

one fixed star to another, distant 90° from the first, would be a tangent to the surface of the earth) it would be infinitely great in comparison with the spaces which occur in daily life."

The above, being the first published, not printed, treatise on the new geometry occupies a unique place in the history of higher mathematics. It gave additional strength to the formative tendencies which characterized this period and marked SCHWEIKART as a constructive and original thinker.

The nascent aspects of this stage received a fruitful contribution when NICOLAI LOBACHEVSKI (1793-1847) created his *Imaginary Geometry* and JANOS BOLYAI (1802-1860) published as an appendix to his father's *Tentamen*, his *Science Absolute of Space*. LOBACHEVSKI and BOLYAI have been called the "Creators of the Non-Euclidean Geometry." And this appellation seems richly to be deserved by these pioneers. Their work gave just the impetus most needed to fix the status of the new line of researches which led to such remarkable discoveries in the more recent years. The *Imaginary Geometry* and the *Science Absolute of Space* were translated by the French mathematician, J. HOÜEL in 1868 and by him elevated out of their forty-five years of obscurity and non-effectiveness to a position where they became available for the mathematical public. To BOLYAI and LOBACHEVSKI, consequently, belong the honor of starting the movement which resulted in the development of metageometry and hence that which has proved to be the gateway of a new mathematical freedom.

GAUSS, SCHWEIKART, LOBACHEVSKI, WOLFGANG and JANOS BOLYAI were the principal figures of the formative period and the value of their work with respect to the formulation of principles upon which was constructed the Temple of Metageometry cannot be overestimated.

The Determinative Period

This period is characterized chiefly by its close relationship to the theory of surfaces. RIEMANN'S Habilitation Lecture on *The Hypotheses Which Constitute the Bases of Geometry* marks the beginning of this epoch. In this dissertation, RIEMANN not only promulgated the system upon which GAUSS had spent more than thirty years of his life in elaborating, for he was a disciple of GAUSS; but he disclosed his own views with respect to space which he regarded as a particular case of manifold. His work contains two fundamental concepts, namely, the *manifold* and the *measure of curvature* of a continuous manifold, possessed of what he called *flatness* in the smallest parts. The conception of the measure of curvature is extended by RIEMANN from surfaces to spaces and a new kind of space, finite, but unbounded, is shown to be possible. He showed that the dimensions of any space are determined by the number of measurements necessary to establish the position of a point in that space. Conceiving, therefore, that space is a manifold of finite, but unbounded, extension, he established the fact that the passage from one element of a manifold to another may be either discrete or continuous and that the manifold is discrete or continuous according

to the manner of passage. Where the manifold is regarded as discrete two portions of it can be compared, as to magnitude, by counting; where continuous, by measurement. If the whole manifold be caused to pass over into another manifold each of its elements passing through a one-dimensional manifold, a two-dimensional manifold is thus generated. In this way, a manifold of n -dimensions can be generated. On the other hand, a manifold of n -dimensions can be analyzed into one of one dimension and one of $(n-1)$ dimensions.

TO RIEMANN, then, is due the credit for first promulgating the idea that space being a special case of manifold is generable, and therefore, *finite*. He laid the foundation for the establishment of a special kind of geometry known as the "elliptic." Space, as viewed by him, possessed the following properties, viz.: generability, divisibility, measurability, ponderability, finity and flexibility.

These are the six pillars upon which rests the structure of hyperspace analyses.³

Generability is that property of geometric space by virtue of which it may be generated, or constructed, by the movement of a line, plane, surface or solid in a direction without itself. *Divisibility* is that property of geometric space by virtue of which it may be segmented or divided into separate parts and superposed, or inserted, upon or between each other. *Measurability* is that property by virtue of

³ Vide *Nature*, Vol. VIII, pp. 14-17; 36, 37 (1873); also *Mathematical Papers*, pp. 65-71.

which geometric space is determined to be a manifold of either a positive or negative curvature, also by which its extent may be measured. *Ponderability* is that property of geometric space by virtue of which it may be regarded as a quantity which can be manipulated, assorted, shelved or otherwise disposed of. *Finiteness* is that property by virtue of which geometric space is limited to the scope of the individual consciousness of a unodim, a duodim or a tridim and by virtue of which it is finite in extent. *Flexibility* is that property by virtue of which geometric space is regarded as possessing curvature, and in consequence of which progress through it is made in a curved, rather than a geodesic line, also by virtue of which it may be flexed without disruption or dilatation.

RIEMANN who thus prepared the way for entrance into a veritable labyrinth of hyperspaces is, therefore, correctly styled "The father of metageometry," and the fourth dimension is his eldest born. He died while but forty years of age and never lived long enough fully to elaborate his theory with respect to its application to the measure of curvature of space. This was left for his very energetic disciple, EUGENIO BELTRAMI (1835-1900) who was born nine years after RIEMANN and lived thirty-four years longer than he. His labors mark the characteristic standpoint of the determinative period. BELTRAMI's mathematical investigations were devoted mainly to the non-Euclidean geometry. These led him to the rather remarkable conclusion that the propositions embodied therein relate to figures lying upon surfaces of constant negative curvature.

BELTRAMI sought to show that such surfaces partake of the nature of the pseudosphere, and in doing so, made use of the following illustration:

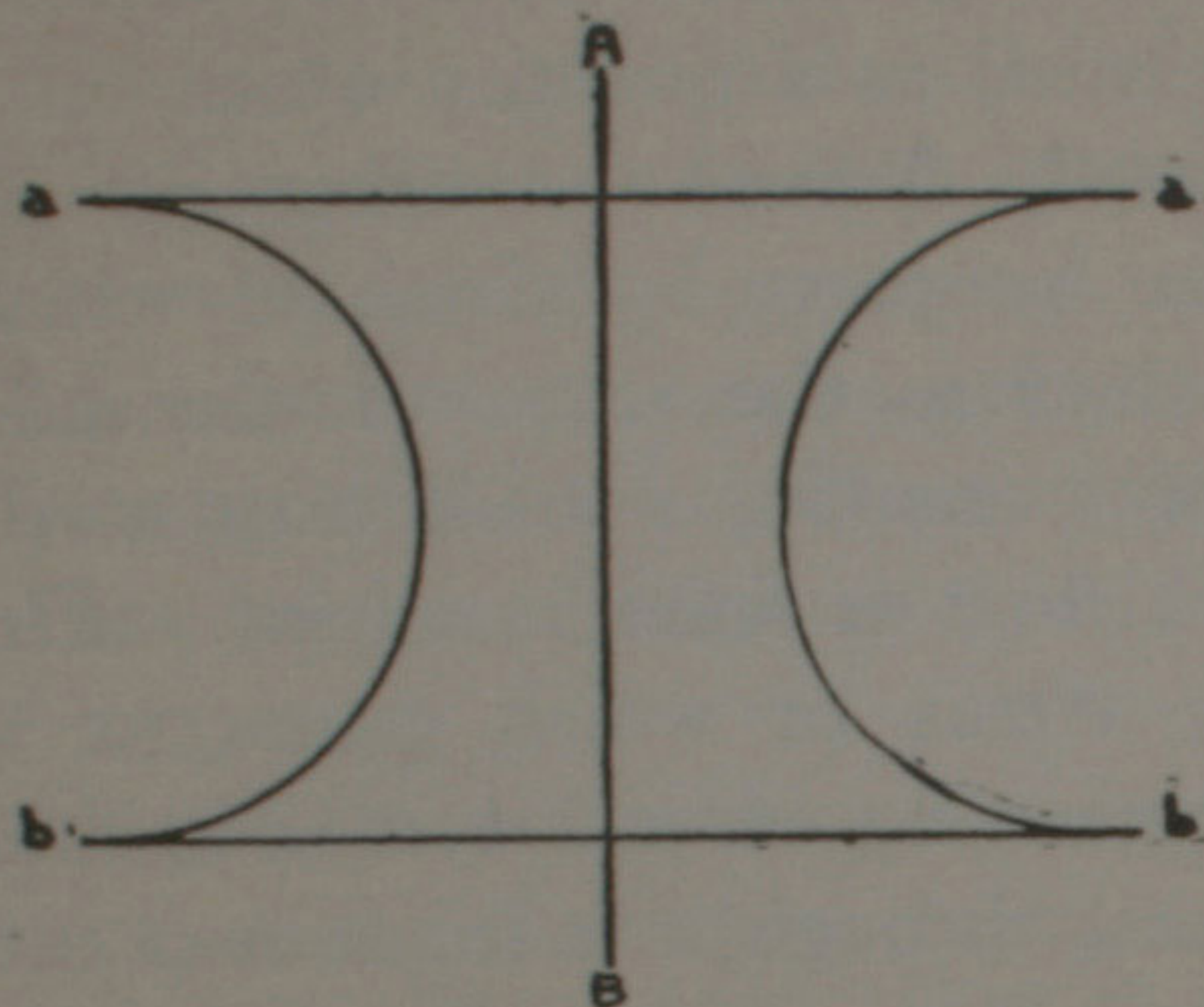


FIG. 3.

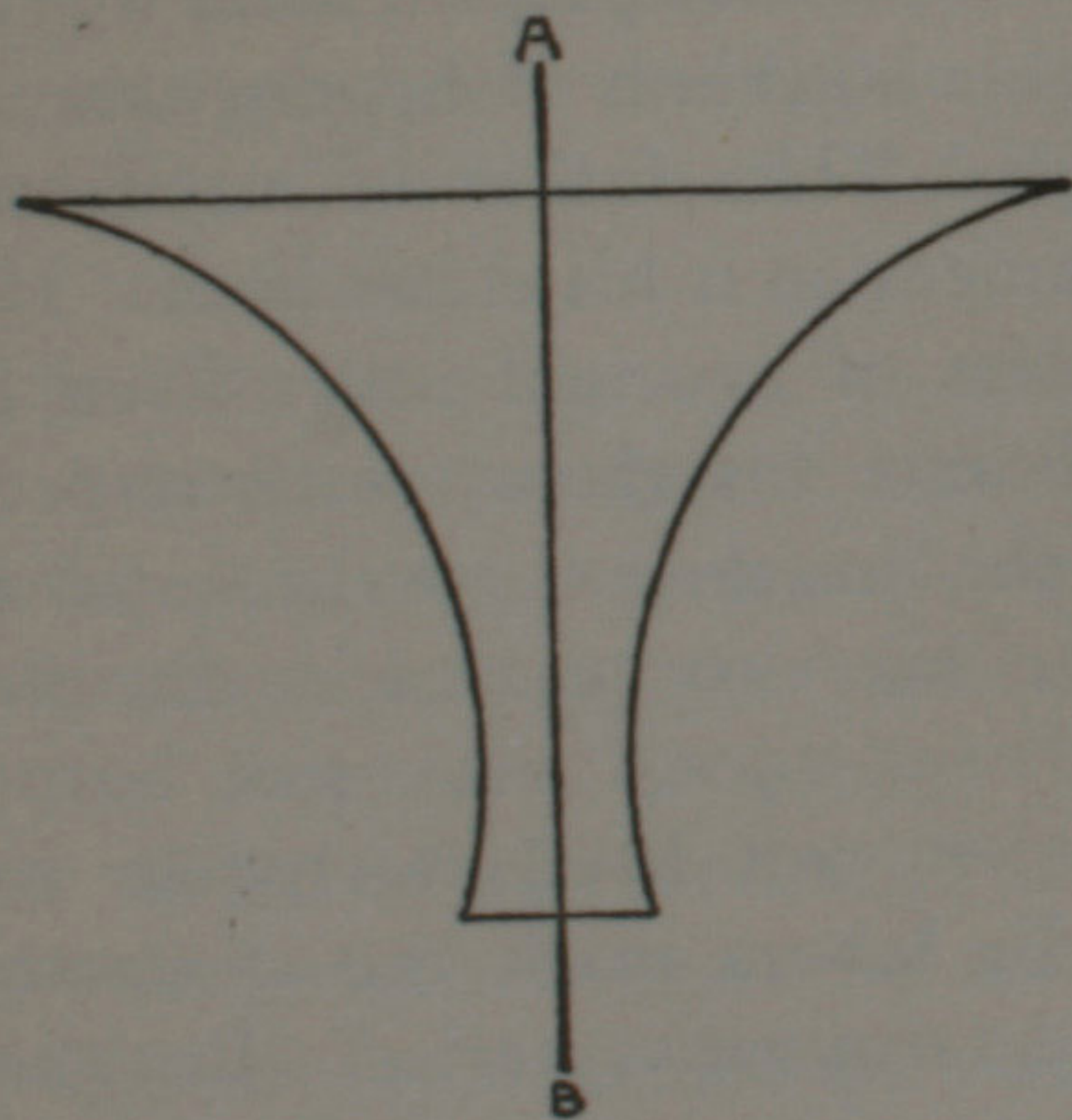


FIG. 4.

If the plane figure $aabb$ is made to revolve upon its axis of symmetry AB the two arcs, ab and ab will describe a pseudospherical concave-convex surface like

that of a solid anchor ring. Above and below, toward *aa* and *bb*, the surface will turn outward with ever-increasing flexure till it becomes perpendicular to the axis and ends at the edge with one curvature infinite. Or, the half of a pseudospherical surface may be rolled up into the shape of a champagne glass, as in Fig. 4. In this way, the two straightest lines of the pseudospherical surface may be indefinitely produced, giving a kind of space (pseudospherical) in which the axiom of parallels does not hold true.

The determinative period marks the most important stage in the development of non-Euclidean geometry and certainly the most significant in the evolution of the idea of hyperspaces and multiple dimensionality. RIEMANN and BELTRAMI are chief among those whose labors characterize the scope of this period. Their work gave direction and general outline for later developments and all subsequent researches along these lines have been conducted in strict conformity with the principles laid down by these pioneer constructionists. They laid out the field and designated its confines beyond which no adventurer has since dared to pass.

The great importance of the work of RIEMANN at this time may be seen further in the fact that it not only marked the beginning of a new epoch in geometry; but his pronouncement of the hypothesis that space is unbounded, though finite, is really the first time in the history of human thought that expression was ever given to the idea that space may yet be only of limited extent. Before that time the minds of all men seemed to have been unanimous in the consideration of space as an illimitable and infinite quantity.

The Elaborative Period

The elaborative stage includes the work of all those who, working upon the bases laid down by LOBACHEVSKI, BOLYAI, SCHWEIKART and RIEMANN, have sought to amplify the conclusions reached by them. Among those whose investigations have greatly multiplied the applications of hyperspace conceptions are HOÜEL (1866) and FLYE ST. MARIE (1871) of France; HELMHOLTZ (1868), FRISCHAUF (1872), KLEIN (1849), and BALTZER (1877) of Germany; BELTRAMI (1872) of Italy; DE TILLY (1879) of Belgium; CLIFFORD and CAYLEY (1821) of England; NEWCOMB (1835) and HALSTEAD of America.

These have been most active in popularizing the subject of non-Euclidean geometry and incidentally the idea of the fourth dimension. The great mass of non-professional mathematical readers, therefore, owe these men an immeasurable debt of gratitude for the work that they have done in the matter of rendering the conceptions which constitute the fabric of meta-geometry understandable and thinkable. A glance at the bibliography appended at the end of this volume will give some idea of the enormous amount of labor that has been expended in an effort to translate the most abstract mathematical principles into a language that could easily be comprehended by the average intelligent person.

The characteristic standpoint of this period is the popular comprehension of the hyperspace concept and the consequent mental liberation which follows. For there is no doubt but that unheard of possibilities of

thought have been revealed by investigations into the nature of space. An entirely new world has been opened to view and only a beginning has been made at the exploration of its extent and resources.

One of the notable incidents of the early years of this period is the position taken by FELIX KLEIN who stands in about the same relation to CAYLEY as BELTRAMI does to RIEMANN, in that he assumed the task of completing the work of his predecessor. KLEIN held that there are only two kinds of RIEMANNIAN *space*—the elliptical and the spherical. Or in other words, that there are only two possible kinds of space in which the propositions announced by RIEMANN could apply. SOPHUS LIE, called the "great comparative anatomist of geometric theories," carried his classifications to a final conclusion in connection with spaces of all kinds and decided that there are possible only four kinds of three dimensional spaces.

But whether men with peering, microscopic, histological vision shall establish the existence of one or many spaces, and regardless of the mathematic rigor with which they shall demonstrate the self-consistency of the doctrines which they hold, the fact remains that the hypotheses thus maintained, while they may be regarded as true descriptions of the spaces concerned, are, nevertheless, incompatible. All of them cannot be valid. It will perhaps be found that none of them are valid, especially objectively so. The only true view, therefore, of these systems of hyperspaces is that which assigns them to their rightful place in the infinitely vast world of pure mathesis where their validity may go unchallenged and their existence unquestioned; for in that domain of unconfined menta-

tion, in that realm of divine intuitability, the marvelous wonderland of ideas and notions, one is not only disinclined to doubt their logical actuality, but is quite willing to accede their claims.

CHAPTER III

ESSENTIALS OF THE NON-EUCLIDEAN GEOMETRY

The Non-Euclidean Geometry Concerned with Conceptual Space Entirely—Outcome of Failures at Solving the Parallel-Postulate—The Basis of the Non-Euclidean Geometry—Space Curvature and Manifoldness—Some Elements of the Non-Euclidean Geometry—Certainty, Necessity and Universality as Bulwarks of Geometry—Some Consequences of Efforts at Solving the Parallel-Postulate—The Final Issue of the Non-Euclidean Geometry—Extended Consciousness.

THE term "non-Euclidean" is used to designate any system of geometry which is not strictly Euclidean in content.

It is interesting to note how the term came to be used. It appears to have been employed first by GAUSS. He did not strike upon it suddenly, however, as in the correspondence between him and WACHTER in 1816 he used the designation "anti-Euclidean" and then, later, following SCHWEIKART, he adopted the latter's terminology and called it "Astral Geometry." This he found in SCHWEIKART's first published *treatise* known by that name and which made its appearance at Marburg in December, 1818. Finally, in his correspondence with TAURINUS in 1824, GAUSS first used the expression "non-Euclidean" to designate the system which he had elaborated and continued to

use it in his correspondence with SCHUMACHER in 1831.

"Non-Legendrean," "semi-Euclidean" and "non-Archimedean" are titles used by M. DEHN to denote all kinds of geometries which represented variations from the hypotheses laid down by LEGENDRE, EUCLID and ARCHIMEDES.

The semi-Euclidean is a system of geometry in which the sum of the angles of a triangle is said to be equal to two right angles, but in which one may draw an infinity of parallels to a straight line through a given point. The non-Euclidean geometry embraces all the results obtained as a consequence of efforts made at finding a satisfactory proof of the parallel-postulate and is, therefore, based upon a conception of space which is at variance with that held by EUCLID. According to the Ionian school space is an infinite continuum possessing uniformity throughout its entire extent. The non-Euclideans maintain that space is not an infinite extension; but a finite though unbounded manifold capable of being generated by the movement of a point, line or plane in a direction without itself. It is also held that space is curved and exists in the shape of a sphere or pseudosphere and is consequently elliptical.

