the angles BAC_1, BAC_2 by A_1, A_2 respectively.



(i) In the $\triangle ABC_1$, $\angle A_1=75^\circ$;

hence $$a_1=\frac{b\sin A_1}{\sin B}=\frac{2\sqrt{2}}{\frac{1}{\sqrt{2}}}\cdot\frac{\sqrt{3}+1}{2\sqrt{2}}=\sqrt{2}(\sqrt{3}+1).$$

(ii) In the $\triangle ABC_2$, $\angle A_2=15^\circ$;

hence $$a_2=\frac{b\sin A_2}{\sin B}=\frac{2\sqrt{2}}{\frac{1}{\sqrt{2}}}\cdot\frac{\sqrt{3}-1}{2\sqrt{2}}=\sqrt{2}(\sqrt{3}-1).$$

Thus the complete solution is $$\begin{cases} C=60^\circ, \text{ or } 120^\circ; \\ A=75^\circ, \text{ or } 15^\circ; \\ a=\sqrt{6}+\sqrt{2}, \text{ or } \sqrt{6}-\sqrt{2}. \end{cases}$$

EXAMPLES. XIII. b.

1. Given $a=1$, $b=\sqrt{3}$, $A=30^\circ$, solve the triangle.

2. Given $b=3\sqrt{2}$, $c=2\sqrt{3}$, $C=45^\circ$, solve the triangle.

3. If $C=60^\circ$, $a=2$, $c=\sqrt{6}$, solve the triangle.

4. If $A=30^\circ$, $a=2$, $c=5$, solve the triangle.

5. If $B=30^\circ$, $b=\sqrt{6}$, $c=2\sqrt{3}$, solve the triangle.

6. If $B=60^\circ$, $b=3\sqrt{2}$, $c=3+\sqrt{3}$, solve the triangle.

7. If $a=3+\sqrt{3}$, $c=3-\sqrt{3}$, $C=15^\circ$, solve the triangle.

8. If $A=18^\circ$, $a=4$, $b=4+\sqrt{80}$, solve the triangle.

9. If $B=135^\circ$, $a=3\sqrt{2}$, $b=2\sqrt{3}$, solve the triangle.

149. Many relations connecting the sides and angles of a triangle may be proved by means of the formulæ we have established.

Example 1. Prove that $(b-c)\cos\frac{A}{2}=a\sin\frac{B-C}{2}$.

Let
$$k=\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C};$$
then
$$a=k\sin A,\quad b=k\sin B,\quad c=k\sin C;$$

$$\begin{aligned}\therefore\ (b-c)\cos\frac{A}{2}&=k(\sin B-\sin C)\cos\frac{A}{2}\\ &=2k\cos\frac{B+C}{2}\sin\frac{B-C}{2}\cos\frac{A}{2}\\ &=2k\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B-C}{2}\\ &=k\sin A\sin\frac{B-C}{2}\\ &=a\sin\frac{B-C}{2}.\end{aligned}$$

Example 2. If $a\cos^2\frac{C}{2}+c\cos^2\frac{A}{2}=\frac{3b}{2}$, shew that the sides of the triangle are in A.P.

Since
$$2a\cos^2\frac{C}{2}+2c\cos^2\frac{A}{2}=3b,$$
$$\therefore\ a(1+\cos C)+c(1+\cos A)=3b,$$
$$\therefore\ a+c+(a\cos C+c\cos A)=3b,$$
$$\therefore\ a+c+b=3b,$$
$$\therefore\ a+c=2b.$$

Thus the sides a, b, c are in A.P.

Example 3. Prove that

$$(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^2)\cot C=0.$$

Let $k=\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$; then

the first side

$$=k^2\left\{(\sin^2 B-\sin^2 C)\frac{\cos A}{\sin A}+\ldots\ldots+\ldots\ldots\right\}$$

$$=k^2\left\{\sin(B+C)\sin(B-C)\frac{\cos A}{\sin A}+\ldots\ldots+\ldots\ldots\right\}. \text{ [Art. 114].}$$

But $\sin(B+C)=\sin A$, and $\cos A=-\cos(B+C)$;

$\therefore$ the first side

$$=-k^2\{\sin(B-C)\cos(B+C)+\ldots\ldots+\ldots\ldots\}$$

$$=-\frac{k^2}{2}\{(\sin 2B-\sin 2C)+(\sin 2C-\sin 2A)+(\sin 2A-\sin 2B)\}$$

$$=0.$$

EXAMPLES. XIII. c.

