Mathematics

F. A. N. Pessôa,
F. A. N. Pessôa,

Durban High School.

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Form VI.

ELEMENTARY TRIGONOMETRY.

ELEMENTARY TRIGONOMETRY

BY

H. S. HALL, M.A.,

FORMERLY SCHOLAR OF CHRIST'S COLLEGE, CAMBRIDGE;
MASTER OF THE MILITARY SIDE, CLIFTON COLLEGE.

AND

S. R. KNIGHT, B.A., M.B., CH.B.,

FORMERLY SCHOLAR OF TRINITY COLLEGE, CAMBRIDGE;
LATE ASSISTANT MASTER AT MARLBOROUGH COLLEGE.

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PREFACE.

THE following pages will be found to comprise all the parts of Elementary Trigonometry which can conveniently be treated without the use of infinite series and imaginary quantities.

The chapters have been subdivided into short sections, and the examples to illustrate each section have been very carefully selected and arranged, the earlier ones being easy enough for any reader to whom the subject is new, while the later ones, and the Miscellaneous Examples scattered throughout the book, will furnish sufficient practice for those who intend to pursue the subject further as part of a mathematical education.

No substantial progress in Trigonometry can be made until the fundamental properties of the Trigonometrical Ratios have been thoroughly mastered. To attain this object very considerable practice in easy Identities and Equations is necessary. We have therefore given special prominence to examples of this kind in the early pages; with the same end in view we have postponed the subject of Radian or Circular Measure to a later stage than is usual, believing that it is in every way more satisfactory to dwell on the properties of the trigonometrical ratios, and to exemplify their use in easy problems, than to bewilder a beginner with an angular system the use of which he cannot appreciate, and which at this stage furnishes nothing but practice in easy Arithmetic.

The subject of Logarithms and their application has been treated very fully, and illustrated by a selection of carefully graduated Examples. It is hoped that the examples worked out in this section may serve as useful models for the student, and may do something to cure that inaccuracy in logarithmic work which is so often due to clumsy arrangement.

In the experience of most teachers it is found extremely difficult to get boys to handle problems in Heights and Distances with any degree of confidence and skill. Accordingly we have devoted much thought to the exposition of this part of the subject, and by careful classification of the Examples we have endeavoured to make Chapters VI. and XVII. as easy and attractive as possible.

Very little advance can be expected in Trigonometry until the principal formulæ can be quoted readily, but whether it is advisable for learners to have lists of formulæ compiled for them, so as to be easily accessible at all times, is a matter upon which teachers hold different views. In our own opinion it is distinctly mischievous to furnish such lists; it encourages indolent habits, and fosters a spurious confidence which leads to disaster when the student has to rely solely upon his own knowledge.

In the general arrangement and succession of the different parts of the subject we have been mainly guided by our own long experience in the class room; but as the manuscript and proof-sheets have been read by several skilled teachers, and have been frequently tested by pupils in all stages of proficiency, the hope is entertained that our treatment is such as to enable beginners to take an intelligent interest in the subject from the first, and to acquire a sound elementary knowledge of practical Trigonometry before they encounter the more theoretical difficulties. At the same time, as each chapter is, as far as possible, complete in itself, it will be easy for teachers to adopt a different order of treatment if they prefer it; the full Table of Contents will facilitate the selection of a suitable course of reading, besides furnishing a useful aid to students who are rapidly revising the subject.

We are indebted to several friends for valuable criticism and advice; in particular, we have to thank Mr T. D. Davies of Clifton College for many useful hints, and for some ingenious examples and solutions in Chapters XXIV. and XXV.

H. S. HALL.
S. R. KNIGHT.

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PREFACE TO THE THIRD EDITION.

THIS edition contains a collection of three hundred Miscellaneous Examples. They are arranged progressively in sets of six, each set being intended as a short revision exercise. In selecting these examples much care has been taken to illustrate every part of the subject, and to fairly represent University and Civil Service examinations.

H. S. HALL.

Sept. 1898.

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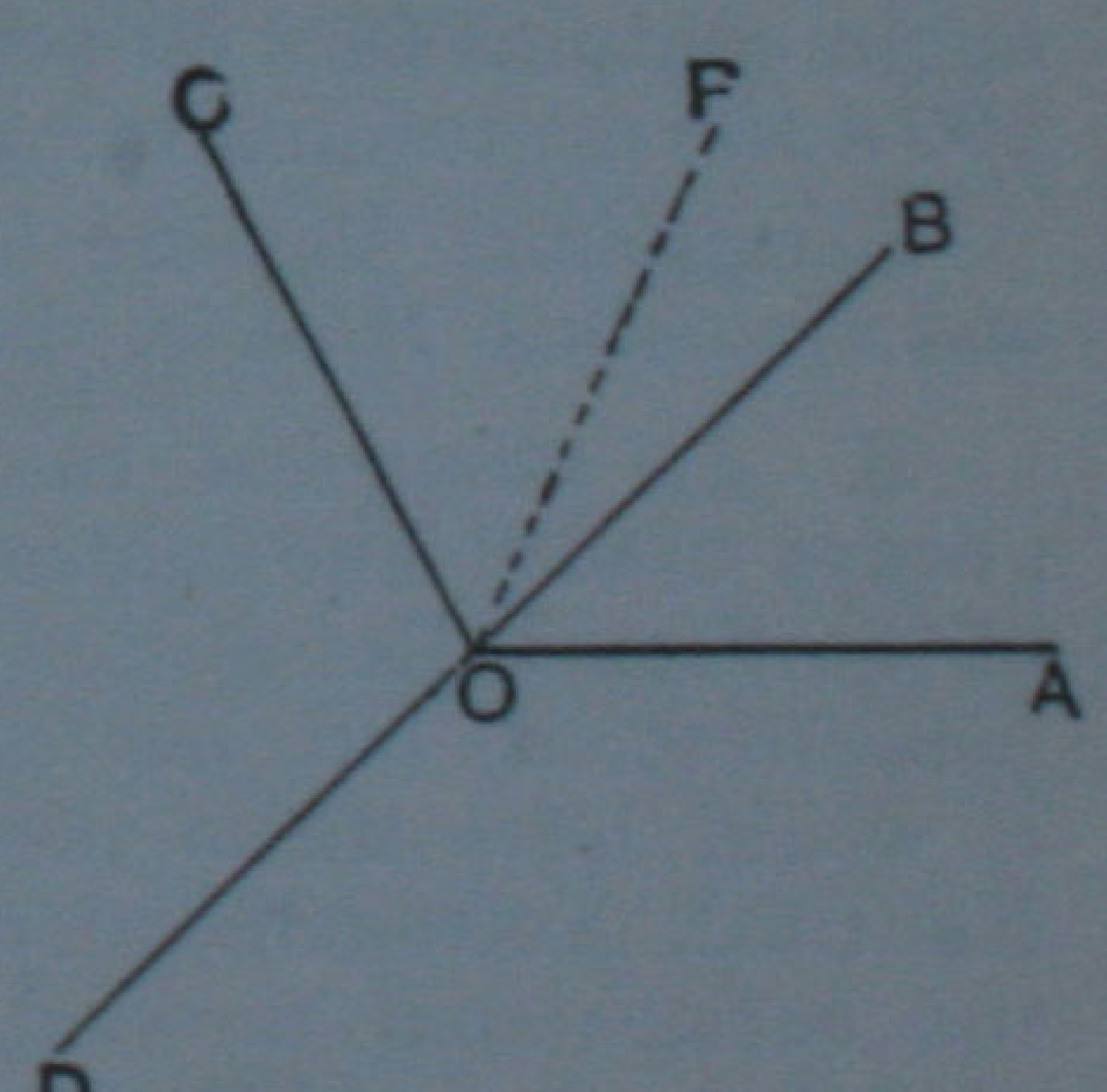
ELEMENTARY TRIGONOMETRY.

CHAPTER I.

MEASUREMENT OF ANGLES.

1. THE word Trigonometry in its primary sense signifies the measurement of triangles. From an early date the science also included the establishment of the relations which subsist between the sides, angles, and area of a triangle; but now it has a much wider scope and embraces all manner of geometrical and algebraical investigations carried on through the medium of certain quantities called trigonometrical ratios, which will be defined in Chap. II. In every branch of Higher Mathematics, whether Pure or Applied, a knowledge of Trigonometry is of the greatest value.

2. Definition of Angle. Suppose that the straight line OP in the figure is capable of revolving about the point O, and suppose that in this way it has passed successively from the position OA to the positions occupied by OB, OC, OD, ..., then the angle between OA and any position such as OC is measured by the *amount of revolution* which the line OP has undergone in passing from its initial position OA into its final position OC.



Moreover the line OP may make any number of complete revolutions through the original position OA before taking up its final position.

It will thus be seen that in Trigonometry angles are not restricted as in Euclid, but may be of any magnitude.

The point O is called the *origin*, and OA the *initial line;* the revolving line OP is known as the *generating line* or the *radius vector.*

3. Measurement of Angles. We must first select some *fixed* unit. The natural unit would be a right angle, but as in practice this is inconveniently large, two systems of measurement have been established, in each of which the unit is a certain fraction of a right angle.

4. Sexagesimal Measure. A right angle is divided into 90 equal parts called *degrees*, a degree into 60 equal parts called *minutes*, a minute into 60 equal parts called *seconds.* An angle is measured by stating the number of degrees, minutes, and seconds which it contains.

For shortness, each of these three divisions, degrees, minutes, seconds, is denoted by a symbol; thus the angle which contains 53 degrees 37 minutes 2·53 seconds is expressed symbolically in the form $53° \, 37' \, 2{\cdot}53''$.

5. Centesimal Measure. A right angle is divided into 100 equal parts called *grades*, a grade into 100 equal parts called *minutes*, a minute into 100 equal parts called *seconds.* In this system the angle which contains 53 grades 37 minutes 2·53 seconds is expressed symbolically in the form $53^g \, 37^` \, 2{\cdot}53^{``}$.

It will be noticed that different accents are used to denote sexagesimal and centesimal minutes and seconds; for though they have the same names, a centesimal minute and second are not the same as a sexagesimal minute and second. Thus a right angle contains 90×60 sexagesimal minutes, whereas it contains 100×100 centesimal minutes.

Sexagesimal Measure is sometimes called the English System, and Centesimal Measure the French System.

6. In *numerical* calculations the sexagesimal measure is always used. The centesimal method was proposed at the time of the French Revolution as part of a general system of decimal measurement, but has never been adopted even in France, as it would have made necessary the alteration of Geographical, Nautical, Astronomical, and other tables prepared according to the sexagesimal method. Beyond giving a few examples in transformation from one system to the other which afford exercise in easy Arithmetic, we shall after this rarely allude to centesimal measure.

In *theoretical* work it is convenient to use another method of measurement, where the unit is the angle subtended at the centre of a circle by an arc whose length is equal to the radius. This system is known as **Circular** or **Radian Measure**, and will be fully explained in Chapter VII.

An angle is usually represented by a single letter, different letters $A, B, C, \ldots, a, \beta, \gamma, \ldots, \theta, \phi, \psi, \ldots$, being used to distinguish different angles. For angles estimated in sexagesimal or centesimal measure these letters are used indifferently, but we shall always denote angles in circular measure by letters taken from the Greek alphabet.

7. *If the number of degrees and grades contained in an angle be D and G respectively, to prove that* $\frac{D}{9}=\frac{G}{10}$.

In sexagesimal measure, the given angle when expressed as the fraction of a right angle is denoted by $\frac{D}{90}$. In centesimal measure, the same fraction is denoted by $\frac{G}{100}$;

$$\therefore \frac{D}{90}=\frac{G}{100}; \text{ that is, } \frac{D}{9}=\frac{G}{10}.$$

8. To pass from one system to the other it is advisable first to express the given angle in terms of a right angle.

In centesimal measure any number of grades, minutes, and seconds may be immediately expressed as the decimal of a right angle. Thus

$$23 \text{ grades} = \tfrac{23}{100} \text{ of a right angle} = \cdot 23 \text{ of a right angle};$$

15 minutes $= \frac{15}{100}$ of a grade $= \cdot 15$ of a grade $= \cdot 0015$ of a right angle;

$$\therefore 23^g\ 15' = \cdot 2315 \text{ of a right angle.}$$

Similarly, $15^g\ 7'\ 53{\cdot}4'' = \cdot 1507534$ of a right angle.

Conversely, any decimal of a right angle can be at once expressed in grades, minutes, and seconds. Thus

$$\begin{aligned} \cdot 2173025 \text{ of a right angle} &= 21{\cdot}73025^g \\ &= 21^g\ 73{\cdot}025' \\ &= 21^g\ 73'\ 2{\cdot}5''. \end{aligned}$$

In practice the intermediate steps are omitted.

Example 1. Reduce $2^{g}\,13^{`}\,4{\cdot}5^{``}$ to sexagesimal measure.

This angle $= {\cdot}0213045$ of a right angle
$= 1°\,55'\,2{\cdot}658''$.

$$\begin{array}{rl} {\cdot}0213045 & \text{of a right angle} \\ 90 & \\ \hline 1{\cdot}917405 & \text{degrees} \\ 60 & \\ \hline 55{\cdot}0443 & \text{minutes} \\ 60 & \\ \hline 2{\cdot}658 & \text{seconds.} \end{array}$$

Obs. In the Answers we shall express the angles to the nearest tenth of a second, so that the above result would be written $1°\,55'\,2{\cdot}7''$.

Example 2. Reduce $12°\,13'\,14{\cdot}3''$ to centesimal measure.

This angle $= {\cdot}13578487\ldots$ of a right angle
$= 13^{g}\,57^{`}\,84{\cdot}9^{``}$.

$$\begin{array}{l} 60\,\underline{)\,14{\cdot}3 \text{ seconds}} \\ 60\,\underline{)\,13{\cdot}238333\ldots \text{minutes}} \\ 90\,\underline{)\,12{\cdot}2206388\ldots \text{degrees}} \\ \quad {\cdot}13578487\ldots \text{of a right angle.} \end{array}$$

EXAMPLES. I.

Express as the decimal of a right angle

1. $67°\,30'$.
2. $11°\,15'$.
3. $37^{g}\,50^{`}$.
4. $2°\,10'\,12''$.
5. $8°\,0'\,36''$.
6. $2^{g}\,4^{`}\,4{\cdot}5^{``}$.

Reduce to centesimal measure

7. $69°\,13'\,30''$.
8. $19°\,0'\,45''$.
9. $50°\,37'\,5{\cdot}7''$.
10. $43°\,52'\,38{\cdot}1''$.
11. $11°\,0'\,38{\cdot}4''$.
12. $142°\,15'\,45''$.
13. $12'\,9''$.
14. $3'\,26{\cdot}3''$.

Reduce to sexagesimal measure

15. $56^{g}\,87^{`}\,50^{``}$.
16. $39^{g}\,6^{`}\,25^{``}$.
17. $40^{g}\,1^{`}\,25{\cdot}4^{``}$.
18. $1^{g}\,2^{`}\,3^{``}$.
19. $3^{g}\,2^{`}\,5^{``}$.
20. $8^{g}\,10^{`}\,6{\cdot}5^{``}$.
21. $6^{`}\,25^{``}$.
22. $37^{`}\,5^{``}$.

23. The sum of two angles is 80^{g} and their difference is $18°$; find the angles in degrees.

24. The number of degrees in a certain angle added to the number of grades in the angle is 152 : what is the angle?

25. If the same angle contains in English measure x minutes, and in French measure y minutes, prove that $50x = 27y$.

26. If s and t respectively denote the numbers of sexagesimal and centesimal seconds in any angle, prove that

$$250s = 81t.$$

CHAPTER II.

TRIGONOMETRICAL RATIOS.

9. DEFINITION. **Ratio** is the relation which one quantity bears to another of the *same* kind, the comparison being made by considering what multiple, part or parts, one quantity is of the other.

To find what multiple or part A is of B we divide A by B; hence the ratio of A to B may be measured by the fraction $\frac{A}{B}$.

In order to compare two quantities they must be expressed in terms of the same unit. Thus the ratio of 2 yards to 27 inches is measured by the fraction $\frac{2 \times 3 \times 12}{27}$ or $\frac{8}{3}$.

OBS. Since a ratio expresses the *number* of times that one quantity contains another, *every ratio is a numerical quantity.*

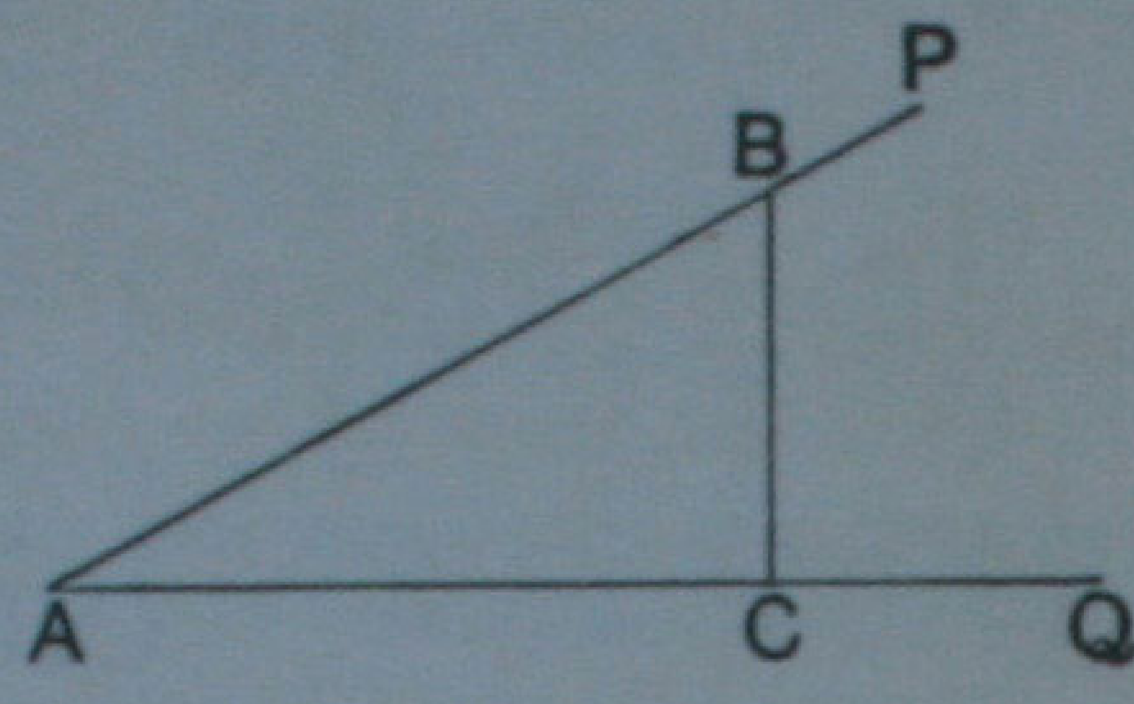
10. DEFINITION. If the ratio of any two quantities can be expressed exactly by the ratio of two integers the quantities are said to be **commensurable**; otherwise, they are said to be **incommensurable.** For instance, the quantities $8\frac{4}{5}$ and $5\frac{1}{3}$ are commensurable, while the quantities $\sqrt{2}$ and 3 are incommensurable. But by finding the numerical value of $\sqrt{2}$ we may express the value of the ratio $\sqrt{2} : 3$ by the ratio of two commensurable quantities to any required degree of approximation. Thus to 5 decimal places $\sqrt{2} = 1{\cdot}41421$, and therefore to the same degree of approximation

$$\sqrt{2} : 3 = 1{\cdot}41421 : 3 = 141421 : 300000.$$

Similarly, for the ratio of any two incommensurable quantities.

Trigonometrical Ratios.

11. Let PAQ be any acute angle; in AP one of the boundary lines take a point B and draw BC perpendicular to AQ. Thus a right-angled triangle BAC is formed.



With reference to the angle A the following definitions are employed.

The ratio $\frac{BC}{AB}$ or $\frac{\textit{opposite side}}{\textit{hypotenuse}}$ is called the **sine of A.**

The ratio $\frac{AC}{AB}$ or $\frac{\textit{adjacent side}}{\textit{hypotenuse}}$ is called the **cosine of A.**

The ratio $\frac{BC}{AC}$ or $\frac{\textit{opposite side}}{\textit{adjacent side}}$ is called the **tangent of A.**

The ratio $\frac{AC}{BC}$ or $\frac{\textit{adjacent side}}{\textit{opposite side}}$ is called the **cotangent of A.**

The ratio $\frac{AB}{AC}$ or $\frac{\textit{hypotenuse}}{\textit{adjacent side}}$ is called the **secant of A.**

The ratio $\frac{AB}{BC}$ or $\frac{\textit{hypotenuse}}{\textit{opposite side}}$ is called the **cosecant of A.**

These six ratios are known as the **trigonometrical ratios.** It will be shewn later that as long as the angle remains the same the trigonometrical ratios remain the same. [Art. 19.]

12. Instead of writing in full the words *sine*, *cosine*, *tangent*, *cotangent*, *secant*, *cosecant*, abbreviations are adopted. Thus the above definitions may be more conveniently expressed and arranged as follows:

$$\sin A = \frac{BC}{AB}, \qquad \operatorname{cosec} A = \frac{AB}{BC},$$

$$\cos A = \frac{AC}{AB}, \qquad \sec A = \frac{AB}{AC},$$

$$\tan A = \frac{BC}{AC}, \qquad \cot A = \frac{AC}{BC}.$$

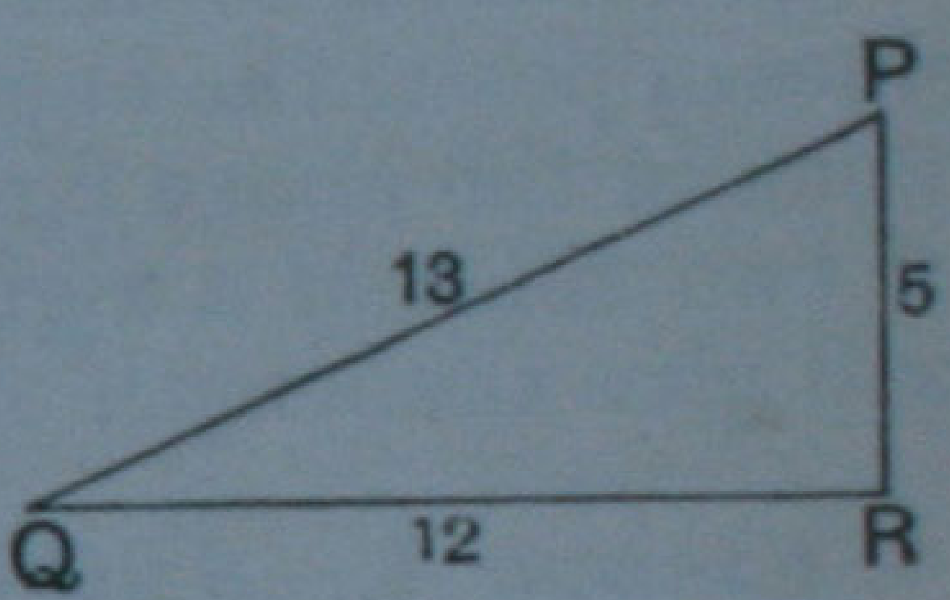
In addition to these six ratios, two others, the *versed sine* and *coversed sine* are sometimes used; they are written vers A and covers A and are thus defined:

$$\text{vers } A = 1 - \cos A, \quad \text{covers } A = 1 - \sin A.$$

13. In Chapter VIII. the definitions of the trigonometrical ratios will be extended to the case of angles of any magnitude, but for the present we confine our attention to the consideration of acute angles.

14. Although the verbal form of the definitions of the trigonometrical ratios given in Art. 11 may be helpful to the student at first, he will gain no freedom in their use until he is able to write down from the figure any ratio at sight.

In the adjoining figure, PQR is a right-angled triangle in which $PQ=13$, $PR=5$, $QR=12$.



Since PQ is the greatest side, R is the right angle. The trigonometrical ratios of the angles P and Q may be written down at once; for example,

$$\sin Q = \frac{PR}{PQ} = \frac{5}{13}, \qquad \cos Q = \frac{QR}{PQ} = \frac{12}{13},$$

$$\tan P = \frac{QR}{PR} = \frac{12}{5}, \qquad \operatorname{cosec} P = \frac{PQ}{QR} = \frac{13}{12}.$$

15. It is important to observe that *the trigonometrical ratios of an angle are numerical quantities.* Each one of them represents the *ratio of one length to another*, and they must themselves never be regarded as lengths.

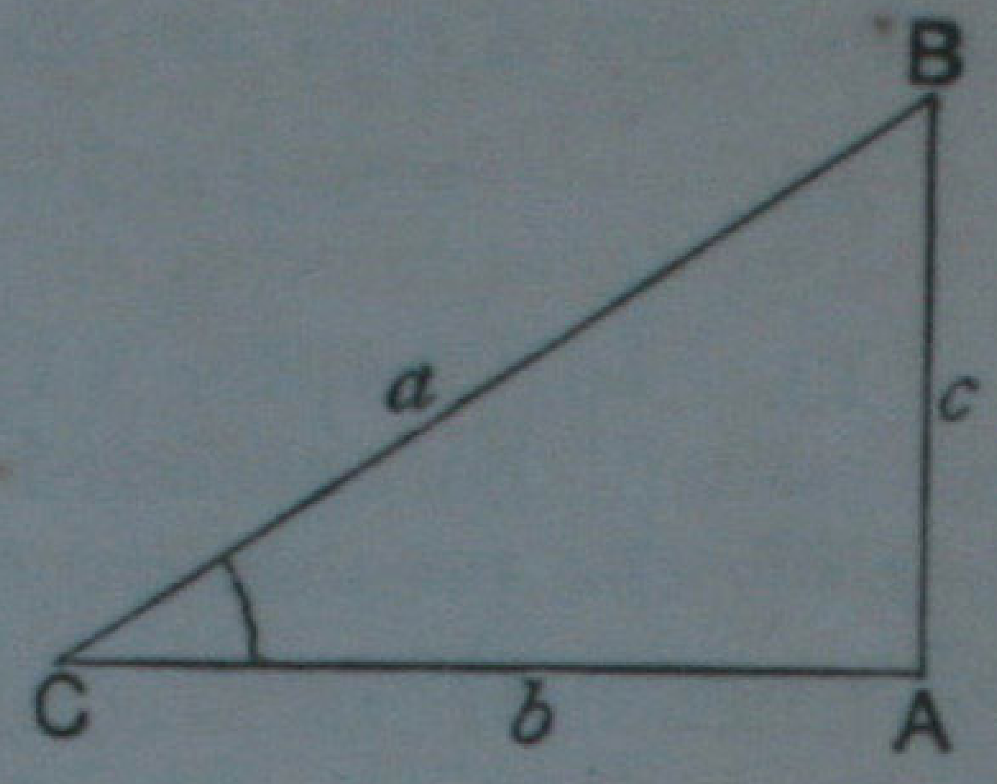
16. In every right-angled triangle the hypotenuse is the greatest side; hence from the definitions of Art. 11 it will be seen that those ratios which have the hypotenuse in the *denominator* can never be greater than unity, while those which have the hypotenuse in the *numerator* can never be less than unity. Those ratios which do not involve the hypotenuse are not thus restricted in value, for either of the two sides which subtend the acute angles may be the greater. Hence

the sine and cosine of an angle can never be greater than 1;

the cosecant and secant of an angle can never be less than 1;

the tangent and cotangent may have any numerical value.

17. Let ABC be a right-angled triangle having the right angle at A; then by Euc. I. 47,



the sq. on BC

=sum of sqq. on AC and AB,

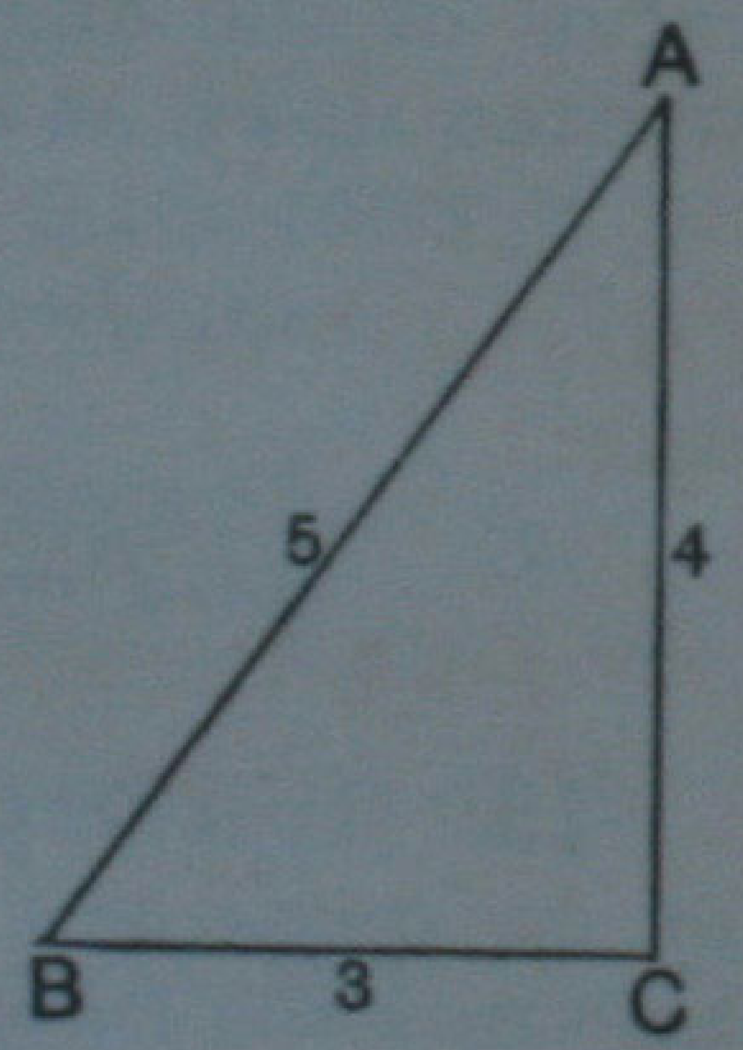
or, more briefly,

$$BC^2 = AC^2 + AB^2.$$

When we use this latter mode of expression it is understood that the sides AB, AC, BC are expressed in terms of some common unit, and the above statement may be regarded as a *numerical relation* connecting the numbers of units of length in the three sides of a right-angled triangle.