The inapplicability of EUCLID's parallel-postulate to lines drawn upon the surface of a sphere suggested the possibility of a space in which the postulate could apply to all possible surfaces or that space itself may be spherical in which case the postulate would be invalidated altogether. Hence, it is quite natural that mathematicians finding themselves unable to prove the postulate with due mathetic precision should turn their

attention to the conceptually possible. In this virtual abandonment of the perceptual for the conceptual lies the fundamental difference between the Euclidean and the non-Euclidean geometries. It may be said to the credit of the Euclideans that they have sought to make their geometric conceptions conform as closely as possible to the actual nature of things in the sensuous world while at the same time they must have perceived that at best their spatial notions were only approximations to the sensuous actuality of objects in space.

On the other hand, non-Euclideans make no pretense at discovering any congruency between their notions and things as they actually are. The attitude of the metageometricians in this respect is very aptly described by CASSIUS JACKSON KEYSER who says:

"He constructs in thought a summitless hierarchy of hyperspaces, an endless series of orderly worlds, worlds that are possible and logically actual, and he is content not to know if any of them be otherwise actual or actualized."¹

The non-Euclidean is, therefore, not concerned about the applicability of ensembles, notions and propositions to real, perceptual space conditions. It is sufficient for him to know that his creations are thinkable. As soon as he can resolve the nebulosity of his consciousness into the conceptual "star-forms" of definite ideas and notions, he sits down to the feast which he finds provided by superfoetated hypotheses fabricated in the deeps of mind and logical actualities

¹ *Mathematics*, by C. J. KEYSER, Adrian Professor of Mathematics, Columbia University.

imperturbed and unmindful of the weal of perceptual space in its homogeneity of form and dimensionality.

Fundamentally, the non-Euclidean geometry is constructed upon the basis of conceptual space almost entirely. Knowledge of its content is accordingly derived from a superperceptual representation of relations and interrelations subsisting between and among notions, ideas, propositions and magnitudes arising out of a conceptual consideration thereof. In other words, representations of the non-Euclidean magnitudes, cannot be said to be strictly perceptual in the same sense that three-space magnitudes are perceived; for three-space magnitudes are really sense objects while hyperspace magnitudes are not sense objects. They are far removed from the sensuous world and in order to conceive them one must raise his consciousness from the sensuous plane to the conceptual plane and become aware of a class of perceptions which are not perceptions in the strict sense of the word, but superperceptions; because they are representations of concepts rather than precepts.

Notions of perceptual space are constituted of the triple presentations arising out of the visual, tactual and motor sensations which are fused together in their final delivery to the consciousness. The synthesis of these three sense-deliveries is accomplished by equilibrating their respective differences and by correcting the perceptions of one sense by those of another in such a way as to obtain a completely reliable perception of the object. This is the manner in which the characteristics of Euclidean space are established.

The characteristics of non-Euclidean space are not arrived at exactly in this way. Being beyond the scope

of the visual, tactile and motor sense apprehensions, it cannot be said to represent judgments derived from any consideration or elaboration of the deliveries presented through these media. Yet, the substance of metageometry, or the science of the measurement of hyperspaces, may not be regarded as an *a priori* substructure upon which the system is founded. That is, the conceptual space of non-Euclidean geometry is not presented to the consciousness as an *a priori* notion. On the other hand, the *a posterioristic* quality of metageometric spaces marks the entire scope of motility of the notions appertaining thereto.

The notions, therefore, of conceptual space are derivable only from the perception of concepts, or, otherwise consist of judgments concerning interconceptual relations. The process of apperception involved in the recognition of relations which may be methodically determined is much removed from the primary procedure of perceiving sense-impressions and fusing them into final deliveries to the consciousness for conceptualization or the elaboration into concepts or general notions. It is a procedure which is in every way superconceptual and extra-sensuous. The metageometrician or analyst in no way relies upon sense-deliveries for the data of his constructions; for, if he did, he should, then, be reduced to the necessity of confining his conclusions to the sphere of motility imposed by the sensible world with the result that we should be able to verify empirically all his postulations. But, contrarily, he goes to the extra-sensuous, and there in the realm of pure conceptuality, he finds the requisite freedom for his theories; thus, environed by a sort of intellectual anarchism, he pursues ana-

lytical pleasures quite unrestrainedly. The difference between the two mental processes—that which leads from the sensible world to conception and that which veers into the fields beyond—is so great that it is hardly permissible to view the results arrived at in the outcome of the separate processes as being identical.

To illustrate this difference, let us draw an analogy. The miner digs the iron ore out of the ground. The iron is separated from the extraneous material and delivered to the furnaces where the metal is melted and turned out as pig iron. It is further treated, and steel, of various grades, cast iron and other kinds of iron are produced. The treatment of the iron ore up to this stage is similar to the treatment of sense-impressions by the Thinker. Steel, cast iron, et cetera, are similar to mental concepts. Later, the steel and other products are converted into instruments and numerous articles. This represents the superperceptual process. Trafficking in iron ore products, such as instruments of precision, watch springs, and the like, represents a stage still farther removed from the primary treatment of the ore and is similar to that to which concepts are treated when the metageometrician manipulates them in the construction of conceptual space-forms. Perception is the dealing with raw iron ore while conception is analogous to the production of the finished product.

Superperception would be analogous to the trafficking in the finished product as such and without any reference to the source or the preceding processes. Thus the notions and judgments of the non-Euclidean geometry are arrived at as a result of a triple process

of perception, conception and superperception the latter being merely superconceived as formal space-notions. But it is obvious that the more complex the processes by which judgments purporting to relate to perceptual things are derived the more likely are those judgments to be at variance with the nature of the things themselves.

In view of the foregoing, the dangers resulting from identifying the products of the two processes are very obvious indeed. But the difference between the two procedures is the difference between Euclidean and non-Euclidean geometries or the difference between perceptual space notions and conceptual space notions. Hence, it is not understood just how or why it has occurred to anyone that the two notions could be made congruent. Magnitudes in perceptual, sensible space are things apart from those that may be said to exist in mathematical space or that space whose qualities and properties have no existence outside of the mind which has conceived them. It is believed to be quite impossible to approach the study of meta-geometrical propositions with a clear, open mind without previously understanding the fundamental distinctions which exist between them.

It follows, therefore, as a logical conclusion that geometric space of whatsoever nature is a purely formal construction of the intellect, and for this reason is completely under the sovereignty of the intellect however whimsical its demands may be. Being thus the creature of the intellect, its possibilities are limited only by the limitations of the intellect itself. Perceptual space, being neither the creature of the intellect nor necessarily an *a priori* notion resident in the mental

substructure, but existing entirely independent of the intellect or its apprehension thereof, cannot be expected to conform to the purely formal restrictions imposed by the mind except in so far as those restrictions may be determined by the nature of perceptual space. And for that matter, it should not be forgotten that, as yet, we have no means of determining whether or not the testimony of the intellect is thoroughly credible simply because there is no other standard by which we may prove its testimony. It is possible to justify the deliveries of the eye by the sense of touch, or vice versa. It is also possible to prove all our sense-deliveries by one or the other of the senses. But we have no such good fortune with the deliveries of the intellect. We have simply to accept its testimony as final; because we cannot do any better. But if it were possible to correct the testimony of the intellect by some other faculty or power which by nature might be more accurate than the intellect it should be found that the intellect itself is sadly limited.

The possible curvature of space is a notion which also characterizes the content of the non-Euclidean geometry. It is upon this notion that the question of the finity and unboundedness of space, in the mathematical sense, rests. In the curved space, the straightest line is a curved line which returns upon itself. Progression eastward brings one to the west; progression northward brings one to the south, et cetera. On this view space is finite, but may not be regarded as possessing boundaries.

Space-curvature, reinforced by the idea that space is also a manifold is the enabling clause of meta-

geometry and without them the analyst dares not proceed. Here again, we are led to the confession that however fantastic these two notions may seem and evidently are, there is nevertheless to be recognized in them a "dim glimpse" of a veritable reality—a slight foreshadowing of the revelation of some great kosmic mystery.

The manifoldness of space is the fiat of analysis. It is the inevitable outcome of the analyst's method of procedure. His education, training and view of things in general inhibit his arriving at any other result and he may be pardoned with good grace for his manufacture of the space-manifold. For by it perhaps a better appreciation of that wonderful extension of consciousness in the nature of which is involved the explanation of the perplexing problems which the manifold and other metageometrical expedients faintly adumbrate may be gained.

It is pertinent, in the light of the above, to examine into some of the relative merits of the three formal bulwarks of geometrical knowledge. These are *certainty*, *necessity* and *universality*.

Geometric certainty is derived solely from the nature of the premises upon which it is based. If the premises be contradictory, it is, of course, defective. But if the premises are non-contradictory or self-evident, then the certainty of geometric notions and conclusions is valid. Another consideration of prime importance in this connection is the *definition*. From it all premises proceed. Hence, the definition is even more important than the premise; for it is the persisting determinant of all geometric conclusions while the premise is dependent upon the limitations of the

definition. The determinative character of the definition has led to its apotheosis; but this, admittedly, has been necessary in order to give stability and permanency to the conclusions which followed. But in spite of this it would appear that the certainty of geometric conclusions is not a quality to be reckoned as absolute or final.

With the same certainty that it can be said the sum of the angles of the triangle is equal to two right angles it may be asserted that that sum is also greater or less than two right angles. Certainty which is based upon the inherent congruity of definitions, premises and propositions is an entirely different matter from that certainty which arises out of the real, abiding validity of a scheme of thought. But this difference is not lessened by the fact that the latter is dependent, in a measure, upon the correct systematization of our spatial experiences by means of methodical processes. Euclidean geometry, accordingly, is not so certain in its applications as it is utilitarian; but non-Euclidean geometry is even less certain than the former and consequently more lacking in its utilitarian possibilities.

The necessity of geometrical determinations is merely the necessity which inheres in logical inferences or deductions. These may or may not be valid. Inasmuch as the necessariness of deductions is primarily based upon the conditional certainty of premises and definitions it appears that this quality is in no way peculiar to geometry whether Euclidean or non-Euclidean. In like manner, the universality of geometric judgments may not properly be regarded as a peculiarity of geometry; but is explicable upon the

basis of the formal character of the assumptions which underlie it. The chief value, then, of non-Euclidean geometry seems to abide in the fact that it clarifies our understanding as to the complex processes by which it is possible to organize and systematize our spatial experiences for assimilation and use in other branches of knowledge.

With the above statement of the case of the non-Euclidean geometry it is now thought permissible to state briefly some of the elements thereof.¹

Below will be found some of the elements obtained as a consequence of efforts made both at proving and disproving the parallel-postulate of Euclid:

"If two points determine a line it is called a straight."

"If two straights make with a transversal equal alternate angles they have a common perpendicular."

"A piece of a straight is called a sect."

"If two equal coplanar sects are erected perpendicular to a straight, if they do not meet, then the sect joining their extremities makes equal angles with them and is bisected by a perpendicular erected midway between their feet."

"The sum of the angles of a rectilineal triangle is a straight angle, in the hypothesis of the right (angle); is greater than a straight angle in the hypo-

¹ The science of pure mathematics is perhaps indebted to no one in so great a degree as to GEORGE BRUCE HALSTEAD, formerly of the University of Texas, whose labors in connection with the popular exposition of the non-Euclidean geometry have been most untiring and effectual. Vide *Popular Astronomy*, Vol. VII and VIII, 1900, Dr. G. B. HALSTEAD.

thesis of the obtuse (angle); is less than a straight angle in the hypothesis of the acute (angle)."

"The hypothesis of right is Euclidean; the hypothesis of the acute is BÖLYAI-LOBACHEVSKIAN; the hypothesis of obtuse is RIEMANNIAN."

"If one straight is parallel to a second the second is parallel to the first."

"Parallels continually approach each other."

"The perpendiculars erected at the middle point of the sides of a triangle are all parallel, if two are parallel."

"If the foot of a perpendicular slides on a straight its extremity describes a curve called an equidistant curve, or an equidistantial."

"An equidistantial will slide on its trace."

"In the hypothesis of the obtuse a straight is of finite size and returns into itself."

"Two straights always intersect."

"Two straights perpendicular to a third straight intersect at a point half a straight from the third either way."

"A pole is half a straight from its polar."

"A polar is the locus of coplanar points half a straight from its pole. Therefore, if the pole of one straight lies on another straight the pole of this second straight is on the first straight."

"The cross of two straights is the pole of the join of their poles."

"Any two straights inclose a plane figure, a digon."

"Two digons are congruent if their angles are equal."

"The equidistantial is a circle with center at the poles of its basal straight."

A typical postulate based upon the BOLYAI hypothesis of the acute angle is the following:

"From any point P drop PC , a perpendicular to any given straight line AB . If D move off indefinitely on the ray CB , the sect will approach as limit PF copunctal with AB at infinity.

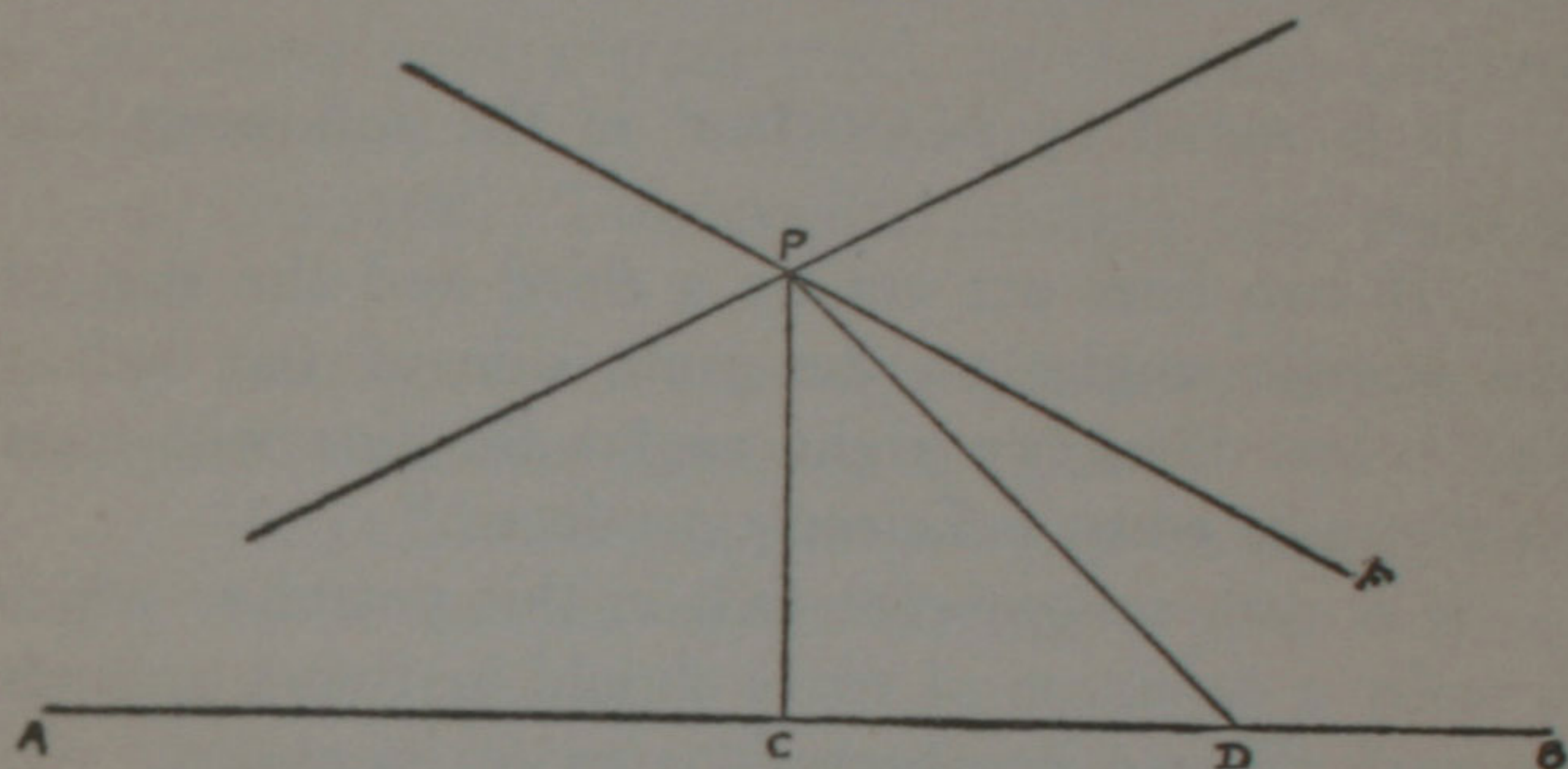


FIG. 5.

PD is said to be at P the parallel to AB toward B . PF makes with PC an angle CPF which is called the angle of parallelism for the perpendicular PC . It is less than a right angle by an amount which is the limit of the deficiency of the triangle PCD . On the other side of PC , an equal angle of parallelism gives the parallel P to BA towards AM .³ Thus at any point there are two parallels to a straight. A straight has, therefore, two separate points at infinity."

"Straights through P which make with PC an angle greater than the angle of parallelism and less than its supplement do not meet the straight AB at all not even at infinity."

³ NOTE.— M may be any point on the line BA indefinitely produced.

The parallel-postulate is stated in the non-Euclidean geometry as follows:

"If a straight line meeting two straight lines make those angles which are inward and upon the same side of it less than two right angles the two straight lines being produced indefinitely will meet each other on this side where the angles are less than two right angles."

It is stated by MANNING⁴ in the following language:

"If two lines are cut by a third and the sum of the interior angles on the same side of the cutting line is less than two right angles the line will meet on that side when sufficiently produced."

It is rather significant that in this postulate which is really a definition of space should be found grounds for such diverse interpretations as to its nature. Of course, the moment the mind seeks to understand the infinite by interpreting it in the unmodified terms of the apparently unchangeable finite it entangles itself into insurmountable difficulties. As a drowning man grasps after straws so the mind, immersed in endless abysses of infinity, fails to conduct itself in a seemly manner; but gasps, struggles and flounders and is happy if it can, in the depths of its perplexity, discover a way of logical escape. The pure mathematician has a hankering after the logically consistent in all his pursuits; to him it is the "Holy Grail" of his highest aspirations. He seeks it as the devotee seeks immortality. It is to him a philosopher's stone, the elixir of perpetual youth, the eternal criterion of all knowledge.

Failures to demonstrate the celebrated postulate of

⁴ Vide *Non-Euclidean Geometry*, p. 91.

EUCLID led, as a matter of course, to the substitution of various other postulates more or less equivalent to it in that each of them may be deduced from the other without the aid of any new hypothesis.

Among those who sought proof by a restatement of the problem are the following:

1. PTOLEMY: The internal angles which two parallels make with a transversal on the same side are supplementary.

2. CLAVIUS: Two parallel straight lines are equidistant.

3. PROCLUS: If a straight line intersects one of two parallels it also intersects the other.

4. WALLIS: A triangle being given another triangle can be constructed similar to the given one and of any size whatever.

5. BOLYAI (W.): Through three points not lying on a straight line a sphere can always be drawn.

6. LORENZ: Through a point between the lines bounding an angle a straight line can always be drawn which will intersect these two lines.