Prove the following identities:

1. $a(\sin B-\sin C)+b(\sin C-\sin A)+c(\sin A-\sin B)=0.$

2. $2(bc\cos A+ca\cos B+ab\cos C)=a^2+b^2+c^2.$

3. $a(b\cos C-c\cos B)=b^2-c^2.$

4. $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+b+c.$

5. $2\left(a\sin^2\dfrac{C}{2}+c\sin^2\dfrac{A}{2}\right)=c+a-b.$

6. $\dfrac{\cos B}{\cos C}=\dfrac{c-b\cos A}{b-c\cos A}.$

7. $\tan A=\dfrac{a\sin C}{b-a\cos C}.$

8. $(b+c)\sin\dfrac{A}{2}=a\cos\dfrac{B-C}{2}.$

9. $\dfrac{a+b}{c}\sin^2\dfrac{C}{2}=\dfrac{\cos A+\cos B}{2}.$

10. $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0.$

11. $\dfrac{\sin(A-B)}{\sin(A+B)}=\dfrac{a^2-b^2}{c^2}.$

12. $\dfrac{c\sin(A-B)}{b\sin(C-A)}=\dfrac{a^2-b^2}{c^2-a^2}.$

[*All articles and examples marked with an asterisk may be omitted on the first reading of the subject.*]

***150.** The *ambiguous case* may also be discussed by first finding the third side.

As before, let a, b, A be given, then

$$\cos A = \frac{b^2+c^2-a^2}{2bc};$$

$$\therefore\ c^2 - 2b\cos A \,.\, c + b^2 - a^2 = 0.$$

By solving this quadratic equation in c, we obtain

$$c = b\cos A \pm \sqrt{b^2\cos^2 A + a^2 - b^2}$$
$$= b\cos A \pm \sqrt{a^2 - b^2\sin^2 A}.$$

(i) When $a < b\sin A$, the quantity under the radical is negative, and the values of c are impossible; so that there is no solution.

(ii) When $a = b\sin A$, the quantity under the radical is zero, and $c = b\cos A$. Since $\sin A < 1$, it follows that $a < b$, and therefore $A < B$. Hence the triangle is impossible unless the given angle A is acute, in which case c is positive and there is one solution.

(iii) When $a > b\sin A$, there are three cases to consider.

(1) Suppose $a < b$, then $A < B$, and as before the triangle is impossible unless A is acute. In this case $b\cos A$ is positive.

Also $\sqrt{a^2 - b^2\sin^2 A}$ is real and $< \sqrt{b^2 - b^2\sin^2 A}$;

that is $\qquad \sqrt{a^2 - b^2\sin^2 A} < b\cos A$;

hence both values of c are real and positive, so that there are two solutions.

(2) Suppose $a > b$, then $\sqrt{a^2 - b^2\sin^2 A} > \sqrt{b^2 - b^2\sin^2 A}$;

that is $\qquad \sqrt{a^2 - b^2\sin^2 A} > b\cos A$;

hence one value of c is positive and one value is negative, whether A is acute or obtuse, and in each case there is only one solution.

(3) Suppose $a = b$, then $\sqrt{a^2 - b^2\sin^2 A} = b\cos A$;

$$\therefore\ c = 2b\cos A \text{ or } 0;$$

hence there is only one solution when A is acute, and when A is obtuse the triangle is impossible.

Example. If b, c, B are given, and if $b<c$, shew that

$$(a_1-a_2)^2+(a_1+a_2)^2\tan^2 B=4b^2,$$

where a_1, a_2 are the two values of the third side.

From the formula $\cos B=\dfrac{c^2+a^2-b^2}{2ca}$,

we have $a^2-2c\cos B \,.\, a+c^2-b^2=0.$

But the roots of this equation are a_1 and a_2; hence by the theory of quadratic equations

$$a_1+a_2=2c\cos B \text{ and } a_1a_2=c^2-b^2.$$

$$\therefore\ (a_1-a_2)^2=(a_1+a_2)^2-4a_1a_2$$
$$=4c^2\cos^2 B-4(c^2-b^2).$$

$$\therefore\ (a_1-a_2)^2+(a_1+a_2)^2\tan^2 B=4c^2\cos^2 B-4(c^2-b^2)+4c^2\cos^2 B\tan^2 B$$
$$=4c^2(\cos^2 B+\sin^2 B)-4c^2+4b^2$$
$$=4c^2-4c^2+4b^2$$
$$=4b^2.$$

*EXAMPLES. XIII. d.