It is usual to denote the numbers of units of length in the sides opposite the angles A, B, C by the letters a, b, c respectively. Thus in the above figure we have $a^2 = b^2 + c^2$, so that if the lengths of two sides of a right-angled triangle are known, this equation will give the length of the third side.

Example 1. ABC is a right-angled triangle of which C is the right angle; if $a=3$, $b=4$, find c, and also $\sin A$ and $\cot B$.



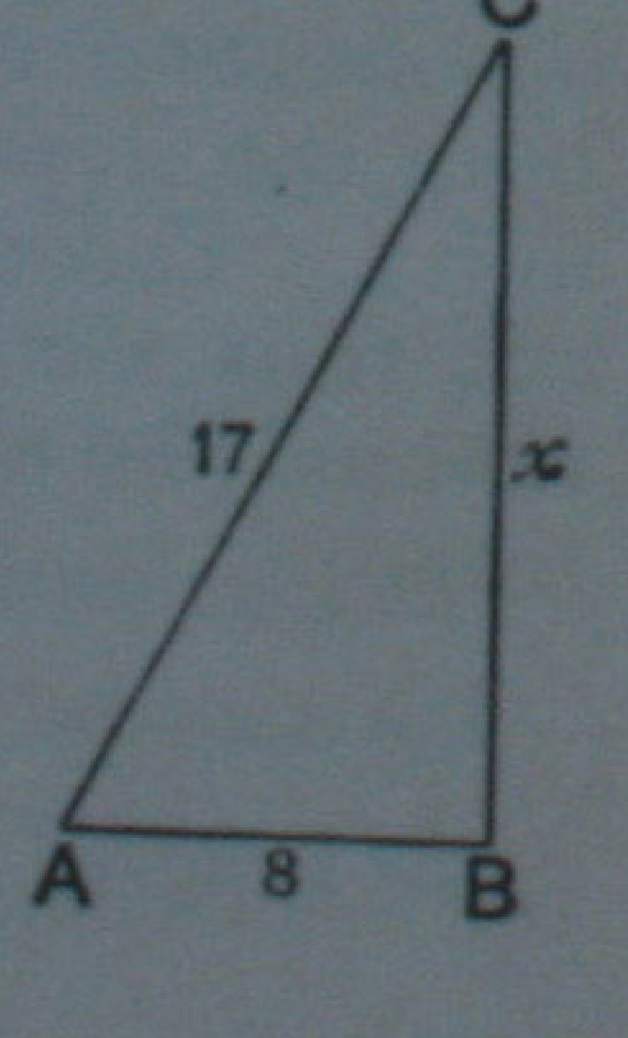
Here $c^2 = a^2 + b^2 = (3)^2 + (4)^2 = 9 + 16 = 25$;

$$\therefore c = 5.$$

Also
$$\sin A = \frac{BC}{AB} = \frac{3}{5};$$

$$\cot B = \frac{BC}{AC} = \frac{3}{4}.$$

Example 2. A ladder 17 ft. long is placed with its foot at a distance of 8 ft. from the wall of a house and just reaches a window-sill. Find the height of the window-sill, and the sine and tangent of the angle which the ladder makes with the wall.



Let AC be the ladder, and BC the wall.

Let x be the number of feet in BC;

then $x^2 = (17)^2 - (8)^2 = (17+8)(17-8) = 25 \times 9$;

$$\therefore x = 5 \times 3 = 15.$$

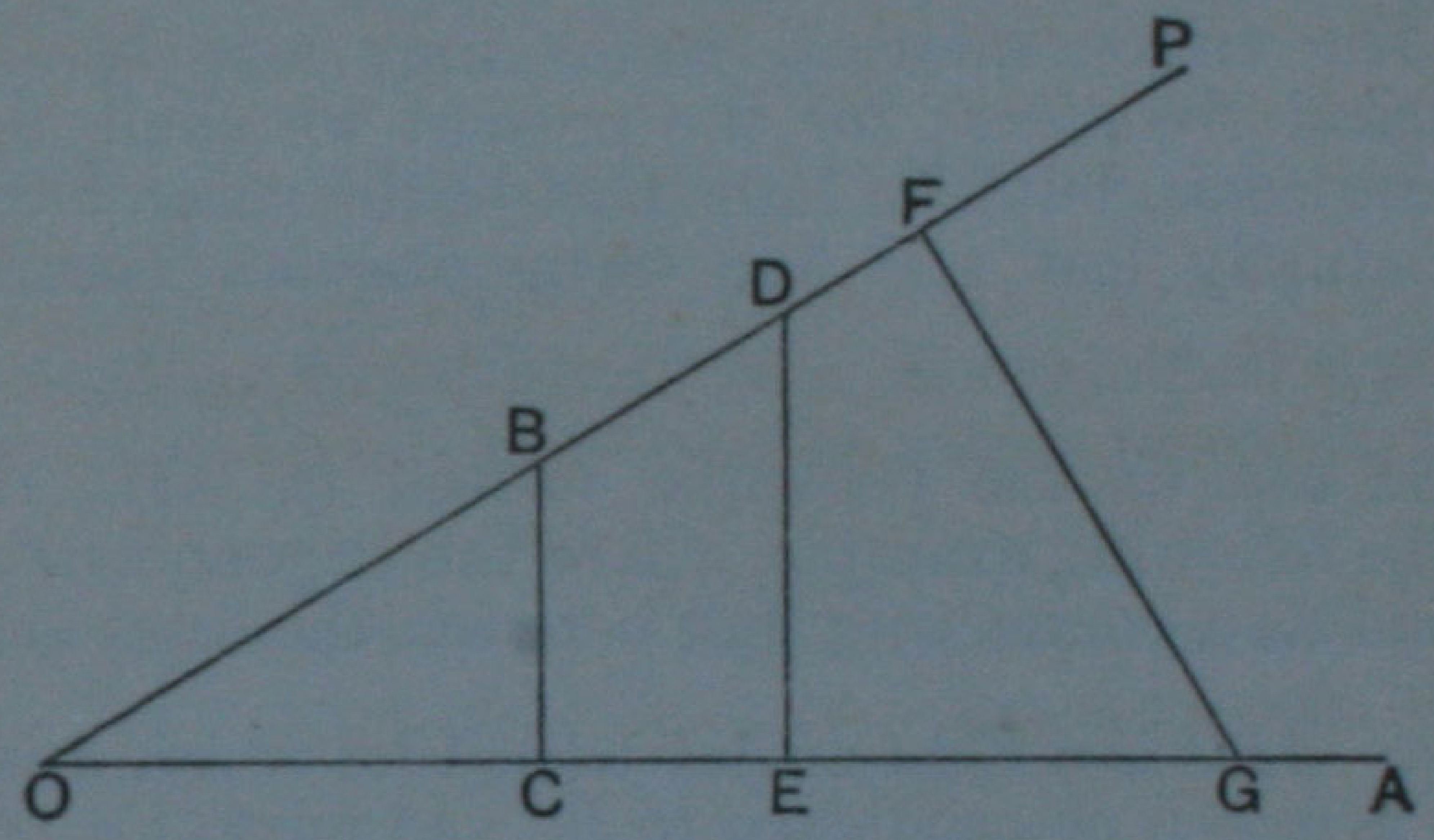
Also
$$\sin C = \frac{AB}{AC} = \frac{8}{17};$$

$$\tan C = \frac{AB}{BC} = \frac{8}{15}.$$

18. The following important proposition depends upon the property of similar triangles proved in Euc. VI. 4. The student who has not read the sixth Book of Euclid should not fail to notice the result arrived at, even if he is unable at this stage to understand the proof.

19. *To prove that the trigonometrical ratios remain unaltered so long as the angle remains the same.*

Let AOP be any acute angle. In OP take any points B and



D, and draw BC and DE perpendicular to OA. Also take any point F in OP and draw FG at right angles to OP.

From the triangle BOC, $\sin POA = \frac{BC}{OB}$;

from the triangle DOE, $\sin POA = \frac{DE}{OD}$;

from the triangle FOG, $\sin POA = \frac{FG}{OG}$.

But the triangles BOC, DOE, FOG are equiangular;

$$\therefore \quad \frac{BC}{OB} = \frac{DE}{OD} = \frac{FG}{OG}. \qquad \text{[Euc. VI. 4.]}$$

Thus the sine of the angle POA is the same whether it is obtained from the triangle BOC, or from the triangle DOE, or from the triangle FOG.

A similar proof holds for each of the other trigonometrical ratios. These ratios are therefore independent of the length of the revolving line and depend only on the magnitude of the angle.

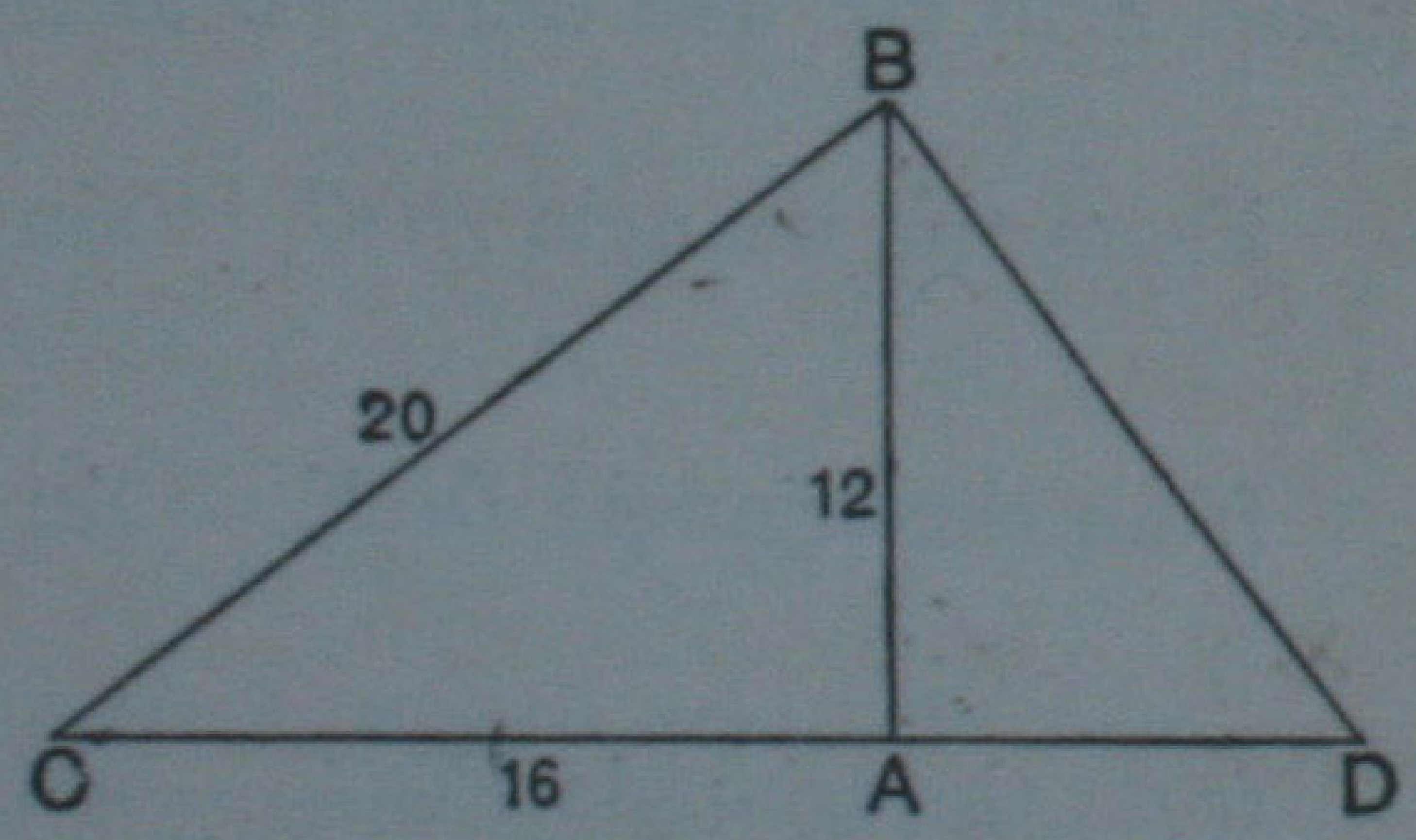
20. If A denote any acute angle, we have proved that all the trigonometrical ratios of A depend only on the magnitude of the angle A and not upon the lengths of the lines which bound the angle. It may easily be seen that a change made in the value of A will produce a consequent change in the values of all the trigonometrical ratios of A. This point will be discussed more fully in Chap. IX.

DEFINITION. Any expression which involves a variable quantity x, and whose value is dependent on that of x is called **a function of x.**

Hence the trigonometrical *ratios* may also be defined as trigonometrical *functions*; for the present we shall chiefly employ the term *ratio*, but in a later part of the subject the idea of ratio is gradually lost and the term *function* becomes more appropriate.

21. The use of the principle proved in Art. 19 is well shewn in the following example, where the trigonometrical ratios are employed as a connecting link between the lines and angles.

Example. ABC is a right-angled triangle of which A is the right angle. BD is drawn perpendicular to BC and meets CA produced in D: if $AB=12$, $AC=16$, $BC=20$, find BD and CD.



From the right-angled triangle CBD,

$$\frac{BD}{BC}=\tan C;$$

from the right-angled triangle ABC,

$$\frac{AB}{AC}=\tan C;$$

$$\therefore \frac{BD}{BC}=\frac{AB}{AC};$$

$$\therefore \frac{BD}{20}=\frac{12}{16}; \text{ whence } BD=15.$$

Again,
$$\frac{CD}{CB}=\sec C=\frac{BC}{CA};$$

$$\therefore \frac{CD}{20}=\frac{20}{16}; \text{ whence } CD=25.$$

The same results can be obtained by the help of Euc. VI. 8.

EXAMPLES. II.

1. The sides AB, BC, CA of a right-angled triangle are 17, 15, 8 respectively; write down the values of $\sin A$, $\sec A$, $\tan B$, $\sec B$.

2. The sides PQ, QR, RP of a right-angled triangle are 13, 5, 12 respectively: write down the values of $\cot P$, $\operatorname{cosec} Q$, $\cos Q$, $\cos P$.

3. ABC is a triangle in which A is a right angle; if $b=15$, $c=20$, find a, $\sin C$, $\cos B$, $\cot C$, $\sec C$.

4. ABC is a triangle in which B is a right angle; if $a=24$, $b=25$, find c, $\sin C$, $\tan A$, $\operatorname{cosec} A$.

5. The sides ED, EF, DF of a right-angled triangle are 35, 37, 12 respectively: write down the values of $\sec E$, $\sec F$, $\cot E$, $\sin F$.

6. The hypotenuse of a right-angled triangle is 15 inches, and one of the sides is 9 inches: find the third side and the sine, cosine and tangent of the angle opposite to it.

7. Find the hypotenuse AB of a right-angled triangle in which $AC=7$, $BC=24$. Write down the sine and cosine of A, and shew that the sum of their squares is equal to 1.

8. A ladder 41 ft. long is placed with its foot at a distance of 9 ft. from the wall of a house and just reaches a window-sill. Find the height of the window-sill, and the sine and cotangent of the angle which the ladder makes with the ground.

9. A ladder is 29 ft. long; how far must its foot be placed from a wall so that the ladder may just reach the top of the wall which is 21 ft. from the ground? Write down all the trigonometrical ratios of the angle between the ladder and the wall.

10. $ABCD$ is a square; C is joined to E, the middle point of AD: find all the trigonometrical ratios of the angle ECD.

11. $ABCD$ is a quadrilateral in which the diagonal AC is at right angles to each of the sides AB, CD: if $AB=15$, $AC=36$, $AD=85$, find $\sin ABC$, $\sec ACB$, $\cos CDA$, $\operatorname{cosec} DAC$.

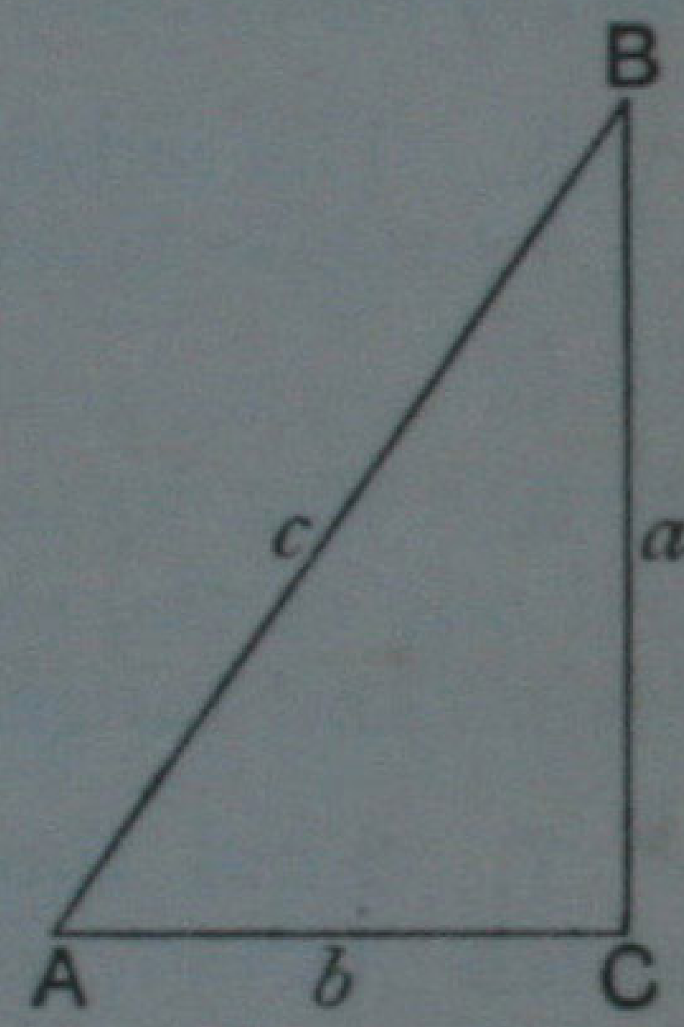
12. $PQRS$ is a quadrilateral in which the angle PSR is a right angle. If the diagonal PR is at right angles to RQ, and $RP=20$, $RQ=21$, $RS=16$, find $\sin PRS$, $\tan RPS$, $\cos RPQ$, $\operatorname{cosec} PQR$.

CHAPTER III.

RELATIONS BETWEEN THE TRIGONOMETRICAL RATIOS.

22. Reciprocal relations between certain ratios.

(1) Let ABC be a triangle, right-angled at C;



then $$\sin A = \frac{BC}{AB} = \frac{a}{c},$$

and $$\operatorname{cosec} A = \frac{AB}{BC} = \frac{c}{a};$$

$$\therefore \sin A \times \operatorname{cosec} A = \frac{a}{c} \times \frac{c}{a} = 1.$$

Thus $\sin A$ and $\operatorname{cosec} A$ are reciprocals;

$$\therefore \sin A = \frac{1}{\operatorname{cosec} A},$$

and $$\operatorname{cosec} A = \frac{1}{\sin A}.$$

(2) Again,

$$\cos A = \frac{AC}{AB} = \frac{b}{c}, \text{ and } \sec A = \frac{AB}{AC} = \frac{c}{b};$$

$$\therefore \cos A \times \sec A = \frac{b}{c} \times \frac{c}{b} = 1;$$

$$\therefore \cos A = \frac{1}{\sec A}, \text{ and } \sec A = \frac{1}{\cos A}.$$

(3) Also

$$\tan A = \frac{BC}{AC} = \frac{a}{b}, \text{ and } \cot A = \frac{AC}{BC} = \frac{b}{a};$$

$$\therefore \tan A \times \cot A = \frac{a}{b} \times \frac{b}{a} = 1;$$

$$\therefore \tan A = \frac{1}{\cot A}, \text{ and } \cot A = \frac{1}{\tan A}.$$

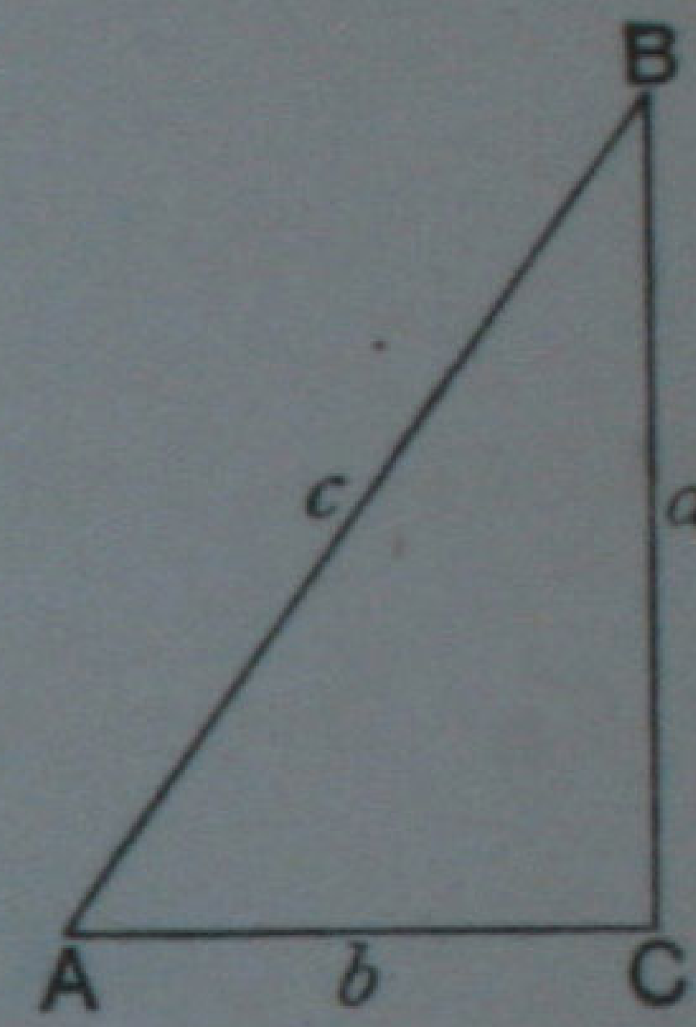
23. *To express* tan A *and* cot A *in terms of* sin A *and* cos A.

From the adjoining figure we have

$$\tan A = \frac{BC}{AC} = \frac{a}{b} = \frac{a}{c} \div \frac{b}{c}$$

$$= \sin A \div \cos A;$$

$$\therefore \quad \tan A = \frac{\sin A}{\cos A}.$$



Again, $$\cot A = \frac{AC}{BC} = \frac{b}{a} = \frac{b}{c} \div \frac{a}{c}$$

$$= \cos A \div \sin A;$$

$$\therefore \quad \cot A = \frac{\cos A}{\sin A};$$

which is also evident from the reciprocal relation $\cot A = \dfrac{1}{\tan A}$.

Example. Prove that $\operatorname{cosec} A \tan A = \sec A$.

$$\operatorname{cosec} A \tan A = \frac{1}{\sin A} \times \frac{\sin A}{\cos A} = \frac{1}{\cos A}$$

$$= \sec A.$$

24. We frequently meet with expressions which involve the square and other powers of the trigonometrical ratios, such as $(\sin A)^2$, $(\tan A)^3$, ... It is usual to write these in the shorter forms $\sin^2 A$, $\tan^3 A$, ...

Thus $$\tan^2 A = (\tan A)^2 = \left(\frac{\sin A}{\cos A}\right)^2$$

$$= \frac{(\sin A)^2}{(\cos A)^2} = \frac{\sin^2 A}{\cos^2 A}.$$

Example. Shew that $\sin^2 A \sec A \cot^2 A = \cos A$.

$$\sin^2 A \sec A \cot^2 A = \sin^2 A \times \frac{1}{\cos A} \times \left(\frac{\cos A}{\sin A}\right)^2$$

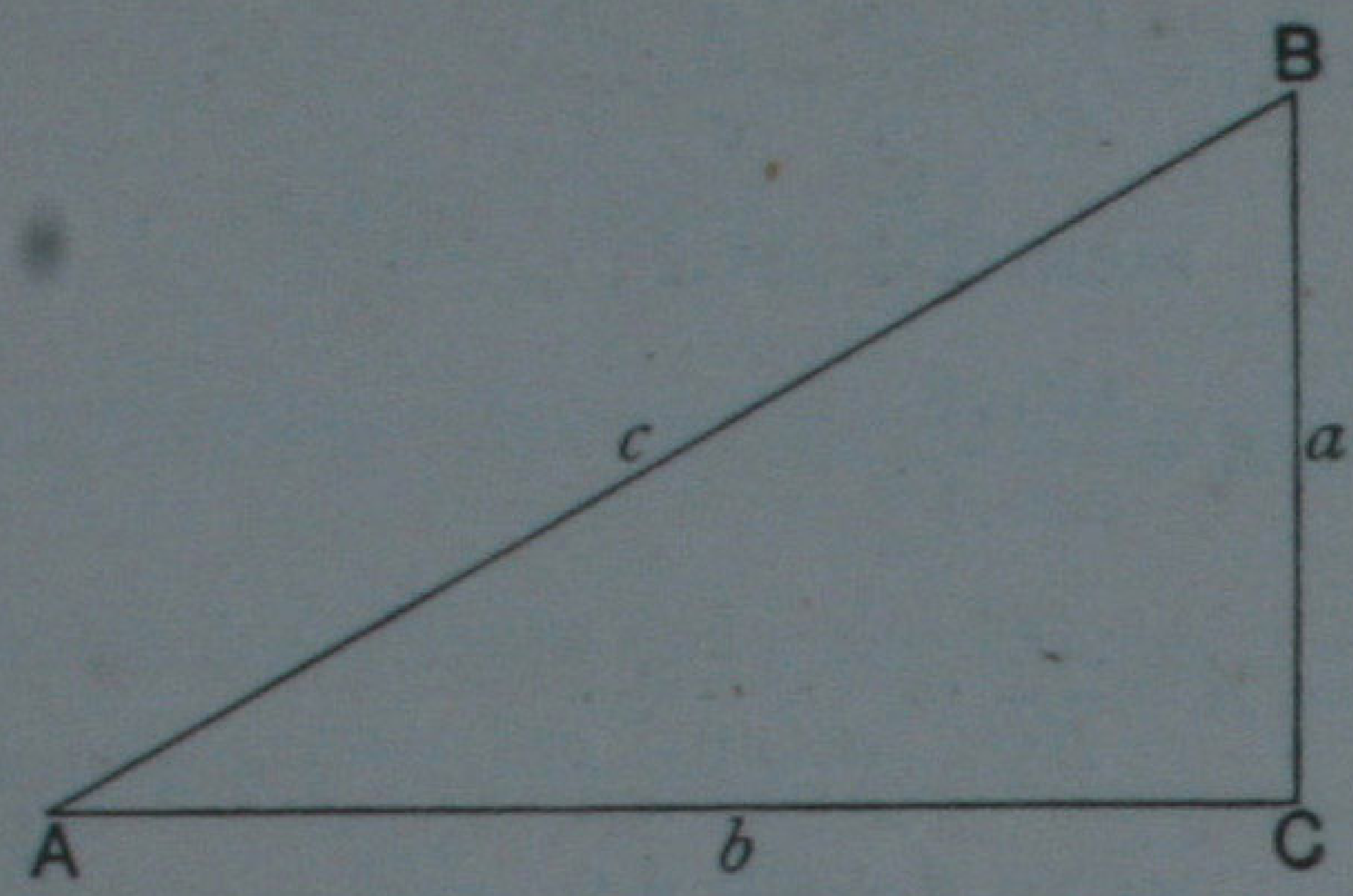
$$= \sin^2 A \times \frac{1}{\cos A} \times \frac{\cos^2 A}{\sin^2 A}$$

$$= \cos A,$$

by cancelling factors common to numerator and denominator.

25. *To prove that* $\sin^2 A + \cos^2 A = 1$.

Let BAC be any acute angle; draw BC perpendicular to



AC, and denote the sides of the right-angled triangle ABC by a, b, c.

By definition, $$\sin A = \frac{BC}{AB} = \frac{a}{c};$$

and $$\cos A = \frac{AC}{AB} = \frac{b}{c};$$

$$\therefore \sin^2 A + \cos^2 A = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2+b^2}{c^2}$$

$$= \frac{c^2}{c^2}$$

$$= 1.$$

Cor. $\sin^2 A = 1 - \cos^2 A$, $\sin A = \sqrt{1-\cos^2 A}$;
$\cos^2 A = 1 - \sin^2 A$, $\cos A = \sqrt{1-\sin^2 A}$.

Example 1. Prove that $\cos^4 A - \sin^4 A = \cos^2 A - \sin^2 A$.

$$\cos^4 A - \sin^4 A = (\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)$$
$$= \cos^2 A - \sin^2 A,$$

since the first factor is equal to 1.

Example 2. Prove that $\cot \alpha \sqrt{1-\cos^2\alpha} = \cos \alpha$.

$$\cot \alpha \sqrt{1-\cos^2\alpha} = \cot \alpha \times \sin \alpha$$

$$= \frac{\cos \alpha}{\sin \alpha} \times \sin \alpha = \cos \alpha.$$

26. *To prove that* $\sec^2 A = 1 + \tan^2 A$.