7. SACCHERI: The sum of the angles of a triangle is equal to two right angles.

There were, of course, many other statements and substitutions used by mathematicians in their endeavors satisfactorily to establish the truth of the parallel-postulate. That their labors should have terminated, first, by doubting it, then by denying, and finally, by building up a system of geometries which altogether ignores the postulate is just what might naturally be expected of these men who have given to the world the non-Euclidean geometry. In doing what they did many, if not all of them, were not aware

in any measure of the proportions of the imposing superstructure that would be built upon their apparent failures. All of them undoubtedly must have sensed the vague adumbrations forecast by the unfolding mysteries which they sought to lay bare; all of them must have felt as they executed the early tasks of those crepuscular days of pure mathematics that the way which they were traveling would lead to the inner shrine of a higher knowledge and a wider freedom; they may have been led by divine intuition to strike out on this new path and yet they could not have known how fully their dreams would be realized by the mathematicians of the twentieth century. If so, they were truly gods and mathesis is their kingdom.

The analyst proceeds upon a basis entirely at variance with that which guides the ordinary investigator in the formulation of his conclusions. The empirical scientist in arriving at his theories or hypotheses is governed at all times by the degree of conformity which his postulates exhibit to the actual phenomena of nature. He endeavors to ascertain just how far or in what degree his hypothesis is congruent with things found in nature. If the dissidence is found to predominate he abandons his theory and makes another statement and again sets out to determine the degree of conformity. If he then finds that the natural phenomena agree with his theory he accepts it as for the time being finally settling the question. In all things he is limited by the answer which nature gives to his queries. Not so with the exponent of pure mathematics. For him the truth of hypotheses and postulates is not dependent upon the fact that physical nature contains phenomena which answer to them. The

sole determining factor for him is whether or not he is able to state with *rational consistency* the assumed first principles and then logically develop their consequences. If he can do this, that is, if he can state his hypotheses with consistency and develop their consequences into a logical system of thought, he is quite satisfied and well pleased with his performances. But the fact that this is true is of vital significance for all who seek clearly to understand the essential character of hyperspatiality.

It appears, therefore, that the science of consequences is the radical essence of pure geometry. The metageometrician enjoys unlimited freedom in the choice of his postulates and suffers curtailment only when it comes to the question of consistency. He is at liberty to formulate as many systems of geometry as the barriers of consistency will permit and these are practically innumerable. So long then as the laws of compatibility remain inviolate his multiplication of postulate-systems may proceed indefinitely. Is it strange then that under conditions where an investigator has such unbridled liberty he should be found indulging in mathetic excesses?

KANT held that the axioms of geometry are synthetic judgments *a priori*; but it appears that in the strictest sense this is not the case. It depends upon the type of mind which is taken as a standard of reference. If it be the uncultivated mind, it is certain that to it the relations expressed by an axiom would never appear spontaneously. If on the other hand, the standard be that of a cultivated mind it is also equally certain that to it these relations would be discovered only after methodical operations. All judgments ar-

rived at as a result of logical processes should, it seems, be regarded as judgments *a posteriori*, i.e., the results of empirical operations. Confessedly, the facts adduced in course of experimentation serve as guides in choosing among all of the many possible logical conventions; but our choice remains untrammelled except by the compulsion arising out of a fear of inconsistency. The real criterion then of all geometries is neither truth, conformability nor necessity, but consistency and convenience.

The difficulty with the non-Euclideans resolved itself into the question as to whether it is more consistent, as well as convenient, to establish a proof of the postulate by taking advantage of the support to be found in other postulates or whether, by seeking a demonstration based upon the deliveries of sense-experience as to the nature of space and its properties, a still more consistent conclusion might be reached. They had further perplexity, however, when it came to a decision as to whether the organic world is produced and maintained in Euclidean space or in a purely conceptual space which alone can be apprehended by the mind's powers of representation. Unwilling to admit the existence of the world in Euclidean space, they turned their attention to the examination of the properties of another kind of space so-called which unlike the space of the Ionian school could be made to answer not only all the purposes of plane and solid figures, but of spherics as well. And so, the manifold space was invented by RIEMANN and later underwent some remarkable improvements at the hands of his disciple, BELTRAMI. But it may be said here, parenthetically, that the truth of the whole matter is that

our world is neither in Euclidean nor non-Euclidean space, both of which, in the last analysis, are conceptual abstractions. Although it may not be denied that the Euclidean space is the more compatible.

The problem of devising a space, if only a very limited portion, in which could be demonstrated the assumed alternative hypothesis and its consequences logically developed, occasioned no inconsiderable concern for the non-Euclidean investigators; but neither LOBACHEVSKI, BOLYAI nor RIEMANN were to be baffled by the difficulties which they met. These only cited them to more laborious toil. Having succeeded in mentally constructing the particular kind of space which was adaptable to their rigorous mathetic requirements it immediately occurred to them that all the qualities of the limited space thus devised might logically be amplified and extended to the entire world of space and that what is true of figures constructed in the segmented portion of space which they used for experimental purposes is also true of figures drawn anywhere in the universe of this space as all lines drawn in the finite, bounded portion could be extended indefinitely and all magnitudes similarly treated. From these results, it was but a single step to the conclusion which followed—that either an entirely new world of space had been discovered or that our notion of the space in which the organic world was produced is wholly wrong and needs revision. But notwithstanding the insurmountable obstacles which stood in the way of the investigators who made the attempt to discover the homology which might exist between the characteristics of the newly fabricated space and the phenomenal world, investigations were carried for-

ward with almost amazing recklessness and loyalty to the mathetic spirit until it was discovered that all efforts to trace out any definite lines of correspondence were futile. Then the policy of ignoring the question of conformability was adopted and has since been pursued with unchangeable regularity by the analytical investigator.

Among the results obtained by the non-Euclidean in their profound researches into the nature of hyper-space are these: 1. It was found that the angular sum of a triangle, being ordinarily assumed to be a variable quantity, is either less or greater than two right angles so that a strictly Euclidean rectangle could not be constructed. 2. The angle sums of two triangles of equal area are equal. 3. No two triangles not equal can have the same angles so that similar triangles are impossible unless they are of the same size. 4. If two equal perpendiculars are erected to the same line, their distance apart increases with their length. 5. A line every point of which is equally distant from a given straight line is a curved line. 6. Any two lines which do not meet, even at infinity, have one common perpendicular which measures their minimum distance. 7. *Lines which meet at infinity are parallel.* But it is apparent that these results have not followed upon any mathematical consequence of other supporting postulates or axioms such as would place them on a coördinate basis with those used as a support for the parallel-postulate; for they are based upon the envisagement of an entirely new principle of space-perception and belong to a wholly different set of space qualities.

The final issue then of the non-Euclidean geometry is neither in the utility of its processes and conclusions

nor in the increscent inclination towards a new outlook upon the world of mathesis; but resides solely in the possibilities yet to be developed in that vast domain of analytical thought which it has discovered and opened to view. To say that it sheds any light upon the nature of the universe is perhaps to take the radical view; yet it cannot be doubted that the researches incident to the formulation of the non-Euclidean geometry have greatly extended the scope of consciousness. Whether the extension is valid and normal or simply a hypertrophic excrescence of mental feverishness; whether by virtue of it we shall more closely approach an understanding of the true nature of the mind of the Infinite, or shall all fall into insanity, are certainly debatable questions. It nevertheless appears evident that humanity has gained something of real, abiding permanence by this new departure. If that something be merely an extended consciousness or an awakening to the fact that there are stages of awareness beyond the strictly sensuous, and every observable evidence points to this, then there has only begun the process by which the faculty of conscious functioning in this new world shall become the normal possession of the human species. But this new world cannot be said to be of mathematical import; for it is doubtful if mathematical laws such as have been devised up to the present time, would obtain therein. So that if anything, it must be psychological and vital.

On this view the worlds of hyperspace inlaid with analytic manifoldnesses and constant curvatures are but the primal excitants which will finally awaken in the mind the faculty of awareness in the new domain of psychological content. Then will come the bloom-

ing of the diurnal flower of the mind's immortality and the outputting of the organ of consciousness where-with the infinite stretches of hyperspaces, the low-lying valleys of reals and imaginaries and the uplifting hills of finites and infinities shall be divested of their mysteries and stand out in their unitariness no longer draped in the veil of the inscrutable and the incomprehensible.

The fourth dimension, regarded by some as a new scope of motion for objects in space, by others as a new and strange direction of spatial extent and by others still as the doorway of the temple of exegesis wherein an explanation may be found for the entire congeries of mysteries and supermysteries which now perplex the human mind, may also be said to be the key to the non-Euclidean geometry. But it really complicates the situation; for one has to be capable of prolonged abstract thought even to envisage is as a conceptual possibility. POINCARÉ⁵ says: "Any one who should dedicate his life to it could, perhaps, eventually imagine the fourth dimension," implying thereby that a lifetime of prolonged abstract thought is necessary to bring the mind to that point of ecstasy where it could even so much as imagine this additional dimension. Nevertheless by it (the fourth dimension) was the non-Euclidean geometry made and without it was not any of the hyperspaces made that were made. It is the view which geometers have taken of space in general that has made the fourth dimension possible, and not only the fourth, but dimensions of all degrees. The basis of the non-Euclidean geometry

⁵ Vide *Nature*, Vol. XLV, 1892.

may be found then in the notion of space which has been predominant in the minds of the investigators.

Finally, it should be pointed out that the non-Euclidean geometry, though a consistent system of postulates, has been constructed upon a misconception based upon the identification of real, perceptual space with systems of space-measurements. Hyperspaces which are not spaces at all should not be confounded with *real space*. But they constitute the substance of non-Euclidean geometry; they are its blood and sinews. Their study is interesting, because of the possibilities of speculation which it offers. No mind that has thought deeply upon the intricacies of the fourth dimension, or hyperspace, remains the same after the process. It is bound to experience a certain sense of humility, and yet some pride born of a knowledge that it has been in the presence of a great mystery and has delved into the fearful deeps of kosmic mind. To the mind that has thus been anointed by the sacred chrism of the inner mysteries of creative mentality there always come that stillness and calm such as characterize the aftermath of reflection upon the incomprehensible and the transfinite.

CHAPTER IV

DIMENSIONALITY

Arbitrary Character of Dimensionality—Various Definitions of Dimension—Real Space and Geometric Space Differentiated—The Finiteness of Space—Difference Between the Purely Formal and the Actual—Space as Dynamic Appearance—The *A Priori* and the *A Posteriori* as Defined by PAUL CARUS.

IN previous chapters we have traced the growth and development of the non-Euclidean geometry showing that the so-called fourth dimension is an aspect thereof. It is now deemed fitting that we should enter into a more detailed study of the question of dimensionality with a view to examining some of the difficulties which encompass it.

The question of dimension is as old as geometry itself. Without it geometric conclusions are void and meaningless. Yet the conception of dimensionality itself is purely conventional. In its application to space there is involved a great deal of confusion because of the inferential character of its definition. For instance, commonly we measure a body in space and arbitrarily assign three elements to determine its position. The simplest standard for this purpose is the cube having three of its edges terminating at one of its corners.

Thus because it is found that the entire volume of a cube is actually comprehended within the directions

indicated by the lines ab , bc and db it is determined that the three coördinates of the point b are necessary and sufficient to establish the dimensions of the cube and consequently of the space in which it rests. The conception may be stated in this way: If a collection of elements, say points or lines, be of such a nature or order that it is sufficient to know a certain definite number of facts about it in order to be able to distinguish every one of the elements from all the others, then

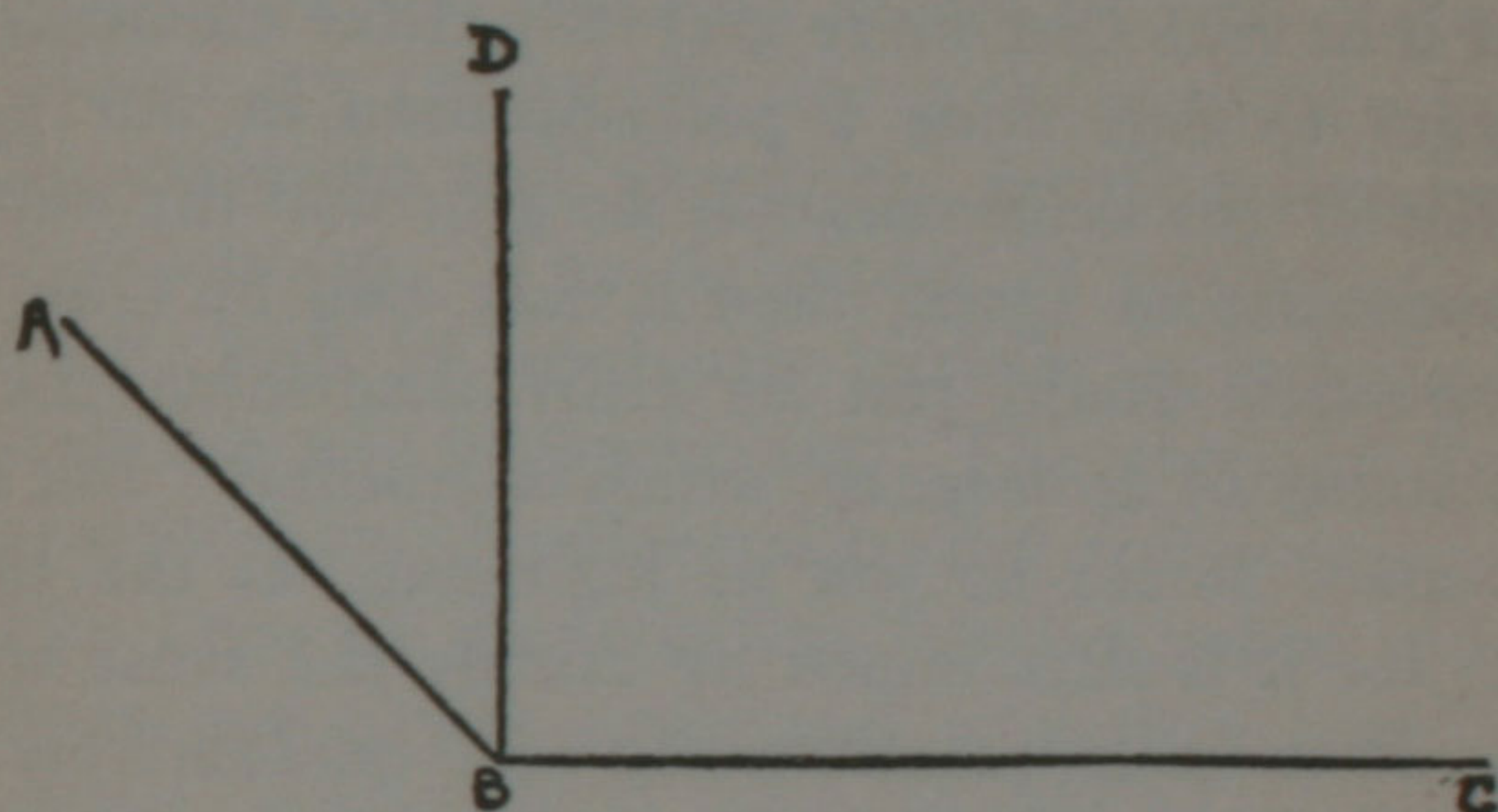


FIG. 6.

the assemblage or collection of elements is said to be of the same number of dimensions as there are elements necessary to its determination. In the above figure there are three elements, namely, the lines ab , bc , and db , which are necessary and sufficient for the determination of the position of the point b . In this way geometers have determined that our space is tri-dimensional; but it is obvious that this conclusion is based not upon any examination of space itself but upon the measurement of bodies in space. Upon this view it is seen that conclusions based upon such a procedure render our notion of the extension of bodies in

space identical with the notion of spatial extensity. In other words, we take bodies in space and by examining their characteristics and properties arrive at an alleged apodeictic judgment of space. It is by means of this conventional norm of geometric knowledge that various other spaces, notably the one-, two-, four- and n -space, have been devised. It would appear that if some more absolute standard of measurement or definition of space were adopted the confusion which now clings to the conception of dimension could be obviated. For if it be true that three and only three elements are necessary to determine a point-position in our space and that in this determination we also find the number of dimensions of space, then it may also be true that n -coördinates would just as truly determine the dimensionality of an n -space, which is granted. But then the n -space would be just as legitimate as the three-space; for it is determined by exactly the same standards. It is both quantitatively and qualitatively the same. If, however, on account of the exigencies that might arise, we are forced to seek solace in the notion of an n -space whither shall we turn for it? It cannot be found; for it is imperceptible, uninhabitable, non-existent, and therefore, absolutely and purely an abstraction. Consequently, there must be something radically wrong with the definition of space or with its determinants.

The purely arbitrary character of dimensionality is very aptly described by CASSIUS JACKSON KEYSER, who says:

“ . . . The dimensionality of a given space is not unique, but depends upon the choice of the geometric entity for primary or generating element.

A space being given, its dimensionality is not therewith determined, but depends upon the will of the investigator who by a proper choice of generating element endows the space with any dimensionality he pleases. That fact is of cardinal significance for science and philosophy."¹

It is a fact of "cardinal significance" for science; because it emphasizes the necessity for some more rational procedure than that of the geometrician in arriving at an absolutely unique method of determining the dimension and essential nature of real space. Its significance for philosophy lies in the need of a logical, rigidly exclusive and absolutely peculiar standard of space definition. The definition of perceptual space should be such as rigorously inhibits its inclusion as a particular in any general class. The necessity for this is warranted by its universality and uniqueness.

The lines of demarkation between what is recognized as perceptual space and what has been called geometric or conceptual space should be very sharply drawn. So that when reference is made to either there will be no doubt as to which is meant. And then, too, conceptual space is no space at all, properly speaking. It is merely a system of space-measurement. And as such has no logical right to be put in the same category as perceptual space.

Real space is unique. Geometric space belongs to a class whose members are capable of indefinite multiplication. It is certainly most illogical to identify them. Perceptual space, figuratively speaking, is a quantity; analytic space is the foot-rule, the yard-stick, the kilometer, by which it is measured and apportioned.

¹ Vide *Monist*, Vol. XVI, 1896, Mathematical Emancipations.

It is logically impossible to predicate the same conclusion for both of them. That is, to do so causes a profound fracture of the fundamental norms of logic. Such conclusions being thus illegitimate it is rather surprising that an error of this nature should have been made. It is perhaps accountable for on the grounds of the geometer's complete *insouciance* as to how his postulates shall stand in their relation to things in the phenomenal world.

It is agreed that as convenient as is Euclid's system of space-measurement it is not by any means congruent with the extension of real space objects. It does, however, approximate congruity with these objects as nearly as possible. How then could it be expected that a system of space-measurement so far removed from this primary congruence as the non-Euclidean system is should exhibit more obvious signs of correspondence? But the advocates of the n -dimensionality of space have illatively asserted the identity of space and its dimensions. Accordingly, there is not recognized any distinction between their conception of space itself and its qualitative peculiarities. They use the terms interchangeably. So that dimension means space and *vice versa*. In this lack of discrimination may be found the source of much of the confusion which attaches to the conception of space.

If it were arguable that the relation between space and its dimensions is the same as that between matter and its properties then the restriction of this relation to three and only three directions of extent would be disallowed; for the reason that if, as is commonly done, dimension be made to mean direction of extent, there would be an unlimited number of directions of extent

and they would all be perceptible. But this is really another fundamental fault. Non-Euclideans have stretched the meaning of the term dimension so that it not only includes the idea of direction but an entirely new class of qualities—the fourth dimension. And despite this reformation of the original conception, they demand that it shall be called space.