1. In a triangle in which each base angle is double of the third angle the base is 2 : solve the triangle.

2. If $B=45°$, $C=75°$, and the perpendicular from A on BC is 3, solve the triangle.

3. If $a=2$, $b=4-2\sqrt{3}$, $c=3\sqrt{2}-\sqrt{6}$, solve the triangle.

4. If $A=18°$, $b-a=2$, $ab=4$, find the other angles.

5. Given $B=30°$, $c=150$, $b=50\sqrt{3}$, shew that of the two triangles which satisfy the data one will be isosceles and the other right-angled.

   Find the third side in the greater of these triangles. Would the solution be ambiguous if the data had been $B=30°$, $c=150$, $b=75$?

6. If $A=36°$, $a=4$, and the perpendicular from C upon AB is $\sqrt{5}-1$, find the other angles.

7. If the angles adjacent to the base of a triangle are $22\frac{1}{2}°$ and $112\frac{1}{2}°$, shew that the altitude is half the base.

8. If $a=2b$ and $A=3B$, find the angles and express c in terms of a.

9. The sides of a triangle are $2x+3$, x^2+3x+3, x^2+2x: shew that the greatest angle is 120°.

Shew that in any triangle

10. $(b-a)\cos C+c(\cos B-\cos A)=c\sin\dfrac{A-B}{2}\operatorname{cosec}\dfrac{A+B}{2}$.

11. $a\sin\left(\dfrac{A}{2}+B\right)=(b+c)\sin\dfrac{A}{2}$.

12. $\sin\left(B+\dfrac{C}{2}\right)\cos\dfrac{C}{2}=\dfrac{a+b}{b+c}\cos\dfrac{A}{2}\cos\dfrac{B-C}{2}$.

13. $\dfrac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\dfrac{a^2+b^2}{a^2+c^2}$.

14. If $c^4-2(a^2+b^2)c^2+a^4+a^2b^2+b^4=0$, prove that C is 60° or 120°.

15. If a, b, A are given, and if c_1, c_2 are the values of the third side in the ambiguous case, prove that if $c_1>c_2$,

(1) $c_1-c_2=2a\cos B_1$.

(2) $\cos\dfrac{C_1-C_2}{2}=\dfrac{b\sin A}{a}$.

(3) $c_1^2+c_2^2-2c_1c_2\cos 2A=4a^2\cos^2 A$.

(4) $\sin\dfrac{C_1+C_2}{2}\sin\dfrac{C_1-C_2}{2}=\cos A\cos B_1$.

16. If $A=45°$, and c_1, c_2 be the two values of the ambiguous side, shew that

$$\cos B_1CB_2=\frac{2c_1c_2}{c_1^2+c_2^2}.$$

17. If $\cos A+2\cos C:\cos A+2\cos B=\sin B:\sin C$, prove that the triangle is either isosceles or right-angled.

18. If a, b, c are in A.P., shew that

$$\cot\frac{A}{2},\ \cot\frac{B}{2},\ \cot\frac{C}{2}\text{ are also in A.P.}$$

19. Shew that

$$\frac{a^2\sin(B-C)}{\sin B+\sin C}+\frac{b^2\sin(C-A)}{\sin C+\sin A}+\frac{c^2\sin(A-B)}{\sin A+\sin B}=0.$$

MISCELLANEOUS EXAMPLES. D.

1. Prove that (1) $\tan 2\theta \cot \theta - 1 = \sec 2\theta$;

(2) $\sin \alpha - \cot \theta \cos \alpha = -\operatorname{cosec} \theta \cos(\alpha + \theta)$.

2. If $a = 48$, $b = 35$, $C = 60°$, find c.

3. If $\cos \alpha = \frac{8}{17}$ and $\cos \beta = \frac{15}{17}$, find

$\tan(\alpha + \beta)$ and $\operatorname{cosec}(\alpha + \beta)$.