With the figure of the previous article, we have

$$\sec A = \frac{AB}{AC} = \frac{c}{b};$$

$$\therefore \quad \sec^2 A = \frac{c^2}{b^2} = \frac{b^2 + a^2}{b^2}$$

$$= 1 + \frac{a^2}{b^2}$$

$$= 1 + \tan^2 A.$$

COR. $\sec^2 A - \tan^2 A = 1, \quad \sec A = \sqrt{1 + \tan^2 A},$

$\tan^2 A = \sec^2 A - 1, \quad \tan A = \sqrt{\sec^2 A - 1}.$

Example. Prove that $\cos A \sqrt{\sec^2 A - 1} = \sin A$.

$$\cos A \sqrt{\sec^2 A - 1} = \cos A \times \tan A$$

$$= \cos A \times \frac{\sin A}{\cos A}$$

$$= \sin A.$$

27. *To prove that* $\operatorname{cosec}^2 A = 1 + \cot^2 A$.

With the figure of Art. 25, we have

$$\operatorname{cosec} A = \frac{AB}{BC} = \frac{c}{a};$$

$$\therefore \quad \operatorname{cosec}^2 A = \frac{c^2}{a^2} = \frac{a^2 + b^2}{a^2}$$

$$= 1 + \frac{b^2}{a^2}$$

$$= 1 + \cot^2 A.$$

COR. $\operatorname{cosec}^2 A - \cot^2 A = 1, \quad \operatorname{cosec} A = \sqrt{1 + \cot^2 A},$

$\cot^2 A = \operatorname{cosec}^2 A - 1, \quad \cot A = \sqrt{\operatorname{cosec}^2 A - 1}.$

Example. Prove that $\cot^4 \alpha - 1 = \operatorname{cosec}^4 \alpha - 2 \operatorname{cosec}^2 \alpha$.

$$\cot^4 \alpha - 1 = (\cot^2 \alpha + 1)(\cot^2 \alpha - 1)$$

$$= \operatorname{cosec}^2 \alpha (\operatorname{cosec}^2 \alpha - 1 - 1)$$

$$= \operatorname{cosec}^2 \alpha (\operatorname{cosec}^2 \alpha - 2)$$

$$= \operatorname{cosec}^4 \alpha - 2 \operatorname{cosec}^2 \alpha.$$

28. The formulæ proved in the last three articles are not independent, for they are merely different ways of expressing in trigonometrical symbols the property of a right-angled triangle proved in Euc. I. 47.

29. It will be useful here to collect the formulæ proved in this chapter.

I. $\operatorname{cosec} A \times \sin A = 1, \quad \operatorname{cosec} A = \dfrac{1}{\sin A}, \quad \sin A = \dfrac{1}{\operatorname{cosec} A};$

$$\sec A \times \cos A = 1, \quad \sec A = \frac{1}{\cos A}, \quad \cos A = \frac{1}{\sec A};$$

$$\cot A \times \tan A = 1, \quad \cot A = \frac{1}{\tan A}, \quad \tan A = \frac{1}{\cot A}.$$

II. $\tan A = \dfrac{\sin A}{\cos A}, \quad \cot A = \dfrac{\cos A}{\sin A}.$

III.
$$\sin^2 A + \cos^2 A = 1,$$
$$\sec^2 A = 1 + \tan^2 A,$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A.$$

Easy Identities.

30. We shall now exemplify the use of these fundamental formulæ in proving *identities.* An identity asserts that two expressions are always equal, and the proof of this equality is called "proving the identity." Some easy illustrations have already been given in this chapter. The general method of procedure is to choose one of the expressions given (usually the more complicated of the two) and to shew by successive transformations that it can be made to assume the form of the other.

Example 1. Prove that $\sin^2 A \cot^2 A + \cos^2 A \tan^2 A = 1$.

Here it will be found convenient to express all the trigonometrical ratios in terms of the sine and cosine.

$$\text{The first side} = \sin^2 A \cdot \frac{\cos^2 A}{\sin^2 A} + \cos^2 A \cdot \frac{\sin^2 A}{\cos^2 A}$$
$$= \cos^2 A + \sin^2 A$$
$$= 1.$$

Example 2. Prove that $\sec^4\theta - \sec^2\theta = \tan^2\theta + \tan^4\theta$.

The form of this identity at once suggests that we should use the secant-tangent formula of Art. 26; hence

$$\begin{aligned}\text{the first side} &= \sec^2\theta(\sec^2\theta - 1)\\ &= (1+\tan^2\theta)\tan^2\theta\\ &= \tan^2\theta + \tan^4\theta.\end{aligned}$$

EXAMPLES. III. a.

Prove the following identities :

1. $\sin A \cot A = \cos A$.
2. $\cos A \tan A = \sin A$.
3. $\cot A \sec A = \operatorname{cosec} A$.
4. $\sin A \sec A = \tan A$.
5. $\cos A \operatorname{cosec} A = \cot A$.
6. $\cot A \sec A \sin A = 1$.
7. $(1-\cos^2 A)\operatorname{cosec}^2 A = 1$.
8. $(1-\sin^2 A)\sec^2 A = 1$.
9. $\cot^2\theta(1-\cos^2\theta) = \cos^2\theta$.
10. $(1-\cos^2\theta)\sec^2\theta = \tan^2\theta$.
11. $\tan\alpha\sqrt{1-\sin^2\alpha} = \sin\alpha$.
12. $\operatorname{cosec}\alpha\sqrt{1-\sin^2\alpha} = \cot\alpha$.
13. $(1+\tan^2 A)\cos^2 A = 1$.
14. $(\sec^2 A - 1)\cot^2 A = 1$.
15. $(1-\cos^2\theta)(1+\tan^2\theta) = \tan^2\theta$.
16. $\cos\alpha \operatorname{cosec}\alpha\sqrt{\sec^2\alpha - 1} = 1$.
17. $\sin^2 A(1+\cot^2 A) = 1$.
18. $(\operatorname{cosec}^2 A - 1)\tan^2 A = 1$.
19. $(1-\cos^2 A)(1+\cot^2 A) = 1$.
20. $\sin\alpha\sec\alpha\sqrt{\operatorname{cosec}^2\alpha - 1} = 1$.
21. $\cos\alpha\sqrt{\cot^2\alpha + 1} = \sqrt{\operatorname{cosec}^2\alpha - 1}$.
22. $\sin^2\theta\cot^2\theta + \sin^2\theta = 1$.
23. $(1+\tan^2\theta)(1-\sin^2\theta) = 1$.
24. $\sin^2\theta\sec^2\theta = \sec^2\theta - 1$.
25. $\operatorname{cosec}^2\theta\tan^2\theta - 1 = \tan^2\theta$.

Prove the following identities :

26. $\frac{1}{\sec^2 A}+\frac{1}{\operatorname{cosec}^2 A}=1.$

27. $\frac{1}{\cos^2 A}-\frac{1}{\cot^2 A}=1.$

28. $\frac{\sin A}{\operatorname{cosec} A}+\frac{\cos A}{\sec A}=1.$

29. $\frac{\sec A}{\cos A}-\frac{\tan A}{\cot A}=1.$

30. $\sin^4 a-\cos^4 a=2\sin^2 a-1=1-2\cos^2 a.$

31. $\sec^4 a-1=2\tan^2 a+\tan^4 a.$

32. $\operatorname{cosec}^4 a-1=2\cot^2 a+\cot^4 a.$

33. $(\tan a \operatorname{cosec} a)^2-(\sin a \sec a)^2=1.$

34. $(\sec\theta\cot\theta)^2-(\cos\theta\operatorname{cosec}\theta)^2=1.$

35. $\tan^2\theta-\cot^2\theta=\sec^2\theta-\operatorname{cosec}^2\theta.$

31. The foregoing examples have required little more than a direct application of the fundamental formulæ; we shall now give some identities offering a greater variety of treatment.

Example 1. Prove that $\sec^2 A+\operatorname{cosec}^2 A=\sec^2 A\operatorname{cosec}^2 A.$

The first side $=\frac{1}{\cos^2 A}+\frac{1}{\sin^2 A}=\frac{\sin^2 A+\cos^2 A}{\cos^2 A\sin^2 A}$

$$=\frac{1}{\cos^2 A\sin^2 A}=\sec^2 A\operatorname{cosec}^2 A.$$

Occasionally it is found convenient to prove the equality of the two expressions by reducing each to the same form.

Example 2. Prove that

$$\sin^2 A\tan A+\cos^2 A\cot A+2\sin A\cos A=\tan A+\cot A.$$

The first side $=\sin^2 A\,.\,\frac{\sin A}{\cos A}+\cos^2 A\,.\,\frac{\cos A}{\sin A}+2\sin A\cos A$

$$=\frac{\sin^4 A+\cos^4 A+2\sin^2 A\cos^2 A}{\sin A\cos A}$$

$$=\frac{(\sin^2 A+\cos^2 A)^2}{\sin A\cos A}=\frac{1}{\sin A\cos A}.$$

The second side $=\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}=\frac{\sin^2 A+\cos^2 A}{\cos A\sin A}$

$$=\frac{1}{\sin A\cos A}.$$

Thus each side of the identity $=\frac{1}{\sin A\cos A}.$

Example 3. Prove that $\dfrac{\tan\alpha-\cot\beta}{\tan\beta-\cot\alpha}=\tan\alpha\cot\beta.$

$$\text{The first side }=\frac{\tan\alpha-\cot\beta}{\dfrac{1}{\cot\beta}-\dfrac{1}{\tan\alpha}}=\frac{\tan\alpha-\cot\beta}{\dfrac{\tan\alpha-\cot\beta}{\tan\alpha\cot\beta}}$$

$$=\frac{\tan\alpha-\cot\beta}{1}\times\frac{\tan\alpha\cot\beta}{\tan\alpha-\cot\beta}$$

$$=\tan\alpha\cot\beta.$$

The transformations in the successive steps are usually suggested by the form into which we wish to bring the result. For instance, in this last example we might have proved the identity by substituting for the tangent and cotangent in terms of the sine and cosine. This however is not the best method, for the form in which the right-hand side is given suggests that we should retain $\tan\alpha$ and $\cot\beta$ unchanged throughout the work.

EXAMPLES. III. b.

Prove the following identities :

1. $\dfrac{\sin\alpha\cot^2\alpha}{\cos\alpha}=\dfrac{1}{\tan\alpha}.$

2. $\dfrac{\sec^2\alpha\cot\alpha}{\operatorname{cosec}^2\alpha}=\tan\alpha.$

3. $1-\operatorname{vers}\theta=\sin\theta\cot\theta.$

4. $\operatorname{vers}\theta\sec\theta=\sec\theta-1.$

5. $\sec\theta-\tan\theta\sin\theta=\cos\theta.$

6. $\tan\theta+\cot\theta=\sec\theta\operatorname{cosec}\theta.$

7. $\sqrt{1+\cot^2 A}\,.\,\sqrt{\sec^2 A-1}\,.\,\sqrt{1-\sin^2 A}=1.$

8. $(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2=2.$

9. $(1+\tan\theta)^2+(1-\tan\theta)^2=2\sec^2\theta.$

10. $(\cot\theta-1)^2+(\cot\theta+1)^2=2\operatorname{cosec}^2\theta.$

11. $\sin^2 A\,(1+\cot^2 A)+\cos^2 A\,(1+\tan^2 A)=2.$

12. $\cos^2 A\,(\sec^2 A-\tan^2 A)+\sin^2 A\,(\operatorname{cosec}^2 A-\cot^2 A)=1.$

13. $\cot^2\alpha+\cot^4\alpha=\operatorname{cosec}^4\alpha-\operatorname{cosec}^2\alpha.$

14. $\dfrac{\tan^2\alpha}{1+\tan^2\alpha}\cdot\dfrac{1+\cot^2\alpha}{\cot^2\alpha}=\sin^2\alpha\sec^2\alpha.$

15. $\dfrac{1}{1-\sin\alpha}+\dfrac{1}{1+\sin\alpha}=2\sec^2\alpha.$

Prove the following identities:

16. $\dfrac{\tan a}{\sec a - 1} + \dfrac{\tan a}{\sec a + 1} = 2 \operatorname{cosec} a.$

17. $\dfrac{1}{1+\sin^2 a} + \dfrac{1}{1+\operatorname{cosec}^2 a} = 1.$

18. $(\sec\theta + \operatorname{cosec}\theta)(\sin\theta + \cos\theta) = \sec\theta \operatorname{cosec}\theta + 2.$

19. $(\cos\theta - \sin\theta)(\operatorname{cosec}\theta - \sec\theta) = \sec\theta \operatorname{cosec}\theta - 2.$

20. $(1 + \cot\theta + \operatorname{cosec}\theta)(1 + \cot\theta - \operatorname{cosec}\theta) = 2\cot\theta.$

21. $(\sec\theta + \tan\theta - 1)(\sec\theta - \tan\theta + 1) = 2\tan\theta.$

22. $(\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2 = \tan^2 A + \cot^2 A + 7.$

23. $(\sec^2 A + \tan^2 A)(\operatorname{cosec}^2 A + \cot^2 A) = 1 + 2\sec^2 A \operatorname{cosec}^2 A.$

24. $(1 - \sin A + \cos A)^2 = 2(1 - \sin A)(1 + \cos A).$

25. $\sin A(1 + \tan A) + \cos A(1 + \cot A) = \sec A + \operatorname{cosec} A.$

26. $\cos\theta(\tan\theta + 2)(2\tan\theta + 1) = 2\sec\theta + 5\sin\theta.$

27. $(\tan\theta + \sec\theta)^2 = \dfrac{1+\sin\theta}{1-\sin\theta}.$

28. $\dfrac{2\sin\theta\cos\theta - \cos\theta}{1 - \sin\theta + \sin^2\theta - \cos^2\theta} = \cot\theta.$

29. $\cot^2\theta \cdot \dfrac{\sec\theta - 1}{1+\sin\theta} + \sec^2\theta \cdot \dfrac{\sin\theta - 1}{1 + \sec\theta} = 0.$

[*The following examples contain functions of two angles; in each case the two angles are quite independent of each other.*]

30. $\tan^2 a + \sec^2\beta = \sec^2 a + \tan^2\beta.$

31. $\dfrac{\tan a + \cot\beta}{\cot a + \tan\beta} = \dfrac{\tan a}{\tan\beta}.$

32. $\dfrac{\tan a - \cot\beta}{\cot a - \tan\beta} = -\dfrac{\cot\beta}{\cot a}.$

33. $\cot a \tan\beta(\tan a + \cot\beta) = \cot a + \tan\beta.$

34. $\sin^2 a\cos^2\beta - \cos^2 a\sin^2\beta = \sin^2 a - \sin^2\beta.$

35. $\sec^2 a\tan^2\beta - \tan^2 a\sec^2\beta = \tan^2\beta - \tan^2 a.$

36. $(\sin a\cos\beta + \cos a\sin\beta)^2 + (\cos a\cos\beta - \sin a\sin\beta)^2 = 1.$

32. By means of the relations collected together in Art. 29, all the trigonometrical ratios can be expressed in terms of any one.

Example 1. Express all the trigonometrical ratios of A in terms of $\tan A$.

We have $\cot A=\frac{1}{\tan A}$;

$$\sec A=\sqrt{1+\tan^2 A};$$

$$\cos A=\frac{1}{\sec A}=\frac{1}{\sqrt{1+\tan^2 A}};$$

$$\sin A=\frac{\sin A}{\cos A}\cos A=\tan A\cos A=\frac{\tan A}{\sqrt{1+\tan^2 A}};$$

$$\operatorname{cosec} A=\frac{1}{\sin A}=\frac{\sqrt{1+\tan^2 A}}{\tan A}.$$

Obs. In writing down the ratios we choose the simplest and most natural order. For instance, $\cot A$ is obtained at once by the *reciprocal relation* connecting the tangent and cotangent: $\sec A$ comes immediately from the tangent-secant formula; the remaining three ratios now readily follow.

Example 2. Given $\cos A=\frac{5}{13}$, find $\operatorname{cosec} A$ and $\cot A$.

$$\operatorname{cosec} A=\frac{1}{\sin A}=\frac{1}{\sqrt{1-\cos^2 A}}$$

$$=\frac{1}{\sqrt{1-\left(\frac{5}{13}\right)^2}}=\frac{1}{\sqrt{1-\frac{25}{169}}}=\frac{1}{\sqrt{\frac{144}{169}}}=\frac{1}{\frac{12}{13}}=\frac{13}{12}.$$

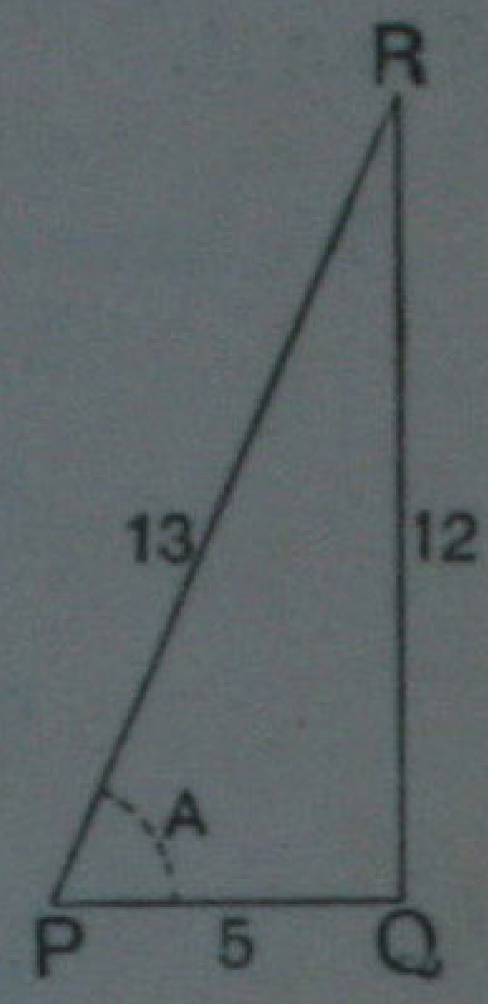
$$\cot A=\frac{\cos A}{\sin A}=\cos A\times\operatorname{cosec} A$$

$$=\frac{5}{13}\times\frac{13}{12}=\frac{5}{12}.$$

33. *It is always possible to describe a right-angled triangle when two sides are given:* for the third side can be found by Euc. I. 47, and the construction can then be effected by Euc. I. 22. We can thus readily obtain all the trigonometrical ratios when one is given, or express all in terms of any one.

Example 1. Given $\cos A = \frac{5}{13}$, find cosec A and cot A.

Take a right-angled triangle PQR, of which Q is the right angle, having the hypotenuse $PR = 13$ units, and $PQ = 5$ units.



Let $QR = x$ units; then

$$x^2 = (13)^2 - (5)^2 = (13+5)(13-5)$$
$$= 18 \times 8 = 9 \times 2 \times 8;$$
$$\therefore\ x = 3 \times 4 = 12.$$

Now $$\cos RPQ = \frac{PQ}{PR} = \frac{5}{13},$$

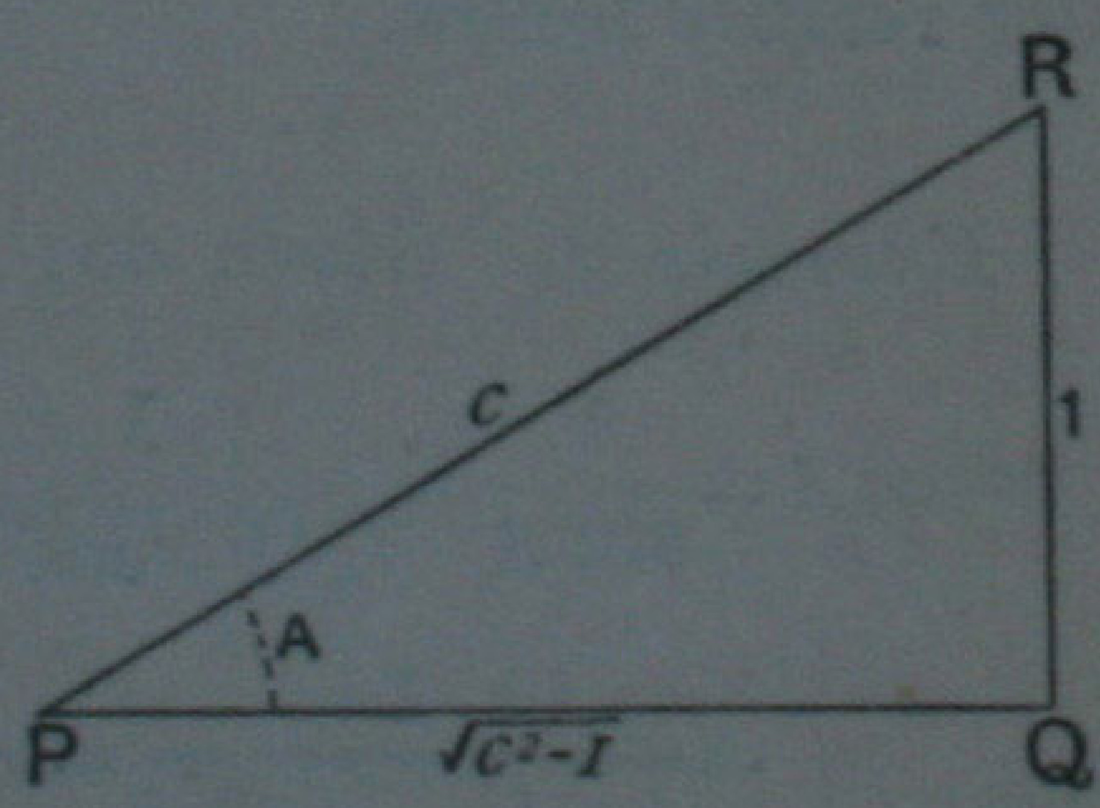
so that $$\angle RPQ = A.$$

Hence $$\text{cosec}\,A = \frac{PR}{QR} = \frac{13}{12},$$

and $$\cot A = \frac{PQ}{QR} = \frac{5}{12}.$$ [Compare Art. 32, Ex. 2.]

Example 2. Find tan A and cos A in terms of cosec A.

Take a triangle PQR right-angled at Q, and having $\angle RPQ = A$. For shortness, denote cosec A by c.



Then $$\text{cosec}\,A = c = \frac{c}{1};$$

but $$\text{cosec}\,A = \frac{PR}{QR};$$

$$\therefore\ \frac{PR}{QR} = \frac{c}{1}.$$

Let QR be taken as the unit of measurement;

then $QR = 1$, and therefore $PR = c$.

Let PQ contain x units; then

$$x^2 = c^2 - 1, \text{ so that } x = \sqrt{c^2-1}.$$

Hence $$\tan A = \frac{QR}{PQ} = \frac{1}{\sqrt{c^2-1}} = \frac{1}{\sqrt{\text{cosec}^2 A - 1}},$$

and $$\cos A = \frac{PQ}{PR} = \frac{\sqrt{c^2-1}}{c} = \frac{\sqrt{\text{cosec}^2 A - 1}}{\text{cosec}\,A}.$$

EXAMPLES. III. c.

1. Given $\sin A=\frac{1}{2}$, find $\sec A$ and $\cot A$.

2. Given $\tan A=\frac{4}{3}$, find $\sin A$ and $\cos A$.

3. Find $\cot\theta$ and $\sin\theta$ when $\sec\theta=4$.

4. If $\tan\alpha=\frac{1}{2}$, find $\sec\alpha$ and $\operatorname{cosec}\alpha$.

5. Find the sine and cotangent of an angle whose secant is 7.

6. If $25\sin A=7$, find $\tan A$ and $\sec A$.

7. Express $\sin A$ and $\tan A$ in terms of $\cos A$.

8. Express $\operatorname{cosec}\alpha$ and $\cos\alpha$ in terms of $\cot\alpha$.

9. Find $\sin\theta$ and $\cot\theta$ in terms of $\sec\theta$.

10. Express all the trigonometrical ratios of A in terms of $\sin A$.

11. Given $\sin A-\cos A=0$, find $\operatorname{cosec} A$.

12. If $\sin A=\frac{m}{n}$, prove that $\sqrt{n^2-m^2}\,.\tan A=m$.

13. If $p\cot\theta=\sqrt{q^2-p^2}$, find $\sin\theta$.

14. When $\sec A=\frac{m^2+1}{2m}$, find $\tan A$ and $\sin A$.

15. Given $\tan A=\frac{2pq}{p^2-q^2}$, find $\cos A$ and $\operatorname{cosec} A$.

16. If $\sec\alpha=\frac{13}{5}$, find the value of $\frac{2\sin\alpha-3\cos\alpha}{4\sin\alpha-9\cos\alpha}$.

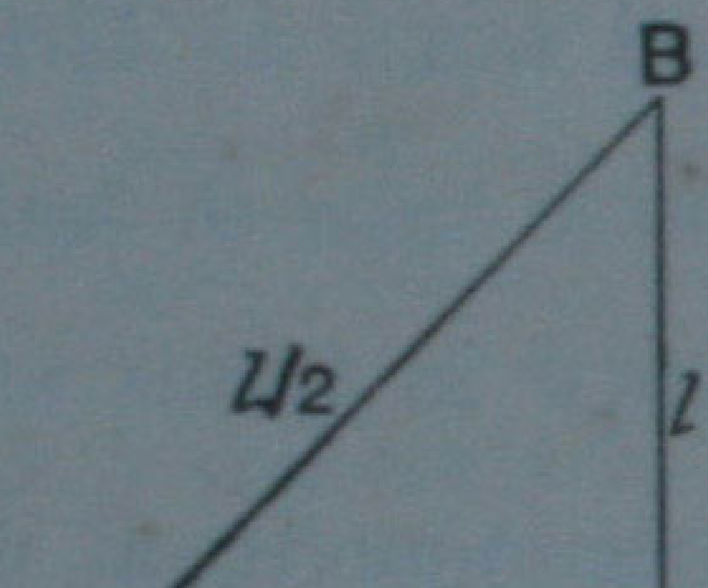
17. If $\cot\theta=\frac{p}{q}$, find the value of $\frac{p\cos\theta-q\sin\theta}{p\cos\theta+q\sin\theta}$.

CHAPTER IV.

TRIGONOMETRICAL RATIOS OF CERTAIN ANGLES.

34. Trigonometrical Ratios of 45°.

Let BAC be a right-angled isosceles triangle, with the right angle at C; so that $B=A=45°$.



Let each of the equal sides contain l units,

then $$AC=BC=l.$$

Also $$AB^2=l^2+l^2=2l^2;$$

$$\therefore\quad AB=l\sqrt{2}.$$

$$\therefore\quad \sin 45°=\frac{BC}{AB}=\frac{l}{l\sqrt{2}}=\frac{1}{\sqrt{2}};$$

$$\cos 45°=\frac{AC}{AB}=\frac{l}{l\sqrt{2}}=\frac{1}{\sqrt{2}};$$

$$\tan 45°=\frac{BC}{AC}=\frac{l}{l}=1.$$

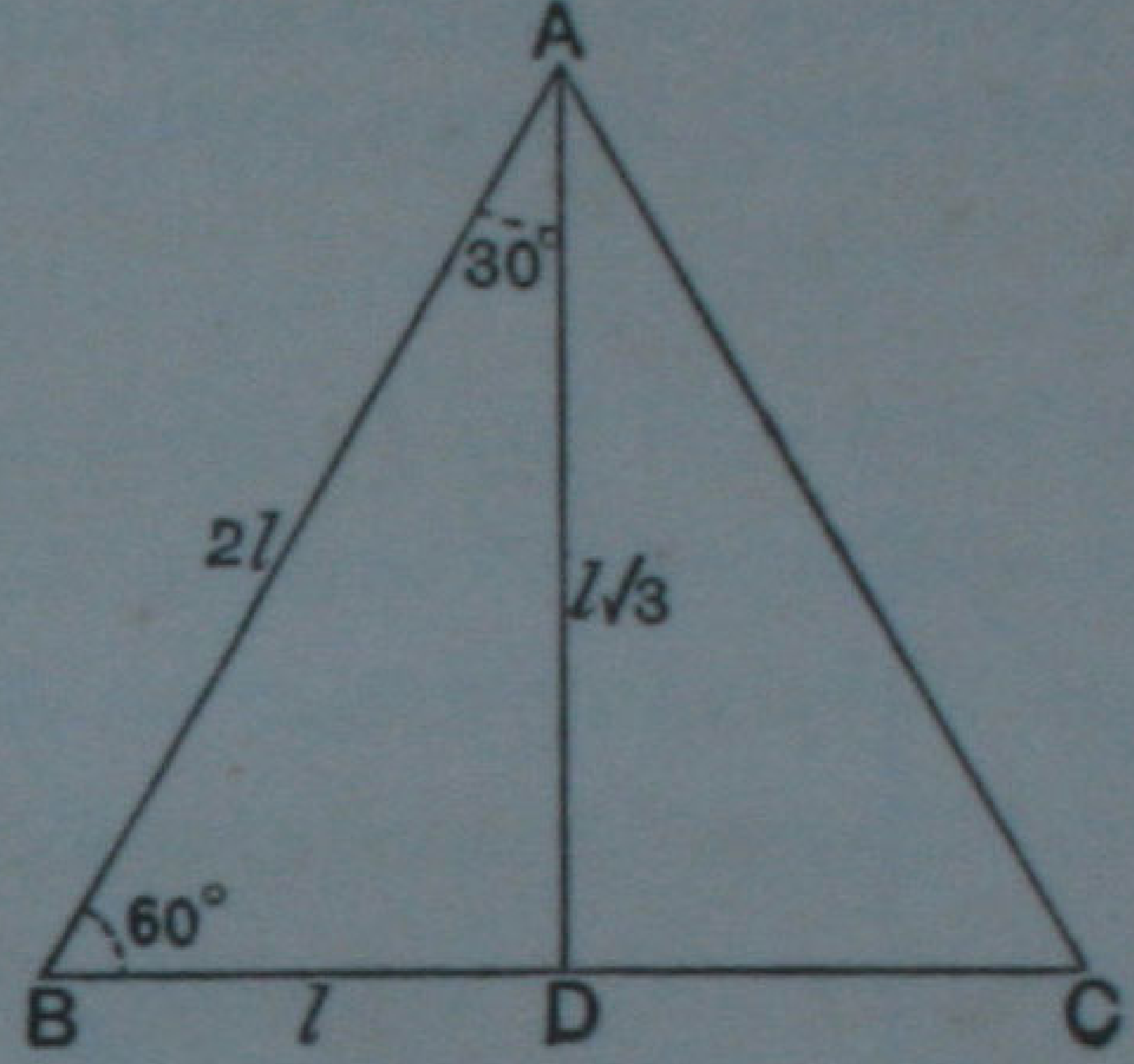
The other three ratios are the reciprocals of these; thus

$$\operatorname{cosec} 45°=\sqrt{2},\quad \sec 45°=\sqrt{2},\quad \cot 45°=1;$$

or they may be read off from the figure.