We have just shown that the generic concept of dimensionality is that three and only three coördinates are necessary and sufficient for its determination. Granting that this is true, are we not compelled consequently to see that we have, by adding a fourth or n -dimensions, involved ourselves into a more complex situation than before? For by postulating a fourth dimension either we have created a new world whose dimensions are four in number or we have explicitly admitted that the three dimensions have a fourth. Aside from the logical difficulties which beset these conclusions there is also set up a condition which is at variance with the most elementary requirements of common sense.

Thus far mathematical thought has not served to clarify our notions of space nor to shed any new light upon the vital processes which are alleged to have their explanation in the new discovery. Simply stated, metageometricians have brought us to the place where we must either recognize that the fourth dimension is another sphere lying dangerously near the earth in which space extends in four primary directions and in which four coördinates are necessary for its determination or we are driven to the other horn of the dilemma where we are brought face to face with the conclusion that the three perceptual space dimensions have in

common a hitherto unknown property or extension in virtue of which it may be viewed as having an unlimited number of dimensions. To accept the latter view is equivalent to saying that, in the above figure, the three lines *ab*, *bc* and *db* have formed a triple *entente* by which they have mutually and severally acquired a new domain, hyperspace, and in which, because of the vast resources of the region, they are able to perform wondrous things.

Let us examine briefly the various current definitions of dimension. It is assumed by not a few that dimension is the same as *direction*. But can we grant this wholly to be true? If so, then a mere child may see that there are and must necessarily be as many dimensions as there are directions. Primarily, there are six directions of space and an unlimited number of subsidiary directions. On this view it is not necessary to invent a new domain of space if the object be merely to discover and utilize a greater number of dimensions than has heretofore been allowed. For the identification of the term dimension with direction already makes available an almost infinite number of dimensions. But this view is objected to by the advocates, for it is contrary to the hypothesis of *n*-dimensionality.

Dimension also means *extent*. This is partially true. It cannot be wholly true. For, if it were, then, space would have only one dimension which is also not allowable under the hypothesis. Then the definition leaves out of account the idea that space is at the same time a direction or collection of directions. The term extension is generic and when applied to space means extension in all possible directions and not in any one

direction. So that it is not permissible to say that space extends in this direction or that because it extends in all directions simultaneously and equally.

Geometers claim that space is a system of coördinates necessary for the establishment of a point-position in it. This view, however, identifies space with a system of space-measurement and is therefore faulty. According to this view there may be as many spaces as there are systems of space-measurement and the latter may be limitless. But if the totality of spaces are to be viewed as one space then we shall have one space with an indefinite number of dimensions; also an indefinite number of space measurements which would be confusing. Much, if not all, of such a system's utility and convenience would be unavailable or useless. That, too, would be in violation of the avowed purpose of these investigations which is to enhance the utility and convenience of mathematic operations.

Now it is evident that space is neither direction, extension, a system of space-measurement nor a system of manifolds whose dimensions are generable. And this is so for the same reason that a piece of cloth is not the elements of measurement—inches, feet, yards—by which it is apportioned. And because we find that the fabric of space lends itself accommodatingly to our conventional norms of measurement is not sufficient reason for identifying it with these norms. Here we have the source of all error in mathematical conclusions about the nature of space; because all such conclusions are based not upon the intrinsic nature of space, but upon artificial forms which we choose to impose upon it for our own convenience. But it should be remembered that the irregularities which we note

are not in space itself but inhere in the forms which we use. For these purposes space is extremely elastic and accommodates itself to the shape and scope of any construction we may decide to try upon it. In this respect it is like water which has no regard for the shape, size or kind of vessel into which it may be posited. There is one thing certain that judging from the above considerations there has been not yet any absolute, all-satisfying definition devised for space by mathematicians.

The best definitions hitherto constructed are purely artificial and arbitrary determinations. It is rather anomalous that there should be so little unanimity about what is the most fundamental consideration of mathematical conclusions which are supposed to be so certain, so necessary and universal as to be incontrovertible. Confessedly, it is a condition which raises again the question as to just what are the limits of mathematical certainty and necessity and just how far we shall depend upon the validity of mathematics to determine for us absolutely certain conclusions about the nature of space. In view of the uncertainty noted, are we justified in following too closely the mathematic lead even in matters of logic, to say nothing of our conception of space? It seems that we shall have necessarily, on account of the recognized limitations of mathematics in this matter, to turn to some more tenable source for the norms of our knowledge concerning space. For in the light of the rather indefensible position which metageometricians have involved themselves there appears to be no hope in this direction.

It is undoubtedly safer not to rely altogether upon

the purely abstract, even in the world of mathesis, for any absolute criterion of knowledge. It is perhaps well that we should expunge the word *absolute* from our vocabularies. It is really a misnomer and has no meaning in the lexicon of nature. There is in reality no *absolute* in the sense of final absolution from all conditions or restrictions.

In the ultimate analysis there is unquestionably no hue, tone, quality, condition nor any imaginable posture of life, being or manifestation that is absolved from every other one of its class or from the totality. All these are relational and interdependent. There is no room for the absolute. In fact, it is a quality which cannot in any way be ascribed to any aspect of kosmic manifestation. It has existence only in the mind and has been devised for the purpose of marking the limits of its scope. All being is relative; all life is relative and is destined to change its qualities as it evolves. All knowledge is also relative and what is true of one state may not be true of another; what is true of one life may not be true of another life; the limitations of one degree of knowledge may not have any bearings upon another degree. The norms of one will not satisfy the conditions of another stage of manifestation. It is always within limits that the criterion of knowledge will be found to satisfy a given set of conditions. Hence within certain limits mathematical conclusions will maintain their validity. Error is committed by pushing the validity of these limits to a position without the sphere of limitations. This seems to be the crux of the whole matter. Mathematicians, notably non-Euclidean, have sought to extend the comparatively small sphere of limits of con-

gruence between mathematic and perceptual space to such an extent as to cause it to encroach upon forbidden territory. In doing this they have erred grievously, causing serious offense to the more sensitive spirit of the high-caste mathematicians among whom are none more truly conservative than PAUL CARUS,² who says:

“Metageometricians are a hot-headed race and display sometimes all the characteristics of sectarian fanatics. To them it is quite clear there may be two straight lines through one and the same point which do not coincide and yet are both parallel to a third line.”

To the student who has carefully followed the development of the non-Euclidean geometry and the notion of hyperspace the above characterization is none too severe nor ill-deserved. Nothing could more vividly yet correctly portray the impious tactics of the metageometrician and establish his perceptual obliquity more surely than the mere fact, mentioned by CARUS, that he can with evident lack of mental perturbation proclaim that two straight lines, noncoincident with each other, may pass through a point and yet be parallel to a third line. But this is a mere trifle, a bagatelle, to the many other infractions of which he is guilty. The wonder is that he is able to secure such obsequious acceptance of his offerings as many of the most serious minded mathematicians are inclined to give. Is it to be wondered at that, despite the profuse protestations of the advocates, many who take up the study of the question of hyperspace should

² Vide *Monist*, Vol. XIX, p. 402 (1909).

experience a deep revulsion from the posture assumed by metageometricians with respect to these queries?

Linked with the idea of dimensionality is the notion that space is infinite. This is a conception which has its roots imbedded in the depths of antiquity. Primitive man, looking up into the heavens at what appeared to him as a never ending extension, was awed by its vastness; but the minds of the most learned of the present-day men are not free from this innate dread of infinity. It permeates the thought life of all alike and none seems to be able to rise above it. Mathematicians, philosophers, scientists all share in the general belief that space is without limit, unending in extent and eternally existent. RIEMANN, whose thought life found its most convenient mode of expression by means of pure mathematics, was the first in the history of human thought to surmise that space is not infinite but limited even though unbounded. But his conception has been much vitiated on account of its entanglement with an *idealized* construction by which space is regarded as a thing to be manipulated and generated by act of thought. Were it not for this his conception would indeed mark the beginning of a new era in psychogenesis. As it is, when all the nonsensical effusions have been cleared away from our space conceptions and men come really to understand something of the essential nature of space this new era will find its true beginnings in the mind of RIEMANN. Although it must be said, as is the case with all progressive movements, the later development of a rationale for this conclusion will vary greatly from his original conception. For he had in mind a space that is generable and therefore a logical construction while ultimately

the mind will swing back to a consideration of real space.

Already men are beginning to see a new light. Already they are beginning to take a new view of space in general. The departure is especially noticeable in the attitude assumed by HIRAM M. STANLEY.* He says:

"If we seek the most satisfactory understanding of space we shall look neither to mathematics nor Psychology but to Physics. The trend of Physics, say with such a representative as OSTWALD, is to make things the expression of force; the constitution and appearance of things are determined by dynamism; and we may best interpret space as a mode of this dynamic appearance."

Space, as a mode of dynamic appearance is a slight improvement upon the old idea of a pure vacuity; for in the light of what we now know about space content much of the dignity of that view has been lost. Men now know that space is not an empty void. They know that the atmosphere fills a great deal of space. They also have extended their conception in this direction to include the ether and occultism goes further and postulates four kinds of ether—the chemical, life, light and psychographic ethers. But it does not stop here. It postulates a series of grades of finer matter than the physical which fills space and permeates its entire extent even to identification with its essential nature.

STANLEY continues:

"Everything does not, as commonly conceived, fall into some pre-existent space convenient for it;

* *Philosophical Review*, Vol. VII (1898).

but everything makes its own spaciousness by its own defensive and offensive force, and the totality of all appearance is space in general."

According to STANLEY, not only do physical, perceptual objects, by their "offensive and defensive force" make their own space but the appearance of that in which no physical object is makes room for itself by its own dynamic force. In other words, that which we call "pure extensity" is by virtue of its dynamism the cause of its own existence.

At first hand there appears to be little worthy of serious consideration in this view of STANLEY; yet, if carried to its logical conclusion, the merit of the hypothesis becomes apparent. Accordingly, interstellar distances which are commonly said to be even without air or life of any kind are really an appearance possessed of a dynamism peculiar to itself. And this very force-appearance, constituting space, is that which makes it perceivable. For instance, let us say the space that exists between the earth and the moon, is not really empty nor does it have an existence prior to itself, but is a mode of dynamic appearance which is the cause of its own existence. Its dynamic character makes it to appear perceptible to our senses. Logically, if the dynamism were removed there would remain neither space nor the appearance of space. If this were true, and it is worthy of serious thought, then space is certainly finite, as in its totality, according to STANLEY's view, it would have to be regarded as a "phenomenon of the inner and finite life of the infinite."

It is believed that we may go a step further and unqualifiedly assert that *space is finite*, even denying its infinity as a "general mode of the activity of the

whole." Yet it is transfinite in the sense that it transcends the comprehension of finite minds or processes. It is *finite* because it is in *manifestation*. Everything that is in manifestation is finite. The infinite is not in manifestation. Infinity has to be limited always to become manifest. The Deity has limited His being in order that there may be a manifested universe. All things, all appearances are finite; because they are phenomena connected with manifestation.

This question may be viewed from another standpoint. All things in manifestation or existence are polar in their constitution. For instance: there cannot be a "here" without a "there." There cannot be an "upper" without a "lower." Right is copolar with wrong; good is copolar with evil; night with day; manifestation with non-manifestation; truth with falsity; infinity with finity and so on, throughout the whole gamut of the pairs of opposites. What is the logical inference? Space is paired with a lack of space. There cannot be what we call *space* without there being at the same time the possibility, at least, of the *lack of space or spacelessness*. This is a conclusion that is rigorously logical and incontrovertible.

But it has been urged that it is impossible for the mind to imagine a condition where there is no space. It even has been asserted that it is contrary to the constitution of the mind itself to imagine "no space." But whether imaginable or not has no effect whatever upon the validity of the conception. Neither, it is said, can we imagine a fourth dimension but the mind has come dangerously near to imagining it. The distance from excogitating upon, discussing and describing the properties of four-space to imagining it is not so great

after all. Truly it is difficult indeed, it seems, to be able to describe a thing yet not be able to imagine or make a mental image of it. There is an evident fallacy here. Either the description of four-space is no description at all or it is a true delineation of an idealized construction which is well within the mind's powers of imagination. Indeed the question of imaginability is not determinative in itself; for what the mind may now be unable to imagine, because of its more or less nebulous character, and owing to its infancy may in the course of time be easily accomplished.

The universe is a compacted *plenum*. It is chock-full of mind, of life, of energy and matter. These four are basically one. They exist, of course, in varying degrees of tenuity and intensity and answer to a wide range of vibrations. Together, in their manifestation of action and interaction, in their *dynamic appearance*, if you please, they constitute space. If these were removed with all that their existence implies there would result a condition of spacelessness in which no one of the appearances which we now perceive would be possible. Even sheer extensity would be nonexistent. All scope of motility would be lacking. Dimension, coördinates, direction, space-relations—all would be impossible.

A straight line is an ideal construction of the mind. It does not exist in nature. It can never be actualized in the phenomenal universe. Between the ideal and the real, or actual, there is a kosmic chasm. It broadens or narrows according as the phenomenal appearance approaches or recedes from the ideal. What, therefore, can be postulated of the one will not apply with equal force to the other. They are not congru-

ent and can never be in the actualized universe. The moment the actual becomes identified with the ideal it ceases to be the actual. The universe does not exist as *pure form*, neither does space. As purely formal constructions of the intellect these can have no perceptible existence. The phenomenal or sensible may not be judged by exactly the same standard as the formal. The phenomenal or sensible represents things as they appear to the senses, or, so far as the actualized universe is concerned, *as they really are*. The *formal* represents things as they are made to appear by the mind. It cannot be actualized. It may be said that the purely formal is the limit of evolution. The phenomenal may approach the ideal as a limit, but can never become fully congruent with it. *The difference between the ideal and the actual is a dynamic one*; it is by virtue of this difference that the universe is held in manifestation. Evolution is the decrement of this difference between the purely formal and the actual. So long then as a kosmic differential is maintained the phenomenal continues to be manifest: when it is finally reduced to nothing it goes out of manifestation. The phenomenal is finite; the ideal infinite.

Wherefore, it is undoubtedly improper to refer to space as being infinite. The term really is inapplicable. Transfinity is much better and more accurate. Space is transfinite because its scope is greater than any finite scope of motility can encompass, because it exceeds finite comprehensibility.

RIEMANN'S notion that space is limited gains weight in the light of the foregoing considerations. But he could not conceive of the limitability and unboundedness of space as such in its pure essence; but

was compelled, by his own limitations, to make an idealized construction in which he could actualize his conception. And for real, dynamic space, he substituted his ideal construction and proceeded upon that basis. And of course, his view while it had no reference to perceptual space nevertheless possessed an illative relation thereto and should be recognized as construable in that light.

The process of squaring the circle recognized as a geometric impossibility is significant of the fluxional nature of the universal residuum perpetually maintained between the archetypal and the manifested kosmos. It seems that there is a profound truth embodied in this problem. There is a lesson that may be learned by mathematicians, philosophers, scientists and thinkers in general. There is an element of eternal necessity and universality about it which is truly symbolic of the finity of the universe and the infinity of the archetypal. Just as a square or a series of polygonal figures inscribed in a circle cannot be made to coincide exactly with the circle so cannot the actual be made to coincide with the ideal. The circumference of the circle is the unapproachable limit of inscribed squares. If it were possible so to multiply squares thus inscribed that a figure coincident with the circumference of a circle might be constructed, such a figure would not be a square but a circle. The manifested universe is like that—the process of inscribing squares within a circle. It is ever *becoming, evolving, developing*, but never quite attains. Infinity is a process. But no single stage in that process is infinite. Each is finite and their totality makes the infinity of the process. The universe manifested to the senses or the intellect is finite.

"Space," says PAUL CARUS, "is the possibility of motion in all directions." ⁴ To be sure, it is admitted that space offers opportunity for motion in all directions. But is space this opportunity of motility? Or is possibility of motion space? The possibility of motion must rest in the thing that moves. It implies a potency in the moving entity, not in space. If it is meant that space is the potency that resides in the moving element it is still more difficult to understand the connotation. But even granting this view, are we not compelled to recognize the dynamism of space as a necessary inference? Another definition which CARUS gives is that space is a "*pure form of extension*." If it be granted that space is a pure form of extension we should have to conclude that it has no actual existence; for *pure form* does not exist except as an idealized construction. It cannot be found in nature. Pure form is *ideal*. Impure or natural form is actual. Therefore the space in which we live and in which the universe exists cannot be a "pure form" because life cannot exist in the purely formal. It is useless to talk about space as mere form so long as it maintains life. The difficulty which this phase of the question presents is another evidence of the inadequacy of our definitions.

It is also found to be impossible to concur in CARUS' conception of knowledge *a priori*. His notion of the *a priori* varies somewhat from the Kantian view. He defines it as an "idealized construction," the "mind made," "abstract thought," and places it in the same category as a concept. This is undoubtedly born of his desire to get rid of KANT's "innate ideas" which

⁴ Vide *Foundations of Mathematics*, p. 107.

seem to be distasteful to him. But in doing so it appears that the real *a priori* has been overlooked. Let us examine for a moment this important question. The *a posteriori* connotes all knowledge gained through the senses, or sense experience. All knowledge therefore whose origin can be traced to the senses is knowledge *a posteriori*. Now, knowledge *a priori* should be just the opposite of this. It should indicate such knowledge as that which does not have its origin in the senses, or which is not dependent upon the ordinary avenues of sense-experience. Abstract thought is as truly experience as smelling, seeing or hearing. It is by traversing its scope of motility that the mind finds out what the norms of logic are. It could not remain quiescent and discover them. It has to be active, examining, comparing and judging. Almost the entire range of thought, its entire scope, is characterized by the *a posterioristic method*. In fact, all thought is *a posterioristic*. Despite the fact that, in thinking in the abstract, it is necessary mentally to remove all elements of concreteness, all materiality and all actuality, the conclusions reached have to be referred to the standards maintained by the actual, the concrete and the material. Then we do not start with the abstract in our thinking. We begin with the concrete and by mentally removing all physical qualities arrive at the abstract.

The mind has a constitution. It acts in a given way because it is its nature so to act. Not because it has learned to act in that manner. It performs certain functions intuitively without previous instruction or experience for the same reason that water dampens or heat warms. It is natural for it to do so.

This naturalness, this performance of function without being taught or without experience constitute the principle of *apriority* in the mind. *Aprioriness* is a principle of mind partaking of the very nature and essence of mind. It is the very mainspring of mentality. Perception and conception are processes which the mind performs intuitively. The mind perceives and conceives because it is impossible for the normal mind to do otherwise. We take a view upon a given question; we assume certain mental attitudes of affirmation, negation or indifference because we have learned to do so by virtue of the tuitional capability of mind. These describe the *a posteriori*. That is, all knowledge obtained as a result of voluntary mental processes constitutes the mass of knowledge *a posteriori*. The *a priori* is what the mind is by nature: the *a posteriori* is what the mind becomes. It is the mind-content.

The *a priori* is not a mental construction; it is an essential principle of mind. It should not be identified with the "purely formal," as is done by PAUL CARUS:⁵

He says:

"The *a priori* is identical with the purely formal which originates in our mind by abstraction. When we limit our attention to the purely relational, dropping all other features out of sight, we produce a field of abstraction in which we can construct purely formal combination, such as numbers, or the ideas of types and species. Thus we create a world of pure thought which has the advantage of being applicable to any purely formal consideration and we

⁵ Vide *Foundations of Mathematics*, p. 42.

work out systems of numbers which, when counting, we can use as standards of reference for our experience in practical life."