4. If $\alpha = \frac{\pi}{21}$, find the value of $\frac{\sin 23\alpha - \sin 7\alpha}{\sin 2\alpha + \sin 14\alpha}$.

5. Prove that $\sin \theta (\cos 2\theta + \cos 4\theta + \cos 6\theta) = \sin 3\theta \cos 4\theta$.

6. If $b = \sqrt{2}$, $c = \sqrt{3} + 1$, $A = 45°$, solve the triangle.

7. Prove that

(1) $2 \sin^2 36° = \sqrt{5} \sin 18°$; (2) $4 \sin 36° \cos 18° = \sqrt{5}$.

8. Prove that $\frac{\sin 3\alpha}{\sin \alpha} + \frac{\cos 3\alpha}{\cos \alpha} = 4 \cos 2\alpha$.

9. If $b = c = 2$, $a = \sqrt{6} - \sqrt{2}$, solve the triangle.

10. Shew that

(1) $\cos 2\alpha - \cot 3\alpha \sin 2\alpha = \tan \alpha (\sin 2\alpha + \cot 3\alpha \cos 2\alpha)$.

(2) $\cos \alpha + \cos 2\alpha + \cos 3\alpha = 4 \cos \alpha \cos \frac{\alpha}{2} \cos \frac{3\alpha}{2} - 1$.

11. In any triangle, prove that

(1) $b^2 \sin 2C + c^2 \sin 2B = 2bc \sin A$;

(2) $\frac{a^2 \sin(B - C)}{\sin A} + \frac{b^2 \sin(C - A)}{\sin B} + \frac{c^2 \sin(A - B)}{\sin C} = 0$.

12. If A, B, C, D are the angles of a quadrilateral, prove that

$$\frac{\tan A + \tan B + \tan C + \tan D}{\cot A + \cot B + \cot C + \cot D} = \tan A \tan B \tan C \tan D.$$

[*Use* $\tan(A + B) = \tan(360° - C - D)$.]

CHAPTER XIV.

LOGARITHMS.

151. DEFINITION. The **logarithm** of any number to a given **base** is the index of the power to which the base must be raised in order to equal the given number. Thus if $a^x = N$, x is called the logarithm of N to the base a.

Example 1. Since $3^4 = 81$, the logarithm of 81 to base 3 is 4.

Example 2. Since $10^1 = 10$, $10^2 = 100$, $10^3 = 1000, \ldots\ldots$ the natural numbers 1, 2, 3,... are respectively the logarithms of 10, 100, 1000,...... to base 10.

Example 3. Find the logarithm of ·008 to base 25.

Let x be the required logarithm; then by definition,

$$25^x = \cdot 008 = \frac{8}{1000} = \frac{1}{125} = \frac{1}{5^3};$$

that is, $(5^2)^x = 5^{-3}$, or $5^{2x} = 5^{-3}$;

whence, by equating indices, $2x = -3$, and $x = -1\cdot 5$.

152. The logarithm of N to base a is usually written $\log_a N$, so that the same meaning is expressed by the two equations

$$a^x = N, \qquad x = \log_a N.$$

From these equations it is evident that $a^{\log_a N} = N$.

Example. Find the value of $\log_{\cdot 01} \cdot 00001$.

Let $\log_{\cdot 01} \cdot 00001 = x$; then $(\cdot 01)^x = \cdot 00001$;

$$\therefore \left(\frac{1}{10^2}\right)^x = \frac{1}{100000}, \text{ or } \frac{1}{10^{2x}} = \frac{1}{10^5}.$$

$$\therefore 2x = 5, \text{ and } x = 2\cdot 5.$$

153. When it is understood that a particular system of logarithms is in use, the suffix denoting the base is omitted.

Thus in arithmetical calculations in which 10 is the base, we usually write log 2, log 3,...... instead of $\log_{10} 2$, $\log_{10} 3$,......

Logarithms to the base 10 are known as **Common Logarithms**; this system was first introduced in 1615 by Briggs, a contemporary of Napier the inventor of Logarithms.

Before discussing the properties of common logarithms we shall prove some general propositions which are true for all logarithms independently of any particular base.

154. *The logarithm of* 1 *is* 0.

For $a^0=1$ for all values of a; therefore $\log 1=0$, whatever the base may be.

155. *The logarithm of the base itself is* 1.

For $a^1=a$; therefore $\log_a a=1$.