35. Trigonometrical Ratios of 60° and 30°.

Let ABC be an equilateral triangle; thus each of its angles is 60°.



Bisect $\angle BAC$ by AD meeting BC at D; then $\angle BAD = 30°$.

By Euc. I. 4, the triangles ABD, ACD are equal in all respects; therefore $BD = CD$, and the angles at D are right angles.

In the right-angled triangle ADB, let $BD = l$; then

$$AB = BC = 2l;$$

$$\therefore \quad AD^2 = 4l^2 - l^2 = 3l^2;$$

$$\therefore \quad AD = l\sqrt{3}.$$

$$\therefore \quad \sin 60° = \frac{AD}{AB} = \frac{l\sqrt{3}}{2l} = \frac{\sqrt{3}}{2};$$

$$\cos 60° = \frac{BD}{AB} = \frac{l}{2l} = \frac{1}{2};$$

$$\tan 60° = \frac{AD}{BD} = \frac{l\sqrt{3}}{l} = \sqrt{3}.$$

Again,

$$\sin 30° = \frac{BD}{AB} = \frac{l}{2l} = \frac{1}{2};$$

$$\cos 30° = \frac{AD}{AB} = \frac{l\sqrt{3}}{2l} = \frac{\sqrt{3}}{2};$$

$$\tan 30° = \frac{BD}{AD} = \frac{l}{l\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

The other ratios may be read off from the figure.

36. The trigonometrical ratios of 45°, 60°, 30° occur very frequently; it is therefore important that the student should be able to quote readily their numerical values. The exercise which follows will furnish useful practice.

At first it will probably be found safer to make use of the accompanying diagrams than to trust to the memory.

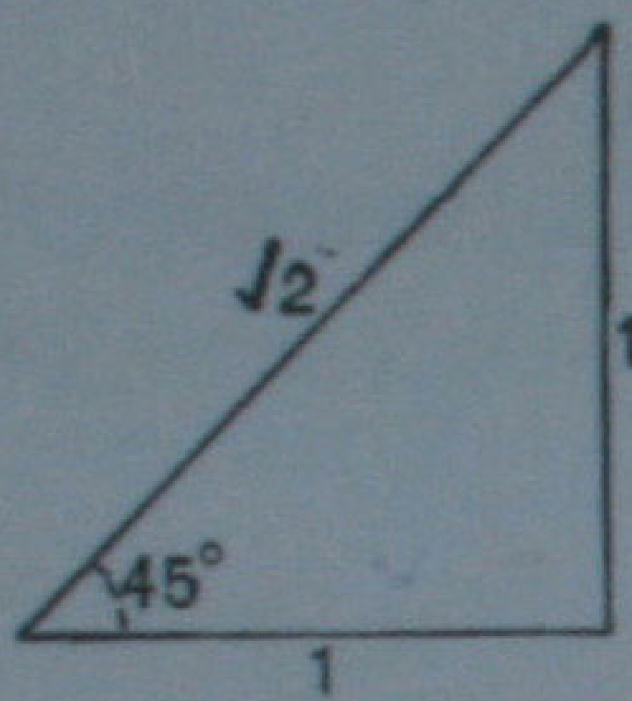


FIG. 1.

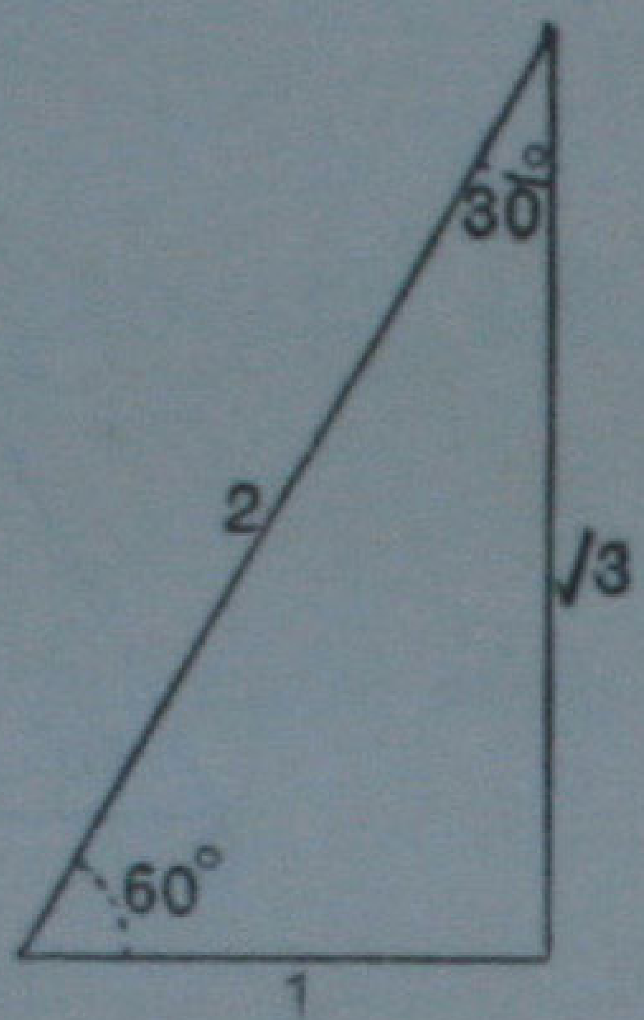


FIG. 2.

The trigonometrical ratios of 45° can be read off from Fig. 1; those of 60° and 30° from Fig. 2.

Example 1. Find the values of $\sec^3 45°$ and $\sin 60° \cot 30° \tan 45°$.

$$\sec^3 45° = (\sec 45°)^3 = (\sqrt{2})^3 = \sqrt{2} \times \sqrt{2} \times \sqrt{2} = 2\sqrt{2}.$$

$$\sin 60° \cot 30° \tan 45° = \frac{\sqrt{3}}{2} \times \sqrt{3} \times 1 = \frac{3}{2}.$$

Example 2. Find the value of

$$2 \cot 45° + \cos^3 60° - 2 \sin^4 60° + \tfrac{3}{4} \tan^2 30°.$$

$$\text{The value} = (2 \times 1) + \left(\frac{1}{2}\right)^3 - 2\left(\frac{\sqrt{3}}{2}\right)^4 + \frac{3}{4}\left(\frac{1}{\sqrt{3}}\right)^2$$

$$= 2 + \frac{1}{8} - 2\left(\frac{3}{4}\right)^2 + \frac{3}{4}\left(\frac{1}{3}\right)$$

$$= 2 + \frac{1}{8} - \frac{9}{8} + \frac{1}{4} = 1\tfrac{1}{4}.$$

EXAMPLES. IV. a.

Find the numerical value of

1. $\tan^2 60° + 2 \tan^2 45°$.
2. $\tan^3 45° + 4 \cos^3 60°$.
3. $2 \operatorname{cosec}^2 45° - 3 \sec^2 30°$.
4. $\cot 60° \tan 30° + \sec^2 45°$.

5. $2\sin 30^\circ \cos 30^\circ \cot 60^\circ$.

6. $\tan^2 45^\circ \sin 60^\circ \tan 30^\circ \tan^2 60^\circ$.

7. $\tan^2 60^\circ + 4\cos^2 45^\circ + 3\sec^2 30^\circ$.

8. $\frac{1}{2}\operatorname{cosec}^2 60^\circ + \sec^2 45^\circ - 2\cot^2 60^\circ$.

9. $\tan^2 30^\circ + 2\sin 60^\circ + \tan 45^\circ - \tan 60^\circ + \cos^2 30^\circ$.

10. $\cot^2 45^\circ + \cos 60^\circ - \sin^2 60^\circ - \frac{3}{4}\cot^2 60^\circ$.

11. $3\tan^2 30^\circ + \frac{4}{3}\cos^2 30^\circ - \frac{1}{2}\sec^2 45^\circ - \frac{1}{3}\sin^2 60^\circ$.

12. $\cos 60^\circ - \tan^2 45^\circ + \frac{3}{4}\tan^2 30^\circ + \cos^2 30^\circ - \sin 30^\circ$.

13. $\frac{1}{3}\sin^2 60^\circ - \frac{1}{2}\sec 60^\circ \tan^2 30^\circ + \frac{4}{3}\sin^2 45^\circ \tan^2 60^\circ$.

14. If $\tan^2 45^\circ - \cos^2 60^\circ = x\sin 45^\circ \cos 45^\circ \tan 60^\circ$, find x.

15. Find x from the equation

$$x\sin 30^\circ \cos^2 45^\circ = \frac{\cot^2 30^\circ \sec 60^\circ \tan 45^\circ}{\operatorname{cosec}^2 45^\circ \operatorname{cosec} 30^\circ}.$$

37. Definition. The **complement** of an angle is its *defect from* a right angle.

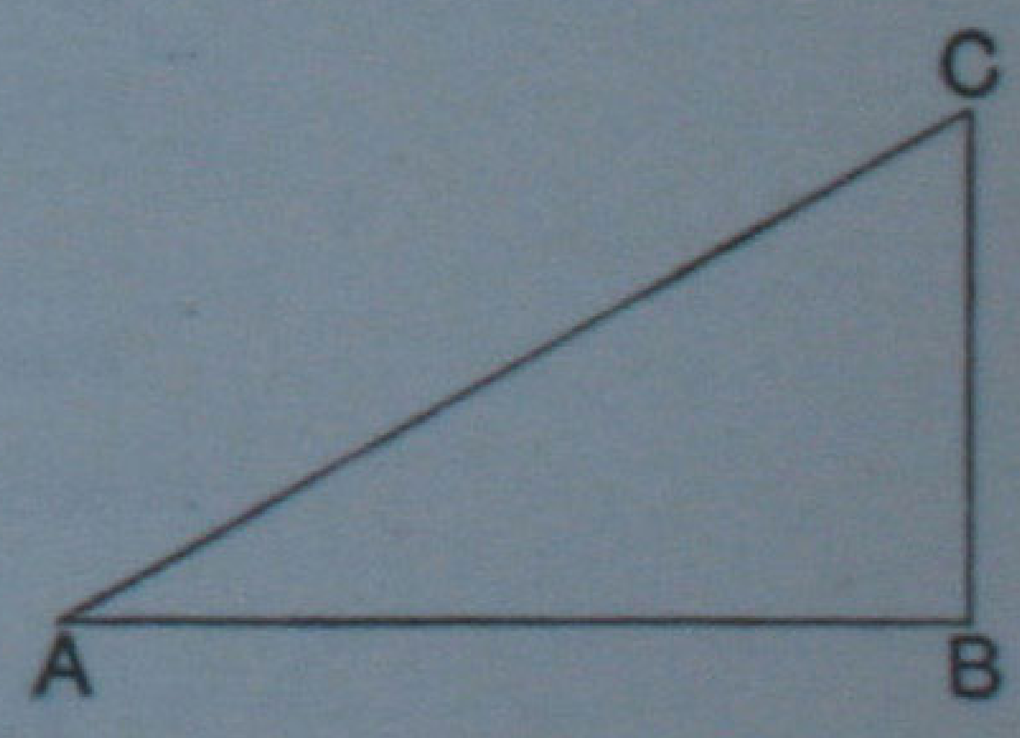
Two angles are said to be **complementary** when their sum is a right angle.

Thus in every right-angled triangle, each acute angle is the complement of the other. For in the figure of the next article, if B is the right angle, the sum of A and C is 90°.

$$\therefore \quad C = 90^\circ - A, \text{ and } A = 90^\circ - C.$$

Trigonometrical Ratios of Complementary Angles.

38. Let ABC be a right-angled triangle, of which B is the right angle; then the angles at A and C are complementary, so that $C = 90^\circ - A$.



$$\therefore \quad \sin(90^\circ - A) = \sin C = \frac{AB}{AC} = \cos A;$$

and $\cos(90^\circ - A) = \cos C = \dfrac{BC}{AC} = \sin A$.

Similarly, it may be proved that

$$\left.\begin{aligned}\tan(90^\circ - A) &= \cot A,\\ \cot(90^\circ - A) &= \tan A;\end{aligned}\right\} \text{ and } \left.\begin{aligned}\sec(90^\circ - A) &= \operatorname{cosec} A,\\ \operatorname{cosec}(90^\circ - A) &= \sec A.\end{aligned}\right\}$$

39. If we define the **co**-sine, **co**-tangent, **co**-secant, as the **co**-functions of the angle, the foregoing results may be embodied in a single statement:

each function of an angle is equal to the corresponding co-function of its complement.

As an illustration of this we may refer to Art. 35, from which it will be seen that

$$\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2};$$

$$\sin 30^\circ = \cos 60^\circ = \frac{1}{2};$$

$$\tan 60^\circ = \cot 30^\circ = \sqrt{3}.$$

Example 1. Find a value of A when $\cos 2A = \sin 3A$.

Since $\cos 2A = \sin(90^\circ - 2A)$,

the equation becomes $\sin(90^\circ - 2A) = \sin 3A$;

$\therefore\ 90^\circ - 2A = 3A$;

whence $A = 18^\circ$.

Thus *one* value of A which satisfies the equation is $A = 18^\circ$. In a later chapter we shall be able to solve the equation more completely, and shew that there are other values of A which satisfy it.

Example 2. Prove that $\sec A \sec(90^\circ - A) = \tan A + \tan(90^\circ - A)$.

Here it will be found easier to begin with the expression on the right side of the identity.

The second side $= \tan A + \cot A$

$$= \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} = \frac{\sin^2 A + \cos^2 A}{\cos A \sin A}$$

$$= \frac{1}{\cos A \sin A}$$

$$= \sec A \operatorname{cosec} A = \sec A \sec(90^\circ - A).$$

EXAMPLES. IV. b.

Find the complements of the following angles:

1. $67^\circ\, 30'$. 2. $25^\circ\, 30''$. 3. $10^\circ\, 1'\, 3''$.

4. $45^\circ - A$. 5. $45^\circ + B$. 6. $30^\circ - B$.

7. In a triangle C is 50° and A is the complement of 10°; find B.

8. In a triangle A is the complement of 40°; and B is the complement of 20°; find C.

Find a value of A in each of the following equations:

9. $\sin A = \cos 4A$.

10. $\cos 3A = \sin 7A$.

11. $\tan A = \cot 3A$.

12. $\cot A = \tan A$.

13. $\cot A = \tan 2A$.

14. $\sec 5A = \operatorname{cosec} A$.

Prove the following identities:

15. $\sin(90^\circ - A)\cot(90^\circ - A) = \sin A$.

16. $\sin A \tan(90^\circ - A)\sec(90^\circ - A) = \cot A$.

17. $\cos A \tan A \tan(90^\circ - A)\operatorname{cosec}(90^\circ - A) = 1$.

18. $\sin A \cos(90^\circ - A) + \cos A \sin(90^\circ - A) = 1$.

19. $\cos(90^\circ - A)\operatorname{cosec}(90^\circ - A) = \tan A$.

20. $\operatorname{cosec}^2(90^\circ - A) = 1 + \sin^2 A \operatorname{cosec}^2(90^\circ - A)$.

21. $\sin A \cot A \cot(90^\circ - A)\sec(90^\circ - A) = 1$.

22. $\sec(90^\circ - A) - \cot A \cos(90^\circ - A)\tan(90^\circ - A) = \sin A$.

23. $\tan^2 A \sec^2(90^\circ - A) - \sin^2 A \operatorname{cosec}^2(90^\circ - A) = 1$.

24. $\tan(90^\circ - A) + \cot(90^\circ - A) = \operatorname{cosec} A \operatorname{cosec}(90^\circ - A)$.

25. $\dfrac{\sin(90^\circ - A)}{\sec(90^\circ - A)} \cdot \dfrac{\tan(90^\circ - A)}{\cos A} = \cos A$.

26. $\dfrac{\operatorname{cosec}^2 A \tan^2 A}{\cot(90^\circ - A)} \cdot \dfrac{\cot A}{\sec^2 A} = \sec^2(90^\circ - A) - 1$.

27. $\dfrac{\cot(90^\circ - A)}{\operatorname{cosec}^2 A} \cdot \dfrac{\sec A \cot^3 A}{\sin^2(90^\circ - A)} = \sqrt{\tan^2 A + 1}$.

28. $\dfrac{\cos^2(90^\circ - A)}{\operatorname{vers} A} = 1 + \sin(90^\circ - A)$.

29. $\dfrac{\cot^2 A \sin^2(90^\circ - A)}{\cot A + \cos A} = \tan(90^\circ - A) - \cos A$.

30. If $x \sin(90^\circ - A)\cot(90^\circ - A) = \cos(90^\circ - A)$, find x.

31. Find the value of x which will satisfy

$$\sec A \operatorname{cosec}(90^\circ - A) - x \cot(90^\circ - A) = 1.$$

Easy Trigonometrical Equations.

40. As a further exercise in using the formulæ of Art. 29 and the numerical values of the functions of 45°, 60°, 30°, we shall now give some examples in trigonometrical equations.

Example 1. Solve $4\cos A=3\sec A$.

By expressing the secant in terms of the cosine, we have

$$4\cos A=\frac{3}{\cos A},$$

$$4\cos^2 A=3,$$

$$\cos A=\pm\frac{\sqrt{3}}{2}.$$

$$\therefore\ \cos A=\frac{\sqrt{3}}{2}\ \ldots\ldots\ldots\ldots(1),$$

or

$$\cos A=-\frac{\sqrt{3}}{2}\ \ldots\ldots\ldots\ldots(2).$$

Since $\cos 30°=\frac{\sqrt{3}}{2}$, we see from (1) that $A=30°$.

The student will be able to understand the meaning of the negative result in (2) after he has read Chap. VIII.

Example 2. Solve $3\sec^2\theta=8\tan\theta-2$.

Since $\sec^2\theta=1+\tan^2\theta$,

we have $3(1+\tan^2\theta)=8\tan\theta-2$,

or $3\tan^2\theta-8\tan\theta+5=0$.

This is a quadratic equation in which $\tan\theta$ is the unknown quantity, and it may be solved by any of the rules for solving quadratic equations.

Thus $(\tan\theta-1)(3\tan\theta-5)=0$,

therefore *either* $\tan\theta-1=0\ \ldots\ldots\ldots\ldots(1)$,

or $3\tan\theta-5=0\ \ldots\ldots\ldots\ldots(2)$.

From (1), $\tan\theta=1$, so that $\theta=45°$.

From (2), $\tan\theta=\frac{5}{3}$, a result which we cannot interpret at present.

41. When an equation involves more than two functions, it will usually be best to express each function in terms of the sine and cosine.

Example. Solve $3\tan\theta+\cot\theta=5\operatorname{cosec}\theta$.

We have
$$\frac{3\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}=\frac{5}{\sin\theta},$$
$$3\sin^2\theta+\cos^2\theta=5\cos\theta,$$
$$3(1-\cos^2\theta)+\cos^2\theta=5\cos\theta,$$
$$2\cos^2\theta+5\cos\theta-3=0,$$
$$(2\cos\theta-1)(\cos\theta+3)=0;$$

therefore *either* $2\cos\theta-1=0$ ....(1),

or $\cos\theta+3=0$ ....(2).

From (1), $\cos\theta=\frac{1}{2}$, so that $\theta=60°$.

From (2), $\cos\theta=-3$, a result which must be rejected as *impossible*, because the numerical value of the cosine of an angle can never be greater than unity. [Art. 16.]

EXAMPLES. IV. c.

Find a solution of each of the following equations:

1. $2\sin\theta=\operatorname{cosec}\theta$.
2. $\tan\theta=3\cot\theta$.
3. $\sec\theta=4\cos\theta$.
4. $\sec\theta-\operatorname{cosec}\theta=0$.
5. $4\sin\theta=3\operatorname{cosec}\theta$.
6. $\operatorname{cosec}^2\theta=4$.
7. $\sqrt{2}\cos\theta=\cot\theta$.
8. $\tan\theta=2\sin\theta$.
9. $\sec^2\theta=2\tan^2\theta$.
10. $\operatorname{cosec}^2\theta=4\cot^2\theta$.
11. $\sec^2\theta=3\tan^2\theta-1$.
12. $\sec^2\theta+\tan^2\theta=7$.
13. $\cot^2\theta+\operatorname{cosec}^2\theta=3$.
14. $2(\cos^2\theta-\sin^2\theta)=1$.
15. $2\cos^2\theta+4\sin^2\theta=3$.
16. $6\cos^2\theta=1+\cos\theta$.
17. $4\sin\theta=12\sin^2\theta-1$.
18. $2\sin^2\theta=3\cos\theta$.
19. $\tan\theta=4-3\cot\theta$.
20. $\cos^2\theta-\sin^2\theta=2-5\cos\theta$.
21. $\cot\theta+\tan\theta=2\sec\theta$.
22. $4\operatorname{cosec}\theta+2\sin\theta=9$.
23. $\tan\theta-\cot\theta=\operatorname{cosec}\theta$.
24. $2\cos\theta+2\sqrt{2}=3\sec\theta$.
25. $2\sin\theta\tan\theta+1=\tan\theta+2\sin\theta$.
26. $6\tan\theta-5\sqrt{3}\sec\theta+12\cot\theta=0$.
27. If $\tan\theta+3\cot\theta=4$, prove that $\tan\theta=1$ or 3.
28. Find $\cot\theta$ from the equation
$$\operatorname{cosec}^2\theta+\cot^2\theta=3\cot\theta.$$

MISCELLANEOUS EXAMPLES. A.

1. Express as the decimal of a right angle

$$(1)\ 25^g\ 37'\ 6{\cdot}4''; \qquad (2)\ 63^\circ\ 21'\ 36''.$$

2. Shew that

$$\sin A \cos A \tan A + \cos A \sin A \cot A = 1.$$

3. A ladder 29 ft. long just reaches a window at a height of 21 ft. from the ground: find the cosine and cosecant of the angle made by the ladder with the ground.

4. If $\operatorname{cosec} A = \frac{17}{15}$, find $\tan A$ and $\sec A$.

5. Shew that $\operatorname{cosec}^2 A - \cot A \cos A \operatorname{cosec} A - 1 = 0$.

---

6. Reduce to sexagesimal measure

$$(1)\ 17^g\ 18'\ 75''; \qquad (2)\ {\cdot}0003 \text{ of a right angle.}$$

7. ABC is a triangle in which B is a right angle; if $c=9$, $a=40$, find b, $\cot A$, $\sec A$, $\sec C$.

8. Which of the following statements are possible and which impossible?

$$(1)\ 4\sin\theta = 1; \quad (2)\ 2\sec\theta = 1; \quad (3)\ 7\tan\theta = 40.$$

9. Prove that $\cos\theta \operatorname{vers}\theta (\sec\theta + 1) = \sin^2\theta$.

10. Express $\sec a$ and $\operatorname{cosec} a$ in terms of $\cot a$.

---

11. Find the numerical value of

$$3\tan^2 30^\circ + \frac{1}{4}\sec 60^\circ + 5\cot^2 45^\circ - \frac{2}{3}\sin^2 60^\circ.$$

12. If $\tan a = \frac{m}{n}$, find $\sin a$ and $\sec a$.

13. If m sexagesimal minutes are equivalent to n centesimal minutes, prove that $m = {\cdot}54n$.

14. If $\sin A = \frac{4}{5}$, prove that $\tan A + \sec A = 3$, when A is an acute angle.

15. Shew that

$$\cot(90° - A) \cot A \cos(90° - A) \tan(90° - A) = \cos A.$$

---

16. PQR is a triangle in which P is a right angle; if $PQ = 21$, $PR = 20$, find $\tan Q$ and $\operatorname{cosec} Q$.

17. Shew that $(\tan a - \cot a) \sin a \cos a = 1 - 2\cos^2 a$.

18. Find a value of θ which satisfies the equation

$$\sec 6\theta = \operatorname{cosec} 3\theta.$$

19. Prove that

$$\tan^2 60° - 2\tan^2 45° = \cot^2 30° - 2\sin^2 30° - \frac{3}{4}\operatorname{cosec}^2 45°.$$

20. Solve the equations:

(1) $3\sin\theta = 2\cos^2\theta$; (2) $5\cot\theta - \operatorname{cosec}^2\theta = 3$.

---

21. Prove that $1 + 2\sec^2 A \tan^2 A - \sec^4 A - \tan^4 A = 0$.

22. In the equation

$$6\sin^2\theta - 11\sin\theta + 4 = 0,$$

shew that one value of θ is impossible, and find the other value.

23. In a triangle ABC right-angled at C, prove that

$$\tan A + \tan B = \frac{c^2}{ab}.$$

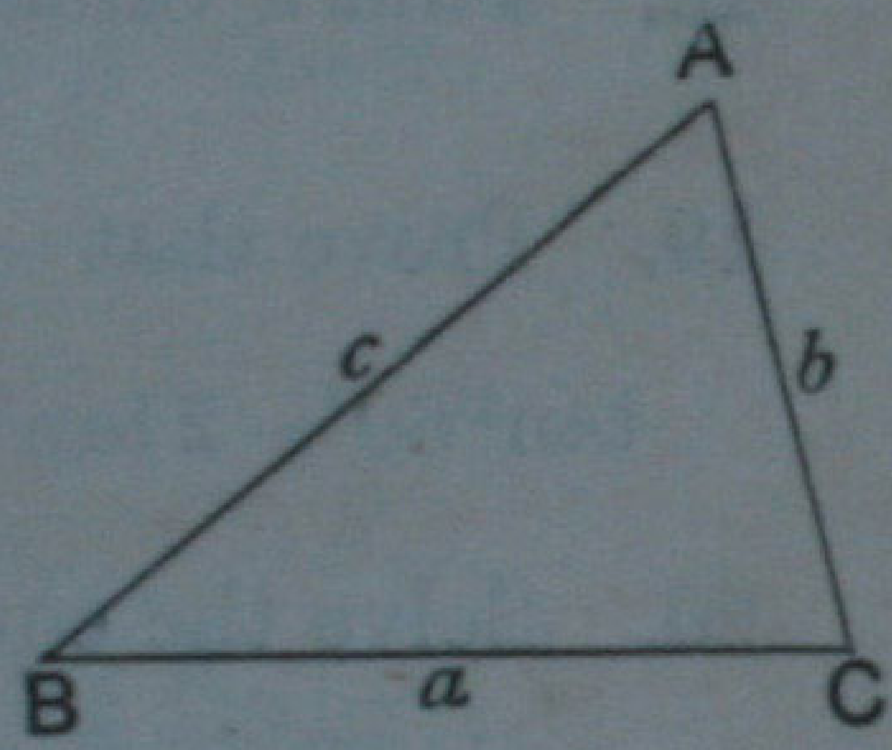
24. If $\cot A = c$, shew that $c + c^{-1} = \sec A \operatorname{cosec} A$.

25. Shew that $\dfrac{\sqrt{2\operatorname{vers} A - \operatorname{vers}^2 A}}{1 - \operatorname{vers} A} = \tan A$.

CHAPTER V.

SOLUTION OF RIGHT-ANGLED TRIANGLES.

42. EVERY triangle has six *parts*, namely, three sides and three angles. In Trigonometry it is usual to denote the three angles by the capital letters A, B, C, and the lengths of the sides respectively opposite to these angles by the letters a, b, c. It must be understood that a, b, c are *numerical quantities* expressing the number of units of length contained in the three sides.



43. We know from Geometry that it is always possible to construct a triangle when any three parts are given, provided that one at least of the parts is a side. Similarly, if the values of suitable parts of a triangle be given, we can by Trigonometry find the remaining parts. The process by which this is effected is called the **Solution of the triangle.**

The *general* solution of triangles will be discussed at a later stage; in this chapter we shall confine our attention to right-angled triangles.

44. From Euc. I. 47, we know that when a triangle is right-angled, if any two sides are given the third can be found. Thus in the figure of the next article, where ABC is a triangle right-angled at A, we have $a^2 = b^2 + c^2$; whence if any two of the three quantities a, b, c are given, the third may be determined.

Again, the two acute angles are complementary, so that if one is given the other is also known.