Thus CARUS definitely links up the *a priori* to a factor which is nothing more nor less than a mental by-product. For such is the category in which would be placed both the process of abstraction and its results. It is therefore exceedingly difficult to understand why so cursory a consideration should have been given to the principle of *apriority* than which no other element of mind is more essentially a part of the mind itself.

The formal is symbolic. It signifies an informing quantity. Pure form itself is but a negation of that which formerly filled it. Then, too, the formal is purely artificial because it is a mental construction. Essentially there is as much difference between the purely formal and the *a priori* as between creator and creature, as between potter and clay. The one is the builder, the other is the material; the one the knower and the other the known. Thus, the only reason that the formal is found to be answerable to the *a priori* at all is due to the fact that it is construable only upon the basis of the *a priori*. But being so is not sufficient warrant for its identification with the *a priori*. The formal merely represents the totality of possibilities in the universe as viewed by the mind; but as the number of possibilities open to the mind is, on account of its nature and purpose limited, it is not to be supposed that it (the mind) shall measure up to all the possibilities offered by the formal. Moreover, it is certain that no sane mind cherishes the hope that there shall ever be found in the universe of life and form a

congruence for all of the possibilities held out by the purely formal.

As an eternal principle of mind, the *a priori* is in agreement with the divine mind of the kosmos. In its *aposteriority* the mind is of diverse tendencies, qualities and characteristics. Apriorily, it acts in unison with the eternal purpose of life and the universal mind. In its *aposteriority*, it often goes awry. In its *apriority* it can never be insane; insanity is a symptom of the morbid *a posteriori*.

The mind in man acts the same as mind in the vegetal and lower animal kingdoms. Metabolism and katabolism, indeed all cell-activity, are *a priori* performances of the mind. Growth and all its phenomena, the cyclicism of natural processes, and every activity connected therewith belong to the category of the *a priori*. Cells multiply, divide, build up and tear down tissues and they do it intuitively. Most certainly these functions are performed without any assistance from the intellect. All the myriad activities in nature with which the intellect in man has not the slightest concern, truly acting in accord with some primordial impetus, are activities *a priori*.

Now what is the attitude of the intellect, in the light of the *a priori*, towards space and the question of dimensionality? It is evident that no matter what this attitude may be it is in agreement with the constitution of things and of the universe. And if so, it is right and without illusion. It is also evident that whatever notion *a posteriori* the intellect may entertain with respect to these questions is unavoidably liable to the illusionary drawbacks common to conclusions based upon limited experience. The geometric

view of space belongs to the category of the *a posteriori*. Hence it is subject to the usual imposition of error.

Tersely stated, KANT's view of space is that it is a form of intuition, a form *a priori*, a transcendental form. As such he considered it to be a native form of perception not belonging to the category of sense-deliveries. Accordingly, space is a form of intuition arising out of and inhering in the constitution of mind. It is a notion which constitutes the universal and eternal prerequisite of mind and is, therefore, intrinsically necessary to all phases of mentation. Now, this being true just what may be said to be the relation of dimensionality to this *a priori* form of space which is found to exist in the mind as an eternal aspect of its nature? Does the mind intuitively measure its contents or its operations by the empirical standard of space-measurement known as dimension? Is the attitude of the mind towards the objectively real one of discrimination *a priori* as to the direction or dimension in which a percept may originate? In other words, does the mind habitually and intuitively refer its data to a system of coördinates for final determination? There is no other answer but that the mind makes no such reference and is dependent upon no kind of coördinate system in any of its operations *a priori*. As a form of intuition, the space notion is present in the mind as a scope of existence, of motility, of being and of sheer roominess. The notion of direction or dimension, being an artificial construction, does not enter into this form of intuition at all. It is only when the mind comes to elaborate upon its perceptive performances and possibilities that the questions of relations, posi-

tions and directions arise. But this latter is a matter separate and distinct from the state of awareness which embodies the notion of space.

Dimension is an arbitrary norm constructed by the mind for the determination of various positions in space. It is an accident or by-product of the process of elaborative cognition, a convenient and appropriate means of measurement for objects in space and their space-relations. But it is no more *a priori* than a foot rule or a square. But being purely an empirical product it may be said to be an aspect of psychogenesis because it relates to the evolutionary aspect of mind. The assumption may therefore be allowed that the mind may, in the course of its evolution, find it convenient and appropriate to devise an additional ordinate or dimension to satisfy the necessities of its more complex ramifications into the nature of things and to determine their greatly increased space-relations. It may be even possible for the mind to function normally in a space of four dimensions. But this would simply be a new adjustment, not a change in the essential nature of mind. It would be like the series of adjustments to environments which man has made in the onward movement of civilization. There has been no serious change in the manhood *per se* of man. That has remained the same; there has been merely a complication of environmental influences. Similarly, in the acquisition of four-dimensional powers, granting that such an acquisition is possible, there is nothing to be added to the *aprioriness* of mind *itself*. Is it not, therefore, logical to assume that the discovery of a fourth coördinate and the consequent conceptualization of the same, point to the development in the mind

of a greatly extended faculty, more keenly penetrative powers of cognition and a further diversification of its environments than it has hitherto enjoyed? Indeed, it seems so.

CHAPTER V

THE FOURTH DIMENSION

The Ideal and the Representative Nature of Objects in the Sensible World—The Psychic Fluxional the Basis of Mental Differences—Natural and Artificial Symbols—Use of Analogies to Prove the Existence of a Fourth Dimension—The Generation of a Hypercube or Tesseract—Possibilities in the World of the Fourth Dimension—Some Logical Difficulties Inhering in the Four-Space Conception—The Fallacy of the Plane-Rotation Hypothesis—C. H. HINTON and Major ELLIS on the Fourth Dimension.

THE world of mathesis is truly a marvelous domain. Vast are its possibilities and vaster still its sweep of conceivability. It is the kingdom of the mind where, in regal freedom, it may perform feats which it is impossible to actualize in the phenomenal universe. In fact, there is no necessity to consider the limitations imposed by the actualities of the sensuous world. Logic is the architect of this region, and for it there is no limit to the admissibility of hypotheses. These may be multiplied at will, and legitimately so. The chief error lies in the attempt to make them appear as actual facts of the physical world.

Mathematicians, speculating upon the possibilities of mathetic constructions and forgetting the necessary distinctions which should be recognized as differentiating the two worlds, in their enthusiasm have been

led into the error of postulating as qualities of the phenomenal world the characteristics of the conceptual. Accordingly, a great deal of confusion as to the proper limits and restrictions of these conceptions has arisen and there still may be found those who are enthusiastically endeavoring to push the actualities of the physical over into the conceptual. But in assuming any attitude towards mathetic propositions, especially with a view to demonstrating their actuality, very careful discrimination as to the essential qualities and their connotations should be made. Hence, before taking up a brief study of the fourth dimension proper, it is deemed fitting to indicate some of the fundamental distinctions which every student of these questions should be able to make with reference to the data which he meets.

All objects of the sensible world have both an essential or ideal nature and a representative or sensuous nature. That is, they may be studied from the standpoint of the ideal as well as the sensuous. The representative nature is that which we recognize as the mode of appearance to our senses which, as KANT held, is not the essential or ideal character of the thing itself. For there is quite as much difference between the sensuous percept and the real thing itself as between an object and its shadow. In fact, a concept viewed in this light, may be seen to have all the characteristics of an ordinary shadow; for instance, the shadow of a tree. View it as the sun is rising; it will then be seen to appear very much elongated, becoming less in length and more distinct in outline as the sun rises to a position directly overhead. The elongation may again be seen when the sun is setting. Throughout the

day as the sun assumes different angles with reference to the tree the proportions and definiteness of the shadow vary accordingly. Thus the angularity of the sun, the intensity and fullness of the light and the shape and size of the tree operate to determine the character of the shadow.

Much the same thing is true of a sensuous representation. If we examine carefully our ideas of geometric quantities and magnitudes, it will be found that the concepts themselves are not identical with the objects of the physical world, but mere mental shadows of them. The angularity of consciousness, or the distinctness of one's state of awareness, being analogous to similar attitudes in the solar influence are the main determinants of the character of the mental shadow or concept. Wherefore mathematical "spaces" or magnitudes are not sensuous things and have therefore no more real existence than a shadow, and strictly speaking not as much; for a shadow may be seen, while such magnitudes can only be conceived. It may be urged that since we can conceive of such things they must have existence of some kind. And so they have, but it is an existence of a different kind from that which we recognize as belonging to things in the sensible world. They have a conceptual existence, but not a sensuous one. Therein lies the great difference.

To be sure, a shadow is a more or less true representation of the thing to which it pertains. That this is true can be established empirically. Similarly, the degree of congruity between objects and concepts likewise may be determined. If this were not true we should be very much disappointed with what we find in the phenomenal world and could never be

quite sure that the mentograph existing in our minds was a faithful representation of the thing which we might be examining. But really the foundation for such a disappointment is present in every concept, every percept with which the mind deals. This disappointment, although in actual experience is reduced to an almost negligible quantity, is due to the failure of sensuous objects to conform wholly to the specific details of the mental shadow or mentograph. This lack of congruence between the mental picture and the object itself is necessary for obvious reasons. It is markedly observable in the early efforts of a child in learning distances, weights, resistances, temperatures and the like. No inconsiderable time is required for the child to be able correctly to harmonize his sense-deliveries with actual conditions. Otherwise, the child would never make any of the ludicrous mistakes of judgment of which it is guilty when trying to get its bearings in the world of the senses. In the course of time the child gradually learns by experience that certain things are true of objects, distances, temperatures, resistances, etc., and that certain things are not true of them. He learns these things by actually contacting various objects. He is then competent to render correct judgments, within certain limits, as to the conditions which he finds in the sensible world. And the allowances, equations and corrections which his motor, sensory and psychic mechanisms learn to make in childhood serve for all subsequent time. And this is important to remember; for the mature mind is apt to forget or overlook the adaptations which the child-mind has made in its growth.

If there were no such differences between the con-

cept and the thing itself, actual physical contact would not be necessary. For one could rely wholly upon the sense-deliveries and each sense might operate entirely independently of all the others as there would be no necessity to correct the delivery of one by those of the others. This, of course, raises the question as to the necessity of sense-experience at all under conditions where there would be no disparity between the thing itself and the ideal representation of it in the mind. The absence of this variable quantity would open to the mind the possibility of really knowing the essential nature of objects in the phenomenal world, a condition of affairs which is admittedly now without the range of the powers of the mind.

At any rate, the essential "thingness" of objects can never be comprehended by the mind until the diminution of this disparity between the object of sense and the mental picture of it which exists in the consciousness has proceeded to such a limit as either completely to have obliterated it or to such an extent that the psychic fluxion is so slight as not to matter.

It is believed that the results of mental evolution, as the mind approaches the transfinite as a limit, will operate to minimize the fluxional quantity which subsists between all objects of sense and their ideal representation as data of consciousness. The conclusion that the mind of early men who lived hundreds of thousands and perhaps millions of years ago on this planet consumed a much longer time in learning the adjustments between the objects which it contacted in the sensuous world and the elementary representations which were registered in its youthful consciousness than is to-day required for similar processes seems to

be demanded, and substantiated as well, by what is known of the phyletic development of the mind in the human race.

In view of the above, it is thought that the duration of such simple mental processes served not only to prolong the physical life of the man of those early days, but may also account for the puerility and incapacity of the mind at that stage. Not that the slow mental processes were active causative agencies in lengthening the life of man, but that they together with the crass physicality of man necessitated a longer physical life. This, perhaps in a larger sense than any other consideration, accounts for the fundamental discrepancies in the mind of the primitive man in comparison with the efficiency of the mind of the present-day man. In view of the potential character of mind and in the light of the well graduated scale of its accomplishments, it is undoubtedly safe to conclude that the quality of mental capacities is proportional to the psychic fluxional which may exist at any time between the ideal and the essential or real. Mental differences and potentialities in general may be due to the magnitude of the psychic fluxional or differential that exists between the conceptual and the perceptual universe. In some minds it may be greater than in others. The chasm between things-in-themselves and the mental notion pertaining thereto may vary in a direct ratio to the individual mind's place in psychogenesis, and therefore, be the key to all mental differences in this respect.

Most certain it is that there may be marked fluctuations in the judicial approach of minds towards any psychic end. In other words, there is not only a fluxional or differential between the object and its rep-

resentation, but also a differential between the approach of one mind and another in the judicial determination of notions concerning ideas. In this way, differences of opinions as to the right and wrong of judgments arise. Indeed, there seem to be zones of affinity for minds of similar characteristics, or minds that have the same degree of differential; so that, in choosing among the many possible judgments predicable upon a species of data, all those minds having the same degree of psychic differential discover a special affinity or agreement among themselves. Hence, we have cults, schools of thought, and various other sectional bodies that find a basis of agreement for their operations in this way. The outcome of this remarkable intellectual phenomenon is that there are as many different kinds of judgments as there are zones of affinity among minds. Various systems of philosophy owe their existence to these considerations, and the considerations themselves flow from the fact that all intellectual operations are essentially superficial; because there is no means by which they may penetrate to the steady flowing stream of reality which pervades and sustains objects in the sensible world.

In view, therefore, of the foregoing and with special reference to geometric constructions, it is necessary in approaching a study of the four-space that it be understood at the outset that the fourth dimension can neither be actualized nor made objectively possible even in the slightest degree in the perceptual world; because it belongs to the world of pure thought and exists there as an "extra personal affair," separate and distinct from the world of the senses.

As says SIMON NEWCOMB :¹

"The experience of the race and all the refinements of modern science may be regarded as showing quite conclusively that, within the limits of our experience, there is no motion of material masses, in the direction of a fourth dimension, no physical agency which we can assume to have its origin in regions to which matter cannot move, when it has three degrees of freedom."

There is, however, no logical objection to the study of the fourth dimension as a purely hypothetical question, if by pursuit of the same an improvement of methods of research and of the outlook upon the field of the actual may be gained. Hence, it is with this attitude of mind that we approach the consideration of the fourth dimension.

Various efforts have been made to render the conception of a fourth dimension of space thinkable. The student of space has reasoned: "We say that there are three dimensions of space. Why should we stop here? May there not be spaces of four dimensions and more?" Or he has said: "If 'A' may represent the side of a square, A^2 its area, and A^3 the volume of a cube with edge equal to A; what may A^4 , A^5 or $A^{n\text{th}}$ represent in our space? The conclusion, with respect to the quantity A^4 , has been that it should represent a space of four dimensions.

Algebraic quantities, however, represent neither objects in space nor space qualities except in a purely conventional manner. All efforts to justify the objective existence of a fourth dimension based upon such

¹ Vide *Science*, Vol. VII, p. 2, No. 158, 1898.

reasoning will, therefore, fail; because the basis of such arguments is itself faulty. In the sentence: "The man loves his bottle," the thing meant is not the bottle, but what the bottle contains. For the purpose of the figure the bottle signifies its contents. There is no more real connection between the bottle and what it contains than between any word and the object for which it stands. Words are said to be symbols of ideas. But they are not natural symbols; they are conventional symbols, made for the purpose of cataloguing, indexing and systematizing our knowledge. Words can be divorced from ideas and objects, or rather have never had any real connection with them. There are two classes of natural symbols, namely; *objects* and *ideas*. These, objects and ideas, symbolize realities. Realities are imperceptible and incomprehensible to the intellect which has aptitude only for a slight comprehension of the symbols of realities. For instance, a tree is a natural symbol. It represents an actuality which is imperceptible to the intellect. The intellect can deal only with the sensible symbol. It is a natural symbol; because it is possible directly to trace a living connection between the tree and the *tree-reality*. That is, it would be possible so to trace out the vital connection between the tree and its reality if the intellect had aptitude for such tracery. But, in reality, since it has no such aptitude, it remains for the work of that higher faculty than the intellect which recognizes both the connection and the intellect's inability to trace it. Further, an object is called a natural symbol because it is the bridge between sensuous representation and reality. It is as if one could begin at the surface of an object and by a subtle proc-

ess of elimination and excortication arrive at the heart of the universum of reality. No such consummation may be reached by dealing with words which have merely an artificial relationship with the objects which they signify. Again, ideas, that is, ideas that are universal in application and have their roots in the great ocean of reality, are natural symbols; because if it were possible to handle an idea with the physical hands it would be possible to arrive at the heart of that which it symbolized without ever losing our connection with the idea itself. In other words, ideas and objects, unlike words, can never be divorced from that which they symbolize. Both, being of the same class, are the opposite poles of realities. This then is the difference between natural symbols and artificial symbols—that a natural symbol, such as objects and ideas, is copolar with reality whereas an artificial symbol, such as words, geometric constructions and the like not only lacks this copolarity but is itself a symbol of natural symbols.

It is, therefore, inconceivable that because the algebraic quantity A^3 has been arbitrarily decreed to be a representation of the volume of a cube, every such quantity in the algebraic series shall actually represent some object or set of objects in the physical world. Even if it be granted that such may be the case, is it not certain that there is a limit to things in the objective universe? Yet there may not be any limit to algebraic or mathematical determinations. The material universe is limited and conditioned; the world of mathesis is unlimited and unconditioned save by its own limitations and conditions. It is irrational

to expect that physical phenomena shall justify all mathematical predicates.

There is perhaps no single mathematical desideratum or consideration which may be said to be the natural symbolism of realities; for the whole of mathematical conclusions is a mass of artificial and arbitrary but concordant symbols of the crasser or nether pole of the antipodes of realism. It is exceedingly dangerous, therefore, to predicate upon such a far-fetched symbolism as mathematics furnishes anything purporting to deal with ultimate realities. And those who insist upon doing so are either blind themselves to these limitations or are madly endeavoring to befog the minds of others who are dependent upon them for leadership in questions of mathematical import.

Analogies have been unsparingly used in efforts to popularize the four-space conception and much of the violence which has been done to the notion is due to this vagary. The mathematical publicist, in trying to give a mental picture of the fourth dimension, examines the appearances of three dimensional beings as they might appear to a two dimensional being or *duodim*. He imagines a race of beings endowed with all the human faculties except that they live in a land of but two dimensions—length and breadth. He thinks of them as shadows of three dimensional beings to whom there are no such conceptions as “up” and “down.” They can see nothing nor sense anything in any way that is without their plane. They can move in any direction within the plane in which they live, but can have no idea of any movement that might carry them without that plane. A house for such beings might be simply a series of rectangles. One of them might

be as safe behind a line as a *tridim* or three dimensional being would be behind a stone wall. A bank safe for the *unodim* would be a mere circle. A *duodim* in any two dimensional prison might be rescued by a *tridim* without the opening of doors or the breaking of walls. An action of a *tridim* performed so as to contact their plane would be to them a miracle, absolutely unaccountable upon the basis of any known fact to the *unodim* or *duodim*. A *tridim* might go into a house where lived a family of *duodims*, appear and disappear without being detected or its ever being discovered how he accomplished such a marvelous feat. Our miracles, after the same fashion, are said to be the antics of some four dimensional being who has similar access to our three dimensional world and whose actions are similarly inexplicable to us. So the analogies have been multiplied. But the temptation to apply the consequences of such reasoning to actual three-space conditions has been so great that many have yielded to it and have consequently sought actually to explain physical phenomena upon the basis of the fourth dimension.