156. *To find the logarithm of a product.*

Let MN be the product; let a be the base of the system, and suppose

$$x=\log_a M, \qquad y=\log_a N;$$

so that

$$a^x=M, \qquad a^y=N.$$

Thus the product $MN=a^x \times a^y=a^{x+y}$;

whence, by definition, $\log_a MN=x+y$

$$=\log_a M+\log_a N.$$

Similarly, $\log_a MNP=\log_a M+\log_a N+\log_a P$; and so on for any number of factors.

Example. $\log 42=\log(2\times3\times7)=\log 2+\log 3+\log 7$.

157. *To find the logarithm of a fraction.*

Let $\frac{M}{N}$ be the fraction, and suppose

$$x=\log_a M, \qquad y=\log_a N;$$

so that

$$a^x=M, \qquad a^y=N.$$

Thus the fraction $\frac{M}{N}=\frac{a^x}{a^y}=a^{x-y}$;

whence, by definition, $\log_a \frac{M}{N}=x-y$

$$=\log_a M-\log_a N.$$

Example. $\log(2\frac{1}{7}) = \log\frac{15}{7} = \log 15 - \log 7$

$$= \log(3\times 5) - \log 7 = \log 3 + \log 5 - \log 7.$$

158. *To find the logarithm of a number raised to any power, integral or fractional.*

Let $\log_a(M^p)$ be required, and suppose

$$x = \log_a M, \text{ so that } a^x = M;$$

then $$M^p = (a^x)^p = a^{px};$$

whence, by definition, $$\log_a(M^p) = px;$$

that is, $$\log_a(M^p) = p\log_a M.$$

Similarly, $$\log_a(M^{\frac{1}{r}}) = \frac{1}{r}\log_a M.$$

159. It follows from the results we have proved that

(1) the logarithm of a product is equal to the sum of the logarithms of its factors;

(2) the logarithm of a fraction is equal to the logarithm of the numerator diminished by the logarithm of the denominator;

(3) the logarithm of the pth power of a number is p times the logarithm of the number;

(4) the logarithm of the rth root of a number is $\frac{1}{r}$ of the logarithm of the number.

Thus by the use of logarithms the operations of multiplication and division may be replaced by those of addition and subtraction; the operations of involution and evolution by those of multiplication and division.

Example. Express $\log\frac{a^5\sqrt{b}}{\sqrt[3]{c^2}}$ in terms of $\log a$, $\log b$, $\log c$.

The expression $= \log(a^5\sqrt{b}) - \log\sqrt[3]{c^2}$

$$= \log a^5 + \log\sqrt{b} - \frac{2}{3}\log c$$

$$= 5\log a + \frac{1}{2}\log b - \frac{2}{3}\log c.$$

160. From the equation $10^x = N$, it is evident that common logarithms will not in general be integral, and that they will not always be positive.

For instance $3154 > 10^3$ and $< 10^4$;

$\therefore \log 3154 = 3 + \text{a fraction}.$

Again, $\cdot 06 > 10^{-2}$ and $< 10^{-1}$;

$\therefore \log \cdot 06 = -2 + \text{a fraction}.$

161. Definition. The integral part of a logarithm is called the **characteristic**, and the decimal part is called the **mantissa**.

The characteristic of the logarithm of any number to the base 10 can be found by inspection, as we shall now shew.

162. *To determine the characteristic of the logarithm of any number greater than unity.*

It is clear that a number with two digits in its integral part lies between 10^1 and 10^2; a number with three digits in its integral part lies between 10^2 and 10^3; and so on. Hence a number with n digits in its integral part lies between 10^{n-1} and 10^n.

Let N be a number whose integral part contains n digits; then

$$N = 10^{(n-1) + \text{a fraction}};$$

$$\therefore \log N = (n-1) + \text{a fraction}.$$

Hence the characteristic is $n-1$; that is, *the characteristic of the logarithm of a number greater than unity is less by one than the number of digits in its integral part, and is positive.*

163. *To determine the characteristic of the logarithm of a decimal fraction.*

A decimal with one cipher immediately after the decimal point, such as $\cdot 0324$, being greater than $\cdot 01$ and less than $\cdot 1$, lies between 10^{-2} and 10^{-1}; a number with two ciphers after the decimal point lies between 10^{-3} and 10^{-2}; and so on. Hence a decimal fraction with n ciphers immediately after the decimal point lies between $10^{-(n+1)}$ and 10^{-n}.