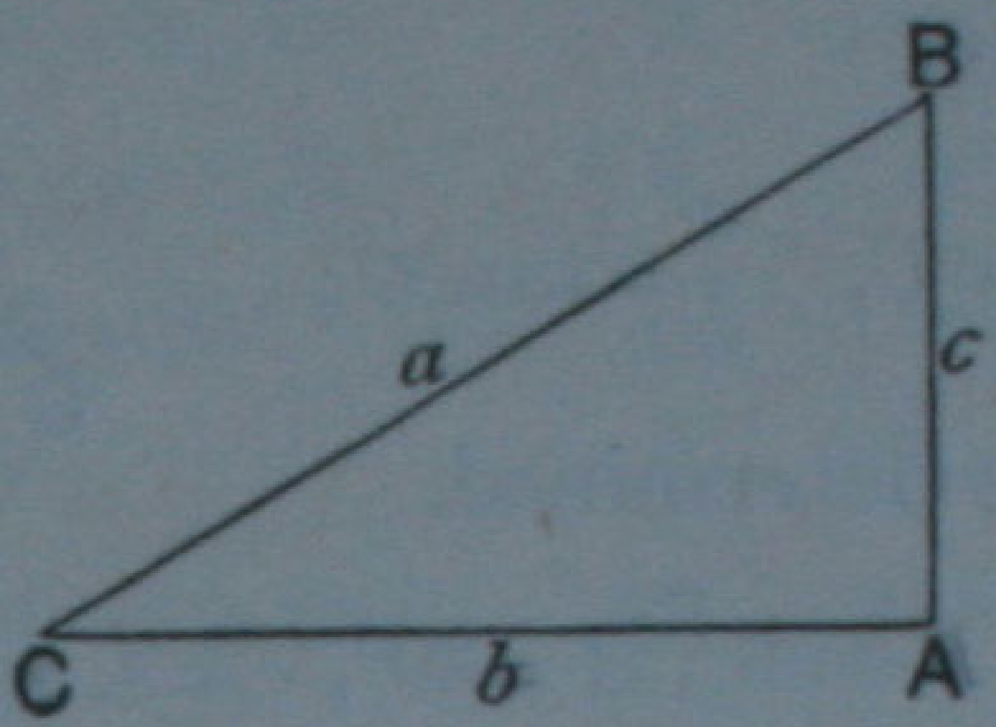
Hence in the solution of right-angled triangles there are really only two cases to be considered :

I. when *any two sides* are given ;

II. when *one side* and *one acute angle* are given.

45. CASE I. *To solve a right-angled triangle when two sides are given.*

Let ABC be a right-angled triangle, of which A is the right angle, and suppose that any two sides are given;



then the third side may be found from the equation

$$a^2 = b^2 + c^2.$$

Also

$$\cos C = \frac{b}{a}, \text{ and } B = 90^\circ - C;$$

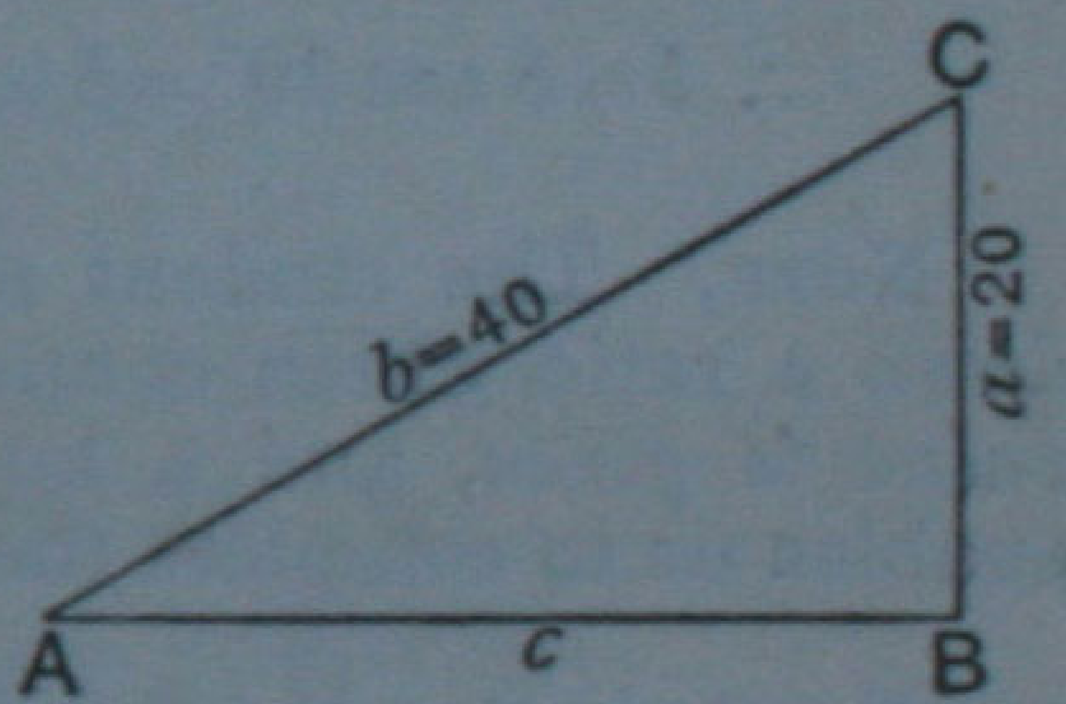
whence C and B may be obtained.

Example. Given $B = 90^\circ$, $a = 20$, $b = 40$, solve the triangle.

Here $c^2 = b^2 - a^2$

$= 1600 - 400 = 1200;$

$\therefore c = 20\sqrt{3}.$



Also $\sin A = \frac{a}{b} = \frac{20}{40} = \frac{1}{2};$

$\therefore A = 30^\circ.$

And $C = 90^\circ - A = 90^\circ - 30^\circ = 60^\circ.$

The solution of a trigonometrical problem may often be obtained in more than one way. In the present case the triangle can be solved without making use of Euc. I. 47.

Another solution may be given as follows:

$$\cos C = \frac{a}{b} = \frac{20}{40} = \frac{1}{2};$$

$$\therefore C = 60^\circ.$$

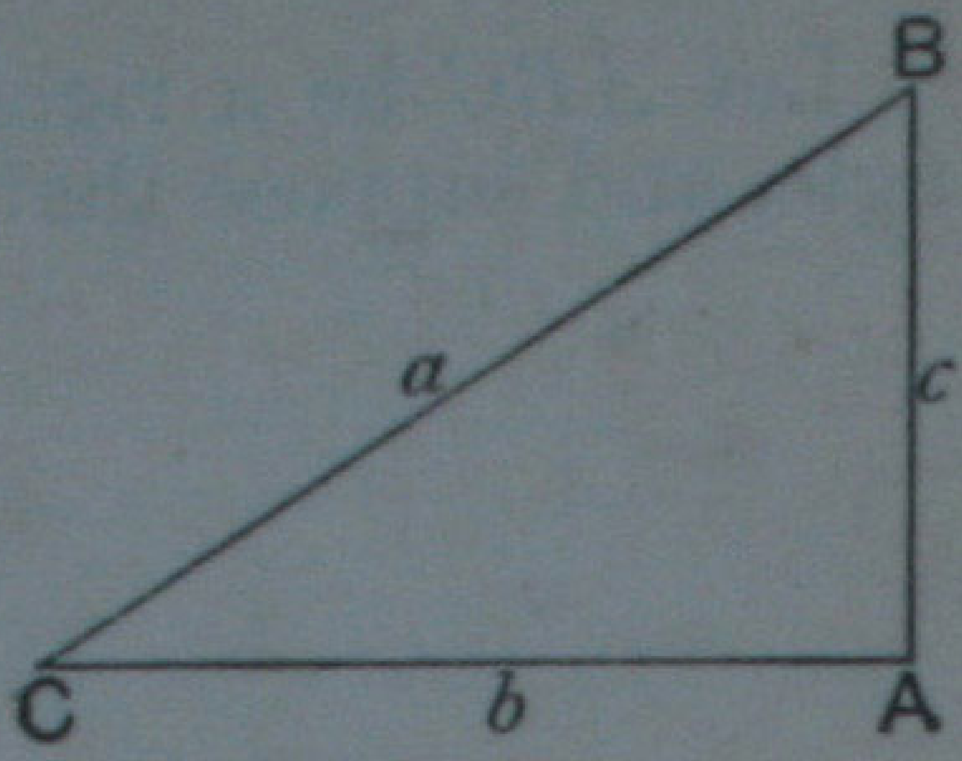
And $$A = 90^\circ - C = 90^\circ - 60^\circ = 30^\circ.$$

Also $$\frac{c}{40} = \cos A = \cos 30^\circ = \frac{\sqrt{3}}{2};$$

$$\therefore c = 20\sqrt{3}.$$

46. CASE II. *To solve a right-angled triangle when one side and one acute angle are given.*

Let ABC be a right-angled triangle of which A is the right angle, and suppose one side b and one acute angle C are given; then

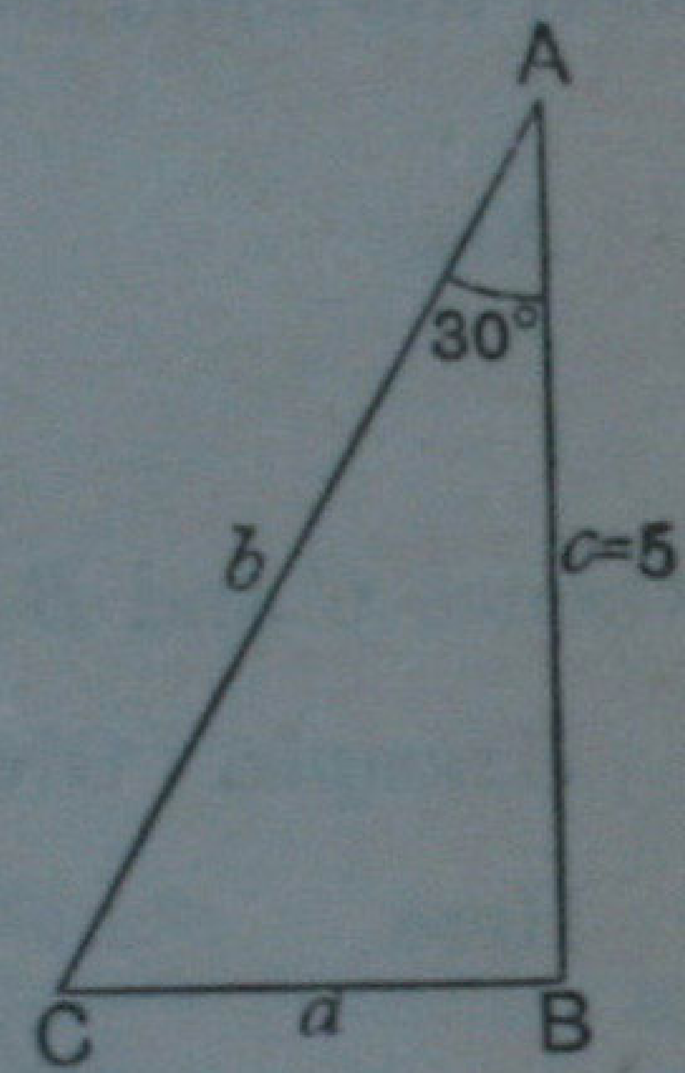


$$B=90^\circ-C, \quad \frac{a}{b}=\sec C, \quad \frac{c}{b}=\tan C;$$

whence B, a, c may be determined.

Example 1. Given $B=90^\circ$, $A=30^\circ$, $c=5$, solve the triangle.

We have $C=90^\circ-A=90^\circ-30^\circ=60^\circ$.



Also $$\frac{a}{5}=\tan 30^\circ;$$

$$\therefore\ a=5\tan 30^\circ=\frac{5}{\sqrt{3}}$$

$$=\frac{5}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{5\sqrt{3}}{3}.$$

Again, $$\frac{b}{5}=\sec 30^\circ;$$

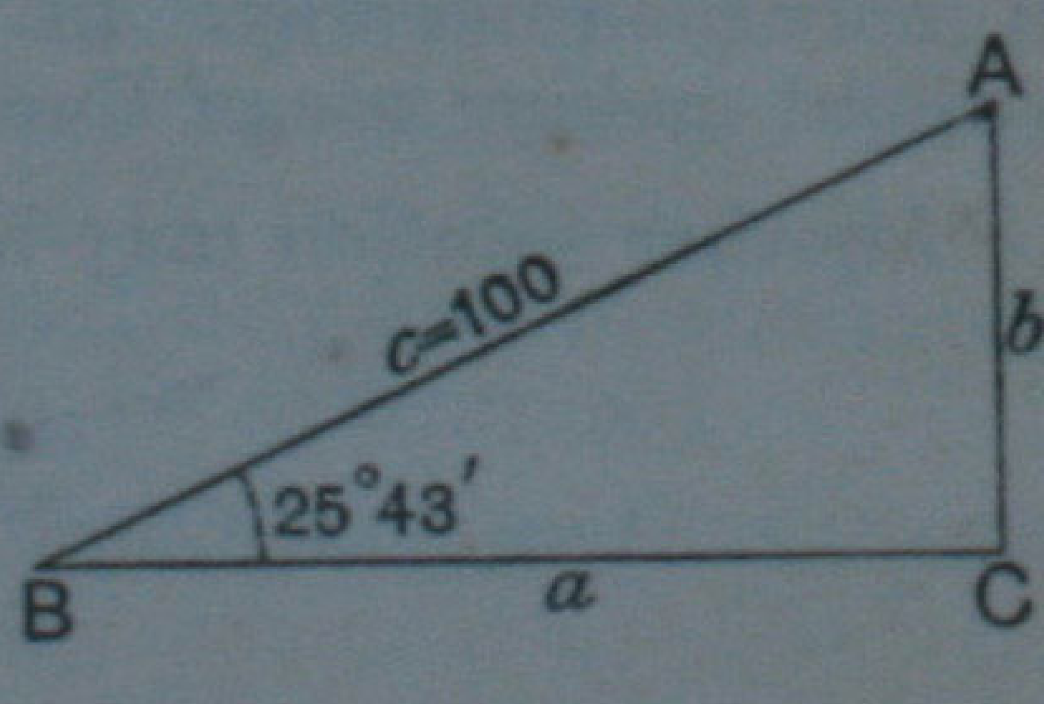
$$\therefore\ b=5\sec 30^\circ=5\times\frac{2}{\sqrt{3}}=\frac{10}{\sqrt{3}}=\frac{10\sqrt{3}}{3}.$$

NOTE. The student should observe that in each case we write down a ratio which connects *the side we are finding with that whose value is given*, and a knowledge of the ratios of the given angle enables us to complete the solution.

Example 2. If $C=90^\circ$, $B=25^\circ 43'$, and $c=100$, solve the triangle, having given $\tan 25^\circ 43'=\cdot 482$ and $\cos 25^\circ 43'=\cdot 901$.

Here $A=90^\circ-B$

$=90^\circ-25^\circ 43'=64^\circ 17'$.



Now $$\frac{a}{c}=\cos B;$$

that is, $$\frac{a}{100}=\cos 25^\circ 43';$$

$$\therefore\ a=100\cos 25^\circ 43'$$

$$=100\times\cdot 901=90\cdot 1.$$

Also $$\frac{b}{a}=\tan B, \text{ or } b=a\tan B;$$

$$\therefore\ b=90\cdot 1\times\tan 25^\circ 43'=90\cdot 1\times\cdot 482=43\cdot 4282.$$

EXAMPLES. V. a.

Solve the triangles in which the following parts are given:

1. $A=90°$, $a=4$, $b=2\sqrt{3}$.

2. $c=6$, $b=12$, $B=90°$.

3. $C=90°$, $b=12$, $a=4\sqrt{3}$.

4. $a=60$, $b=30$, $A=90°$.

5. $a=20$, $c=20$, $B=90°$.

6. $a=5\sqrt{3}$, $b=15$, $C=90°$.

7. $b=c=2$, $A=90°$.

8. $2c=b=6\sqrt{3}$, $B=90°$.

9. $C=90°$, $a=9\sqrt{3}$, $A=30°$.

10. $A=90°$, $B=30°$, $a=4$.

11. $A=60°$, $c=8$, $C=90°$.

12. $A=60°$, $C=30°$, $b=6$.

13. $B=90°$, $C=60°$, $b=100$.

14. $A=30°$, $B=60°$, $b=20\sqrt{3}$.

15. $B=C=45°$, $c=4$.

16. $2B=C=60°$, $a=8$.

17. If $C=90°$, $\cot A=\cdot07$, $b=49$, find a.

18. If $C=90°$, $A=38°\ 19'$, $c=50$, find a;
given $\sin 38°\ 19'=\cdot62$.

19. If $a=100$, $B=90°$, $C=40°\ 51'$, find c;
given $\tan 40°\ 51'=\cdot8647$.

20. If $b=20$, $A=90°$, $C=78°\ 12'$, find a;
given $\sec 78°\ 12'=4\cdot89$.

21. If $B=90°$, $A=36°$, $c=100$, solve the triangle;
given $\tan 36°=\cdot73$, $\sec 36°=1\cdot24$.

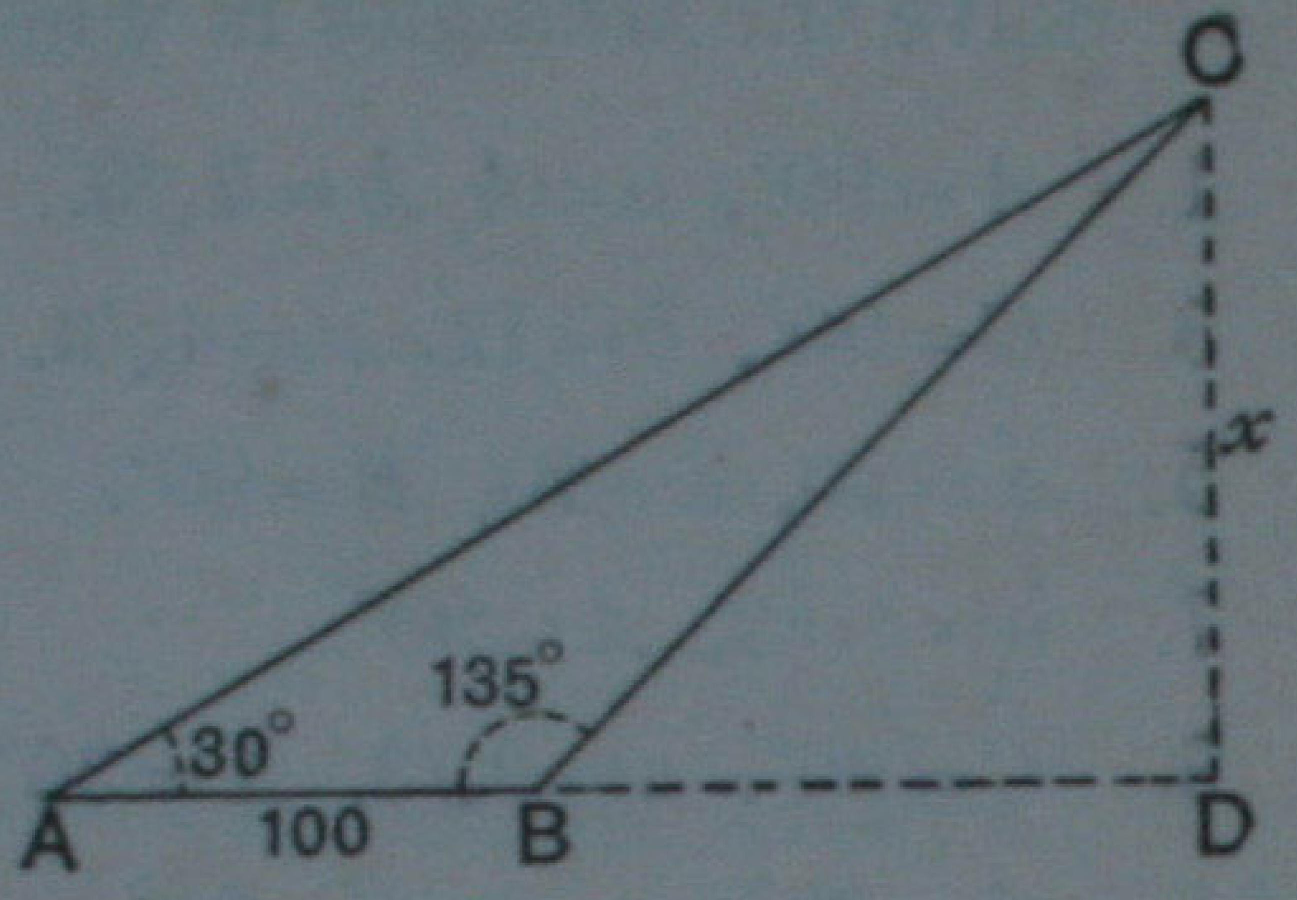
22. If $A=90°$, $c=37$, $a=100$, solve the triangle;
given $\sin 21°\ 43'=\cdot37$, $\cos 21°\ 43'=\cdot93$.

23. If $A=90°$, $B=39°\ 24'$, $b=25$, solve the triangle;
given $\cot 39°\ 24'=1\cdot2174$, $\operatorname{cosec} 39°\ 24'=1\cdot5755$.

24. If $C=90°$, $a=225$, $b=272$, solve the triangle;
given $\tan 50°\ 24'=1\cdot209$.

47. It will be found that all the varieties of the solution of right-angled triangles which can arise are either included in the two cases of Arts. 45 and 46, or in some modification of them. Sometimes the solution of a problem may be obtained by solving *two* right-angled triangles. The two examples we give as illustrations will in various forms be frequently met with in subsequent chapters.

Example 1. In the triangle ABC, the angles A and B are equal to $30°$ and $135°$ respectively, and the side AB is 100 feet; find the length of the perpendicular from C upon AB produced.



Draw CD perpendicular to AB produced, and let $CD=x$.

Then $\angle CBD=180°-135°=45°$;

$$\therefore BD=CD=x.$$

Now in the right-angled triangle ADC,

$$\frac{CD}{AD}=\tan DAC=\tan 30°;$$

that is,

$$\frac{x}{x+100}=\frac{1}{\sqrt{3}};$$

$$\therefore x\sqrt{3}=x+100;$$

$$x(\sqrt{3}-1)=100,$$

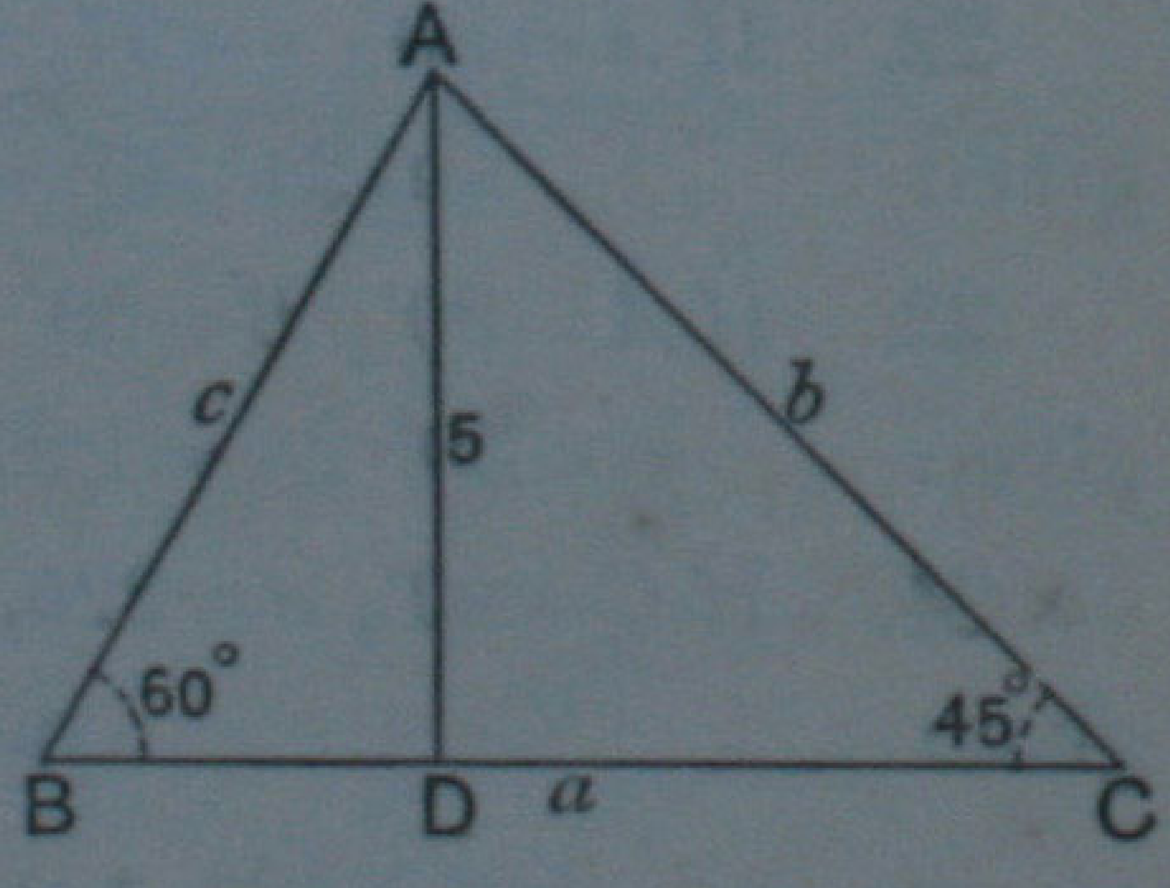
$$x=\frac{100}{\sqrt{3}-1}=\frac{100(\sqrt{3}+1)}{3-1};$$

$$\therefore x=50(\sqrt{3}+1).$$

Thus the distance required is $50(\sqrt{3}+1)$ feet.

Example 2. In the triangle ABC, AD is drawn perpendicular to BC; solve the triangle, having given

$$AD=5, \quad \angle ABD=60°, \quad \angle ACD=45°.$$



In the right-angled triangle ABD,

$$\frac{AB}{AD}=\operatorname{cosec} ABD;$$

$$\therefore AB=AD\operatorname{cosec} ABD=5\operatorname{cosec}60°$$

$$=5\times\frac{2}{\sqrt{3}}=\frac{10}{\sqrt{3}}=\frac{10\sqrt{3}}{3}.$$

Also $\dfrac{BD}{AD}=\cot ABD$;

$$\therefore BD=AD\cot ABD$$

$$=5\cot 60°=\frac{5}{\sqrt{3}}=\frac{5\sqrt{3}}{3}.$$

In the right-angled triangle ADC,

$$\angle DAC=45°=\angle DCA;$$

$$\therefore DC=DA=5.$$

Thus $$BC = CD + BD = 5 + \frac{5\sqrt{3}}{3} = \frac{15+5\sqrt{3}}{3}.$$

And $$\frac{AC}{AD} = \operatorname{cosec} ACD;$$

$$\therefore\ AC = AD \operatorname{cosec} ACD = 5 \operatorname{cosec} 45° = 5\sqrt{2}.$$

Finally, $$\angle BAC = 180° - 60° - 45° = 75°.$$

Thus $$a = \frac{15+5\sqrt{3}}{3},\quad b = 5\sqrt{2},\quad c = \frac{10\sqrt{3}}{3},\quad A = 75°.$$

EXAMPLES. V. b.

1. ABC is a triangle, and BD is perpendicular to AC produced: find BD, given

$$A = 30°,\ C = 120°,\ AC = 20.$$

2. If BD is perpendicular to the base AC of a triangle ABC, find a and c, given

$$A = 30°,\ C = 45°,\ BD = 10.$$

3. In the triangle ABC, AD is drawn perpendicular to BC making BD equal to 15 ft.: find the lengths of AB, AC, and AD, given that B and C are equal to 30° and 60° respectively.

4. In a right-angled triangle PQR, find the segments of the hypotenuse PR made by the perpendicular from Q; given

$$QR = 8,\quad \angle QRP = 60°,\quad \angle QPR = 30°.$$

5. If PQ is drawn perpendicular to the straight line QRS, find RS, given

$$PQ = 36,\quad \angle RPQ = 30°,\quad \angle SPQ = 60°.$$

6. If PQ is drawn perpendicular to the straight line QRS, find RS, given

$$PQ = 20,\quad \angle PRS = 135°,\quad \angle PSR = 30°.$$

7. In the triangle ABC, the angles B and C are equal to 45° and 120° respectively; if $a = 40$ find the length of the perpendicular from A on BC produced.