The utilitarian side of the question of hyperspace has not been neglected either. And so, early in the development of the hypothesis and its various connotations, the attention of investigators was turned to this aspect of the inquiry. Strange possibilities were revealed as a result. For instance, it was found that an expert fourth dimensional operator is possessed of extraordinary advantages over ordinary tridimensional beings. Operating from his mysterious hiding place in hyperspace, he could easily appear and disappear in so mysterious a manner that even the most

strongly sealed chests of treasures would be easily and entirely at his disposal. No city police, Scotland Yard detective nor gendarme could have any terrors for him. DRS. JEKYLL and MESSRS. HYDE might abound everywhere without fear of detection. Objects as well as persons might be made to pass into or out of closed rooms "without penetrating the walls," thus making escape easy for the imprisoned. No tridimensional state, condition or system of arrangements, accordingly, would be safe from the ravages of evilly inclined four dimensional entities. Objects that now are limited to a point or line rotation could in the fourth dimension rotate about a plane and thus further increase the perplexities of our engineering and mechanical problems; four lines could be erected perpendicular to each other whereas in three space only three such lines can be erected; the right hand could be maneuvered into the fourth dimension and be recovered as a left hand; the mysteries of growth, decay and death would find a satisfactory explanation on the basis of the fourth dimensional hypothesis and many, if not all, of the perplexing problems of physiology, chemistry, physics, astronomy, anthropology and psychology would yield up their mysteries to the skill of the fourth dimensional operator. Marvelous possibilities these and much to be desired! But the most remarkable thing about these so-called possibilities is their impossibility. It is this kind of erratic reasoning that has brought the conception of a fourth dimension into general disrepute with the popular mind. It is to be regretted, too, for the notion is a perfectly legitimate one in the domain of mathesis where it originated and rightly belongs.

It is not to be wondered at that metageometricians and others should at first surmise that, in the four-space, they had found the key to the deep mysteries of nature in all branches of inquiry. For so vast was the domain and so marvelous were the possibilities which the new movement revealed that it was to be expected that those who were privileged to get the first glimpses thereof would not be able to realize fully their significance. But the stound of their minds and the attendant magnification of the elements which they discovered were but incidents in the larger and more comprehensive process of adjustment to the great outstanding facts of psychogenesis which is only faintly foreshadowed in the so-called hyperdimensional. The whole scope of inquiry connected with hyperspace is not an end in itself. It is merely a means to an end. And that is the preparation of the human mind for the inborning of a new faculty and consequently more largely extended powers of cognition. Metageometrical discoveries are therefore the excrescences of a deeper, more significant world process of mental unfoldment. They belong to the matutinal phenomena incident to this new stage of mental evolution. All such investigations are but the preliminary exercises which give birth to new tendencies which are destined to flower forth into additional faculties and capacities. So that it is well that the evolutionary aspect of the question be not overlooked; for there is danger of this on account of the magnitude and kosmic importance of its scope of motility.

A geometric line is said to be a space of one dimension. A plane is a space of two dimensions and a cube, a space of three dimensions. In figure 7 below,

the line ab is said to be one dimensional; because only one coördinate is necessary to locate a point-position in it. The plane, $abcd$, figure 8, is said to be two dimensional because two coördinates, ab and db are required to locate a point, as the point b . The cube $abcdefgh$, figure 9, is said to be tridimensional, because, in order to locate the point b , for instance, it is necessary to have three coördinates, ab , bc and gb . The tesseract is said to be four dimensional, because, in order to locate the point b , in the tesseract, it is necessary to have four coördinates, ab , bc , bb' and $h'b$, figure 10.

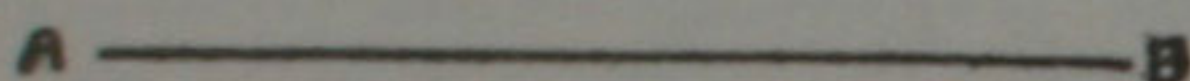


FIG. 7.

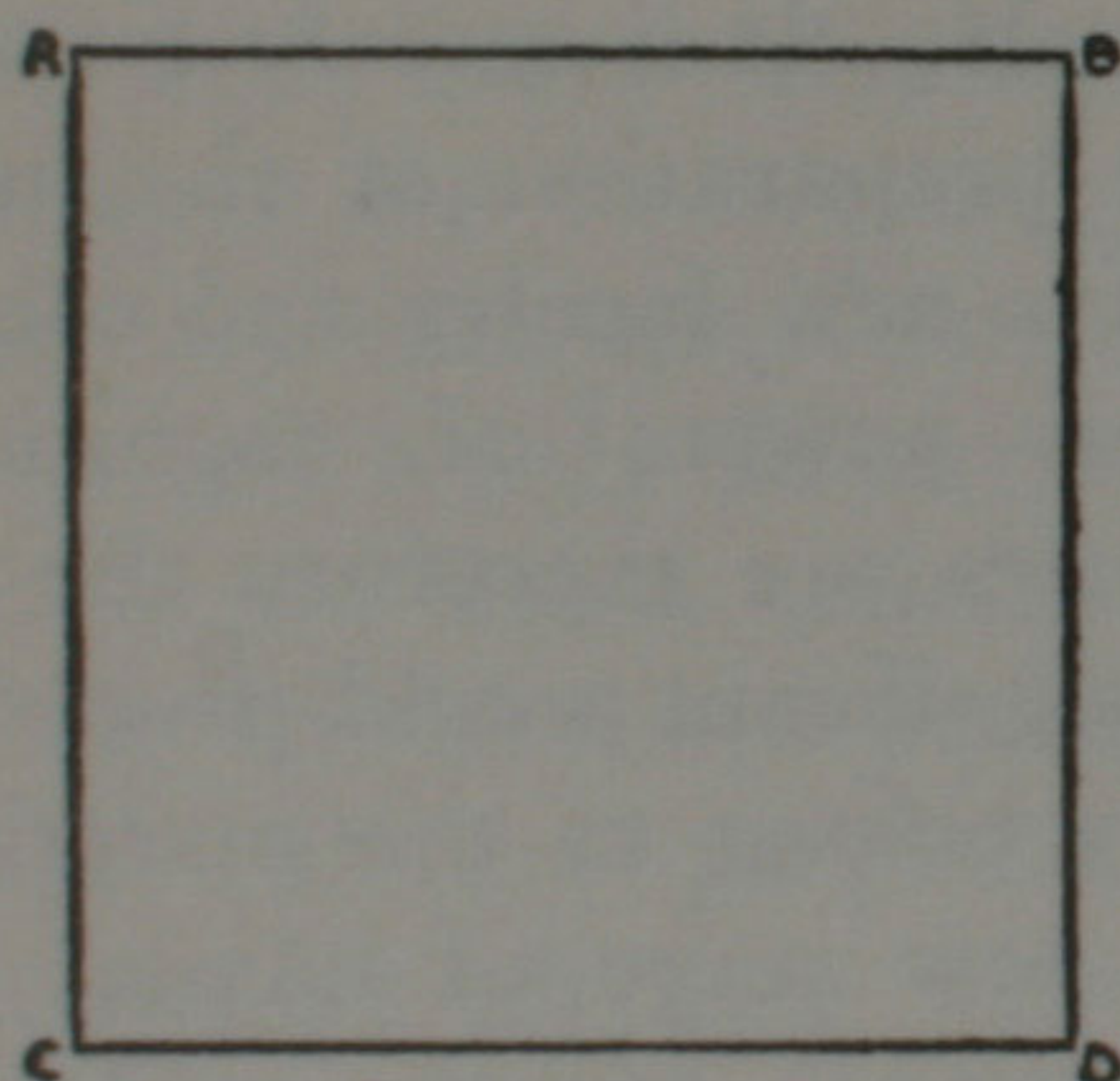


FIG. 8.

It will be noted that in figures 8, 9 and 10, the element of perpendicularity enters as a necessary determination. In figure 8, the lines ab and bd are perpendicular to each other. Similarly, in Fig. 10, lines ab , bc , bb' and $h'b$ are perpendicular to one another. That is, at their intersections, they make right angles. Similarly, figures representing any number of dimensions may be constructed.

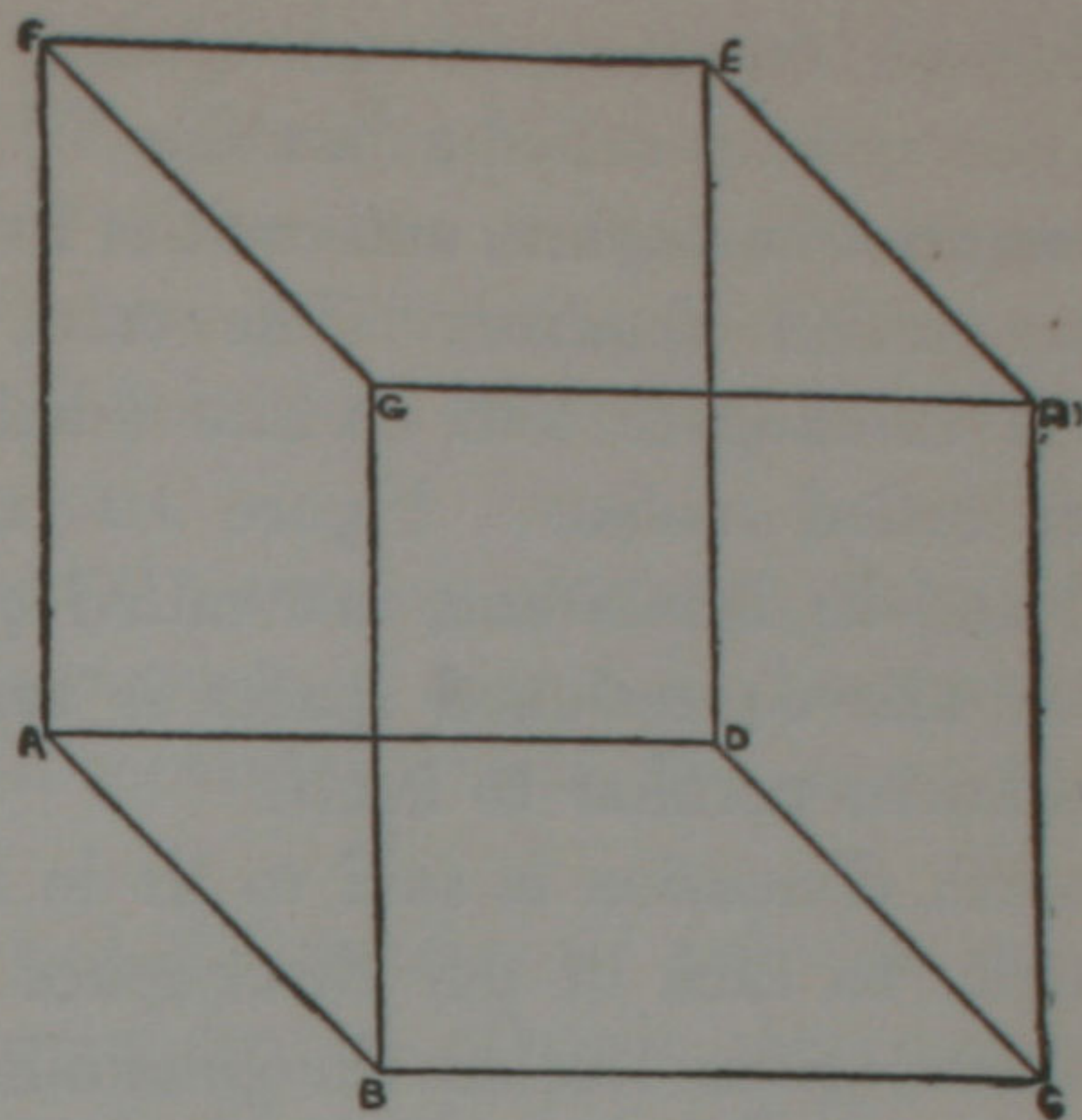


FIG. 9.

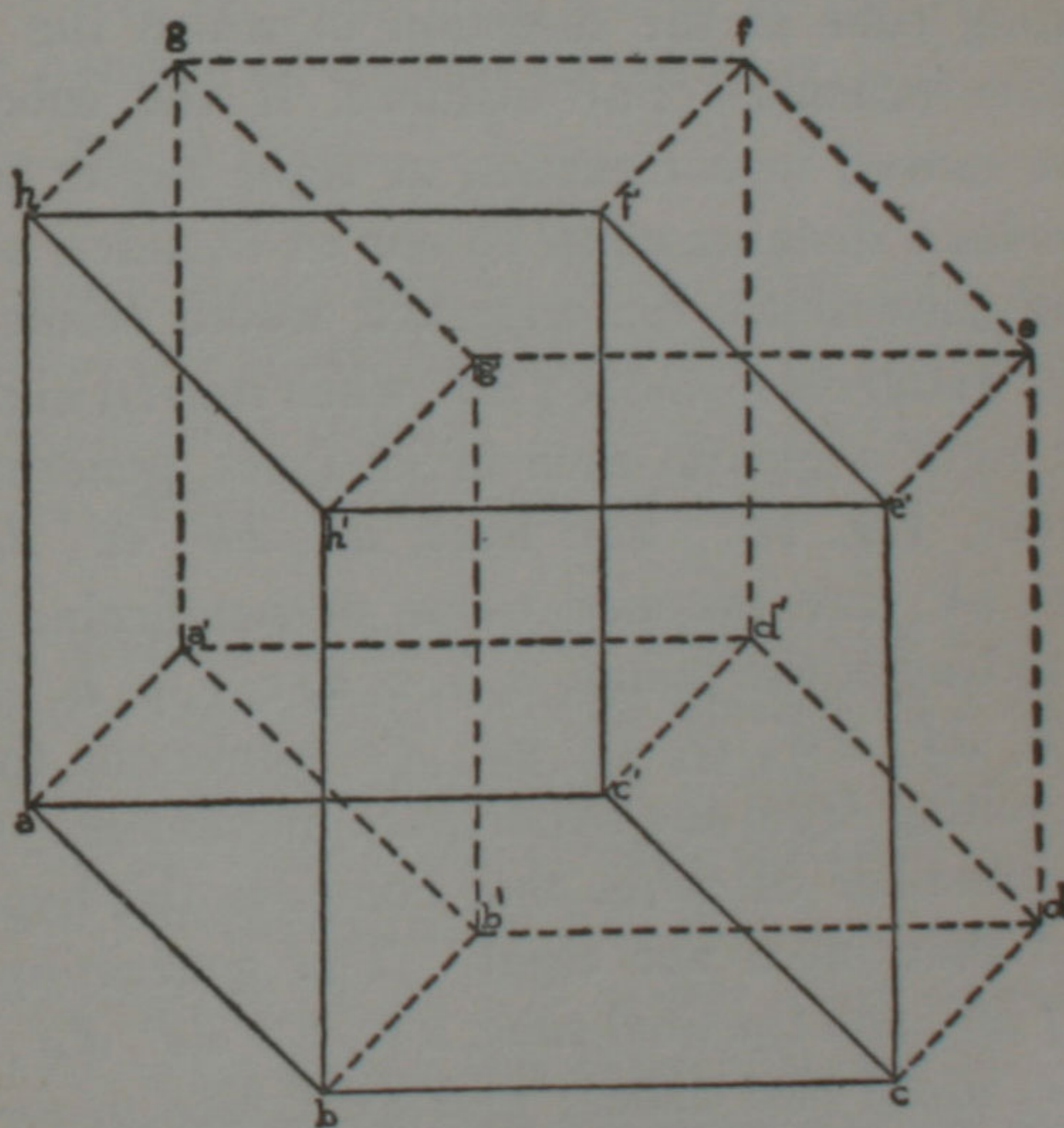


FIG. 10.—The Tesseract.

The line ab represents a one-space. An entity living in a one space is called a "unodim." The plane, $abcd$, represents a two-space, and entities living in such a space are called *duodims*. The cube, $abcdefgh$, represents a three-space and entities inhabiting such a space are called *tridims*. Figure 10 represents a four-space, and its inhabitants are called *quartodims*. Each of the above-mentioned spaces is said to have certain limitations peculiar to itself.

The fourth dimension is said to lie in a direction at right angles to each of our three-space directions. This, of course, gives rise to the possibility of generating a new kind of volume, the hypervolume. The hypercube or tesseract is described by moving the generating cube in the direction in which the fourth dimension extends. For instance, if the cube, Fig. 9, were moved in a direction at right angles to each of its sides a distance equal to one of its sides, a figure of four dimensions, the tesseract, would result.

The initial cube, $abcc'e'f'hh'$, when moved in a direction at right angles to each of its faces, generates the hypercube, Fig. 10. The lines, aa' , bb' , cc' , dd' , ee' , ff' , gg' , hh' , are assumed to be perpendicular to the lines meeting at the points, a , b , c , d , e , f , g , h . Hence $a'b'$, $b'd$, dd' , $d'a'$, ef , fg , gg' , $g'e$, represent the final cube resulting from the hyperspace movement. Counting the number of cubes that compose the hypercube we find that there are eight. The generating cube, $abcc'e'f'hh'$, and the final cube, $a'b'$, $b'd$, dd' , $d'a'$, ef , fg , gg' , $g'e$, make two cubes; and each face generates a cube making eight in all. A tesseract, therefore, is a figure bounded by eight cubes.

To find the different elements of a tesseract, the following rules will apply:

1. *To find the number of lines:* Multiply the number of lines in the generating cube by two, and add a line for each point or corner in it. E.g., $2 \times 12 = 24 + 8 = 32$.

2. *To find the number of planes, faces or squares:* Multiply the number of planes in the generating cube by 2 and add a plane for each line in it. E.g., $2 \times 6 + 12 = 24$.

3. *To find the number of cubes in a hypercube:* Multiply the number of cubes in the generating cube, one, by two and add a cube for each plane in it. E.g., $2 \times 1 + 6 = 8$.

4. *To find the number of points or corners:* Multiply the number of corners in the generating cube by 2. E.g., $2 \times 8 = 16$.

In a plane there may be three points each equally distant from one another. These may be joined, forming an equilateral triangle in which there are three vertices or points, three lines or sides and one surface.

In three-space there may be four points each equidistant from the others. At the vertices of a regular tetrahedron may be found such points. The tetrahedron has four points, one at each vertex, 6 lines and 4 equilateral triangles, as in Fig. 11.