Let D be a decimal beginning with n ciphers; then

$$D = 10^{-(n+1) + \text{a fraction}};$$

$$\therefore \log D = -(n+1) + \text{a fraction}.$$

Hence the characteristic is $-(n+1)$; that is, *the characteristic of the logarithm of a decimal fraction is greater by unity than the number of ciphers immediately after the decimal point and is negative.*

164. The logarithms to base 10 of all integers from 1 to 200000 have been found and tabulated; in most Tables they are given to seven places of decimals.

The base 10 is chosen on account of two great advantages.

(1) From the results already proved it is evident that the characteristics can be written down by inspection, so that only the mantissæ have to be registered in the Tables.

(2) The mantissæ are the same for the logarithms of all numbers which have the same significant digits; so that it is sufficient to tabulate the mantissæ of the logarithms of *integers.*

This proposition we proceed to prove.

165. Let N be any number, then since multiplying or dividing by a power of 10 merely alters the position of the decimal point without changing the sequence of figures, it follows that $N\times 10^p$, and $N\div 10^q$, where p and q are any integers, are numbers whose significant digits are the same as those of N.

Now $$\log(N\times 10^p)=\log N+p\log 10$$
$$=\log N+p \quad\text{.......................}(1).$$

Again, $$\log(N\div 10^q)=\log N-q\log 10$$
$$=\log N-q \quad\text{.......................}(2).$$

In (1) an integer is added to $\log N$, and in (2) an integer is subtracted from $\log N$; that is, the mantissa or decimal portion of the logarithm remains unaltered.

In this and the three preceding articles the mantissæ have been supposed positive. In order to secure the advantages of Briggs' system, we arrange our work so as *always to keep the mantissa positive*, so that when the mantissa of any logarithm has been taken from the Tables the characteristic is prefixed with its appropriate sign, according to the rules already given.

166. In the case of a negative logarithm the minus sign is written *over the characteristic*, and not before it, to indicate that the characteristic alone is negative, and not the whole expression.

Thus $\overline{4}\cdot30103$, the logarithm of $\cdot0002$, is equivalent to $-4+\cdot30103$, and must be distinguished from $-4\cdot30103$, an expression in which both the integer and the decimal are negative. In working with negative logarithms an arithmetical artifice will sometimes be necessary in order to make the mantissa positive. For instance, a result such as $-3\cdot69897$, in which the whole expression is negative, may be transformed by subtracting 1 from the integral part and adding 1 to the decimal part. Thus

$$-3\cdot69897=-4+(1-\cdot69897)=\overline{4}\cdot30103.$$

Example 1. Required the logarithms of $\cdot0002432$.

In the Tables we find that 3859636 is the mantissa of log 2432 (the decimal point as well as the characteristic being omitted); and, by Art. 163, the characteristic of the logarithm of the given number is -4;

$$\therefore \log \cdot0002432=\overline{4}\cdot3859636.$$

Example 2. Find the cube root of $\cdot0007$, having given

$$\log 7=\cdot8450980, \quad \log 887904=5\cdot9483660.$$

Let x be the required cube root; then

$$\log x=\frac{1}{3}\log(\cdot0007)=\frac{1}{3}(\overline{4}\cdot8450980)=\frac{1}{3}(\overline{6}+2\cdot8450980);$$

that is, $$\log x=\overline{2}\cdot9483660;$$

but $$\log 887904=5\cdot9483660;$$

$$\therefore x=\cdot0887904.$$

167. The logarithm of 5 and its powers can easily be obtained from log 2; for

$$\log 5=\log\frac{10}{2}=\log 10-\log 2=1-\log 2.$$

Example. Find the value of the logarithm of the reciprocal of $324\sqrt[5]{125}$, having given $\log 2=\cdot3010300$, $\log 3=\cdot4771213$.

Since $\log\frac{1}{a}=-\log a$, the required value

$$=-\log(324\sqrt[5]{125})=-\log(2^2\times3^4\times5^{\frac{3}{5}})$$

$$=-\left(2\log 2+4\log 3+\frac{3}{5}\log 5\right)$$

$$=-2\cdot9299272$$

$$=\overline{3}\cdot0700728.$$

$$2\log 2=\cdot6020600$$
$$4\log 3=1\cdot9084852$$
$$\frac{6}{10}\log 5=\cdot4193820$$
$$\overline{\quad 2\cdot9299272}$$

EXAMPLES. XIV. a.