8. If CD is drawn perpendicular to the straight line DBA, find DC and BD, given

$$AB = 59,\quad \angle CBD = 45°,\quad \angle CAB = 32°\,50',\quad \cot 32°\,50' = 1{\cdot}59.$$

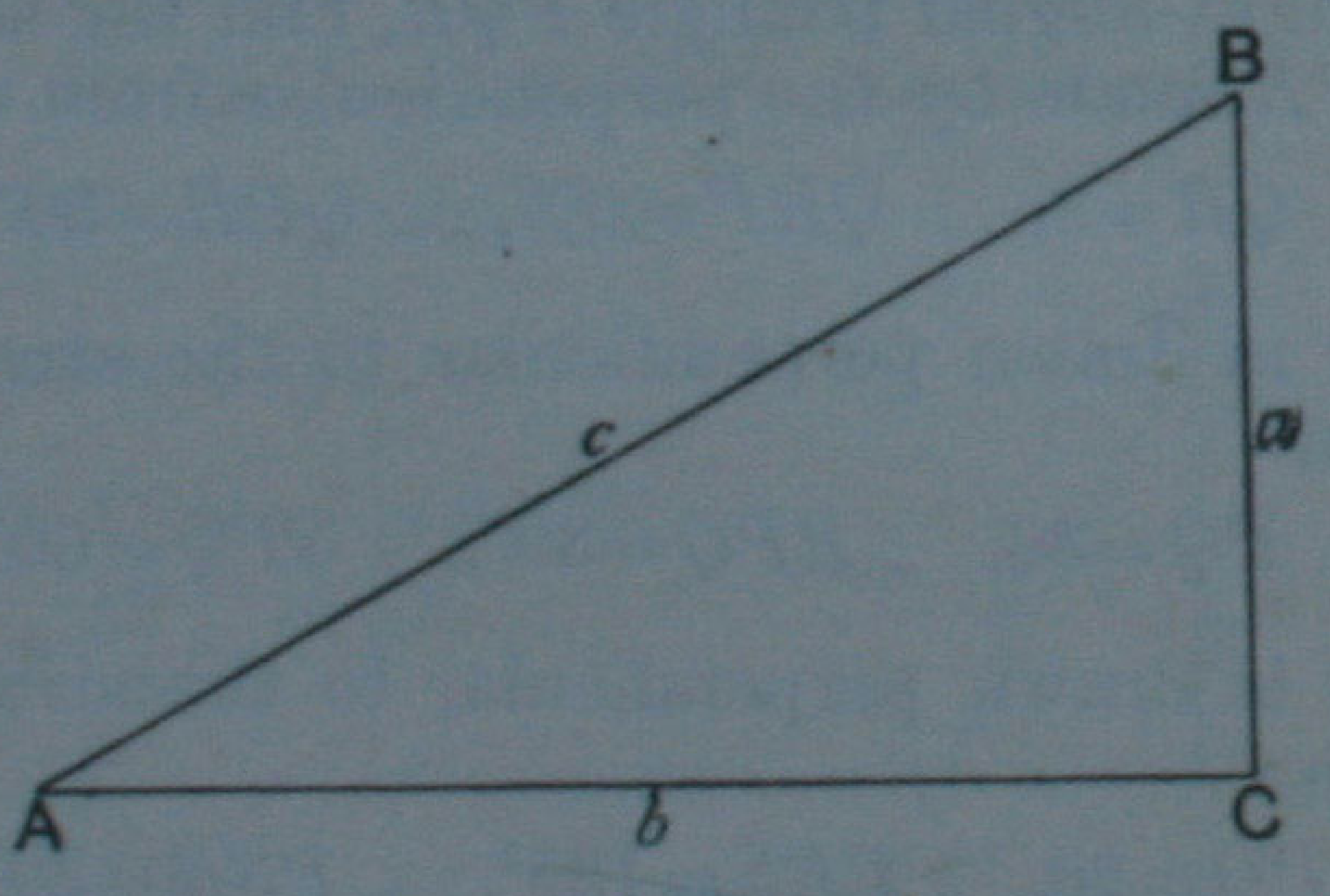
CHAPTER VI.

EASY PROBLEMS.

48. THE principles explained in the previous chapters may now be applied to the solution of problems in heights and distances. It will be assumed that by the use of suitable instruments the necessary lines and angles can be measured with sufficient accuracy for the purposes required.

After the practice afforded by the examples in the last chapter, the student should be able to write down at once any side of a right-angled triangle in terms of another through the medium of the functions of either acute angle. In the present and subsequent chapters it is of great importance to acquire readiness in this respect.

For instance, from the adjoining figure, we have

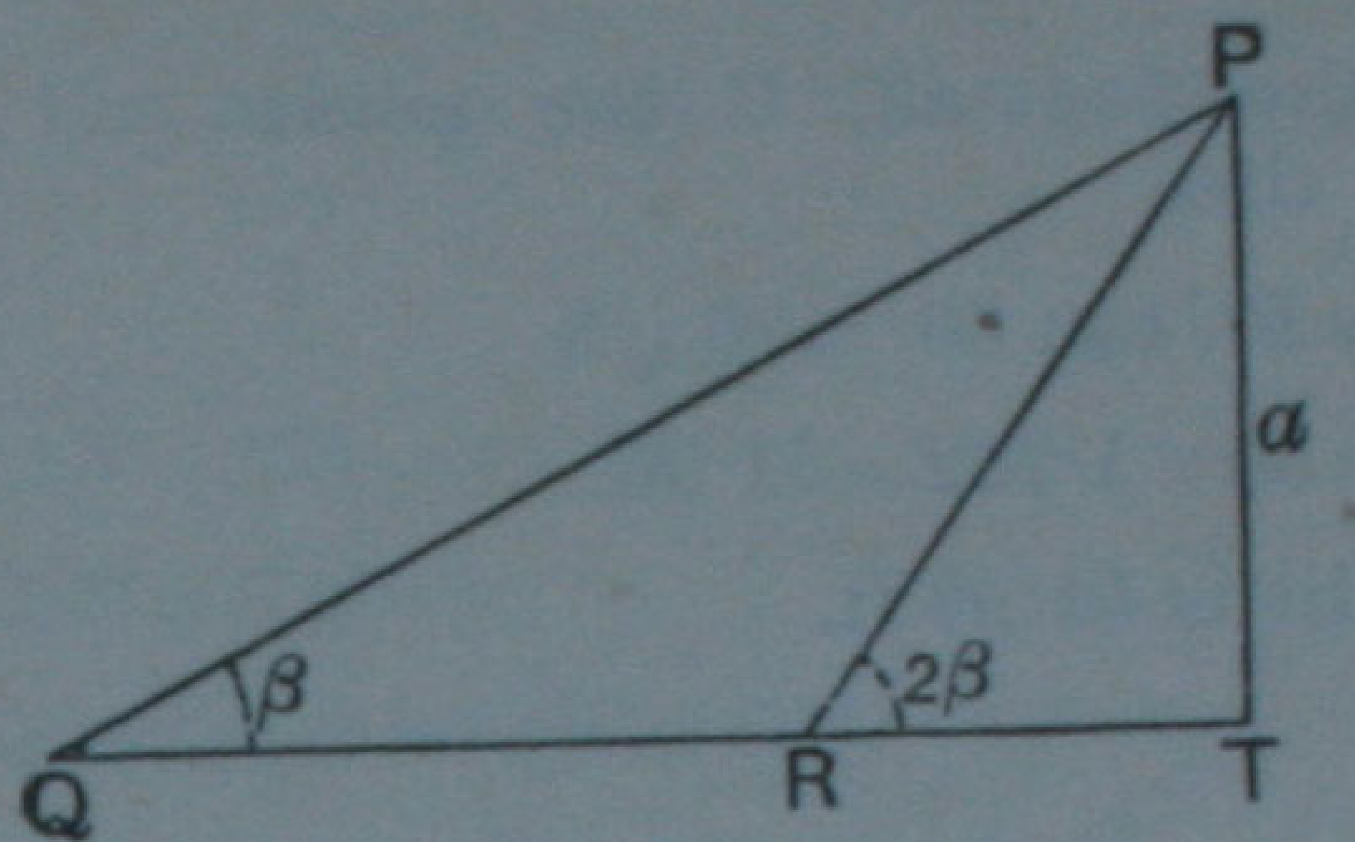


$$a=c\sin A,\quad a=c\cos B,\quad a=b\cot B,$$

$$a=b\tan A,\quad c=a\sec B,\quad b=a\tan B.$$

These relations are not to be committed to memory but in each case should be read off from the figure. There are several other similar relations connecting the parts of the above triangle, and the student should practise himself in obtaining them quickly.

Example. Q, R, T are three points in a straight line, and TP is drawn perpendicular to QT. If $PT=a$, $\angle PQT=\beta$, $\angle PRT=2\beta$, express the lengths of all the lines of the figure in terms of a and β.



By Euc. I. 32,

$$\angle QPR = \angle PRT - \angle PQR;$$
$$\therefore \angle QPR = 2\beta - \beta = \beta = \angle PQR;$$
$$\therefore QR = PR.$$

In the right-angled triangle PRT,

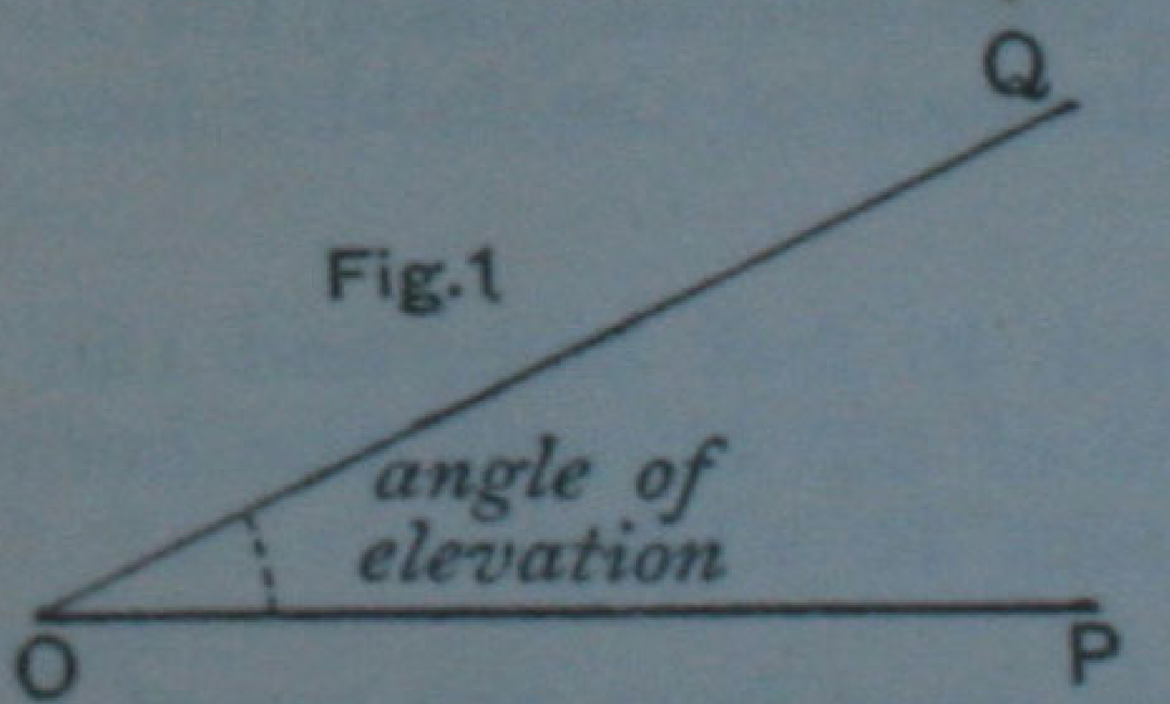
$$PR = a \operatorname{cosec} 2\beta;$$
$$\therefore QR = a \operatorname{cosec} 2\beta.$$

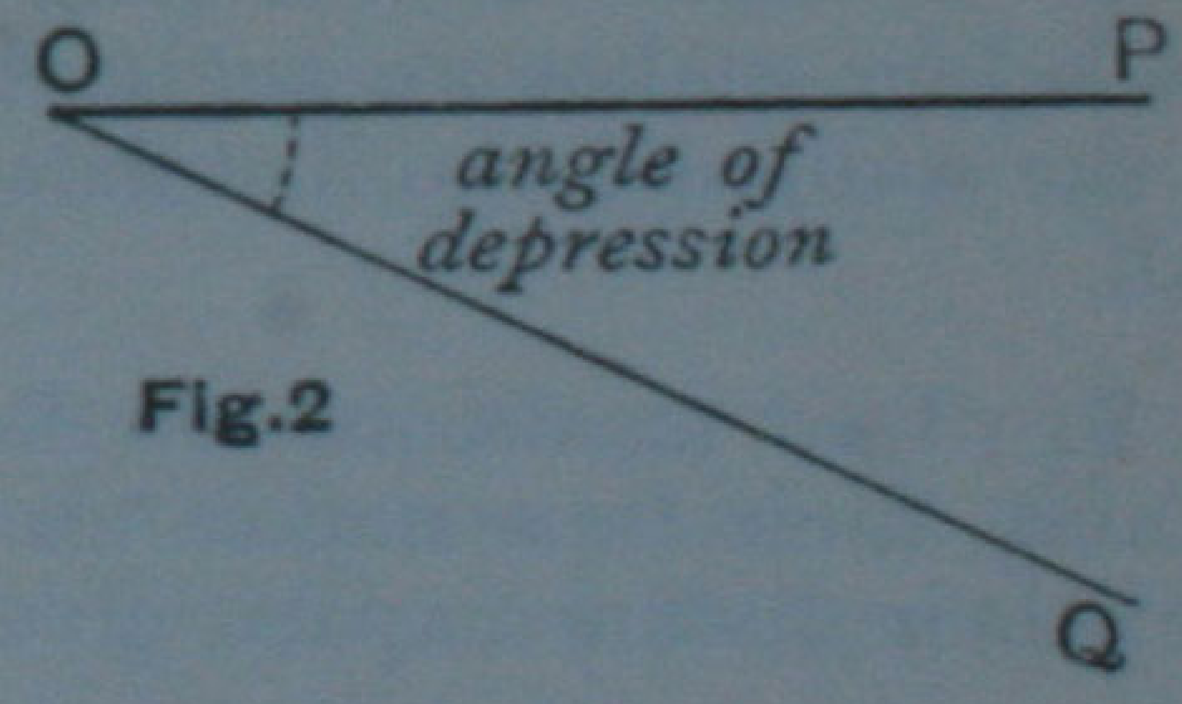
Also $$TR = a \cot 2\beta.$$

Lastly, in the right-angled triangle PQT,

$$QT = a \cot \beta,$$
$$PQ = a \operatorname{cosec} \beta.$$

49. Angles of elevation and depression. Let OP be a horizontal line in the same vertical plane as an object Q, and let OQ be joined.

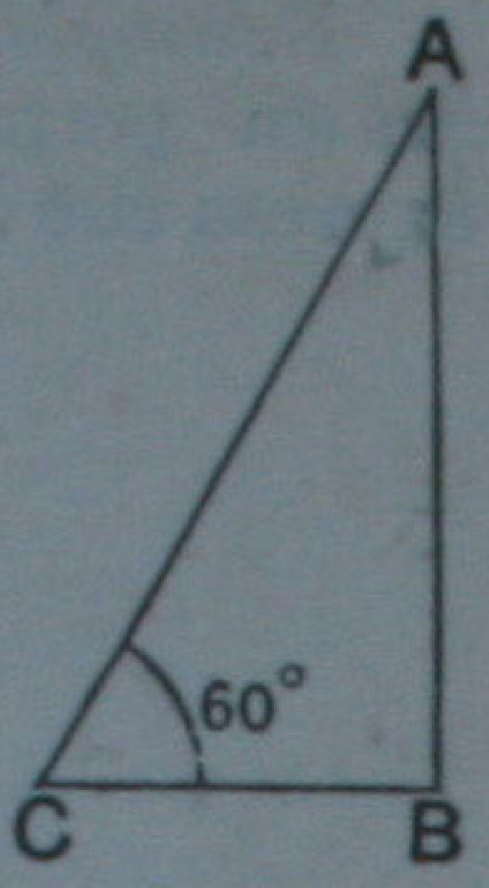




In Fig. 1, where the object Q is *above* the horizontal line OP, the angle POQ is called the **angle of elevation** of the object Q as seen from the point O.

In Fig. 2, where the object Q is *below* the horizontal line OP, the angle POQ is called the **angle of depression** of the object Q as seen from the point O.

Example I. A flagstaff stands on a horizontal plane, and from a point on the ground at a distance of 30 ft. its angle of elevation is 60°: find its height.



Let AB be the flagstaff, C the point of observation; then

$$AB = BC \tan 60° = 30\sqrt{3}$$
$$= 30 \times 1{\cdot}732 = 51{\cdot}96.$$

Thus the height is 51·96 ft.

EXAMPLES. VI. a.

[*The results should be expressed in a form free from surds by using the approximations* $\sqrt{2} = 1{\cdot}414$, $\sqrt{3} = 1{\cdot}732$.]

1. The angle of elevation of the top of a chimney at a distance of 300 feet is 30°: find its height.

2. From a ship's masthead 160 feet high the angle of depression of a boat is observed to be 30°: find its distance from the ship.

3. Find the angle of elevation of the sun when the shadow of a pole 6 feet high is $2\sqrt{3}$ feet long.

4. At a distance 86·6 feet from the foot of a tower the angle of elevation of the top is 30°. Find the height of the tower and the observer's distance from the top.

5. A ladder 45 feet long just reaches the top of a wall. If the ladder makes an angle of 60° with the wall, find the height of the wall, and the distance of the foot of the ladder from the wall.

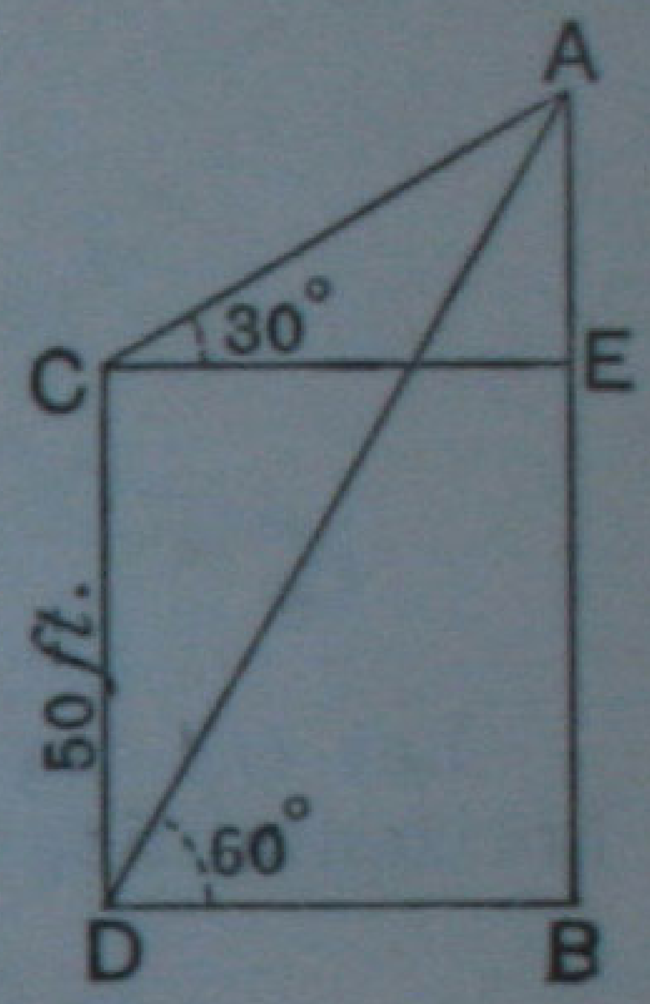
6. Two masts are 60 feet and 40 feet high, and the line joining their tops makes an angle of 33° 41′ with the horizon: find their distance apart, given $\cot 33° 41' = 1{\cdot}5$.

7. Find the distance of the observer from the top of a cliff which is 132 yards high, given that the angle of elevation is 41° 18′, and that $\sin 41° 18' = {\cdot}66$.

8. One chimney is 30 yards higher than another. A person standing at a distance of 100 yards from the lower observes their tops to be in a line inclined at an angle of 27° 2′ to the horizon: find their heights, given $\tan 27° 2' = {\cdot}51$.

Example II. From the foot of a tower the angle of elevation of the top of a column is 60°, and from the top of the tower, which is 50 ft. high, the angle of elevation is 30°: find the height of the column.

Let AB denote the column and CD the tower; draw CE parallel to DB.



Let $AB = x$;

then $AE = AB - BE = x - 50$.

Let $DB = CE = y$.

From the right-angled triangle ADB,

$$y = x \cot 60° = \frac{x}{\sqrt{3}}.$$

From the right-angled triangle ACE,

$$y = (x - 50) \cot 30° = \sqrt{3}\,(x - 50).$$

$$\therefore \frac{x}{\sqrt{3}} = \sqrt{3}\,(x - 50),$$

$$x = 3\,(x - 50);$$

whence $x = 75$.

Thus the column is 75 ft. high.

9. The angle of elevation of the top of a tower is 30°; on walking 100 yards nearer the elevation is found to be 60°: find the height of the tower.

10. A flagstaff stands upon the top of a building; at a distance of 40 feet the angles of elevation of the tops of the flagstaff and building are 60° and 30°: find the length of the flagstaff.

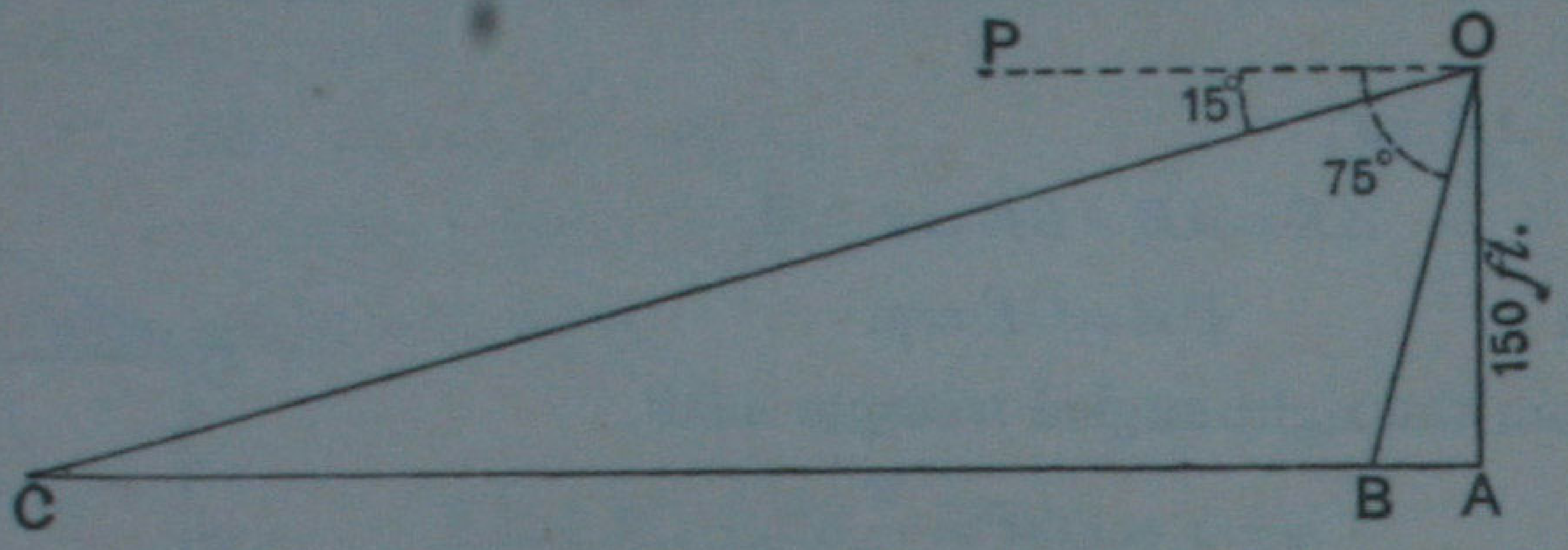
11. The angles of elevation of a spire at two places due east of it and 200 feet apart are 45° and 30°: find the height of the spire.

12. From the foot of a post the elevation of the top of a steeple is 45°, and from the top of the post, which is 30 feet high, the elevation is 30°; find the height and distance of the steeple.

13. The height of a hill is 3300 feet above the level of a horizontal plane. From a point A on this plane the angular elevation of the top of the hill is 60°. A balloon rises from A and ascends vertically upwards at a uniform rate; after 5 minutes the angular elevation of the top of the hill to an observer in the balloon is 30°: find the rate of the balloon's ascent in miles per hour.

Example III. From the top of a cliff 150 ft. high the angles of depression of two boats which are due South of the observer are 15° and 75°: find their distance apart, having given

$$\cot 15° = 2 + \sqrt{3} \text{ and } \cot 75° = 2 - \sqrt{3}.$$



Let OA represent the cliff, B and C the boats. Let OP be a horizontal line through O; then

$$\angle POC = 15° \text{ and } \angle POB = 75°;$$
$$\therefore \angle OCA = 15° \text{ and } \angle OBA = 75°.$$

Let $CB = x$, $AB = y$; then $CA = x + y$.

From the right-angled triangle OBA,

$$y = 150 \cot 75° = 150(2 - \sqrt{3}) = 300 - 150\sqrt{3}.$$

From the right-angled triangle OCA,

$$x + y = 150 \cot 15° = 150(2 + \sqrt{3}) = 300 + 150\sqrt{3}.$$

By subtraction, $\quad x = 300\sqrt{3} = 519{\cdot}6.$

Thus the distance between the boats is 519·6 ft.

14. From the top of a monument 100 feet high, the angles of depression of two objects on the ground due west of the monument are 45° and 30°: find the distance between them.

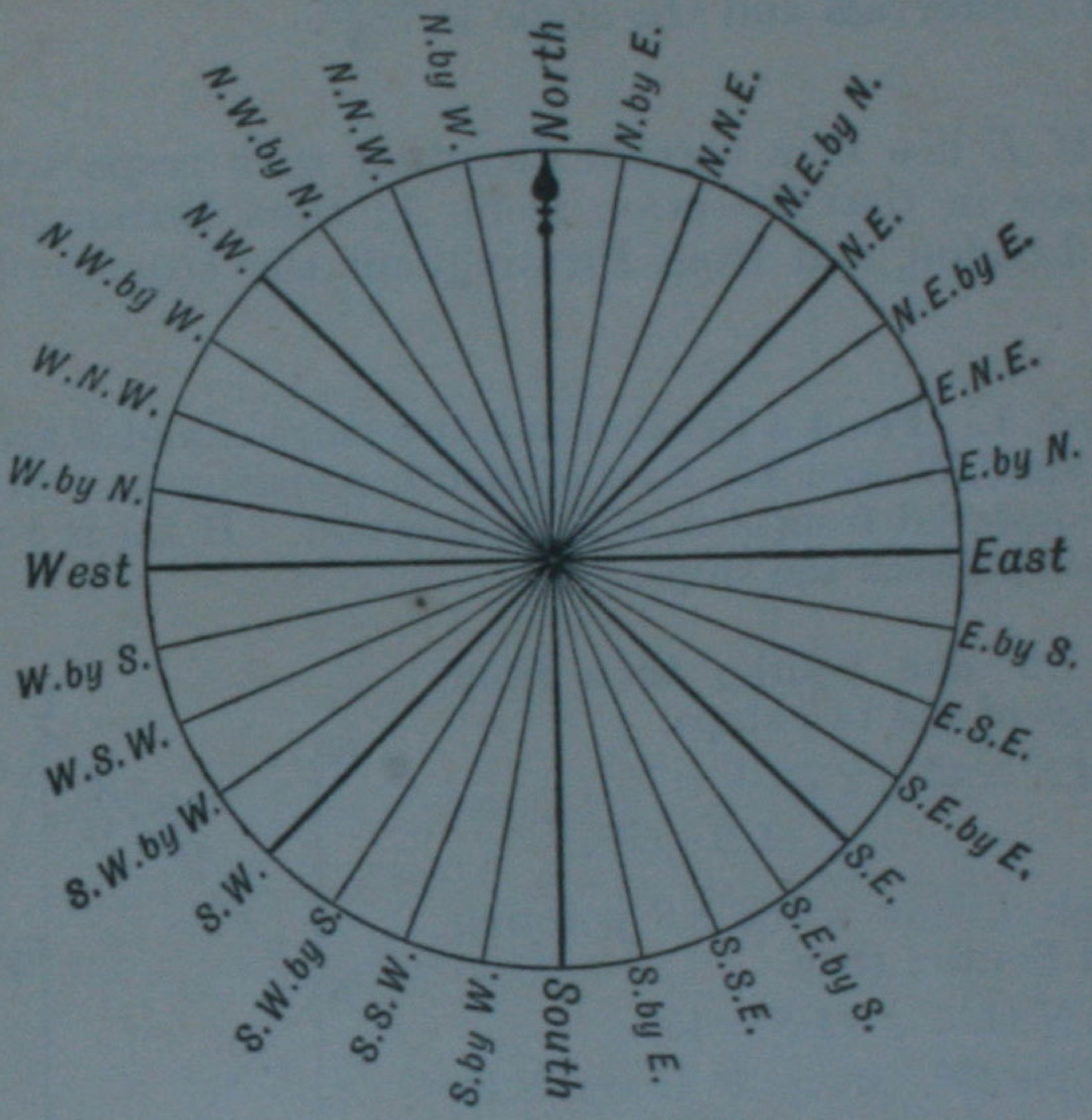
15. The angles of depression of the top and foot of a tower seen from a monument 96 feet high are 30° and 60°: find the height of the tower.

16. From the top of a cliff 150 feet high the angles of depression of two boats at sea, each due north of the observer, are 30° and 15°: how far are the boats apart?

17. From the top of a hill the angles of depression of two consecutive milestones on a level road running due south from the observer are 45° and 22° respectively. If cot 22° = 2·475 find the height of the hill in yards.

18. From the top of a lighthouse 80 yards above the horizon the angles of depression of two rocks due west of the observer are 75° and 15°: find their distance apart, given cot 75° = ·268 and cot 15° = 3·732.

50. Trigonometrical Problems sometimes require a knowledge of the **Points of the Mariner's Compass**, which we shall now explain.



In the above figure, it will be seen that 32 points are taken at equal distances on the circumference of a circle, so that the arc between any two consecutive points subtends at the centre of the circle an angle equal to $\frac{360}{32}^\circ$, that is to $11\frac{1}{4}^\circ$.