In four-space, we have 5 points each equidistant from all the rest, giving the hypertetrahedron. This four dimensional figure may be generated by moving the tetrahedron in the direction of the fourth dimension, as in Fig. 12. If a plane be passed through each

of the six edges of the tetrahedron and the new vertex there will be six new planes or faces, making 10 in all, counting the original four. From the new vertex there is also a tetrahedron resting upon each base of the original tetrahedron so that there are five tetrahedra in all. *A hypertetrahedron is a four-dimensional figure consisting of five tetrahedra, ten faces, 10 lines and 5 points.*

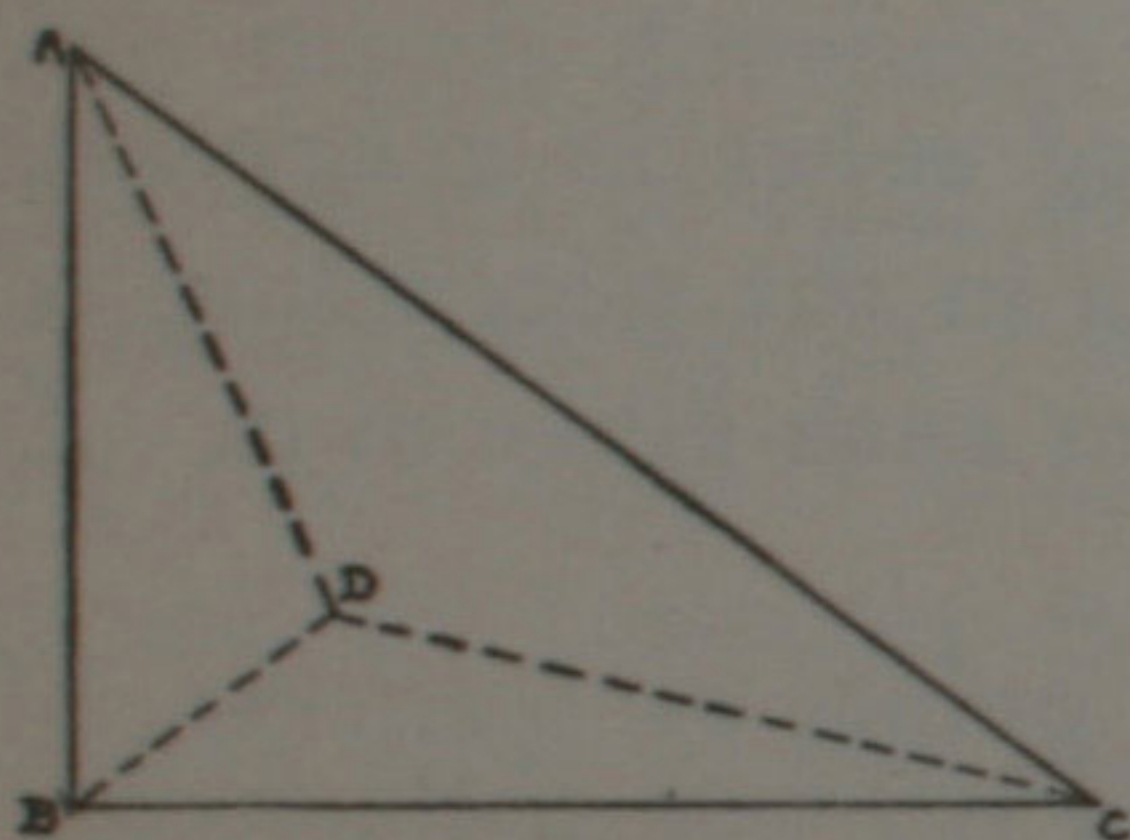


FIG. 11.—Tetrahedron.

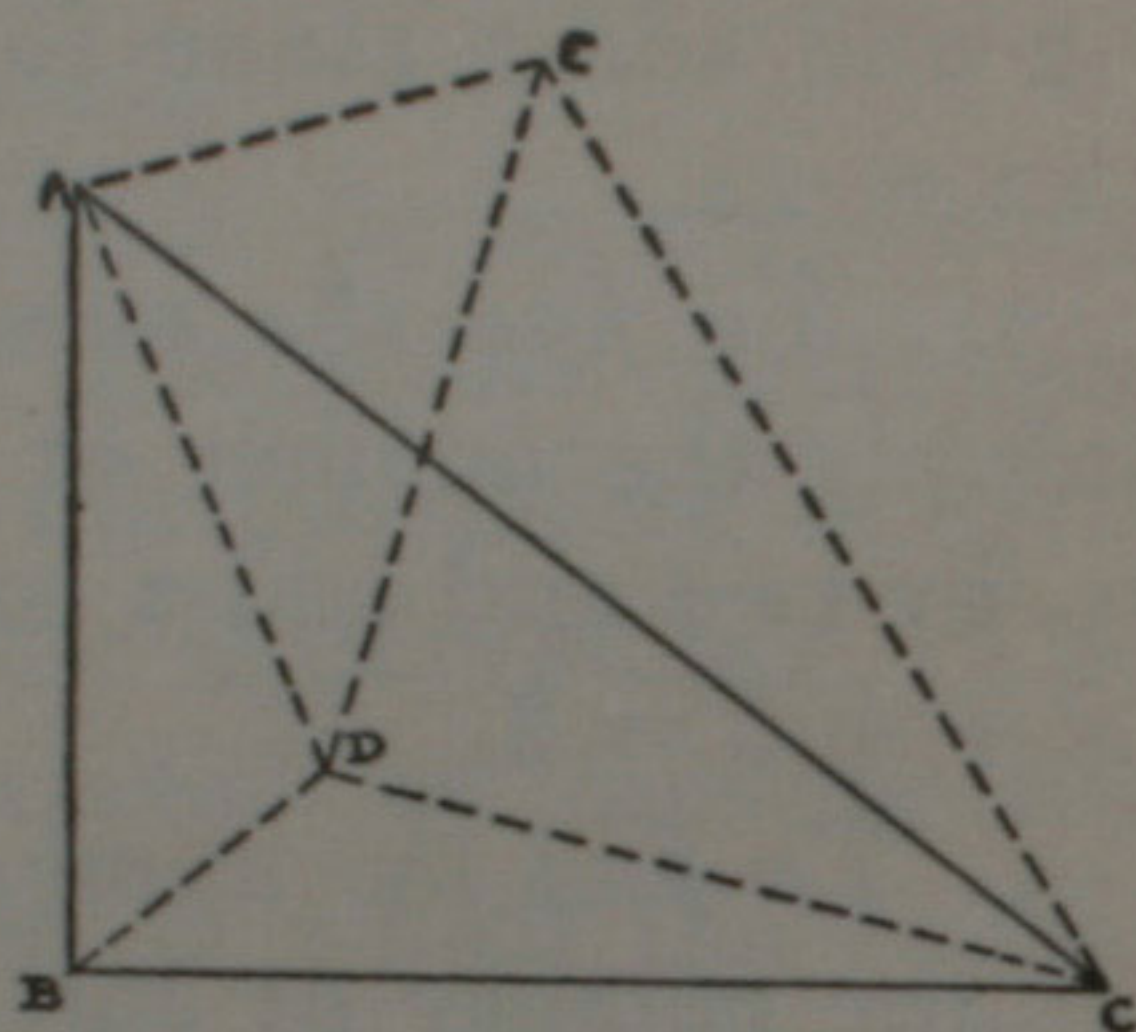


FIG. 12.—Hypertetrahedron.

PAUL CARUS² suggests the use of mirrors so arranged that they give eight representations of a cube when placed at their point of intersection. He says:

“If we build up three mirrors at right angles and place any object in the intersecting corner we shall see the object not once, but eight times. The body is reflected below and the object thus doubled is mirrored not only on both upright sides but in addition in the corner beyond, appearing in either of the upright mirrors coincidingly in the same place. Thus the total multiplication of our tridimensional boundaries of a four dimensional complex is rendered eight-fold.

“We must now bear in mind that this representation of a fourth dimension suffers from all the

² Vide *Foundations of Mathematics*, pp. 93-94.

faults of the analogous figure of a cube in two dimensional space. The several figures are not eight independent bodies but are mere boundaries and the four dimensional space is conditioned by their interrelation. It is that unrepresentable something which they inclose, or in other words, of which they are assumed to be boundaries. If we were four dimensional beings we could naturally and easily enter into the mirrored space and transfer tridimensional bodies or parts of them into those other objects reflected here in the mirrors representing the boundaries of the four dimensional object. While thus on the one hand the mirrored pictures would be as real as the original object, they would not take up the space of our three dimensions, and in this respect, our method of representing the fourth dimension by mirrors would be quite analogous to the cube pictured on a plane surface, for the space to which we (being limited to our tridimensional space-conception), would naturally relegate the seven additional mirrored images is unoccupied and if we should make the trial, we would find it empty."

The utility of such a representation as that which CARUS outlines in the above is granted, i.e., so far as the purpose which it serves in giving a general idea of what a four-space object might be imagined to be like, but the illustration does not demonstrate the existence of a fourth dimension. It only shows what might be if there were a four-space in which objects could exist and be examined. We, of course, have no right to assume that because it can be shown by analogous reasoning that certain characteristics of the fourth dimensional object can be represented in three-space the possible existence of such an object is thereby

established. Not at all. For there is no imaginable condition of tridimensional mechanics in which an object may be said to have an objective existence similar to that represented by the mirrored cube.

But there are discrepancies in this representation which well might be considered. They have virtually the force of invalidating somewhat the conception which the analogy is designed to illustrate. For instance, in the case of the mirrored object placed at the point of intersection of the three mirrors built up at right angles to each other. Upon examination of such a construction it is found that the reflection of the object in the mirrors has not any perceptible connection with the object itself. And this, too, despite the fact that they are regarded as boundaries of the hypercube; especially is this true when it is noted that these reflections are called upon to play the part of real, palpable boundaries. If a fourth dimensional object were really like the mirror-representation it would be open to serious objections from all viewpoints. The replacement of any of the boundaries required in the analogy would necessarily mean the replacement of the hypercube itself. In other words, if the real cube be removed from its position at the intersection of the mirrors no reflection will be seen, and hence no boundaries and no hypercube. The analogy while admittedly possessing some slight value in the direction meant, is nevertheless valueless so far as a detailed representation is concerned. So the analogy falls down; but once again is the question raised as to whether the so-called fourth dimension can be established or proven at all upon purely mathematical grounds. It also emphasizes the necessity for

a clearer conception of the meaning of dimension and space.

The logical difficulties which beset the hyperspace conception are dwelt upon at length by JAMES H. HYSLOP. He says:³

"The supposition that there are three dimensions instead of one, or that there are only three dimensions is purely arbitrary, though convenient for certain practical purposes. Here the supposition expresses only differences of directions from an assumed point. Thus what would be said to lie in a plane in one relation would lie in the third dimension in another. There is nothing to determine absolutely what is the first, second, or third dimension. If the plane horizontal to the sensorium be called plane dimension, the plane vertical to it will be called solid, or the third dimension, but a change of position will change the names of these dimensions without involving the slightest qualitative change or difference in meaning.

"Moreover, we usually select three lines or planes terminating vertically at the same point, the lines connecting the three surfaces of a cube with the same point, as the representative of what is meant by three dimensions, and reduce all other lines and planes to these. But interesting facts are observable here. 1. If the vertical relation between two lines be necessary for defining a dimension, then all other lines than the specified ones are either not in any dimension at all, or they are outside the three given dimensions. This is denied by all parties, which only shows that a vertical relation to other lines is not necessary to the determination of a dimension. 2. If lines outside the three vertically intersecting lines still lie in dimension or

³Vide *Philosophical Review*, Vol. V, 1896, p. 352, et. seq.

are reducible to the other dimensions they may lie in more than one dimension at the same time which after all is a fact. This only shows that qualitatively all three dimensions are the same and that any line outside of another can only represent a dimension in the sense of *direction* from a given point or line, and we are entitled to assume as many dimensions as we please, all within three dimensions.

"This mode of treatment shows the source of the illusion about the 'fourth dimension.' The term in its generic import denotes commensurable quality and denotes only one such quality, so that the property supposed to determine non-Euclidean geometry must be qualitatively different from this, if its figures involve the necessary qualitative differentiation from Euclidean mathematics. But this would shut out the idea of 'dimension' as its basis which is contrary to the supposition. On the other hand, the term has a specific meaning which as different qualitatively from the generic includes a right to use the generic term to describe them differentially, but if used only quantitatively, that is, to express direction as it, in fact, does in these cases, involves the admission of the actual, not a supposititious, existence of a fourth dimension which again is contrary to the supposition of the non-Euclidean geometry. Stated briefly, dimension as commensurable quality makes the existence of the fourth dimension a transcendental problem, but as mere direction, an empirical problem. And the last conception satisfies all the requirements of the case because it conforms to the purely quantitative differences which exist between Euclidean and non-Euclidean geometry as the very language about 'surfaces,' 'triangles,' etc., in spite of the prefix 'pseudo,' necessarily implies."

Thus it would seem that those who have been most diligent in constructing the hyperspace conception have been the least careful of the logical difficulties which beset the elaboration of their assumptions. Yet it sometimes requires the illogical, the absurd and the aberrant to bring us to a right conception of the truth, and when we come to a comparison of the two, truth and absurdity, we are the more surprised that error could have gained so great foothold in face of so overwhelming evidences to the contrary.

The entire situation is, accordingly, aptly set forth by HYSLOP when he says, continuing:

"There are either a confusion of the abstract with the concrete or of quantitative with qualitative logic, . . . so that all discussion about a fourth dimension is simply an extended mass of equivocations turning upon the various meanings of the term 'dimension.' This when once discovered, either makes the controversy ridiculous or the claim for non-Euclidean properties a mere truism, but effectually explodes the logical claims for a new dimensional quality of space as a piece of mere jugglery in which the juggler is as badly deceived as his spectators. It simply forces mathematics to transcend its own functions as defined by its own advocates and to assume the prerogatives of metaphysics."

Shall we, therefore, assent to the imperialistic policy of mathematicians who would fain usurp the preserves of the metaphysician in order that they may exploit a superfoetated hypothesis? It is not believed that the harshness of HYSLOP's judgment in this respect is undeserved. It is, however, regretted that the

notions of mathematicians have been so inchoate as to justify this rather caustic, though appropriate criticism. For it does appear that the moment the mathematician deserts the province of his restricted sphere of motility and enters the realm of the transcendental, that moment he loses his way and becomes an inexperienced mariner on an uncharted sea.

It is interesting to note that CASSIUS JACKSON KEYSER,⁴ while recognizing the purely arbitrary character of the so-called dimensionality of space, nevertheless lends himself to the view that "if we think of the line as generating element we shall find that our space has four dimensions. That fact may be seen in various ways, as follows:

"A line is determined by any two of its points. Every line pierces every plane. By joining the points of one plane to all the points of another, all the lines of space are obtained. To determine a line, it is, then, enough to determine two of its points, one in the one plane and one in the other. For each of these determinations two data, as before explained, are necessary and sufficient. The position of the line is thus seen to depend upon four independent variables, and the four dimensionality of our space *in lines* is obvious."

Similarly he argues for the four dimensionality of space in spheres:

"We may view our space as an assemblage of its spheres. To distinguish a sphere from all other spheres, we need to know four and but four independent facts about it, as say, three that shall de-

⁴ Vide *Monist*, Vol. XVI, 1896, Mathematical Emancipations.

termine its center and one its size. Hence our space is four dimensional also in spheres. In circles, its dimensionality is six; in surfaces of second order (those that are pierced by a straight line in two points), nine; and so on ad infinitum."

The view taken by KEYSER is a typical one. It is the mathematical view and is characterized by a certain lack of restraint which is found to be peculiar to the whole scheme of thought relating to hyperspace. It is clear that the kind of space that will permit of such radical changes in its nature as to be at one time three dimensional, at another time four dimensional, then six, nine and even n -dimensional is not the kind of space in which the objective world is known to exist. Indeed, it is not the kind of space that really exists at all. In the first place, a line cannot generate perceptual space. Neither can a circle, nor a sphere nor any other geometrical construction. It is, therefore, not permissible, except mathematically, to view our space either as "an assemblage of its spheres," its circles or its surfaces; for obviously perceptual space is not a geometrical construction even though the intellect naturally finds inhering in it a sort of latent geometrism which is kosmical. For there is a wide difference between that kosmic order which is space and the finely elaborated abstraction which the geometer deceives himself into identifying with space. There is absolutely neither perceptible nor imperceptible means by which perceptual space in anywise can be affected by an act of will, ideation or movement. Just why mathematicians persist in vagarizing upon the generability of space by movement of lines, circles, planes, etc., is confessedly not easily understood espe-

cially when the natural outcome of such procedure is self-stultification. It is far better to recognize, as a guiding principle in all mathematical disquisitions respecting the nature of space that the possibilities found to inhere in an idealized construction cannot be objectified in kosmic, sensible space. The line of demarkation should be drawn once for all, and all metageometrical calculations and theories should be prefaced by the remark that: "if objective space were amenable to the peculiarities of an idealized construction such and such a result would be possible," or words to that effect. This mode of procedure would serve to clarify many if not all of the hyperspace conceptions for the non-mathematician as well as for the metageometricians themselves, especially those who are unwilling to recognize the utter impossibility of their constructions as applied to perceptual space. We should then cease to have the spectacle of otherwise well-demeanored men committing the error of trying to realize abstractions or abstractionizing realities. Herein is the crux of the whole matter, that mathematicians, rather than be content with realities as they find them in the kosmos, should seek to reduce them to abstractions, or, on the other hand, make their abstractions appear to be realities.

KEYSER proceeds to show how the concept of the generability of hyperspace may be conceived by beginning with the point, moving it in a direction without itself and generating a line; beginning with the line, treating it similarly, and generating a plane; taking the plane, moving it in a direction at right angles to itself and generating a cube; finally, using the cube as generating element and constructing

a four-space figure, the tesseract. Now, as a matter of fact, a point being intangible cannot be moved in any direction neither can a point-portion of sensible space be removed. Nevertheless, we quite agree with him when he asserts:

"Certainly there is naught of absurdity in supposing that *under suitable stimulation the human mind may, in the course of time, speedily develop a spatial intuition of four or more dimensions.*" (The italics in the above quotation are ours.)

Here we have a tacit implication that the notion which geometers have heretofore designated as "dimension" really is a matter of consciousness, of intuition, and therefore, determinable only by the limitations of consciousness and the deliveries of our intuitive cognitions. As a more detailed discussion of this phase of the subject shall be entered into when we come to a consideration of Chapter VI on "Consciousness as the Norm of Space Determinations" further comment is deferred until then.

Now, as it appears certain that what geometers are accustomed to call "dimension" is both relative and interchangeable in meaning—the one becoming the other according as it is viewed—the conclusion very naturally follows that neither constructive nor symbolic geometry is based upon dimension as commensurable quality. The real basis of the non-Euclidean geometry is dimension as direction. For whatever else may be said of the fourth dimension so-called it is certainly unthinkable, even to the meta-geometricians, when it is absolved from direction although no specific direction can be assigned to it. It

is agreed perhaps among all non-Euclidean publicists that the fourth dimension must lie in a "direction which is at right angles to all the three dimensions." But if they are asked how this direction may be ascertained or even imagined they are nonplused because they simply do not know. The difficulty in this connection seems to hinge about the question of identifying the conditions of the world of phantasy with those of the world of sense. There are distortions, ramifications, submersibles, duplex convolutions and other mathetic acrobatics which can be performed in the realm of the conceptual the execution of which could never be actualized in the objective world. Because these antics are possible in the premises of the mathematical imagination is scarce justification for the attempts at reproduction in an actualized and phenomenal universe.

One of the proudest boasts of the fourth dimensionist is that hyperspace offers the possibility of a new species of rotation, namely, *rotation about a plane*. He refers to the fact that in the so-called one-space, rotation can take place only about a point. For instance in Figure 7, the line *ab* represents a one-space in which rotation can take place only about one of the two points *a* and *b*. In Figure 8 which represents a two-space, rotation may take place about the line *ab* or the line *cd*, etc., or, in other words, the plane *abcd* can be rotated on the axial line *ab* in the direction of the third dimension. In tridimensional space only two kinds of rotation are possible, namely, rotation about a point and about a line. In the fourth dimension it is claimed that rotation can take place about a plane. For example, the cube in Figure 9, by

manipulation in the direction of the fourth dimension, can be made to rotate about the side $abgf$.