1. Find the logarithms respectively

of the numbers 1024, 81, ·125, ·01, $\dot{\cdot 3}$, 100,

to the bases 2, $\sqrt{3}$, 4, ·001, $\dot{\cdot 1}$, ·01.

2. Find the values of

$$\log_8 16, \quad \log_{81} 243, \quad \log_{\cdot 01} 10, \quad \log_{49} 343\sqrt{7}.$$

3. Find the numbers whose logarithms respectively

to the bases 49, ·25, ·03, 1, ·64, 100, ·1,

are 2, $\frac{1}{2}$, -2, -1, $-\frac{1}{2}$, 1·5, -4.

4. Find the respective characteristics

of the logarithms of 325, 1603, 2400, 10000, 19,

to the bases 3, 11, 7, 9, 21.

5. Write down the characteristics of the common logarithms of 3·26, 523·1, ·03, 1·5, ·0002, 3000·1, ·1.

6. The mantissa of log 64439 is ·8091488, write down the logarithms of ·64439, 6443900, ·00064439.

7. The logarithm of 32·5 is 1·5118834, write down the numbers whose logarithms are

$$\cdot 5118834, \quad 2 \cdot 5118834, \quad \bar{4} \cdot 5118834.$$

[*When required the following logarithms may be used*

$\log 2 = \cdot 3010300$, $\log 3 = \cdot 4771213$, $\log 7 = \cdot 8450980$.]

Find the value of

8. $\log 768$. 9. $\log 2352$. 10. $\log 35 \cdot 28$.

11. $\log \sqrt{6804}$. 12. $\log \sqrt[5]{\cdot 00162}$. 13. $\log \cdot 021\dot{7}$.

14. $\log \cos 60°$. 15. $\log \sin^3 60°$. 16. $\log \sqrt[3]{\sec 45°}$.

Find the numerical value of

17. $2 \log \frac{15}{8} - \log \frac{25}{162} + 3 \log \frac{4}{9}$.

18. Evaluate $16\log\frac{10}{9}-4\log\frac{25}{24}-7\log\frac{80}{81}$.

19. Find the seventh root of 7,

given $\log 1{\cdot}320469={\cdot}1207283$.

20. Find the cube root of ·00001764,

given $\log 260315=5{\cdot}4154995$.

21. Given $\log 3571=3{\cdot}5527899$, find the logarithm of

$$3{\cdot}571\times{\cdot}03571\times\sqrt[3]{3571}.$$

22. Given $\log 11=1{\cdot}0413927$, find the logarithm of

$$({\cdot}00011)^{\frac{1}{3}}\times(1{\cdot}21)^2\times(13{\cdot}31)^{\frac{4}{3}}\div 12100000.$$

23. Find the number of digits in the integral parts of

$$\left(\frac{21}{20}\right)^{300} \text{ and } \left(\frac{126}{125}\right)^{1000}.$$

24. How many positive integers have characteristic 3 when the base is 7?

168. Suppose that we have a table of logarithms of numbers to base a and require to find the logarithms to base b.

Let N be one of the numbers, then $\log_b N$ is required.

Let $b^y=N$, so that $y=\log_b N$.

$$\therefore \log_a(b^y)=\log_a N;$$

that is,

$$y\log_a b=\log_a N;$$

$$\therefore y=\frac{1}{\log_a b}\times\log_a N,$$

or

$$\log_b N=\frac{1}{\log_a b}\times\log_a N \qquad \text{(1)}.$$

Now since N and b are given, $\log_a N$ and $\log_a b$ are known from the Tables, and thus $\log_b N$ may be found.

Hence it appears that to transform logarithms from base a to base b we have only to multiply them all by $\frac{1}{\log_a b}$; this is a constant quantity and is given by the Tables; it is known as the *modulus*.

If in equation (1) we put a for N, we obtain

$$\log_b a = \frac{1}{\log_a b} \times \log_a a = \frac{1}{\log_a b};$$

$$\therefore \log_b a \times \log_a b = 1.$$

169. The following examples further illustrate the great use of logarithms in arithmetical work.

Example 1. Given $\log 2 = \cdot 3010300$ and $\log 4844544 = 6\cdot 6852530$, find the value of $(6\cdot 4)^{\frac{1}{10}} \times (\sqrt[4]{\cdot 256})^3 \div \sqrt{80}$.