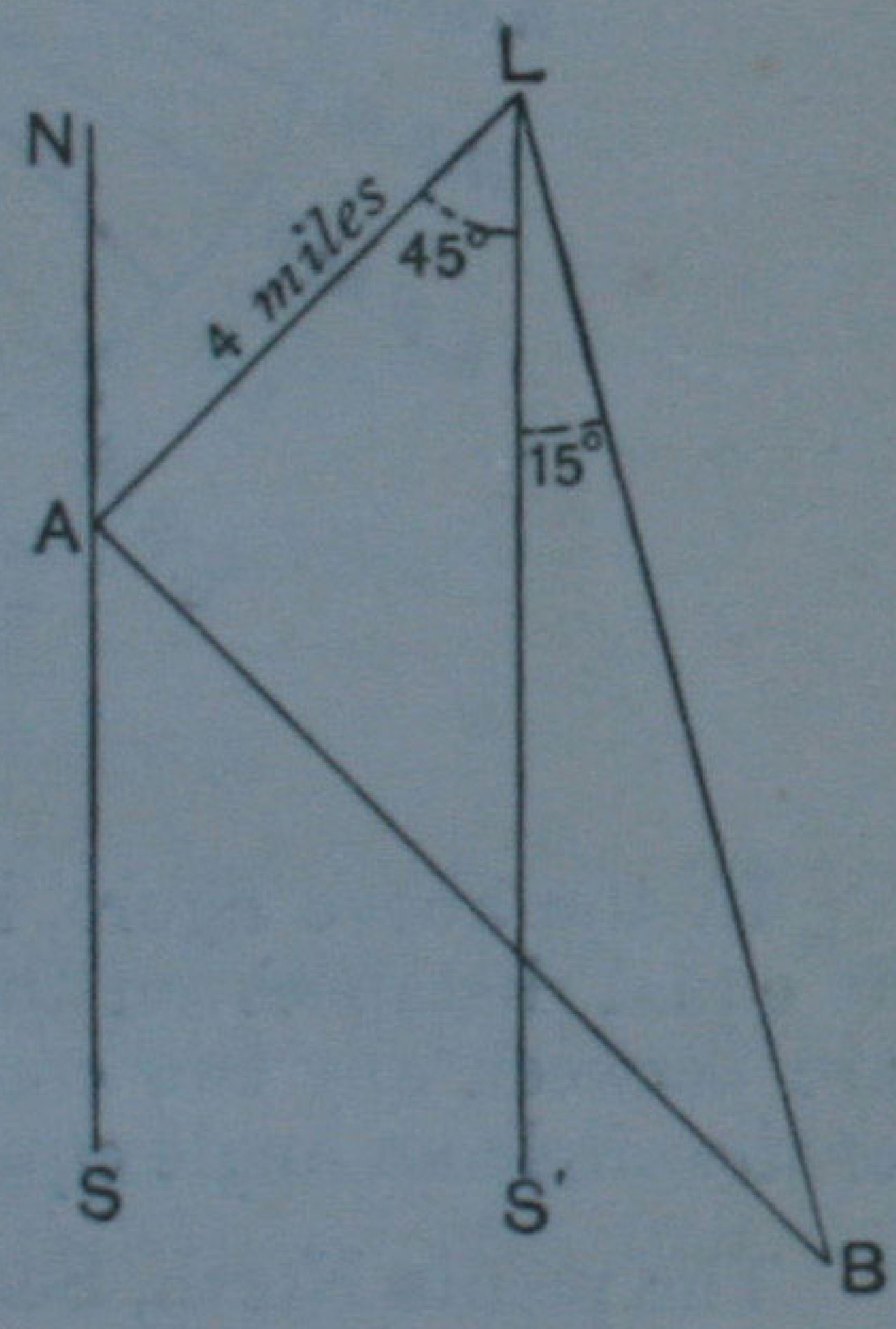
The points North, South, East, West are called the **Cardinal Points**, and with reference to them the other *points* receive their names. The student will have no difficulty in learning these if he will carefully notice the arrangement in any one of the principal quadrants.

51. Sometimes a slightly different notation is used; thus N. $11\frac{1}{4}^\circ$ E. means a direction $11\frac{1}{4}^\circ$ east of north, and is therefore the same as N. by E. Again S.W. by S. is 3 *points* from south and may be expressed by S. $33\frac{3}{4}^\circ$ W., or since it is 5 *points* from west it can also be expressed by W. $56\frac{1}{4}^\circ$ S. In each of these cases it will be seen that the angular measurement is made from the direction which is first mentioned.

52. The angle between the directions of any two points is obtained by multiplying $11\frac{1}{4}°$ by the number of intervals between the points. Thus between S. by W. and W.S.W. there are 5 intervals and the angle is $56\frac{1}{4}°$; between N.E. by E. and S.E. there are 7 intervals and the angle is $78\frac{3}{4}°$.

53. If B lies in a certain direction with respect to A, it is said to *bear* in that direction from A; thus Birmingham *bears* N.W. of London, and from Birmingham the *bearing* of London is S.E.

Example 1. From a lighthouse L two ships A and B are observed in directions S.W. and 15° East of South respectively. At the same time B is observed from A in a S.E. direction. If LA is 4 miles find the distance between the ships.



Draw LS' due South; then from the bearings of the two ships,

$$\angle ALS' = 45°, \quad \angle BLS' = 15°,$$

so that $$\angle ALB = 60°.$$

Through A draw a line NS pointing North and South; then

$$\angle NAL = \angle ALS' = 45°,$$

and $\angle BAS = 45°$, since B bears S.E. from A;

hence $\angle BAL = 180° - 45° - 45° = 90°$.

In the right-angled triangle ABL,

$$AB = AL \tan ALB = 4 \tan 60°$$
$$= 4\sqrt{3} = 6{\cdot}928.$$

Thus the distance between the ships is 6·928 miles.

Example 2. At 9 A.M. a ship which is sailing in a direction E. 40° S. at the rate of 8 miles an hour observes a fort in a direction 50° North of East. At 11 A.M. the fort is observed to bear N. 20° W.: find the distance of the fort from the ship at each observation.

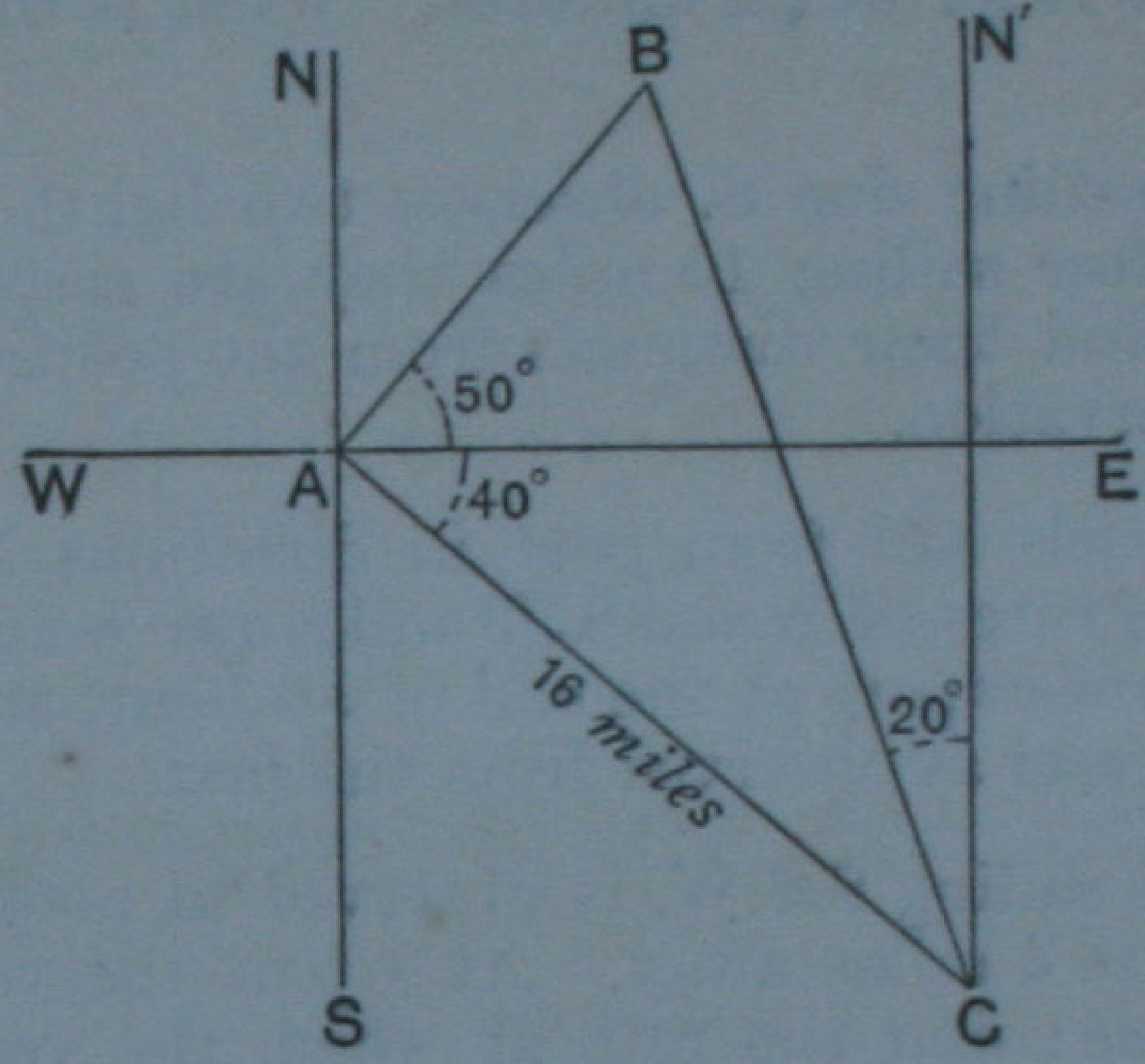
Let A and C be the first and second positions of the ship; B the fort.

Through A draw lines towards the cardinal points of the compass.

From the observations made

$$\angle EAC = 40°, \quad \angle EAB = 50°, \text{ so that } \angle BAC = 90°.$$

Through C draw CN' towards the North; then $\angle BCN'=20°$, for the bearing of the fort from C is N. 20° W.



Also $$\angle ACN' = \angle CAS = 90° - 40° = 50°;$$
$$\therefore \angle ACB = \angle ACN' - \angle BCN' = 50° - 20° = 30°.$$

In the right-angled triangle ACB,

$$AB = AC \tan ACB = 16 \tan 30° = \frac{16}{\sqrt{3}} = \frac{16\sqrt{3}}{3} = 9{\cdot}237 \text{ nearly};$$

and $$BC = AC \sec ACB = 16 \sec 30° = 16 \times \frac{2}{\sqrt{3}} = \frac{32\sqrt{3}}{3} = 18{\cdot}475 \text{ nearly}.$$

Thus the distances are 9·237 and 18·475 miles nearly.

EXAMPLES. VI. b.

1. A person walking due E. observes two objects both in the N.E. direction. After walking 800 yards one of the objects is due N. of him, and the other lies N.W.: how far was he from the objects at first?

2. Sailing due E. I observe two ships lying at anchor due S.; after sailing 3 miles the ships bear 60° and 30° S. of W.; how far are they now distant from me?

3. Two vessels leave harbour at noon in directions W. 28° S. and E. 62° S. at the rates 10 and $10\frac{1}{2}$ miles per hour respectively. Find their distance apart at 2 p.m.

4. A lighthouse facing N. sends out a fan-shaped beam extending from N.E. to N.W. A steamer sailing due W. first sees the light when 5 miles away from the lighthouse and continues to see it for $30\sqrt{2}$ minutes. What is the speed of the steamer?

5. A ship sailing due S. observes two lighthouses in a line exactly W. After sailing 10 miles they are respectively N.W. and W.N.W.; find their distances from the position of the ship at the first observation.

6. Two vessels sail from port in directions N. 35° W. and S. 55° W. at the rates of 8 and $8\sqrt{3}$ miles per hour respectively. Find their distance apart at the end of an hour, and the bearing of the second vessel as observed from the first.

7. A vessel sailing S.S.W. is observed at noon to be E.S.E. from a lighthouse 4 miles away. At 1 p.m. the vessel is due S. of the lighthouse: find the rate at which the vessel is sailing. Given $\tan 67\frac{1}{2}° = 2{\cdot}414$.

8. A, B, C are three places such that from A the bearing of C is N. 10° W., and the bearing of B is N. 50° E.; from B the bearing of C is N. 40° W. If the distance between B and C is 10 miles, find the distances of B and C from A.

9. A ship steaming due E. sights at noon a lighthouse bearing N.E., 15 miles distant; at 1.30 p.m. the lighthouse bears N.W. How many knots per day is the ship making? Given 60 knots = 69 miles.

10. At 10 o'clock forenoon a coaster is observed from a lighthouse to bear 9 miles away to N.E. and to be holding a south-easterly course; at 1 p.m. the bearing of the coaster is 15° S. of E. Find the rate of the coaster's sailing and its distance from the lighthouse at the time of the second observation.

11. The distance between two lighthouses, A and B, is 12 miles and the line joining them bears E. 15° N. At midnight a vessel which is sailing S. 15° E. at the rate of 10 miles per hour is N.E. of A and N.W. of B: find to the nearest minute when the vessel crosses the line joining the lighthouses.

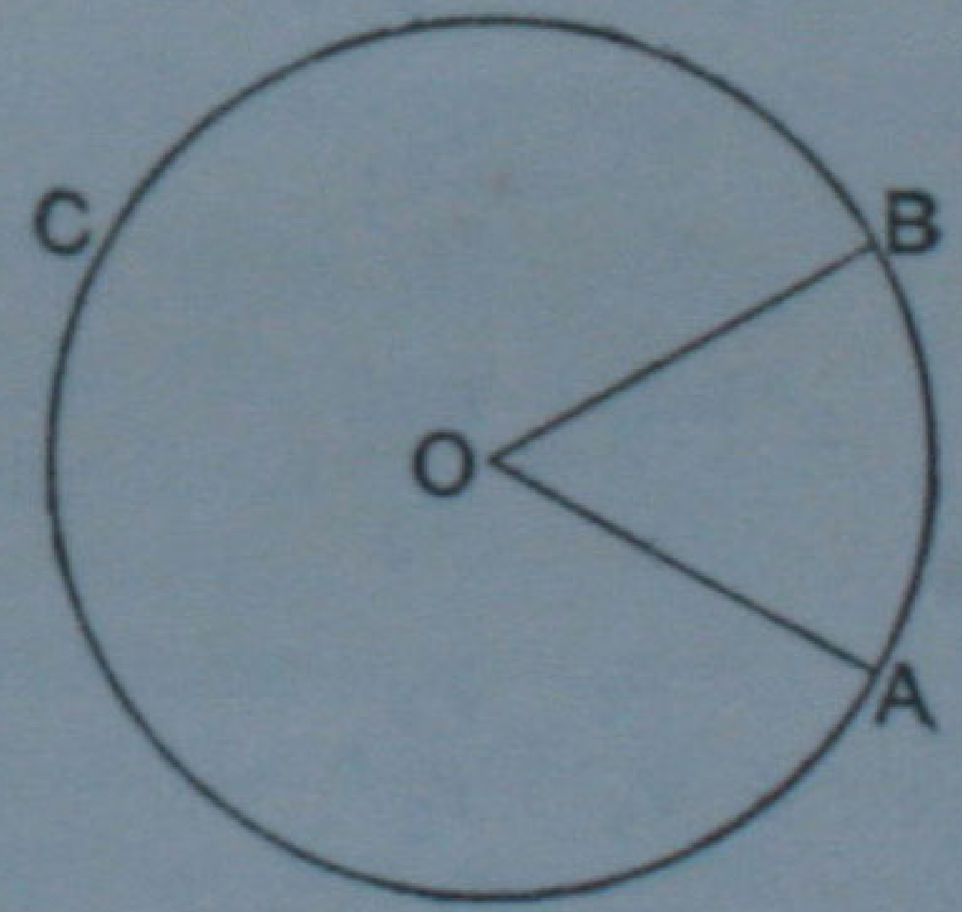
12. From A to B, two stations of a railway, the line runs W.S.W. At A a person observes that two spires, whose distance apart is 1·5 miles, are in the same line which bears N.N.W. At B their bearings are N. $7\frac{1}{2}°$ E. and N. $37\frac{1}{2}°$ E. Find the rate of a train which runs from A to B in 2 minutes.

CHAPTER VII.

RADIAN OR CIRCULAR MEASURE.

54. WE shall now return to the system of measuring angles which was briefly referred to in Art. 6. In this system angles are not measured in terms of a submultiple of the right angle, as in the sexagesimal and centesimal methods, but a certain angle known as a *radian* is taken as the standard unit, in terms of which all other angles are measured.

55. DEFINITION. A **radian** is the angle subtended at the centre of any circle by an arc equal in length to the radius of the circle.

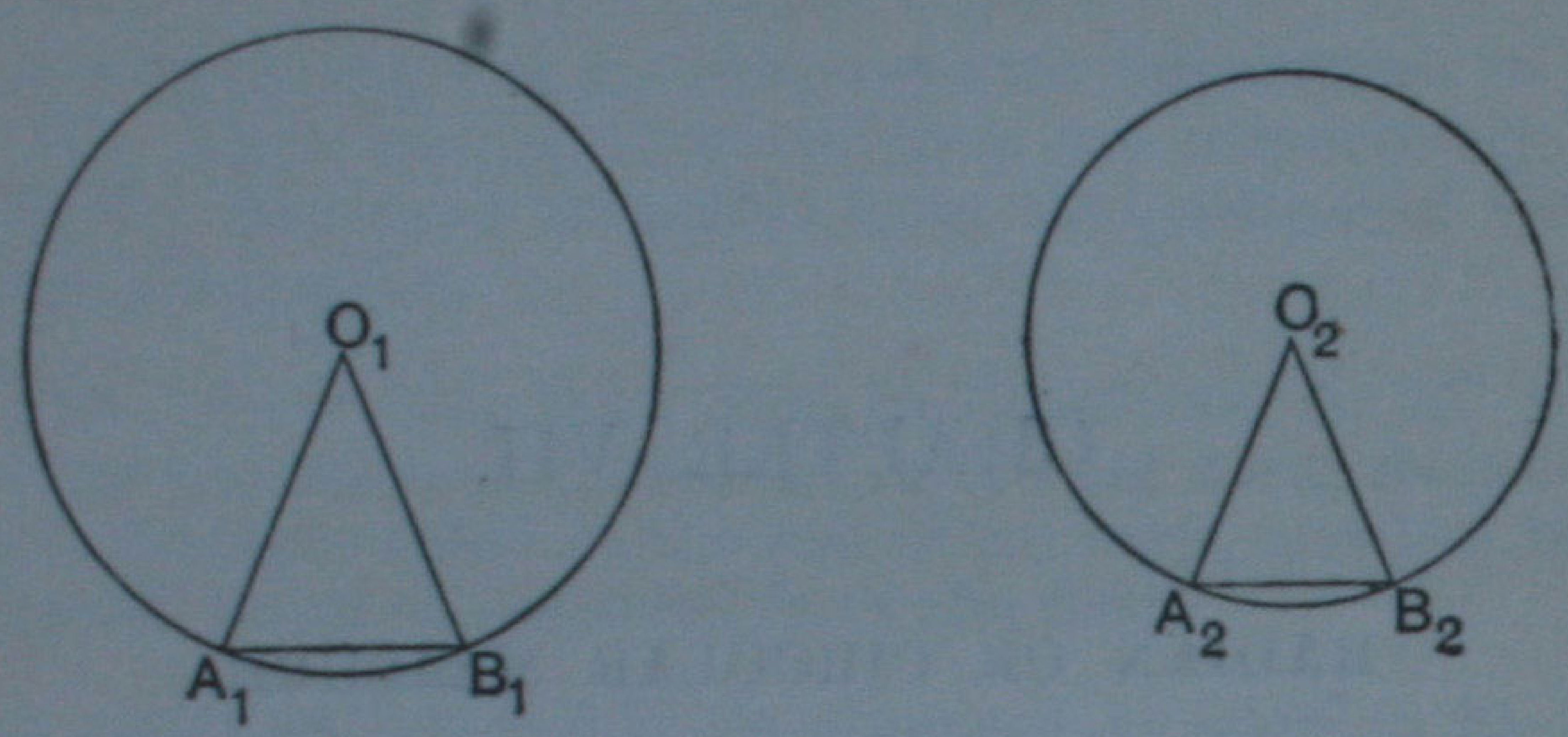


In the above figure, ABC is a circle, and O its centre. If on the circumference we measure an arc AB equal to the radius and join OA, OB, the angle AOB is a radian.

56. In any system of measurement it is essential that the unit should be always the same. In order to shew that a radian, constructed according to the above definition, is of constant magnitude, we must first establish an important property of the circle.

57. *The circumferences of circles are to one another as their radii.*

Take *any* two circles whose radii are r_1 and r_2, and in each circle let a regular polygon of n sides be described.



Let A_1B_1 be a side of the first, A_2B_2 a side of the second polygon, and let their lengths be denoted by a_1, a_2. Join their extremities to O_1 and O_2 the centres of the circles. We thus obtain two isosceles triangles whose vertical angles are equal, each being $\frac{1}{n}$ of four right angles.

Hence the triangles are equiangular, and therefore we have by Euc. VI. 4,

$$\frac{A_1B_1}{O_1A_1} = \frac{A_2B_2}{O_2A_2};$$

that is,

$$\frac{a_1}{r_1} = \frac{a_2}{r_2};$$

$$\therefore \frac{na_1}{r_1} = \frac{na_2}{r_2};$$

that is,

$$\frac{p_1}{r_1} = \frac{p_2}{r_2},$$

where p_1 and p_2 are the perimeters of the polygons. This is true whatever be the number of sides in the polygons. By taking n sufficiently large we can make the perimeters of the two polygons differ from the circumferences of the corresponding circles by as small a quantity as we please; so that ultimately

$$\frac{c_1}{r_1} = \frac{c_2}{r_2},$$

where c_1 and c_2 are the circumferences of the two circles.

58. It thus appears that *the ratio of the circumference of a circle to its radius is the same whatever be the size of the circle;* that is,

$$\textit{in all circles } \frac{\textit{circumference}}{\textit{diameter}} \textit{ is a constant quantity.}$$

This constant is incommensurable and is always denoted by the Greek letter π. Though its numerical value cannot be found exactly, it is shewn in a later part of the subject that it can be obtained to any degree of approximation. To ten decimal places its value is 3·1415926536. In many cases $\pi = \frac{22}{7}$, which is true to two decimal places, is a sufficiently close approximation; where greater accuracy is required the value 3·1416 may be used.

59. If c denote the circumference of the circle whose radius is r, we have

$$\frac{\text{circumference}}{\text{diameter}} = \pi;$$

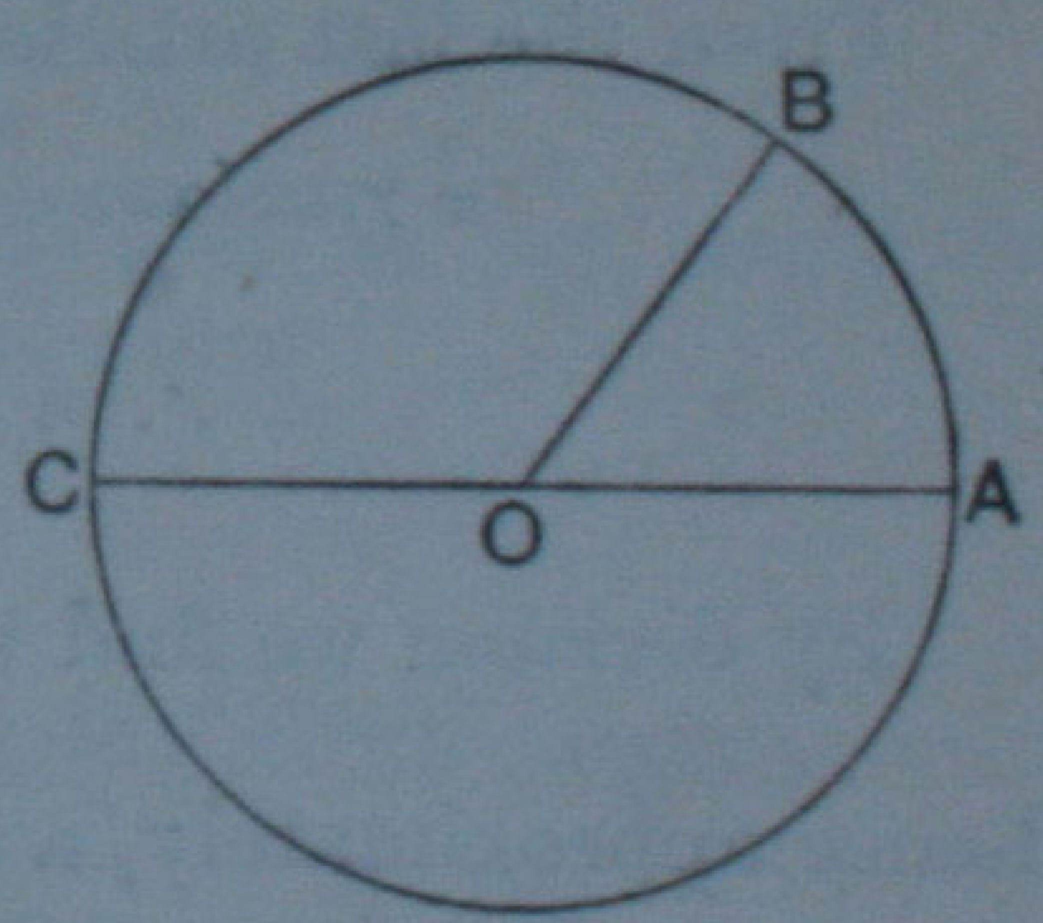
$$\therefore \quad \frac{c}{2r} = \pi,$$

or

$$c = 2\pi r.$$

60. *To prove that all radians are equal.*

Draw *any* circle; let O be its centre and OA a radius. Let the arc AB be measured equal in length to OA. Join OB; then $\angle AOB$ is a radian. Produce AO to meet the circumference in C. By Euc. VI. 33, angles at the centre of a circle are proportional to the arcs on which they stand; hence



$$\frac{\angle AOB}{\text{two right angles}} = \frac{\text{arc } AB}{\text{arc } ABC}$$

$$= \frac{\text{radius}}{\text{semi-circumference}} = \frac{r}{\pi r} = \frac{1}{\pi},$$

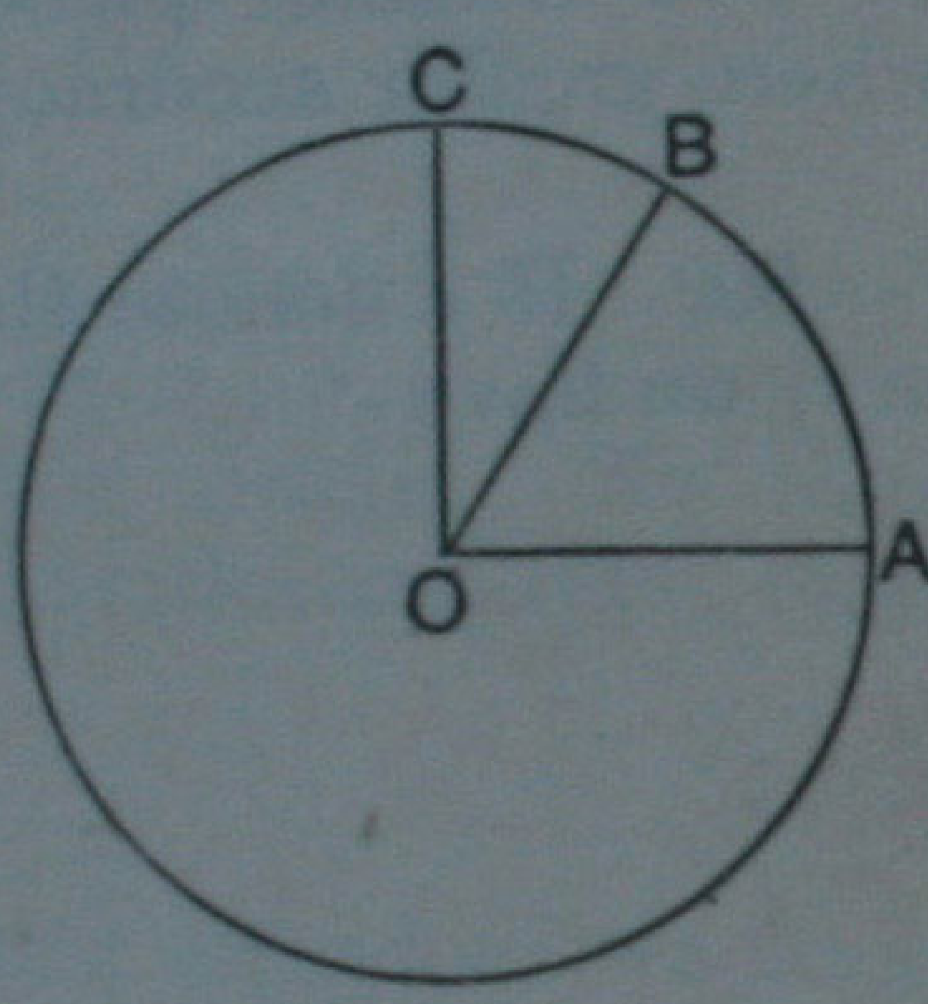
which is constant; that is, *a radian always bears the same ratio to two right angles, and therefore is a constant angle.*

61. Since a radian is constant it is taken as a standard unit, and the *number of radians* contained in any angle is spoken of as its **radian measure** or **circular measure.** [See Art. 71.] In this system, an angle is usually denoted by a *mere number*, the unit being implied. Thus when we speak of an angle 2·5, it is understood that its radian measure is 2·5, or, in other words, that the angle contains $2\frac{1}{2}$ radians.

Where it is desirable to refer to the unit expressly, a radian may be denoted by the letter ρ, thus 3ρ represents an angle which contains three radians.

62. *To find the radian measure of a right angle.*

Let AOC be a right angle at the centre of a circle, and AOB a radian; then



the radian measure of $\angle AOC$

$$=\frac{\angle AOC}{\angle AOB}=\frac{\text{arc } AC}{\text{arc } AB}$$

$$=\frac{\frac{1}{4}\text{(circumference)}}{\text{radius}}=\frac{\frac{1}{4}(2\pi r)}{r}$$

$$=\frac{\pi}{2};$$

that is, a right angle contains $\frac{\pi}{2}$ radians.