A very ingenious argument is used to show how rotation about a plane is thinkable and possible in hyperspace. But with this, as with the entire fabric of hyperspace speculations, dependence is placed almost entirely upon analogous and symbolic conceptions for evidence as to the consistency and rationality of the conclusions arrived at.

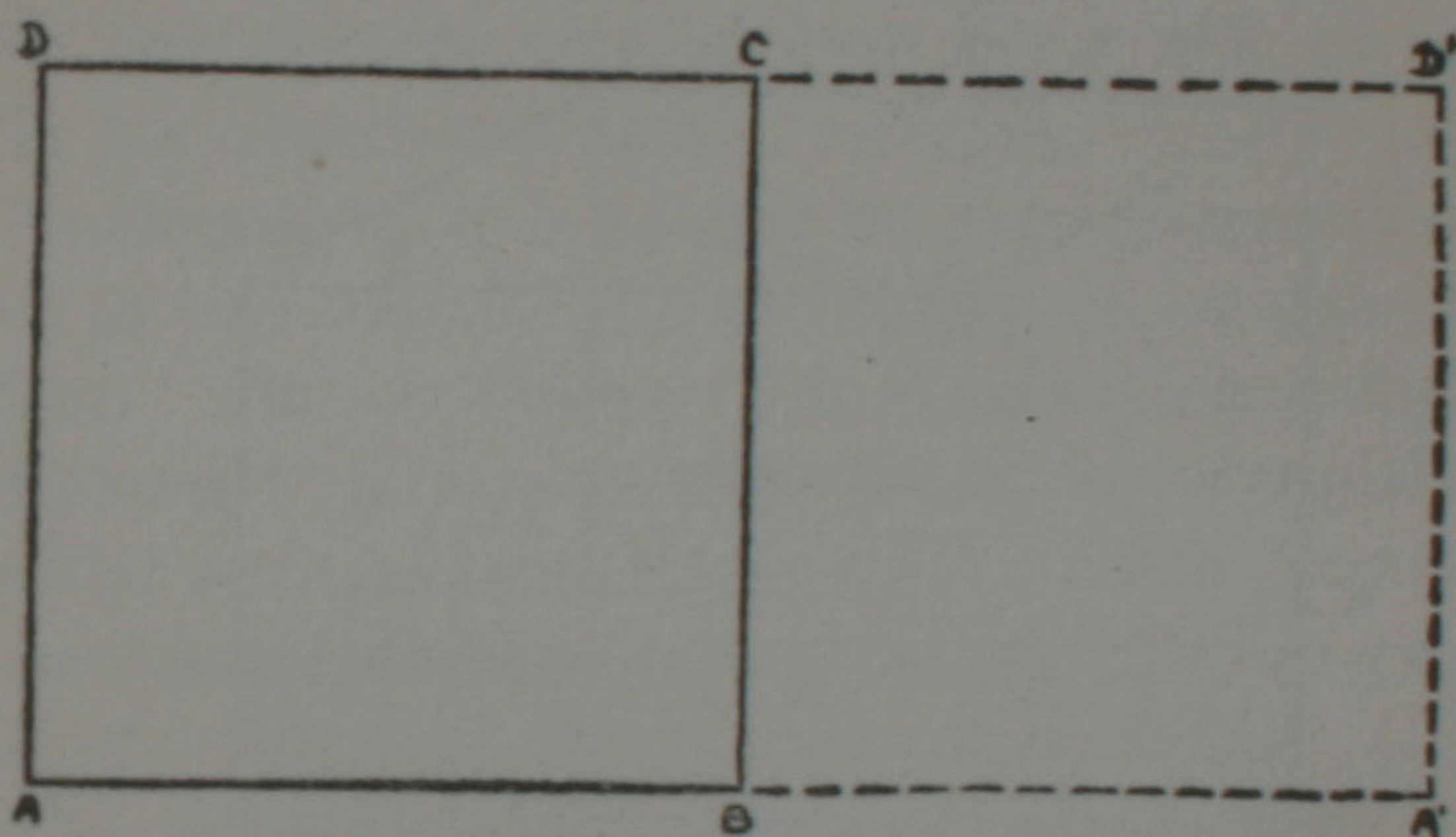


FIG. 13.

It is urged that inasmuch as the rotation about the line bc in Figure 13 would be incomprehensible or unimaginable to a plane being for the reason that such a rotation involves a movement of the plane into the third dimension, a dimension of which the plane being has no knowledge, in like manner rotation about a plane is also unimaginable or incomprehensible to a tridim or a three dimensional being. It is shown, however, that the plane being, by making use of the possibilities of an "assumed" tridimension, could arrive at a rational explanation of line rotation.

Figure 14 offers an illustration by means of which a two dimensional mathematician could demonstrate the possibility of line rotation. He is already acquainted with rotation about a point; for it is the only possible rotation that is observable in his two dimensional world. By conceiving of a line as an infinity or succession of points extending in the same direc-

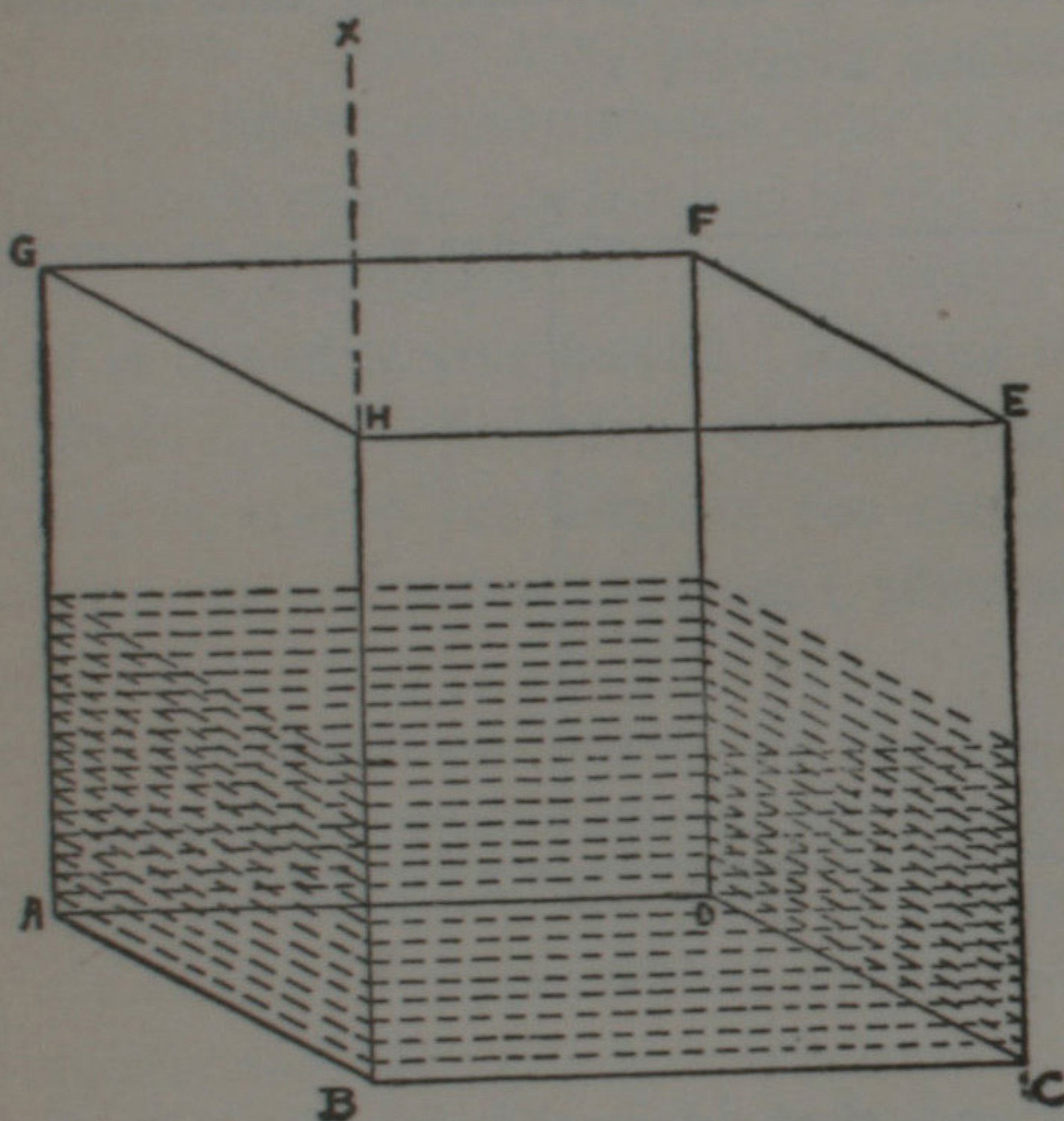


FIG. 14.

tion; by imagining the movement of his plane in the direction of the third dimension thereby generating a cube and at the same time assuming that the lines thus generated were merely successions of points extending in the same direction, he could demonstrate that the entire cube Figure 14, could be rotated about the line BHX used as an axis. For upon this hypothesis it would be arguable that a cube is a succession of planes piled one upon the other and limited only

by the length of the cube which would be extending in the, to him, unknown direction of the third dimension. He could very logically conclude that as a plane can rotate about a point, a succession of planes constituting a tridimensional cube, could also be conceived as rotating about a line which would be a succession of points under the condition of the hypothesis. His demonstration, therefore, that the cube, Figure 14, can be made to rotate around the line *BHX* would be thoroughly rational. He could thus prove line-rotation without even being able to actualize in his experience such a rotation.

Analogously, it is sought by metageometricians to prove in like manner the possibility of rotation about a plane. Thus in Figure 16 is shown a cube which has been rotated about one of its faces and changed from its initial position to the position it would occupy when the rotation had been completed or its final position attained.

The gist of the arguments put forward as a basis for plane-rotation is briefly stated thus: The face *cefg* is conceived as consisting of an infinity of lines. A cube, as in Figure 15, is imagined or assumed to be sected into an infinity of such lines, each line being the terminus of one of the planes which make up the cube. Each one of the constituting planes is thought of as rotating about its line-boundary which intersects the side of the cube. The process is continued indefinitely until the entire series of planes is rotated, one by one, around the series of lines which constitute the axial plane. Hence, in order that the cube, Figure 16, may change from its initial position to its final position each one of the infinitesimal planes of which

the cube is assumed to be composed must be made to rotate about each one of the infinitesimal lines of

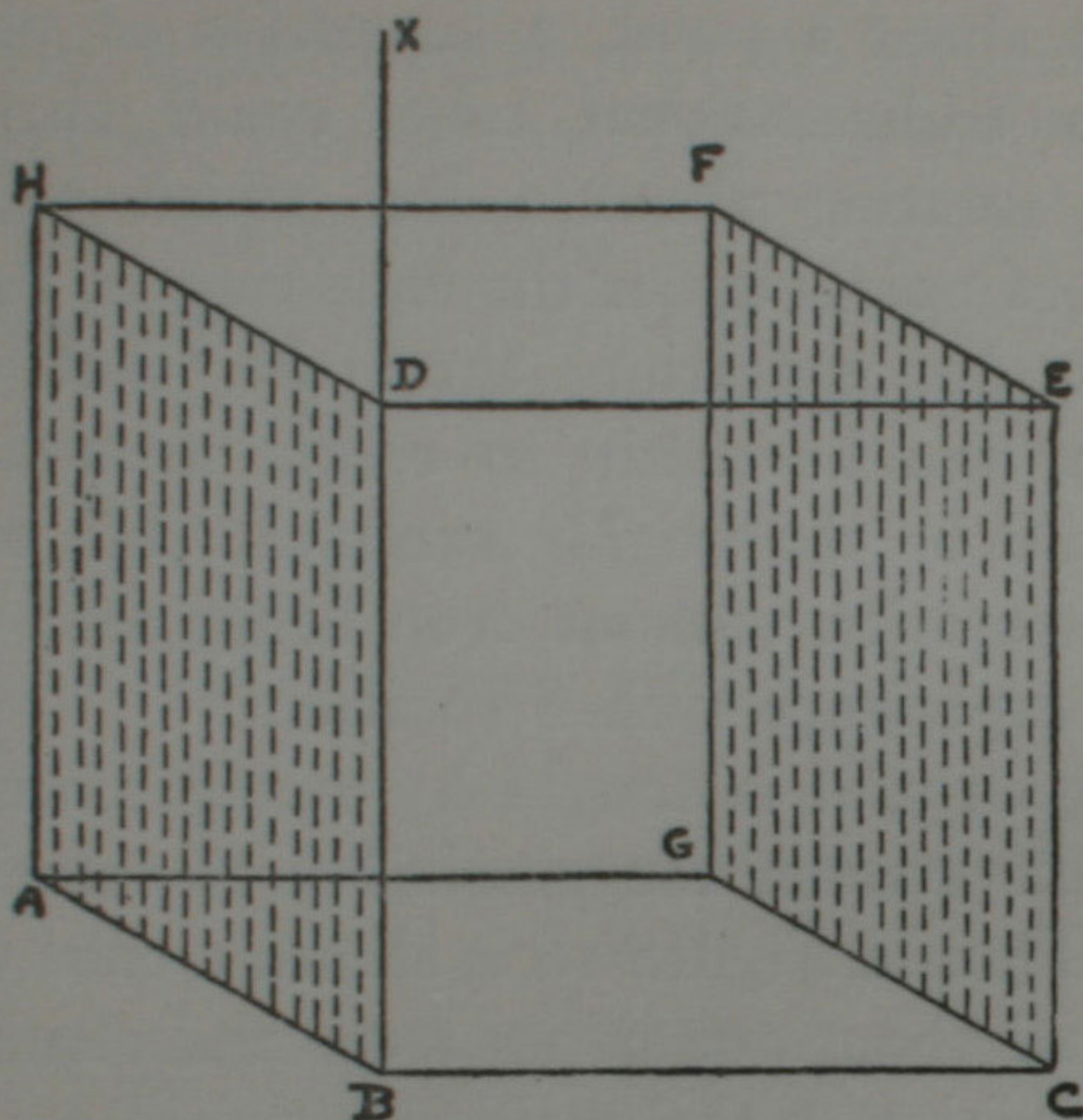


FIG. 15.

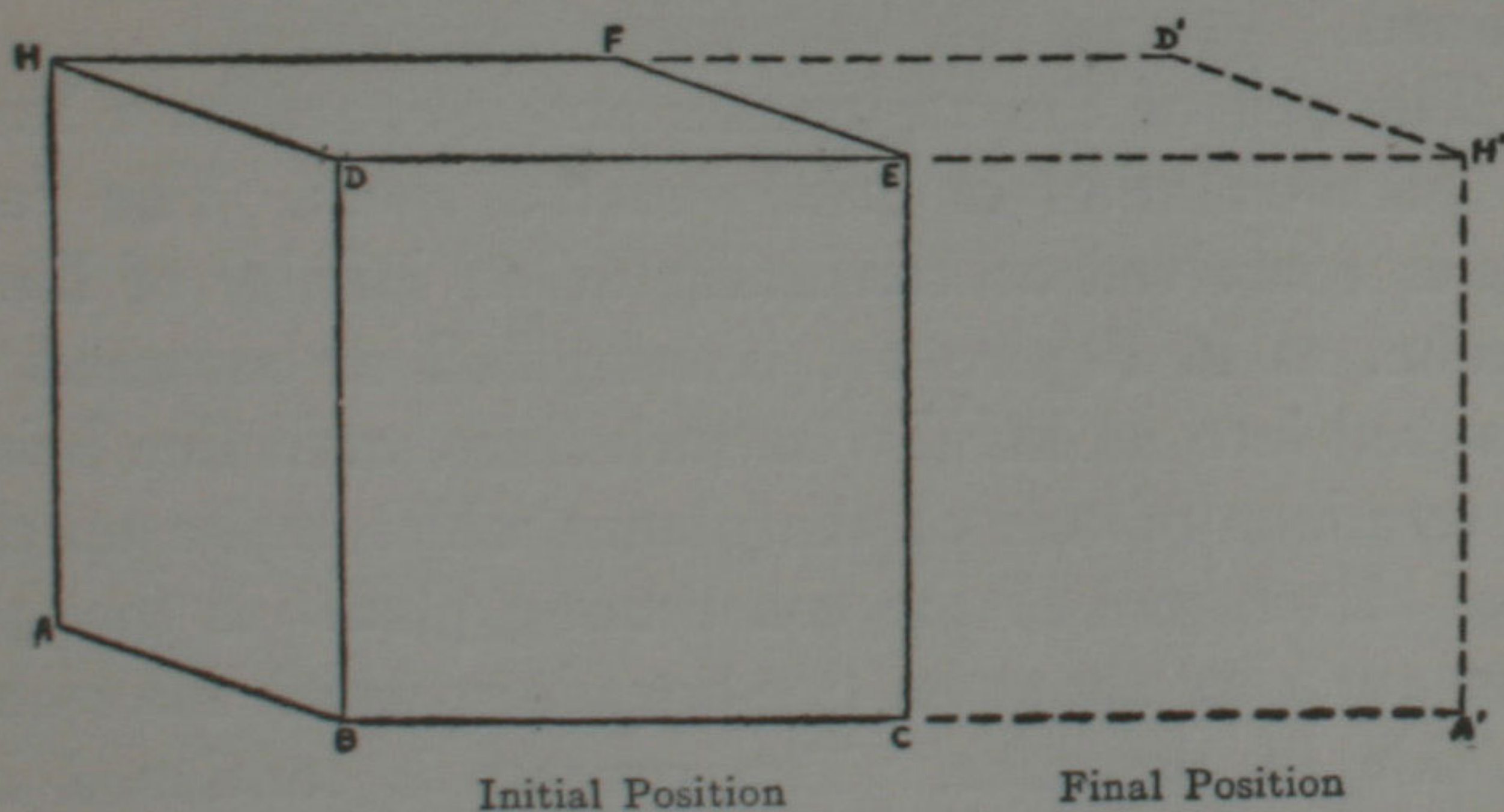


FIG. 16.—Plane Rotation

which the plane used as an axis is composed. In this way, it is shown that the entire cube has been made to rotate about its face, *cefg*. This concisely, is the

"quod erat demonstrandum" of the metageometrician who sets out to prove rotation about a plane. Thus it is made to appear that in order that tridimensional beings may be enabled to conceive of four-space rotation, as in Figures 15 and 16, in which the rotation must also be thought of as taking place in the direction of the fourth dimension, they must adopt the same tactics that a two dimensional being would use to understand some of the possibilities of the tridimensional world.

It is, of course, unwise to assume that because a thing can be shown to be possible by analogical reasoning its actuality is thereby established. This consideration cannot be too emphatically insisted upon; for many have been led into the error by relying too confidentially upon results based upon this line of argumentation. There is a vast difference between mentally doing what may be assumed to be possible, the hypothetical, and the doing of what is actually possible, the practical.

In the first place, plane-rotation in the actual universe is a structural impossibility. The very nature and constitution of material bodies will not admit of such contortion as that required by the rotation of a body, say a cube, about one of its faces. Let us examine some of the results of plane rotation. 1. The rotation must take place in the direction of the fourth dimension. Now, as it is utterly impossible for any one, whether layman or metageometrician, even to imagine or conceive, in any way that is practical, the direction of the fourth dimension it is also impossible for one to move or rotate a plane, surface, line or any other body in that direction. We are in the very