Let x be the value of the expression; then

$$\log x = \frac{1}{10}\log\frac{64}{10} + \frac{3}{4}\log\frac{256}{1000} - \frac{1}{2}\log 80$$

$$= \frac{1}{10}(\log 2^6 - 1) + \frac{3}{4}(\log 2^8 - 3) - \frac{1}{2}(\log 2^3 + 1)$$

$$= \left(\frac{6}{10} + 6 - \frac{3}{2}\right)\log 2 - \left(\frac{1}{10} + \frac{9}{4} + \frac{1}{2}\right)$$

$$= \left(5 + \frac{1}{10}\right)\log 2 - 2\tfrac{17}{20}$$

$$= 1\cdot 5051500 + \cdot 0301030 - 2\cdot 85.$$

Thus $\log x = \overline{2}\cdot 6852530.$

But $\log 4844544 = 6\cdot 6852530,$

$$\therefore x = \cdot 04844544.$$

Example 2. Find how many ciphers there are between the decimal point and the first significant digit in $(\cdot 0504)^{10}$; having given

$$\log 2 = \cdot 301, \ \log 3 = \cdot 477, \ \log 7 = \cdot 845.$$

Denote the expression by E; then

$$\log E = 10 \log \frac{504}{10000}$$

$$= 10(\log 504 - 4)$$

$$= 10\{\log(2^3 \times 3^2 \times 7) - 4\}$$

$$= 10\{3\log 2 + 2\log 3 + \log 7 - 4\}$$

$$= 10(2\cdot 702 - 4) = 10(\overline{2}\cdot 702)$$

$$= \overline{20} + 7\cdot 02 = \overline{13}\cdot 02.$$

$3\log 2 =$	$\cdot 903$
$2\log 3 =$	$\cdot 954$
$\log 7 =$	$\cdot 845$
	$2\cdot 702$

Thus the number of ciphers is 12. [Art. 163.]

Exponential equations.

170. If in an equation the unknown quantity appears as an exponent, the solution may be effected by the help of logarithms.

Example 1. Solve the equation $8^{5-3x} = 12^{4-2x}$, having given

$$\log 2 = \cdot 30103, \text{ and } \log 3 = \cdot 47712.$$

From the given equation, by taking logarithms, we have

$$(5-3x)\log 8 = (4-2x)\log 12;$$

$$\therefore 3(5-3x)\log 2 = (4-2x)(2\log 2 + \log 3);$$

$$\therefore 15\log 2 - 8\log 2 - 4\log 3 = x(9\log 2 - 4\log 2 - 2\log 3);$$

$$\therefore x = \frac{7\log 2 - 4\log 3}{5\log 2 - 2\log 3} = \frac{\cdot 19873}{\cdot 55091}.$$

Thus $x = \cdot 36$ nearly.

$$\begin{array}{r} 7\log 2 = 2\cdot 10721 \\ 4\log 3 = 1\cdot 90848 \\ \hline \cdot 19873 \end{array}$$

$$\begin{array}{r} 5\log 2 = 1\cdot 50515 \\ 2\log 3 = \cdot 95424 \\ \hline \cdot 55091 \end{array}$$

$$55091\ )\ 198730\ (\ 36$$
$$\begin{array}{r} 165273 \\ \hline 334570 \end{array}$$

Example 2. Given $\log 2 = \cdot 30103$, solve the simultaneous equations

$$2^x . 5^y = 1, \qquad 5^{x+1} . 2^y = 2.$$

Take logarithms of the given equations;

$$\therefore x\log 2 + y\log 5 = 0, \qquad (x+1)\log 5 + y\log 2 = \log 2.$$

For shortness, put $\log 2 = a$, $\log 5 = b$.

Thus $$ax + by = 0,$$

and $$b(x+1) + ay = a, \text{ or } bx + ay = a - b.$$

By eliminating y, $x(a^2 - b^2) = -b(a-b)$,

$$\therefore x = -\frac{b}{a+b} = -\frac{\log 5}{\log 2 + \log 5} = -\frac{\log 5}{\log 10} = -\log 5 = -\cdot 69897.$$

And $$y = -\frac{ax}{b} = \frac{a}{b}\log 5 = a = \log 2 = \cdot 30103.$$