63. *To find the number of degrees in a radian.*

From the last article it follows that

$$\pi \textit{ radians} = 2 \textit{ right angles} = 180 \textit{ degrees.}$$

$$\therefore \text{ a radian} = \frac{180}{\pi} \text{ degrees.}$$

By division we find that $\frac{1}{\pi} = \cdot 31831$ nearly;

hence approximately, a radian $= 180 \times \cdot 31831 = 57\cdot 2958$ degrees.

64. The formula

$$\pi \textit{ radians} = 180 \textit{ degrees}$$

connecting the sexagesimal and radian measures of an angle, is a useful result which enables us to pass readily from one system to the other.

Example. Express 75° in radian measure, and $\frac{\pi}{54}$ in sexagesimal measure.

(1) Since 180 degrees $=\pi$ radians,

$$75 \text{ degrees} = \frac{75}{180}\pi \text{ radians} = \frac{5\pi}{12} \text{ radians}.$$

Thus the radian measure is $\frac{5\pi}{12}$.

(2) Since π radians $=180$ degrees,

$$\frac{\pi}{54} \text{ radians} = \frac{180}{54} \text{ degrees}.$$

Thus the angle $=\frac{10}{3}$ degrees $=3^\circ\ 20'$.

65. It may be well to remind the student that the symbol π always denotes a *number*, viz. 3·14159.... When the symbol stands alone, without reference to any angle, there can be no ambiguity; but even when π is used to denote an angle, it must still be remembered that π is a *number*, namely, the number of radians in two right angles.

NOTE. It is not uncommon for beginners to make statements such as "$\pi=180$" or "$\frac{\pi}{2}=90$." Without some modification this mode of expression is quite incorrect. It is true that π *radians* are equal to 180 *degrees*, but the statement '$\pi=180$' is no more correct than the statement "$20=1$" to denote the equivalence of 20 shillings and 1 sovereign.

66. *If the number of degrees and radians in an angle be represented by D and θ respectively, to prove that*

$$\frac{D}{180}=\frac{\theta}{\pi}.$$

In sexagesimal measure, the ratio of the given angle to two right angles is expressed by $\frac{D}{180}$.

In radian measure, the ratio of these same two angles is expressed by $\frac{\theta}{\pi}$.

$$\therefore \frac{D}{180}=\frac{\theta}{\pi}.$$

Example 1. What is the radian measure of 45° 13′ 30″?

If D be the number of degrees in the angle, we have $D = 45{\cdot}225$.

$$\begin{array}{r} 60\,)\;30 \\ 60\,)\;\overline{13{\cdot}5} \\ \overline{{\cdot}225} \end{array}$$

Let θ be the number of radians in the given angle, then

$$\frac{\theta}{\pi} = \frac{45{\cdot}225}{180} = \frac{1{\cdot}005}{4};$$

$$\therefore\ \theta = \frac{\pi}{4} \times 1{\cdot}005 = \frac{3{\cdot}1416}{4} \times 1{\cdot}005$$

$$= {\cdot}7854 \times 1{\cdot}005 = {\cdot}789327.$$

Thus the radian measure is ·789327.

Example 2. Express in sexagesimal measure the angle whose radian measure is 1·309.

Let D be the number of degrees; then

$$\frac{D}{180} = \frac{1{\cdot}309}{\pi};$$

$$\therefore\ D = \frac{180 \times 1{\cdot}309}{3{\cdot}1416} = \frac{180 \times 1309 \times 10}{31416}$$

$$= \frac{180 \times 10}{24} = 75.$$

Thus the angle is 75°.

EXAMPLES. VII. a.

[*Unless otherwise stated* $\pi = 3{\cdot}1416$.

It should be noticed that $31416 = 8 \times 3 \times 7 \times 11 \times 17$.]

Express in radian measure as fractions of π:

1. 45°. 2. 30°. 3. 105°. 4. 22° 30′.
5. 18°. 6. 57° 30′. 7. 14° 24′. 8. 78° 45′.

Find numerically the radian measure of the following angles:

9. 25° 50′. 10. 37° 30′. 11. 82° 30′.
12. 68° 45′. 13. 157° 30′. 14. 52° 30′.

Express in sexagesimal measure:

15. $\frac{3\pi}{4}$. 16. $\frac{7\pi}{45}$. 17. $\frac{5\pi}{27}$. 18. $\frac{5\pi}{24}$.

19. $\cdot 3927\rho$. 20. $\cdot 5236\rho$. 21. $\cdot 6545\rho$. 22. $2\cdot 8798\rho$.

Taking $\pi = \frac{22}{7}$, find the radian measure of:

23. $36° \, 32' \, 24''$. 24. $70° \, 33' \, 36''$.

25. $116° \, 2' \, 45\cdot 6''$. 26. $171° \, 41' \, 50\cdot 4''$.

27. Taking $\frac{1}{\pi} = \cdot 31831$, shew that a radian contains 206265 seconds approximately.

28. Shew that a second is approximately equal to $\cdot 0000048$ of a radian.

67. The angles $\frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{6}$ are the equivalents in radian measure of the angles 45°, 60°, 30° respectively.

Hence the results of Arts. 34 and 35 may be written as follows:

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \qquad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \qquad \tan\frac{\pi}{4} = 1;$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \qquad \cos\frac{\pi}{3} = \frac{1}{2}, \qquad \tan\frac{\pi}{3} = \sqrt{3};$$

$$\sin\frac{\pi}{6} = \frac{1}{2}, \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}.$$

Example. Find the value of

$$3\tan^2\frac{\pi}{6} + \frac{4}{3}\cos^2\frac{\pi}{6} - \frac{1}{2}\cot^3\frac{\pi}{4} - \frac{2}{3}\sin^2\frac{\pi}{3} + \frac{1}{8}\sec^4\frac{\pi}{3}.$$

$$\text{The value} = 3\left(\frac{1}{\sqrt{3}}\right)^2 + \frac{4}{3}\left(\frac{\sqrt{3}}{2}\right)^2 - \frac{1}{2}(1)^3 - \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{8}(2)^4$$

$$= \left(3 \times \frac{1}{3}\right) + \left(\frac{4}{3} \times \frac{3}{4}\right) - \frac{1}{2} - \left(\frac{2}{3} \times \frac{3}{4}\right) + \left(\frac{1}{8} \times 16\right)$$

$$= 1 + 1 - \frac{1}{2} - \frac{1}{2} + 2 = 3.$$

68. When expressed in radian measure the complement of θ is $\frac{\pi}{2}-\theta$, and corresponding to the formulæ of Art. 38 we now have relations of the form

$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta, \qquad \tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta.$$

Example. Prove that

$$(\cot\theta+\tan\theta)\cot\left(\frac{\pi}{2}-\theta\right)=\operatorname{cosec}^2\left(\frac{\pi}{2}-\theta\right).$$

$$\begin{aligned}\text{The first side} &= (\cot\theta+\tan\theta)\tan\theta\\ &= \cot\theta\tan\theta+\tan^2\theta\\ &= 1+\tan^2\theta=\sec^2\theta\\ &= \operatorname{cosec}^2\left(\frac{\pi}{2}-\theta\right).\end{aligned}$$

69. By means of Euc. I. 32, it is easy to find the number of radians in each angle of a regular polygon.

Example. Express in radians the interior angle of a regular polygon which has n sides.

The sum of the *exterior* angles = 4 right angles. [Euc. I. 32 Cor.]

Let θ be the number of radians in an exterior angle; then

$$n\theta=2\pi, \text{ and therefore } \theta=\frac{2\pi}{n}.$$

But interior angle = two right angles − exterior angle

$$=\pi-\theta=\pi-\frac{2\pi}{n}.$$

Thus each interior angle $=\dfrac{(n-2)\pi}{n}$.

EXAMPLES. VII. b.

Find the numerical value of

1. $\sin\frac{\pi}{3}\cos\frac{\pi}{6}\cot\frac{\pi}{4}$.

2. $\tan\frac{\pi}{6}\cot\frac{\pi}{3}\cos\frac{\pi}{4}$.

3. $\frac{1}{2}\cos\frac{\pi}{3}+2\operatorname{cosec}\frac{\pi}{6}$.

4. $2\sin\frac{\pi}{4}+\frac{1}{2}\sec\frac{\pi}{4}$.

Find the numerical value of

5. $\cot^2\frac{\pi}{6}+4\cos^2\frac{\pi}{4}+3\sec^2\frac{\pi}{6}$.

6. $3\tan^2\frac{\pi}{6}-\frac{1}{3}\sin^2\frac{\pi}{3}-\frac{1}{2}\operatorname{cosec}^2\frac{\pi}{4}+\frac{4}{3}\cos^2\frac{\pi}{6}$.

7. $\left(\sin\frac{\pi}{6}+\cos\frac{\pi}{6}\right)\left(\sin\frac{\pi}{3}-\cos\frac{\pi}{3}\right)\sec\frac{\pi}{3}$.

Prove the following identities :

8. $\sin\theta\sec\left(\frac{\pi}{2}-\theta\right)-\cot\theta\cot\left(\frac{\pi}{2}-\theta\right)=0.$

9. $\sin^2\left(\frac{\pi}{2}-\theta\right)\operatorname{cosec}\theta-\tan^2\left(\frac{\pi}{2}-\theta\right)\sin\theta=0.$

10. $\dfrac{\sin^2\left(\frac{\pi}{2}-\theta\right)}{\operatorname{cosec}\theta}\cdot\dfrac{\sec\theta}{\cot\left(\frac{\pi}{2}-\theta\right)}=\cos^2\theta.$

11. $\tan\theta+\tan\left(\frac{\pi}{2}-\theta\right)=\sec\theta\sec\left(\frac{\pi}{2}-\theta\right)$.

12. $\sec^2\theta+\sec^2\left(\frac{\pi}{2}-\theta\right)=(1+\tan^2\theta)\sec^2\left(\frac{\pi}{2}-\theta\right)$.

13. Find the number of radians in each exterior angle of
(1) a regular octagon, (2) a regular quindecagon.

14. Find the number of radians in each interior angle of
(1) a regular dodecagon, (2) a regular heptagon.

15. Shew that

$$\tan^2\frac{\pi}{3}-\cot^2\frac{\pi}{3}=\frac{\cos^2\frac{\pi}{6}-\cos^2\frac{\pi}{3}}{\cos^2\frac{\pi}{3}\cos^2\frac{\pi}{6}}.$$

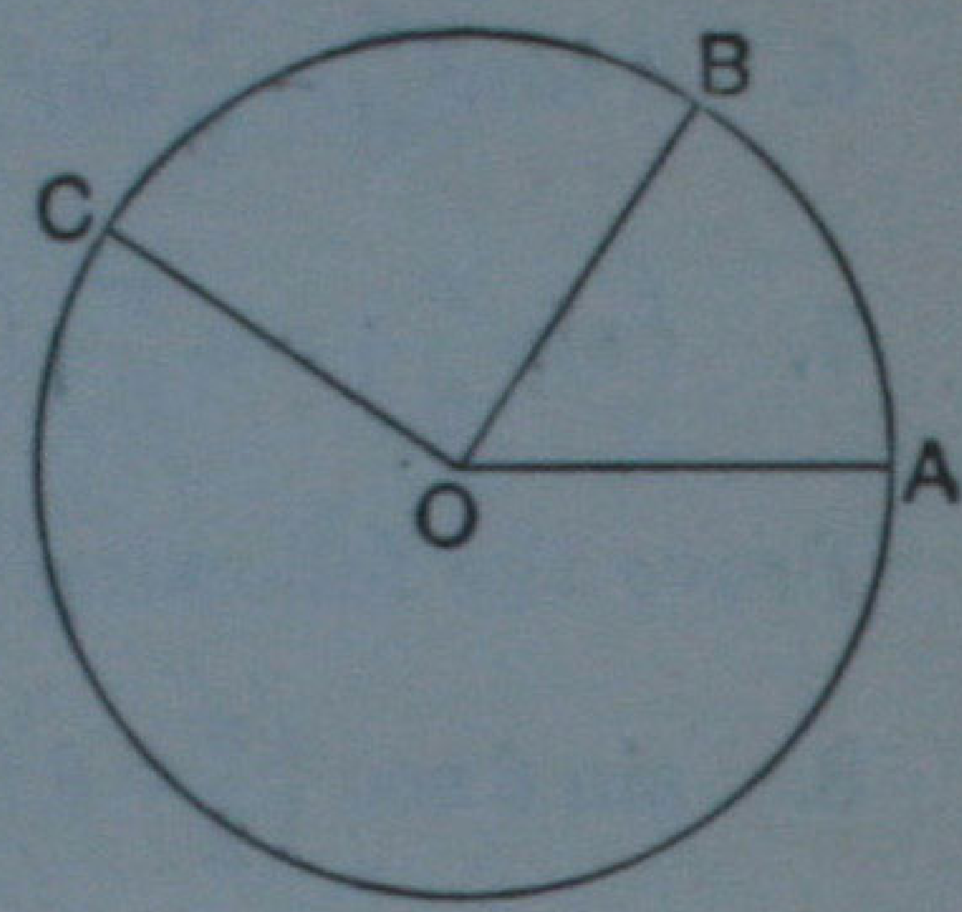
16. Shew that the sum of the squares of

$$\sin\theta+\sin\left(\frac{\pi}{2}-\theta\right) \text{ and } \cos\theta-\cos\left(\frac{\pi}{2}-\theta\right)$$

is equal to 2.

70. *To prove that the radian measure of any angle at the centre of a circle is expressed by the fraction* $\dfrac{\textit{subtending arc}}{\textit{radius}}$.

Let AOC be any angle at the centre of a circle, and AOB a radian; then



radian measure of $\angle AOC$

$$=\frac{\angle AOC}{\angle AOB}$$

$$=\frac{\text{arc } AC}{\text{arc } AB}$$

$$=\frac{\text{arc } AC}{\text{radius}},$$

since arc AB=radius;

that is, the radian measure of $\angle AOC=\dfrac{\text{subtending arc}}{\text{radius}}$.

71. If a be the length of the arc which subtends an angle of θ radians at the centre of a circle of radius r, we have seen in the preceding article that

$$\theta=\frac{a}{r}, \text{ and therefore } a=r\theta.$$

The fraction $\dfrac{\text{arc}}{\text{radius}}$ is usually called the *circular measure* of the angle at the centre of the circle subtended by the arc.

The *circular measure* of an angle is therefore equal to its *radian measure*, each denoting the *number of radians* contained in the angle. We have preferred to use the term *radian measure* exclusively, in order to keep prominently in view the unit of measurement, namely the radian.

NOTE. The term *circular measure* is a survival from the times when Mathematicians spoke of the trigonometrical functions of the *arc*. [See page 80.]

Example 1. Find the angle subtended by an arc of 7·5 feet at the centre of a circle whose radius is 5 yards.

Let the angle contain θ radians; then

$$\theta=\frac{\text{arc}}{\text{radius}}=\frac{7\cdot5}{15}=\frac{1}{2}.$$

Thus the angle is half a radian.

Example 2. In running a race at a uniform speed on a circular course, a man in each minute traverses an arc of a circle which subtends $2\frac{6}{7}$ radians at the centre of the course. If each lap is 792 yards, how long does he take to run a mile? $\left[\pi=\frac{22}{7}\right]$.

Let r yards be the radius of the circle; then

$$2\pi r = \text{circumference} = 792;$$

$$\therefore\ r=\frac{792}{2\pi}=\frac{792\times 7}{2\times 22}=126.$$

Let a yards be the length of the arc traversed in each minute; then from the formula $a=r\theta$,

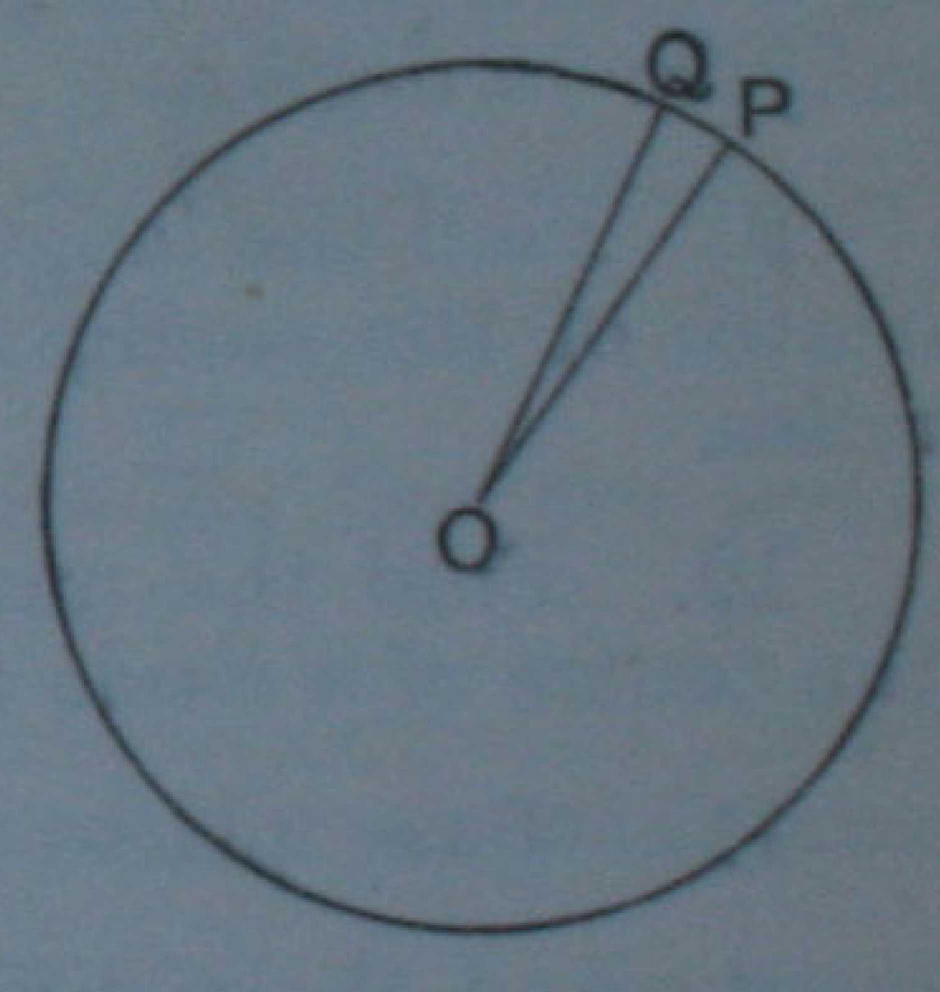
$$a=126\times 2\tfrac{6}{7}=\frac{126\times 20}{7}=360;$$

that is, the man runs 360 yds. in each minute.

$$\therefore\ \text{the time}=\frac{1760}{360}\ \text{or}\ \frac{44}{9}\ \text{minutes.}$$

Thus the time is 4 min. $53\frac{1}{3}$ sec.

Example 3. Find the radius of a globe such that the distance measured along its surface between two places on the same meridian whose latitudes differ by $1°\,10'$ may be 1 inch, reckoning that $\pi=\frac{22}{7}$.



Let the adjoining figure represent a section of the globe through the meridian on which the two places P and Q lie. Let O be the centre, and denote the radius by r inches.

Now $\frac{\text{arc}\,PQ}{\text{radius}}=$ number of radians in $\angle POQ$;

but arc $PQ=1$ inch, and $\angle POQ=1°\,10'$;

$$\therefore\ \frac{1}{r}=\text{number of radians in } 1\tfrac{1}{6}°$$

$$=1\tfrac{1}{6}\times\frac{\pi}{180}=\frac{7}{6}\times\frac{22}{7}\times\frac{1}{180}=\frac{11}{540};$$

whence $$r=\frac{540}{11}=49\tfrac{1}{11}.$$

Thus the radius is $49\frac{1}{11}$ inches.

EXAMPLES. VII. c.

1. Find the radian measure of the angle subtended by an arc of 1·6 yards at the centre of a circle whose radius is 24 feet.

2. An angle whose circular measure is ·73 subtends at the centre of a circle an arc of 219 feet; find the radius of the circle.

3. An angle at the centre of a circle whose radius is 2·5 yards is subtended by an arc of 7·5 feet; what is the angle?

4. What is the length of the arc which subtends an angle of 1·625 radians at the centre of a circle whose radius is 3·6 yards?

5. An arc of 17 yds. 1 ft. 3 in. subtends at the centre of a circle an angle of 1·9 radians; find the radius of the circle in inches.

6. The flywheel of an engine makes 35 revolutions in a second; how long will it take to turn through 5 radians? $\left[\pi = \frac{22}{7}\right]$.

7. The large hand of a clock is 2 ft. 4 in. long; how many inches does its extremity move in 20 minutes? $\left[\pi = \frac{22}{7}\right]$.

8. A horse is tethered to a stake; how long must the rope be in order that, when the horse has moved through 52·36 yards at the extremity of the rope, the angle traced out by the rope may be 75 degrees?

9. Find the length of an arc which subtends 1 minute at the centre of the earth, supposed to be a sphere of diameter 7920 miles.

10. Find the number of seconds in the angle subtended at the centre of a circle of radius 1 mile by an arc $5\frac{1}{2}$ inches long.

11. Two places on the same meridian are 145·2 miles apart; find their difference in latitude, taking $\pi = \frac{22}{7}$, and the earth's diameter as 7920 miles.

12. Find the radius of a globe such that the distance measured along its surface between two places on the same meridian whose latitudes differ by $1\frac{3}{11}°$ may be 1 foot, taking $\pi = \frac{22}{7}$.

MISCELLANEOUS EXAMPLES. B.

1. Express in degrees the angle whose circular measure is ·15708.

2. If $C=90°$, $A=30°$, $c=110$, find b to two decimal places.

3. Find the number of degrees in the unit angle when the angle $\frac{12\pi}{25}$ is represented by $1\frac{3}{5}$.

4. What is the radius of the circle in which an arc of 1 inch subtends an angle of 1′ at the centre?

5. Prove that

(1) $(\sin a+\cos a)(\tan a+\cot a)=\sec a+\operatorname{cosec} a$;

(2) $(\sqrt{3}+1)(3-\cot 30°)=\tan^3 60°-2\sin 60°$.

6. Find the angle of elevation of the sun when a chimney 60 feet high throws a shadow $20\sqrt{3}$ yards long.

---

7. Prove the identities:

(1) $(\tan\theta+2)(2\tan\theta+1)=5\tan\theta+2\sec^2\theta$;

(2) $1+\frac{\cot^2 a}{1+\operatorname{cosec} a}=\operatorname{cosec} a$.

8. One angle of a triangle is 45° and another is $\frac{5\pi}{8}$ radians; express the third angle both in sexagesimal and radian measure.

9. The number of degrees in an angle exceeds 14 times the number of radians in it by 51. Taking $\pi=\frac{22}{7}$, find the sexagesimal measure of the angle.

10. If $B=30°$, $C=90°$, $b=6$, find a, c, and the perpendicular from C on the hypotenuse.

11. Shew that

(1) $\cot\theta+\cot\left(\frac{\pi}{2}-\theta\right)=\operatorname{cosec}\theta\operatorname{cosec}\left(\frac{\pi}{2}-\theta\right)$;

(2) $\operatorname{cosec}^2\theta+\operatorname{cosec}^2\left(\frac{\pi}{2}-\theta\right)=\operatorname{cosec}^2\theta\operatorname{cosec}^2\left(\frac{\pi}{2}-\theta\right)$.

12. The angle of elevation of the top of a pillar is 30°, and on approaching 20 feet nearer it is 60°: find the height of the pillar.

---

13. Shew that $\tan^2 A - \sin^2 A = \sin^4 A \sec^2 A$.

14. In a triangle the angle A is $3x$ degrees, the angle B is x grades, and the angle C is $\dfrac{\pi x}{300}$ radians: find the number of degrees in each of the angles.

15. Find the numerical value of

$$\sin^3 60^\circ \cot 30^\circ - 2\sec^2 45^\circ + 3\cos 60^\circ \tan 45^\circ - \tan^2 60^\circ.$$

16. Prove the identities:

(1) $(1+\tan A)^2 + (1+\cot A)^2 = (\sec A + \operatorname{cosec} A)^2$;

(2) $(\sec a - 1)^2 - (\tan a - \sin a)^2 = (1-\cos a)^2$.

17. Which of the following statements is possible and which impossible?

(1) $\operatorname{cosec}\theta = \dfrac{a^2+b^2}{2ab}$; (2) $2\sin\theta = a + \dfrac{1}{a}$.

18. A balloon leaves the earth at the point A and rises at a uniform pace. At the end of 1·5 minutes an observer stationed at a distance of 660 feet from A finds the angular elevation of the balloon to be 60°; at what rate in miles per hour is the balloon rising?

---

19. Find the number of radians in the angles of a triangle which are in arithmetical progression, the least angle being 36°.

20. Shew that

$$\sin^2 a \sec^2\beta + \tan^2\beta \cos^2 a = \sin^2 a + \tan^2\beta.$$

21. In the triangle ABC if $A = 42°$, $B = 116° 33'$, find the perpendicular from C upon AB produced; given

$$c = 55, \quad \tan 42^\circ = \cdot 9, \quad \tan 63^\circ 27' = 2.$$

22. Prove the identities:

(1) $\cot a + \dfrac{\sin a}{1+\cos a} = \operatorname{cosec} a$;

(2) $\operatorname{cosec} a (\sec a - 1) - \cot a (1 - \cos a) = \tan a - \sin a$.

23. Shew that $\left(\dfrac{1+\cot 60^\circ}{1-\cot 60^\circ}\right)^2 = \dfrac{1+\cos 30^\circ}{1-\cos 30^\circ}$.

24. A man walking N.W. sees a windmill which bears N. 15° W. In half-an-hour he reaches a place which he knows to be W. 15° S. of the windmill and a mile away from it. Find his rate of walking and his distance from the windmill at the first observation.

---

25. Find the number of radians in the complement of $\dfrac{3\pi}{8}$.

26. Solve the equations:

(1) $3 \sin\theta + 4\cos^2\theta = 4\frac{1}{2}$; (2) $\tan\theta + \sec 30^\circ = \cot\theta$.

27. If $5 \tan a = 4$, find the value of

$$\frac{5 \sin a - 3 \cos a}{\sin a + 2 \cos a}.$$

28. Prove that

$$\frac{1 - \sin A \cos A}{\cos A (\sec A - \operatorname{cosec} A)} \times \frac{\sin^2 A - \cos^2 A}{\sin^3 A + \cos^3 A} = \sin A.$$

29. Find the distance of an observer from the top of a cliff which is 195·2 yards high, given that the angle of elevation is 77° 26′, and that sin 77° 26′ = ·976.

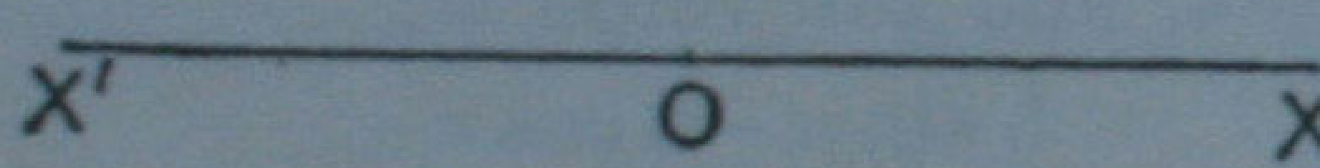
30. A horse is tethered to a stake by a rope 27 feet long. If the horse moves along the circumference of a circle always keeping the rope tight, find how far it will have gone when the rope has traced out an angle of 70°. $\left[\pi = \dfrac{22}{7}\right]$

CHAPTER VIII.

TRIGONOMETRICAL RATIOS OF ANGLES OF ANY MAGNITUDE.

72. In the present chapter we shall find it necessary to take account not only of the magnitude of straight lines, but also of the direction in which they are measured.

Let O be a fixed point in a horizontal line XX', then the position of any other point P in the line, whose distance from O is a given length a, will not be determined unless we know on which side of O the point P lies.

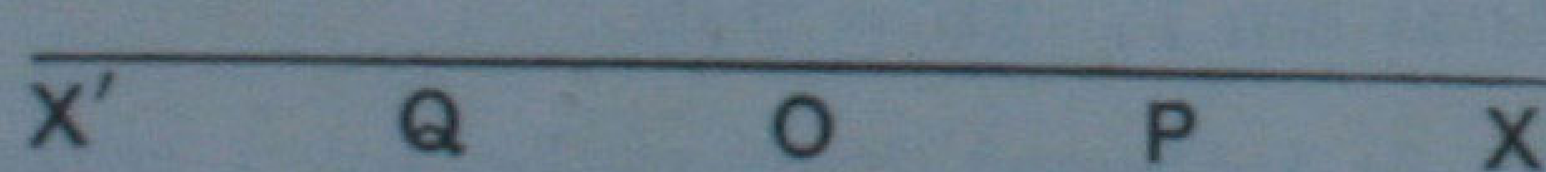


But there will be no ambiguity if it is agreed that distances measured in one direction are positive and distances measured in the opposite direction are negative.

Hence the following **Convention of Signs** is adopted:

lines measured from O to the right are positive,

lines measured from O to the left are negative.



Thus in the above figure, if P and Q are two points on the line XX' at a distance a from O, their positions are indicated by the statements

$$OP = +a, \quad OQ = -a.$$

73. A similar *convention of signs* is used in the case of a plane surface.

Let O be any point in the plane; through O draw two straight lines XX' and YY' in the horizontal and vertical direction respectively, thus dividing the plane into four *quadrants*.